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Behavior of the Kobayashi distance near boundary of pseudoconvex Reinhardt domains

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BEHAVIOR OF THE KOBAYASHI DISTANCE NEAR BOUNDARY OF PSEUDOCONVEX REINHARDT DOMAINS

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Abstract. We prove that the Kobayashi distance near boundary of a pseudoconvex Reinhardt domain D increases asymptotically at most like $-\log d_D + C$. Moreover, for boundary points from $\mathrm{int}\overline{D}$ the growth does not exceed $\frac{1}{2}\log(-\log d_D) + C$. The lower estimate by $-\frac{1}{2}\log d_D + C$ is obtained under additional assumptions of $\mathcal{C}^1$-smoothness of a domain and a non-tangential convergence.

1. Introduction and results

The problem of a boundary behavior of the Kobayashi (pseudo)distance in pseudoconvex Reinhardt domains is connected with studying their Kobayashi completeness. The qualitative condition for the k-completeness of a bounded domain D is

$$k_D(z_0, z) \to \infty \text{ as } z \to \partial D.$$

The main fact is that if a pseudoconvex Reinhardt domain D is hyperbolic then it is k-complete. At first Pflug [7] proved it for bounded complete domains. A second step was done by Fu for bounded domains in [2]. The general case was finally solved by Zwonek in [8].

Hence it is natural to ask about a quantitative behavior of the function $k_D(z_0, \cdot)$. Forstnerič and Rosay estimated it from below on bounded strongly pseudoconvex domains. Namely, it was proved in [1] that

$$k_D(z_1, z_2) \geq -\frac{1}{2}\log d_D(z_1) - \frac{1}{2}\log d_D(z_2) + C$$

for z_j near two distinct points $\zeta_j \in \partial D$, $j = 1, 2$. In the same paper the authors showed the opposite estimate for $\mathcal{C}^{1+\varepsilon}$-smooth domains with z_1, z_2 near $\zeta_0 \in \partial D$. This estimate in the bounded case follows from the inequality for the Lempert function of bounded $\mathcal{C}^{1+\varepsilon}$-smooth domains obtained by Nikolov, Pflug and Thomas

$$\widetilde{k}_D(z_1, z_2) \leq -\frac{1}{2}\log d_D(z_1) - \frac{1}{2}\log d_D(z_2) + C, \; z_1, z_2 \in D$$

in [6]. It was also proved that the above estimate fails in the $\mathcal{C}^1$-smooth case. The other general version of an upper estimate, for $\mathcal{C}^2$-smooth domains, can be found in [3]. The case of bounded convex domains was investigated by Mercer in [5]. For such domains we have

$$-\frac{1}{2}\log d_D(z) + C' \leq k_D(z_0, z) \leq -\alpha \log d_D(z) + C$$

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with $\alpha > \frac{1}{2}$ and z close to $\zeta_0 \in \partial D$ (the constant α can not be replaced with $\frac{1}{2}$). An example

$$D_\beta := \{(z, w) \in \mathbb{C}^2 : |z|^\beta + |w|^\beta < 1\},\ 0 < \beta < 1$$

shows that the lower estimate by $-\alpha \log d_D(z) + C$, where $\alpha > 0$ — a constant independent on a domain, is not true for complete pseudoconvex Reinhardt domains. Easy calculations lead to

$$k_{D_\beta}((0,0),(z,0)) \le -\frac{\beta}{2} \log d_{D_\beta}(z,0) + C$$

if $0 < z < 1$ and $(z, 0)$ tends to $(1, 0)$.

In the paper we prove the following theorems.

Theorem 1. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Fix $z_0 \in D$ and $\zeta_0 \in \partial D$. Then for some constant C the inequality*

$$k_D(z_0, z) \le -\log d_D(z) + C$$

holds if $z \in D$ tends to ζ_0. Additionally, for $\zeta_0 \in \mathbb{C}^n_*$ the estimate can be improved to

$$k_D(z_0, z) \le -\frac{1}{2} \log d_D(z) + C'$$

where C' is a constant.

Theorem 2. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Fix $z_0 \in D$ and $\zeta_0 \in \partial D \cap \operatorname{int}\overline{D}$. Then for some constant C the inequality*

$$k_D(z_0, z) \le \frac{1}{2} \log(-\log d_D(z)) + C$$

holds if $z \in D$ tends to ζ_0.

Theorem 3. *Let $D \subset \mathbb{C}^n$ be a $\mathcal{C}^1$-smooth pseudoconvex Reinhardt domain. Fix $z_0 \in D$ and $\zeta_0 \in \partial D$. Then for some constant C the inequality*

$$k_D(z_0, z) \ge -\frac{1}{2} \log d_D(z) + C$$

holds if $z \in D$ tends non-tangentially to ζ_0.

