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On relative ranks of Lipschitz mappings

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Abstract

Let X be a countably infinite subset of $(0,\infty)$. In this paper we give a condition on the metric of X under which relative rank of all functions on X modulo Lipschitz mappings is equal to $\mathfrak{d}$.

A semigroup is an algebraic structure consisting of a set together with an associative binary operation. A monoid is a semigroup with an identity element. If S is a semigroup and A is a subset of S, by $\langle A\rangle$ we denote the subsemigroup generated by A. The cardinal number $\operatorname{rank}(S\colon A)=\min\{|B|\colon \langle A\cup B\rangle = S\}$ is called *the relative rank of S with respect to A*. In this paper we will consider only monoids of mappings with the composition of mappings as a monoid operation.

Let $S_n \subset \mathbb{N}^{\mathbb{N}}$ consists of all the functions f such that

$$f[\{n(i-1)+1,\dots,ni\}]\subset\{n(i-1)+1,\dots,ni\}$$

for all $i\in\mathbb{N}$.

Our first aim is to strengthen the following proposition. For proof see [1, Theorem 2.1].

Proposition 1. *We have*

$$\operatorname{rank}(\mathbb{N}^{\mathbb{N}}\colon S_2)=\mathfrak{c}.$$

We will prove our generalization in the same manner as in [1]. First we will formulate two lemmas.

Lemma 1. *Let $k\in\mathbb{N}$ and let $A_1, A_2, \dots$ be subsets of $\mathbb{N}$ each of cardinality not exceeding k. Then there is no uncontable family F of functions $f\colon\mathbb{N}\to\mathbb{N}$ satisfying $f(i)\in A_i$ for each $i\in\mathbb{N}$, such that for any two functions $f,g\in F$ the set $\{i\colon f(i)=g(i)\}$ is finite.*

Proof. The proof is an induction with respect to k. For $k=1$ this statement is clear. Assume that the statement is true for some k and sets $A_1, A_2, \dots$ are of cardinalities not exceeding k+1. Assume for contradiction that such an uncountable family F does exist. Let us take one function $f\in F$. Each

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function g from F has now different values than f from some point on, say n_g. Thus

$$F = \bigcup_{i=1}^{\infty} \{g \in F \colon n_g = i\},$$

whence there must be $i_0 \in \mathbb{N}$ such that the set

$$G = \{g \in F \colon n_g = i_0\}$$

is uncountable. Let us see that the family $\{g|_{\{i_0+1, i_0+2, \ldots\}} \colon g \in G\}$ satisfies now the induction hypothesis for k and thus it cannot be uncountable. □

To formulate the second one, we need one more definition. We say that a family of sets is *almost disjoint* if any two of its elements have finite intersection.

Lemma 2. *If $|A| = \aleph_0$ then there exists an almost disjoint family of infinite subsets of A of cardinality continuum.*

It is well known that such a family does exists; see, for example [2, Theorem 1.3]. Now we are ready to prove our generalization of Proposition 1.

Proposition 2. *We have*

$$\mathrm{rank}(\mathbb{N}^{\mathbb{N}} \colon \bigcup_{n \in \mathbb{N}} S_{2^n}) = \mathfrak{c}.$$

Proof. First see that $S_{2^i} \subset S_{2^j}$ for $i \leq j$. Now for contradiction assume that there exists $F \subset \mathbb{N}^{\mathbb{N}}$ such that $\langle S_2 \cup F \rangle = \mathbb{N}^{\mathbb{N}}$ and $|F| = \kappa < \mathfrak{c}$. Let $m, n \in \mathbb{N}$ and $f_1, \ldots, f_n$ be any functions from F. Let

$$F_{(f_1, \ldots, f_n; m)} = \{\varphi_{n+1} \circ f_n \circ \varphi_n \circ \ldots \circ f_1 \circ \varphi_1 \colon \varphi_{n+1}, \ldots, \varphi_1 \in S_{2^m}\}.$$

Now we are going to prove via induction with respect to n that the family $F_{(f_1, \ldots, f_n; m)}$ satisfies the hypothesis of Lemma 1. Let n = 0. Then the claim is satisfied with $k = 2^m$ because we deal only with $\varphi_1 \in S_{2^m}$. Assume now that the claim is satisfied for $n \in \mathbb{N} \cup \{0\}$. Thus

$$\varphi_n \circ f_{n-1} \circ \ldots \circ f_1 \circ \varphi_1(i) \in A'_i$$

for some $|A'_i| \leq k$. Then

$$f_n \circ \varphi_n \circ f_{n-1} \circ \ldots \circ f_1 \circ \varphi_1(i) \in f_n[A'_i]$$

and $|f_n[A'_i]| \leq |[A'_i]|$. By the properties of mappings from S_{2^m}

$$\varphi_{n+1}[f_n[A'_i]] \subset A_i,$$

where

$$|A_i| \leq 2^m |f_n[A'_i]| \leq 2^m |A'_i| = 2^m k$$

and the claim is proved. By Lemma 2 there exists an almost disjoint family Z of infinite subsets of $\mathbb{N}$. Let $A \in Z$ and let $f_A : \mathbb{N} \to A$ be any strictly increasing function. The set $\{i \colon f_A(i) = f_B(i)\}$ is finite for any sets $A, B \in Z$ and any $i \in \mathbb{N}$, for otherwise the sets A and B would have infinite intersection which would contradict almost disjointness of Z. Thus, by Lemma 1, only countably many functions f_A, $A \in Z$ may belong to $F_{(f_1,\dots,f_n;m)}$. As there are only κ families $F_{(f_1,\dots,f_n;m)}$ and $\mathfrak{c}$ functions f_A some functions f_A do not belong to any family $F_{(f_1,\dots,f_n;m)}$ and thus to $\langle F \cup \bigcup_{n\in\mathbb{N}} S_{2^n} \rangle = \mathbb{N}^{\mathbb{N}}$. $\square$

A dominating family is any subfamily A of $\mathbb{N}^{\mathbb{N}}$ such that for any $f \in \mathbb{N}^{\mathbb{N}}$ there exists $g \in A$ such that $g(n) \geq f(n)$ for all $n \in \mathbb{N}$. Let us define the cardinal

$$\mathfrak{d} = \min\{|A| \colon A \text{ is dominating family }\}.$$

Of course $\mathfrak{d} > \aleph_0$. Furthermore, it is known that both statements $\mathfrak{d} < \mathfrak{c}$ and $\mathfrak{d} = \mathfrak{c}$ are consistent with ZFC.

Let $\Delta = \{f \in \mathbb{N}^{\mathbb{N}} \colon \forall_{n\in\mathbb{N}} f(n) \leq n\}$.

Lemma 3. *We have*

$$\operatorname{rank}(\mathbb{N}^{\mathbb{N}} \colon \Delta) \leq \mathfrak{d}$$

Proof. Let F be a dominating family and $|F| = \mathfrak{d}$. For $f \in \mathbb{N}^{\mathbb{N}}$ let $g \in F$ be such that $f(n) \leq g(n)$ for all $n \in \mathbb{N}$. We can choose $\varphi \in \Delta$ which satisies $\varphi(g(n)) = f(n)$ for all $n \in \mathbb{N}$. Then $\varphi \circ g = f$ and this ends the proof. $\square$

Let $a_i > 0$ for $i \in \mathbb{N}$. We will consider space $X = \{\sum_{i=1}^{n} a_i \colon i \in \mathbb{N}\}$ with discrete topology. It is trivial that $\iota \colon X \to \mathbb{N}$ defined by $\iota\left(\sum_{i=1}^{n} a_i\right) = n$ is an order isomorphism.

Let $\mathcal{L}(\mathbb{N}; M)$ denote the family of functions from $\mathbb{N}^{\mathbb{N}}$ satisfying the Lipschitz condition with constant M. Before proving the main result of the paper we need a simple lemma which provides a usefull tool to create Lipschitz functions.

Lemma 4. *Let $f \colon X \to X$ and $y_i = \sum_{n=1}^{i} a_i$ for all $i \in \mathbb{N}$. If M is such that $|f(y_{i+1}) - f(y_i)| \leq L|y_{i+1} - y_i|$ for all $i \in \mathbb{N}$ then $f \in \mathcal{L}(X, M)$.*

Proof. Let i, j be positive integers and assume that $i > j$. We see that

$$|f(y_i) - f(y_j)| = |\sum_{k=j}^{i-1} f(y_{k+1}) - f(y_k)| \leq \sum_{k=j}^{i-1} |f(y_{k+1}) - f(y_k)|$$

$$\leq \sum_{k=j}^{i-1} L|y_{k+1} - y_k| = L|\sum_{k=j}^{i-1} y_{k+1} - y_k| = L|y_i - y_j|.$$

□

As an immediate consequence of this lemma we have the following.