2. Notations and definitions

By D we denote a domain in $\mathbb{C}^n$. The *Kobayashi (pseudo)distance* is defined as

$k_D(w, z) := \sup\{d_D(w, z) : (d_D)$ is a family of holomorphically invariant pseudodistances less than or equal to $\widetilde{k}_D\}$,

where

$$\widetilde{k}_D(w, z) := \inf\{p(\lambda, \mu) : \lambda, \mu \in \mathbb{D} \text{ and } \exists f \in \mathcal{O}(\mathbb{D}, D) :\ f(\lambda) = w,\ f(\mu) = z\}$$

is the *Lempert function* of D, $\mathbb{D} \subset \mathbb{C}$ — the unit disc and p — the Poincaré distance on $\mathbb{D}$. For general properties of functions k_D one can see [3].

Denote z_j as the j-th coordinate of point $z \in \mathbb{C}^n$. A domain D is called a *Reinhardt domain* if $(\lambda_1 z_1, \dots, \lambda_n z_n) \in D$ for all numbers $\lambda_1, \dots, \lambda_n \in \partial\mathbb{D}$ and points $z \in D$. A Reinhardt domain D is *complete in j-th direction* if

$$(\{1\}^{j-1} \times \overline{\mathbb{D}} \times \{1\}^{n-j}) \cdot D \subset D,$$

where $A \cdot B := \{(a_1b_1, \ldots, a_nb_n) : a \in A,\, b \in B\}$. Define subspaces $V_j^n := \{z \in \mathbb{C}^n : z_j = 0\}$ for $j = 1, \ldots, n$. If a Reinhardt domain D is complete in the j-th direction for all j such that $D \cap V_j^n \neq \emptyset$ then D is called *relatively complete*.

Let us denote $A_* := A \setminus \{0\}$ for a set $A \subset \mathbb{C}$ and $\mathbb{C}_*^n := (\mathbb{C}_*)^n$. By $d_D(z)$ denote a distance of a point $z \in D$ to ∂D (here, exceptionally, D can be a domain in $\mathbb{R}^n$) and by $\zeta_D(z)$ — one of points admitting a distance of a point $z \in D$ to ∂D.

We will use the following main branch of the power $z^\alpha := e^{\alpha \log z} = e^{\alpha(\log|z| + i \operatorname{Arg} z)}$, where the main argument $\operatorname{Arg} z \in (-\pi, \pi]$. Define $z^\alpha := z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}$, $|z|^\alpha := |z_1|^{\alpha_1} \cdot \ldots \cdot |z_n|^{\alpha_n}$ for $z \in \mathbb{C}_*^n$ and $\alpha \in \mathbb{R}^n$. Moreover, let $|z| := (|z_1|, \ldots, |z_n|)$ for $z \in \mathbb{C}^n$, $\log|z| := (\log|z_1|, \ldots, \log|z_n|)$ for $z \in \mathbb{C}_*^n$ and $\log D := \{\log|z| : z \in D \cap \mathbb{C}_*^n\}$ — a *logarithmic image* of D. We use C to denote constants not necessarily the same in different places. We also need notations $f \lesssim g$ if there exists $C > 0$ such that $f \le Cg$; $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

We call D a $\mathcal{C}^k$*-smooth* domain if for any point $\zeta_0 \in \partial D$ there exist its open neighbourhood $U \subset \mathbb{C}^n$ and a $\mathcal{C}^k$-smooth function $\rho : U \longrightarrow \mathbb{R}$ such that

(1) $U \cap D = \{z \in U : \rho(z) < 0\}$;
(2) $U \setminus \overline{D} = \{z \in U : \rho(z) > 0\} \neq \emptyset$;
(3) $\nabla\rho := \left(\frac{\partial\rho}{\partial\bar{z}_1}, \ldots, \frac{\partial\rho}{\partial\bar{z}_n}\right) \neq 0$ on U.

The function ρ is called a *local defining function* for D at the point ζ_0.

For a $\mathcal{C}^1$-smooth domain D we define a *normal vector* to ∂D at a point $\zeta_0 \in \partial D$ as

$$\nu_D(\zeta_0) := \frac{\nabla\rho(\zeta_0)}{\|\nabla\rho(\zeta_0)\|},$$

where ρ is a local defining function for D at ζ_0. Clearly

$$z = \zeta_D(z) - d_D(z)\nu_D(\zeta_D(z))$$

for $z \in D$ and

$$\lim_{D \ni z \to \zeta_0} \nu_D(\zeta_D(z)) = \nu_D(\zeta_0)$$

for every choice of $\zeta_D(z)$. For the transparent notation we shorten the symbol $\nu_D(\zeta_D(z))$ to $\nu_D(z)$.

To define a non-tangential convergence we need a concept of a *cone* with a vertex $x_0 \in \mathbb{R}^n$, a semi-axis $\nu \in (\mathbb{R}^n)_*$ and an angle $\alpha \in (0, \frac{\pi}{2})$. It is a set of $x \in \mathbb{R}^n \setminus \{x_0\}$ such that an angle between vectors ν and $x - x_0$ does not exceed α. Let D be a $\mathcal{C}^1$-smooth domain and $\zeta_0 \in \partial D$. We say that $z \in D$ tends *non-tangentially* to ζ_0 if there exist a cone $\mathcal{A} \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ with a vertex ζ_0, a semi-axis $-\nu_D(\zeta_0)$ and an angle $\alpha \in (0, \frac{\pi}{2})$ and an open neighbourhood $U \subset \mathbb{C}^n$ of ζ_0 such that $U \cap \mathcal{A} \subset D$ and z tends to ζ_0 in $U \cap \mathcal{A}$.