Corollary 1. *Let $(x_i)_{i\in\mathbb{N}}$ be strictly increasing sequence of points of X and $f\colon X \to X$ be a function. If $f|_{\{x_i,\ldots,x_{i+1}\}}$ satisfy Lipschitz condition with constant M for all $i \in \mathbb{N}$ then $f \in \mathcal{L}(X, M)$.*

Theorem 1. *Let $M > 0$. If there exist functions $f, g \in \mathcal{L}(X, M)$ such that*

$$\forall_{n\in\mathbb{N}} \exists_{x\in X} |\iota(f(x)) - \iota(f(y))| \geq n \tag{1}$$

$$\forall_{n\in\mathbb{N}} \exists_{x\in X,\ \iota(x)\geq n} f(x) = g(x) \tag{2}$$

then $\operatorname{rank}(\mathcal{C}(X), \mathcal{L}(X, M)) = \operatorname{rank}(\mathcal{C}(X), \mathcal{L}(X)) = \mathfrak{d}$

Proof. It is known that if $\sum_{i\in\mathbb{N}} a_i = \infty$ then $\operatorname{rank}(\mathcal{C}(X), \mathcal{L}(X)) \geq \mathfrak{d}$ (see [3, Theorem 3.1]). Hence, it is enough to show that $\operatorname{rank}(\mathcal{C}(X), \mathcal{L}(X)) \leq \mathfrak{d}$.

Let f and g have properties (1) and (2). First we will prove that

$$\forall_{n\in\mathbb{N}} \exists_{x\in X,\ \iota(x)\geq n} |\iota(f(x)) - \iota(g(x))| \geq n \tag{3}$$

Let $n \in \mathbb{N}$. Take $k = \max\{|\iota(f(x)) - \iota(g(x))|\colon x \in X \wedge \iota(x) \leq n\} + n$. By (1) there exists $x \in X$ such that

$$|\iota(f(x)) - \iota(g(x))| \geq k \geq n$$

from definition of k we obtain that $\iota(x) > n$.

Let us now define two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of points of X. We do it inductively. Let x_1, y_1 be such that $f(y_1) = g(y_1)$, $x_1 \geq y_1$, $|\iota(f(x_1)) - \iota(g(x_1))| \geq 1$. Once $x_1, y_1, \ldots, x_{n-1}, y_{n-1}$ have been defined we choose x_n, y_n such that $f(y_n) = g(y_n)$, $x_n \geq y_n$, $|\iota(f(x_n)) - \iota(g(x_n))| \geq n$. Now we can define sets

$$A_n = \iota^{-1}(\{\min\{\iota(f(x_n)), \iota(g(x_n))\}, \ldots, \max\{\iota(f(x_n)), \iota(g(x_n))\}\})$$

for all $n \in \mathbb{N}$. Clearly, $|A_n| \geq n$.

Now fix $n \in \mathbb{N}$ and $a \in A_n$. We will show that there exists function $h_n\colon \{y_n, \dots, y_{n+1}\} \to X$ satisfying Lipschitz condition with constant M such that $f(x_n) = a$.

If $a = f(x_n)$ or $a = g(x_n)$ then this is trivial. In other case assume without loss of generality that $g(x_n) < a < f(x_n)$. Moreover let us assume $f(y_n) \leq a$ and $f(y_{n+1}) \geq a$. Cases where $f(y_n) \geq a$ or $f(y_{n+1}) \leq a$ are similar. Let $a_1 = \max\{x \in X\colon x < x_n \wedge f(x) < a\}$ and $a_2 = \min\{x \in X\colon x > x_n \wedge g(x) < a\}$. We define h_n as follows:

$$h_n(x) = \begin{cases} f(x) & \text{if } x \leq a_1, \\ a & \text{if } a_1 < x < a_2, \\ g(x) & \text{if } a_2 \leq x. \end{cases}$$

Of course $h_n(x_n) = a$. Now it suffices to show that $|h_n(\min\{x\colon x > a_1\}) - h_n(a_1)| \leq M|\min\{x\colon x > a_1\} - a_1|$ and $|h_n(\max\{x\colon x < a_1\}) - h_n(a_1)| \leq M|\max\{x\colon x < a_1\} - a_1|$. We will show only the first inequality. Prove of the second is similar.

$$|h_n(\min\{x\colon x > a_1\}) - h_n(a_1)| = |a - f(a_1)| = a - f(a_1)$$

$$\leq f(\min\{x\colon x > a_1\}) - f(a_1) \leq M|\min\{x\colon x > a_1\} - a_1|$$

This and Corollary 2 gives us the following statement:
For any sequence $(b_n)_{n\in\mathbb{N}}$ such that $b_n \in A_n$ for all $n \in \mathbb{N}$ exists function $h\colon X \to X$ satisfying Lipschitz condition with constant M such that $f(x_n) = b_n$ for all $n \in \mathbb{N}$.