We say that a Reinhardt domain D satisfies the *Fu condition* if for any $j \in \{1, \ldots, n\}$ the following implication holds

$$\partial D \cap V_j^n \neq \emptyset \Longrightarrow D \cap V_j^n \neq \emptyset.$$

The following well-known properties of pseudoconvex Reinhardt domains will be used in the paper (see e.g. [4]).

Fact 1. *A Reinhardt domain D is pseudoconvex if and only if $\log D$ is convex and D relatively complete.*

Fact 2. *A $\mathcal{C}^1$-smooth Reinhardt domain satisfies the Fu condition.*

3. Proofs

Proof of Theorem 1. We proceed as follows. The first step is to simplify the general case to 'real' coordinates, further we consider some parallelepipeds contained in the given domain and use the decreasing property of the Kobayashi distance. Finally, we explicitly calculate and estimate it in other domains — cartesian products of a belt and annuli in $\mathbb{C}$. To improve the estimate for a boundary point with all non-zero coordinates we use similar methods, but with intervals instead of parallelepipeds.

Using some biholomorphism of the form

$$w \ni \mathbb{C}^n \longmapsto (a_1 w_1, \ldots, a_n w_n) \in \mathbb{C}^n,\ a \in \mathbb{C}^n_*$$

and the triangle inequality for k_D, we can assume that $z_0 = (1, \ldots, 1)$ and $|\zeta_{0j}| \neq 1$ for $j = 1, \ldots, n$. Notice that the proof can be reduced to $z \in D \cap \mathbb{C}^n_*$ near ζ_0 and next to the case

$$z \in D \cap (0, \infty)^n \text{ near } \zeta_0 \in \partial D \cap ([0, \infty) \setminus \{1\})^n.$$

Indeed, the first reduction follows from the continuity of k_D and the triangle inequality for k_D. Now, if $z \to \zeta_0$ then $|z| \to |\zeta_0| \in \partial D$ and

$$k_D(z_0, z) = k_D(\widetilde{z_0}, |z|),$$

where

$$\widetilde{z_0} := \left(\frac{|z_1|}{z_1} z_{01}, \ldots, \frac{|z_n|}{z_n} z_{0n}\right) \in T := \{(\lambda_1 z_{01}, \ldots, \lambda_n z_{0n}) : \lambda_1, \ldots, \lambda_n \in \partial\mathbb{D}\}.$$

The continuity of k_D gives

$$\max_{T \times T} k_D =: C < \infty$$

and therefore

$$k_D(\widetilde{z_0}, |z|) \leq k_D(\widetilde{z_0}, z_0) + k_D(z_0, |z|) \leq k_D(z_0, |z|) + C.$$

The property $d_D(|z|) = d_D(z)$ finishes this reduction. In what follows, we assume that points $z \in D \cap (0, \infty)^n$ are sufficiently close to $\zeta_0 \in \partial D \cap ([0, \infty) \setminus \{1\})^n$.

Observe that

$$d_{\log D}(\log z) \geq \varepsilon d_D(z)$$

for some $\varepsilon > 0$. Indeed, for $u \in \mathbb{R}^n$, $\|u\| < 1$ and $0 \leq t \leq \varepsilon d_D(z)$, where

$$\varepsilon := \frac{1}{3(\|\zeta_0\| + 1)},$$

we have

$$\log z + tu \in \log D$$

if and only if

$$\left(z_1 e^{tu_1}, \ldots, z_n e^{tu_n}\right) \in D$$

but this property follows from

$$\left\|(z_j e^{tu_j})_{j=1}^n - z\right\| \leq \sqrt{\sum_{j=1}^n z_j^2 (2t)^2} \leq 2t(\|\zeta_0\| + 1) < d_D(z).$$

Moreover, for $\zeta_0 = 0$ a similar consideration leads to

$$d_{\log D}(\log z) \geq \varepsilon' \frac{d_D(z)}{\|z\|}$$

for sufficiently small $\varepsilon' > 0$. Indeed, there exists $\varepsilon' \in \left(0, \frac{1}{2}\right)$ such that the inequalities

$$\left|e^{tu_j} - 1\right| \le 2t,\ j = 1, \dots, n$$

hold for $0 \le t \le \varepsilon'$. Hence for $0 \le t \le \varepsilon' \frac{d_D(z)}{\|z\|}$ we have

$$\left\|(z_j e^{tu_j})_{j=1}^n - z\right\| \le \sqrt{\sum_{j=1}^n z_j^2 (2t)^2} \le 2\varepsilon' \frac{d_D(z)}{\|z\|} \|z\| < d_D(z).$$

Denote

$$\widetilde{d}_D(z) := \begin{cases} \varepsilon d_D(z),\ \zeta_0 \neq 0 \\ \varepsilon'' \frac{d_D(z)}{\|z\|},\ \zeta_0 = 0, \end{cases}$$

where $\varepsilon'' := \varepsilon' d_{\log D}(0)$. Let us define

$$m_z := \min\{0, \log z_1\},\ M_z := \max\{0, \log z_1\}$$

and consider the set

$$D_z := \{w \in \mathbb{C}^n : m_z - \widetilde{d}_D(z) < \log|w_1| < M_z + \widetilde{d}_D(z),$$
$$\frac{\log z_j}{\log z_1} \log|w_1| - \widetilde{d}_D(z) < \log|w_j| < \frac{\log z_j}{\log z_1} \log|w_1| + \widetilde{d}_D(z),\ j = 2, \dots, n\}.$$

Then $\log D_z$ is a domain in $\mathbb{R}^n$ containing points 0 and $\log z$ but contained in a convex domain $\log D$. Define also

$$G_z := \{v \in \mathbb{C}^n : m_z - \widetilde{d}_D(z) < \operatorname{Re} v_1 < M_z + \widetilde{d}_D(z),$$
$$-\widetilde{d}_D(z) < \log|v_j| < \widetilde{d}_D(z),\ j = 2, \dots, n\}.$$

Hence the holomorphic map

$$f_z(v) := \left(e^{v_1}, v_2 e^{v_1 \frac{\log z_2}{\log z_1}}, \dots, v_n e^{v_1 \frac{\log z_n}{\log z_1}}\right),\ v \in G_z$$

has values in D_z. Moreover

$$w = f_z\left(\log w_1, \frac{w_2}{w_1^{\frac{\log z_2}{\log z_1}}}, \dots, \frac{w_n}{w_1^{\frac{\log z_n}{\log z_1}}}\right) \text{ for } w \in D_z.$$

Therefore

$$k_D(z_0, z) \le k_{D_z}(z_0, z) = k_{D_z}\left(f_z(0, 1, \dots, 1), f_z\left(\log z_1, \frac{z_2}{z_1^{\frac{\log z_2}{\log z_1}}}, \dots, \frac{z_n}{z_1^{\frac{\log z_n}{\log z_1}}}\right)\right) =$$
$$= k_{D_z}(f_z(0, 1, \dots, 1), f_z(\log z_1, 1, \dots, 1)) \le k_{G_z}((0, 1, \dots, 1), (\log z_1, 1, \dots, 1)) =$$
$$= \max\{k_{S_z}(0, \log z_1), k_{A_z}(1, 1), \dots, k_{A_z}(1, 1)\} = k_{S_z}(0, \log z_1),$$

where

$$S_z := \left\{\lambda \in \mathbb{C} : m_z - \widetilde{d}_D(z) < \operatorname{Re}\lambda < M_z + \widetilde{d}_D(z)\right\}$$

and

$$A_z := \left\{\lambda \in \mathbb{C} : -\widetilde{d}_D(z) < \log|\lambda| < \widetilde{d}_D(z)\right\}.$$

Using suitable biholomorphisms, we calculate

$$k_{S_z}(0, \log z_1) = p\left(\frac{i - \exp \pi i P(z)}{i + \exp \pi i P(z)}, \frac{i - \exp \pi i Q(z)}{i + \exp \pi i Q(z)}\right),$$

where
$$P(z) := \frac{\widetilde{d}_D(z) - m_z}{2\widetilde{d}_D(z) + M_z - m_z},\ Q(z) := \frac{\log z_1 + \widetilde{d}_D(z) - m_z}{2\widetilde{d}_D(z) + M_z - m_z}.$$

Analogously, after changing the index 1 to any of $2, \dots, n$, we get
$$k_D(z_0, z) \le \min_{j=1,\dots,n} k_{S_z^{(j)}}(0, \log z_j),$$
where
$$k_{S_z^{(j)}}(0, \log z_j) = p\left(\frac{i - \exp \pi i P^{(j)}(z)}{i + \exp \pi i P^{(j)}(z)}, \frac{i - \exp \pi i Q^{(j)}(z)}{i + \exp \pi i Q^{(j)}(z)}\right)$$
and
$$S_z^{(j)} := \left\{\lambda \in \mathbb{C} : m_z^{(j)} - \widetilde{d}_D(z) < \operatorname{Re} \lambda < M_z^{(j)} + \widetilde{d}_D(z)\right\},$$
$$m_z^{(j)} := \min\{0, \log z_j\},\ M_z^{(j)} := \max\{0, \log z_j\},\ j = 1, \dots, n,$$
$$P^{(j)}(z) := \frac{\widetilde{d}_D(z) - m_z^{(j)}}{2\widetilde{d}_D(z) + M_z^{(j)} - m_z^{(j)}},\ Q^{(j)}(z) := \frac{\log z_1 + \widetilde{d}_D(z) - m_z^{(j)}}{2\widetilde{d}_D(z) + M_z^{(j)} - m_z^{(j)}}.$$