Without loss of generality we may assume $|A_n| = n$. Let us choose any function $\psi\colon X \to X$ such that $|\psi[A_i]| = i$ and $\psi[A_i] \subset \psi[A_{i+1}]$ for all $i \in \mathbb{N}$. Making use of this and Lemma 3 we obtain that $\operatorname{rank}(\mathcal{C}(X), \mathcal{L}(X, M)) \leq \mathfrak{d}$. $\square$

Now we present an example of a space which satisfies the assumption of Theorem 1 above, but does not satisfy the assumptions of [3, Theorem 5.4].

Example 1. *Let* $a_i = \frac{1}{l+m}$ *for* $l, m \in \mathbb{N}$ *such that* $i = l10^m$, $10 \nmid l$. $\operatorname{rank}(\mathcal{C}(X), \mathcal{L}(X, 1)) = \mathfrak{d}$.

Proof. For any natural numbers l, s, t such that $l > s > t$ we have $x_{10^{l+s}} = \frac{1}{s+l} \geq \frac{1}{2l}$ and $x_{10^{l+s}+t} \leq \frac{1}{10^l}$. Furthermore, $\lim_{l\to\infty} \frac{10^l}{2l} = \infty$. Hence we can choose sequence $(n_i)_i \in \mathbb{N}$ such that $n_{i+1} - n_i > i$ and $\sum_{k=1}^{i} a_{n_i+k} \leq a_{n_i}$. Let $x_n = \sum_{i=1}^{n} a_i$. Now let us define functions

$$g(x_n) = \begin{cases} x_{n_i+i} & \text{if } n = n_i + t \text{ for some } t \in \{0, \dots, i\}, \\ x_n & \text{otherwise} \end{cases}$$

and f equal identity on X.

An easy computation shows that f and g satisfy conditions of Theorem 1. $\square$

It is easy to verify that this space does not hold the assumption of [3, Theorem 5.4].

Theorem 2. *If* $\sum_{i\in\mathbb{N}} a_i = \infty$ *and* $a_i \geq a_{i+1}$ *for all* $i \in \mathbb{N}$ *then* $\operatorname{rank}(\mathcal{C}(X), \mathcal{L}(X, M)) = \mathfrak{d}$ *for every* $M \geq 2$.

Proof. Let $x_n = \sum_{i=1}^{n} a_i$ for all $n \in \mathbb{N}$. Now we can define f and g as follows.

$$f(x_n) = \begin{cases} x_{2n-(k-1)k} & \text{for } k \in \mathbb{N} \text{ and } (k-1)k \leq n < k^2, \\ x_{k^2+k} & \text{for } k \in \mathbb{N} \text{ and } k^2 \leq n < k(k+1). \end{cases}$$

$$g(x_n) = x_n$$

for all $n \in \mathbb{N}$.

We claim that $f \in \mathcal{L}(X, 2)$. Indeed, let $n \in \mathbb{N}$ if there exists $k \in \mathbb{N}$ such that $(k-1)k \leq n < k^2$ then

$$|f(x_{n+1}) - f(x_n)| = |x_{2n+2-(k-1)k} - x_{2n-(k-1)k}| = |a_{2n+2-(k-1)k} + a_{2n+1-(k-1)k}|$$

$$\leq |a_{2n+2-n} + a_{2n+1-n}| \leq 2|a_{n+1}| = 2|x_{n+1} - x_n|.$$

If such $k \in \mathbb{N}$ does not exist then $l^2 \leq n < l(l+1)$ for some $l \in \mathbb{N}$. But then

$$|f(x_{n+1}) - f(x_n)| = 0 \leq 2|x_{n+1} - x_n|.$$

By Theorem 1 it suffices to show (1) and (2) to finish this proof. However, for all $j \in \mathbb{N}$ we have

$$|\iota(f(x_{j^2})) - \iota(g(x_{j^2}))| = |j^2 + j - j^2| = j$$

and

$$|f(x_{j^2-j}) - g(x_{j^2-j})| = |x_{j^2-j} - x_{j^2-j}| = 0.$$

This ends the proof. $\square$

References

[1] J. D. Mitchell & M. Morayne & Y. Péresse & M. R. Quick, Generating transformation semigroups using endomorphisms of preorders, graphs, and tolerances, *Annals of Pure and Applied Logic* 161 (2010), 1471-1485.

[2] K. Kunen, *Set Theory, in: Studies in Logic*, North Holland, Amsterdam, New York, Oxford, Tokyo, 1980.

[3] J. Cichoń & J. D. Mitchell & M. Morayne & Y. Péresse, Relative ranks of Lipschitz mappings on countable discrete metric spaces, *Topology and its Applications* 158 (2011), 412–423.

[4] S. Banach, Sur un theorème de M. Sierpiński, *Fund. Math.* 25 (1935) 5–6.

[5] W. Sierpiński, Sur les suites infinies de fonctions définies dans les ensembles quelconques, *Fund. Math.* 24 (1935) 209–212.