Consider two cases: $\zeta_0 \ne 0$ and $\zeta_0 = 0$. If $\zeta_0 \ne 0$ then choose $j \in \{1, \dots, n\}$ such that $\zeta_{0j} \ne 0$ (recall that $\zeta_{0j} = |\zeta_{0j}| \ne 1$). In the case of $\zeta_{0j} > 1$ we obtain
$$k_{S_z^{(j)}}(0, \log z_j) = p\left(\frac{i - \exp \pi i T^{(j)}(z)}{i + \exp \pi i T^{(j)}(z)}, \frac{i - \exp \pi i U^{(j)}(z)}{i + \exp \pi i U^{(j)}(z)}\right) \le \tag{1}$$
$$\le p\left(0, \frac{i - \exp \pi i T^{(j)}(z)}{i + \exp \pi i T^{(j)}(z)}\right) + p\left(0, \frac{i - \exp \pi i U^{(j)}(z)}{i + \exp \pi i U^{(j)}(z)}\right),$$
where
$$T^{(j)}(z) := \frac{\varepsilon d_D(z)}{2\varepsilon d_D(z) + \log z_j},\ U^{(j)}(z) := \frac{\log z_j + \varepsilon d_D(z)}{2\varepsilon d_D(z) + \log z_j}.$$
We have, by Taylor expansion
$$\frac{i - \exp \pi i T^{(j)}(z)}{i + \exp \pi i T^{(j)}(z)} = i - \pi i T^{(j)}(z) + O\left(d_D(z)^2\right).$$
Hence
$$p\left(0, \frac{i - \exp \pi i T^{(j)}(z)}{i + \exp \pi i T^{(j)}(z)}\right) = p\left(0, i - \pi i T^{(j)}(z) + O\left(d_D(z)^2\right)\right) \le$$
$$\le \frac{\log 2}{2} - \frac{1}{2} \log\left(1 - \left|i - \pi i T^{(j)}(z) + O\left(d_D(z)^2\right)\right|\right) \le$$
$$\le \frac{\log 2}{2} - \frac{1}{2} \log\left(1 - \left|i - \pi i T^{(j)}(z)\right| - \left|O\left(d_D(z)^2\right)\right|\right) =$$
$$= \frac{\log 2}{2} - \frac{1}{2} \log\left(\pi \frac{\varepsilon d_D(z)}{2\varepsilon d_D(z) + \log z_j} - O\left(d_D(z)^2\right)\right) \le -\frac{1}{2} \log d_D(z) + C.$$
Similarly
$$\frac{i - \exp \pi i U^{(j)}(z)}{i + \exp \pi i U^{(j)}(z)} = -i + \pi i T^{(j)}(z) + O\left(d_D(z)^2\right),$$
which gives the same estimation for the second summand.

Otherwise if $\zeta_{0j} < 1$, we have
$$k_{S_z^{(j)}}(0, \log z_j) = p\left(\frac{i - \exp \pi i V^{(j)}(z)}{i + \exp \pi i V^{(j)}(z)}, \frac{i - \exp \pi i W^{(j)}(z)}{i + \exp \pi i W^{(j)}(z)}\right), \tag{2}$$

where

$$V^{(j)}(z) := \frac{\varepsilon d_D(z) - \log z_j}{2\varepsilon d_D(z) - \log z_j},\ W^{(j)}(z) := \frac{\varepsilon d_D(z)}{2\varepsilon d_D(z) - \log z_j}.$$

We see that the expression in (2) is the expression in (1) after substitute $\log z_j \rightsquigarrow -\log z_j$ and the estimates stay true.

Assume $\zeta_0 = 0$. We have for $j = 1, \ldots, n$

$$k_{S_z^{(j)}}(0, \log z_j) = p\left(\frac{i - \exp \pi i X^{(j)}(z)}{i + \exp \pi i X^{(j)}(z)}, \frac{i - \exp \pi i Y^{(j)}(z)}{i + \exp \pi i Y^{(j)}(z)}\right) \leq$$

$$\leq p\left(0, \frac{i - \exp \pi i X^{(j)}(z)}{i + \exp \pi i X^{(j)}(z)}\right) + p\left(0, \frac{i - \exp \pi i Y^{(j)}(z)}{i + \exp \pi i Y^{(j)}(z)}\right),$$

where

$$X^{(j)}(z) := \frac{\varepsilon'' d_D(z)\|z\|^{-1} - \log z_j}{2\varepsilon'' d_D(z)\|z\|^{-1} - \log z_j},\ Y^{(j)}(z) := \frac{\varepsilon'' d_D(z)\|z\|^{-1}}{2\varepsilon'' d_D(z)\|z\|^{-1} - \log z_j}.$$

Putting

$$\delta^{(j)}(z) := \frac{\varepsilon'' d_D(z)}{\|z\| \log z_j}$$

we have

$$X^{(j)}(z) = \frac{\delta^{(j)}(z) - 1}{2\delta^{(j)}(z) - 1},\ Y^{(j)}(z) = \frac{\delta^{(j)}(z)}{2\delta^{(j)}(z) - 1}$$

and $\delta^{(j)}(z) \to 0$ as $z \to 0$. The analogous calculations as in the first case give

$$\frac{i - \exp \pi i X^{(j)}(z)}{i + \exp \pi i X^{(j)}(z)} = -i + \pi i Y^{(j)}(z) + O\left(\delta^{(j)}(z)^2\right)$$

and

$$\frac{i - \exp \pi i Y^{(j)}(z)}{i + \exp \pi i Y^{(j)}(z)} = i - \pi i Y^{(j)}(z) + O\left(\delta^{(j)}(z)^2\right).$$

Therefore

$$p\left(0, \frac{i - \exp \pi i X^{(j)}(z)}{i + \exp \pi i X^{(j)}(z)}\right) \leq \frac{\log 2}{2} - \frac{1}{2}\log\left(\pi \frac{\delta^{(j)}(z)}{2\delta^{(j)}(z) - 1} - O\left(\delta^{(j)}(z)^2\right)\right) \leq$$

$$\leq -\frac{1}{2}\log(-\delta^{(j)}(z)) + C$$

and similarly

$$p\left(0, \frac{i - \exp \pi i Y^{(j)}(z)}{i + \exp \pi i Y^{(j)}(z)}\right) \leq -\frac{1}{2}\log(-\delta^{(j)}(z)) + C.$$

Finally

$$\min_{j=1,\ldots,n} k_{S_z^{(j)}}(0, \log z_j) \leq \min_{j=1,\ldots,n} -\log(-\delta^{(j)}(z)) + C =$$

$$= -\log d_D(z) + \log\|z\| + \min_{j=1,\ldots,n} \log(-\log z_j) + C =$$

$$= -\log d_D(z) + \log\|z\| + \log\left(-\log \max_{j=1,\ldots,n} z_j\right) + C \leq$$

$$\leq -\log d_D(z) + \log\|z\| + \log(-\log\|z\|) + C \leq -\log d_D(z) + C.$$

For improving the estimate in the case of $\zeta_0 \in \partial D \cap \mathbb{C}_*^n$, we may assume that $z_0 \in \mathbb{C}_*^n$ and $|z_{0j}|, |\zeta_{0j}| \neq 1$ for $j = 1, \ldots, n$. Since $\log D$ is a convex domain, the interval

$$I_z := \{t \log|z| + (1-t)\log|z_0| : t \in (-\varepsilon(z), 1 + \delta(z))\}$$

is contained in $\log D$ for some positive numbers $\delta(z)$, $\varepsilon(z)$. The number $\varepsilon(z)$ can be chosen as a sufficiently small positive constant ε independent of z. Indeed,

$$t\log|z|+(1-t)\log|z_0|=\log|z_0|+t(\log|z|-\log|z_0|)$$

and $\|\log|z|-\log|z_0|\|$ is bounded, say by M. Hence

$$\varepsilon:=\frac{d_{\log D}(\log|z_0|)}{2M}$$

is good. Analogously,

$$\frac{d_{\log D}(\log|z|)}{2M}$$

is a candidate for $\delta(z)$. We have

$$\frac{d_{\log D}(\log|z|)}{2M}\geq\delta d_D(z)$$

for some $\delta>0$ (in fact, "$\geq$" can be replaced with "$\approx$"). Thus we can choose $\delta(z):=\delta d_D(z)$.

From the inclusion $I_z\subset\log D$ it follows that

$$\exp I_z\subset D$$

i.e.

$$\left(\left|\frac{z_1}{z_{01}}\right|^t|z_{01}|,\dots,\left|\frac{z_n}{z_{0n}}\right|^t|z_{0n}|\right)\in D$$

for $t\in(-\varepsilon,1+\delta d_D(z))$. Hence the holomorphic map

$$f_z(\lambda):=\left(e^{i\arg z_1}\left|\frac{z_1}{z_{01}}\right|^\lambda|z_{01}|,\dots,e^{i\arg z_n}\left|\frac{z_n}{z_{0n}}\right|^\lambda|z_{0n}|\right)$$

leading from the strip

$$S_z:=\{\lambda\in\mathbb{C}:-\varepsilon<\operatorname{Re}\lambda<1+\delta d_D(z)\}$$

has values in D. Moreover $f_z(1)=z$ and $f_z(0)$ lies on the torus

$$T:=\{(\lambda_1z_{01},\dots,\lambda_nz_{0n}):\lambda_1,\dots,\lambda_n\in\partial\mathbb{D}\}.$$

Therefore

$$k_D(z_0,z)\leq k_D(z_0,f_z(0))+k_D(f_z(0),z)\leq$$
$$\leq k_D(f_z(0),f_z(1))+\max_{T\times T}k_D\leq k_{S_z}(0,1)+\max_{T\times T}k_D.$$

Calculating $k_{S_z}(0,1)$ we get

$$k_{S_z}(0,1)=p\left(\frac{i-\exp\pi iP^{(j)}(z)}{i+\exp\pi iP^{(j)}(z)},\frac{i-\exp\pi iQ^{(j)}(z)}{i+\exp\pi iQ^{(j)}(z)}\right),$$

where

$$P^{(j)}(z):=\frac{\varepsilon}{1+\varepsilon+\delta d_D(z)},\ Q^{(j)}(z):=\frac{1+\varepsilon}{1+\varepsilon+\delta d_D(z)}.$$

Certainly, first of the above argument of the function p tends to some point from the unit disc. For the second we have

$$\frac{i-\exp\pi iQ^{(j)}(z)}{i+\exp\pi iQ^{(j)}(z)}=-i+\pi i\frac{\delta d_D(z)}{1+\varepsilon+\delta d_D(z)}+O\left(d_D(z)^2\right).$$

Consequently

$$p\left(0,\frac{i-\exp\pi iQ^{(j)}(z)}{i+\exp\pi iQ^{(j)}(z)}\right)\leq\frac{\log 2}{2}-\frac{1}{2}\log\left(\pi\frac{\delta d_D(z)}{1+\varepsilon+\delta d_D(z)}-O\left(d_D(z)^2\right)\right)\leq$$

$$\leq-\frac{1}{2}\log d_D(z)+C.$$

The triangle inequality for p finishes the proof. □

Proof of Theorem 2. The proof is based on decreasing and product properties of the Kobayashi distance and need to consider some cases which form, in fact, an induction.

Note that if $E\subset\mathbb{R}^n$ is a convex domain then $E=\operatorname{int}\overline{E}$. The condition $\zeta_0\in\partial D\cap\operatorname{int}\overline{D}$ implies $\zeta_0\notin\mathbb{C}^n_*$. To see this, assume that $\zeta_0\in\mathbb{C}^n_*$. An easy topological argument shows that

$$\log|\zeta_0|\in(\partial\log D)\cap\operatorname{int}\overline{\log D}=(\partial\log D)\cap\log D=\emptyset.$$

Assume, without loss of generality, that

$$\zeta_0=(\zeta_{01},\dots,\zeta_{0k},0,\dots,0),$$

where $0\leq k\leq n-1$ and $\zeta_{0j}\neq 0$, $j\leq k$. Let $r>0$ be such that an open polydisc $P(\zeta_0,r)$ is contained in $\overline{D}$. Then $\log P(\zeta_0,r)\subset\log\overline{D}$. Taking interiors of both sides we get

$$\log P(\zeta_0,r)\subset\operatorname{int}\log\overline{D}=\operatorname{int}\overline{\log D}=\log D.$$

Therefore

$$P(\zeta_0,r)\cap\mathbb{C}^n_*\subset D. \tag{3}$$

Clearly (for fixed small r)

$$P(\zeta_0,r)\cap\mathbb{C}^n_*=\mathbb{D}(\zeta_{01},r)\times\ldots\times\mathbb{D}(\zeta_{0k},r)\times(r\mathbb{D}_*)^{n-k},$$

where $\mathbb{D}(\zeta_{0j},r)$ is a disc in $\mathbb{C}$ centered at ζ_{0j} with radius r. Hence, choosing any $z_0\in P(\zeta_0,r)\cap\mathbb{C}^n_*$, we have

$$k_D(z_0,z)\leq\max\left\{\max_{j=1,\dots,k}k_{\mathbb{D}(\zeta_{0j},r)}(z_{0j},z_j),\max_{j=k+1,\dots,n}k_{r\mathbb{D}_*}(z_{0j},z_j)\right\}$$

for $z\in D\cap\mathbb{C}^n_*$ near ζ_0. For $j=1,\dots,k$ the numbers z_j tend to ζ_{0j}, so the first of the above maxima is bounded by a constant. The well-known estimate for the punctured disc gives us

$$k_{r\mathbb{D}_*}(z_{0j},z_j)\leq\frac{1}{2}\log(-\log d_{r\mathbb{D}_*}(z_j))+C=\frac{1}{2}\log(-\log|z_j|)+C$$

for $j=k+1,\dots,n$. Therefore

$$k_D(z_0,z)\leq\frac{1}{2}\log\left(-\log\min_{j=k+1,\dots,n}|z_j|\right)+C. \tag{4}$$

The above estimate is not sufficiently good yet. Denote $z':=(z_1,\dots,z_k)$. Note that

$$(z',0,\dots,0)\in\partial D. \tag{5}$$

Indeed, $(z',0,\dots,0)\in\overline{D}$. If $(z',0,\dots,0)\in D$ then D is complete in the directions $k+1,\dots,n$ (Fact 1). Moreover, $(\zeta_{01},\dots,\zeta_{0k},r/2,\dots,r/2)\in D$, which implies $(\zeta_{01},\dots,\zeta_{0k},0,\dots,0)\in D$ — a contradiction.

We claim that for all $k+1 \le p < q \le n$

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0) \in \partial D \text{ or } (z', 0, \dots, 0, \underline{z_q}, 0, \dots, 0) \in \partial D, \tag{6}$$

where the symbol $\underline{z_j}$ means that z_j is on the j-th place. If (6) it is not true then both points belong to D (recall that $P(\zeta_0, r) \subset \overline{D}$). Hence D is complete in the directions $k+1, \dots, n$ and $(z', 0, \dots, 0) \in D$, which contradicts (5).

Therefore all points

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0),\ p = k+1, \dots, n,$$

except possibly one, belong to ∂D. Consider the following cases.

Case 1.1. One of above points, say $(z', 0, \dots, 0, z_n)$, does not belong to ∂D. Then it belongs to D. Hence D is complete in the directions $k+1, \dots, n-1$. Now the inclusion (3) can be improved to

$$P(\zeta_0, r) \cap (\mathbb{C}^{n-1} \times \mathbb{C}_*) \subset D$$

and

$$P(\zeta_0, r) \cap (\mathbb{C}^{n-1} \times \mathbb{C}_*) = \mathbb{D}(\zeta_{01}, r) \times \ldots \times \mathbb{D}(\zeta_{0k}, r) \times (r\mathbb{D})^{n-k-1} \times r\mathbb{D}_*.$$

The estimate for $k_D(z_0, z)$ is improved to

$$\max\left\{\max_{j=1,\dots,k} k_{\mathbb{D}(\zeta_{0j},r)}(z_{0j}, z_j), \max_{j=k+1,\dots,n-1} k_{r\mathbb{D}}(z_{0j}, z_j),\ k_{r\mathbb{D}_*}(z_{0n}, z_n)\right\} =$$

$$= k_{r\mathbb{D}_*}(z_{0n}, z_n) \le \frac{1}{2}\log(-\log|z_n|) + C.$$

It remains to notice that

$$(z', z_{k+1}, \dots, z_{n-1}, 0) \in \partial D$$

since in the opposite case the domain D would be complete in n-th direction and the property $(z', 0, \dots, 0, z_n) \in D$ would imply $(z', 0, \dots, 0) \in D$ — a contradiction with (5). Thus

$$d_D(z) \le \|z - (z', z_{k+1}, \dots, z_{n-1}, 0)\| = |z_n|,$$

which let us estimate

$$\frac{1}{2}\log(-\log|z_n|) + C \le \frac{1}{2}\log(-\log d_D(z)) + C.$$

Case 1.2. All the points

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0),\ p = k+1, \dots, n$$

belong to ∂D. We claim that for all $k+1 \le p < q \le n$ and $k+1 \le p' < q' \le n$ with $\{p, q\} \ne \{p', q'\}$

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0, \underline{z_q}, 0, \dots, 0) \in \partial D \text{ or } (z', 0, \dots, 0, \underline{z_{p'}}, 0, \dots, 0, \underline{z_{q'}}, 0, \dots, 0) \in \partial D.$$

Analogously as before we use an argument of completeness in the suitable directions to get

$$(z', 0, \dots, 0, \underline{z_j}, 0, \dots, 0) \in D$$

for some $j \in \{p, q, p', q'\}$ — a contradiction with the assumption of the case 1.2. Therefore all points

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0, \underline{z_q}, 0, \dots, 0),\ k+1 \le p < q \le n,$$

except possibly one, belong to ∂D. Again we consider two cases.

Case 2.1. One of above points, say $(z', 0, \dots, 0, z_{n-1}, z_n)$, does not belong to ∂D. Then it belongs to D. We see, analogously as in the case 1.1, that

$$P(\zeta_0, r) \cap (\mathbb{C}^{n-2} \times \mathbb{C}_*^2) \subset D,$$

$$k_D(z_0, z) \le \frac{1}{2} \log\left(-\log \min_{j=n-1,n} |z_j|\right) + C,$$

$$(z', z_{k+1}, \dots, z_{n-2}, z_{n-1}, 0),\ (z', z_{k+1}, \dots, z_{n-2}, 0, z_n) \in \partial D,$$

$$d_D(z) \le \min_{j=n-1,n} |z_j|.$$

Case 2.2. All the points

$$(z', 0, \dots, 0, \underline{z_p}, 0, \dots, 0, \underline{z_q}, 0, \dots, 0),\ k+1 \le p < q \le n$$

belong to ∂D. We see, by induction, that in the s-th step $(s = 3, \dots, n-k-1)$ all points

$$(z', 0, \dots, 0, \underline{z_{p_1}}, 0, \dots, 0, \underline{z_{p_s}}, 0, \dots, 0),\ k+1 \le p_1 < \dots p_s \le n,$$

except possibly one, belong to ∂D.

If one of these points, say $(z', 0, \dots, 0, z_{n-s+1}, \dots, z_n)$, does not belong to ∂D then it belongs to D and

$$P(\zeta_0, r) \cap (\mathbb{C}^{n-s} \times \mathbb{C}_*^s) \subset D,$$

$$k_D(z_0, z) \le \frac{1}{2} \log(-\log \min_{j=n-s+1,\dots,n} |z_j|) + C,$$

$$(z', z_{k+1}, \dots, z_{n-s}, z_{n-s+1}, \dots, z_{j-1}, 0, z_{j+1}, \dots, z_n) \in \partial D,\ j = n-s+1, \dots, n,$$

$$d_D(z) \le \min_{j=n-s+1,\dots,n} |z_j|,$$

which finishes the proof in the case $s.1$.

If all the points

$$(z', 0, \dots, 0, \underline{z_{p_1}}, 0, \dots, 0, \underline{z_{p_s}}, 0, \dots, 0),\ k+1 \le p_1 < \dots p_s \le n$$

belong to ∂D then we "jump" from the case $s.2$ to the case $(s+1).1$, getting finally in the case $(n-k-1).1$

$$(z', 0, z_{k+2}, \dots, z_n) \in D,$$

$$P(\zeta_0, r) \cap (\mathbb{C}^{k+1} \times \mathbb{C}_*^{n-k-1}) \subset D,$$

$$k_D(z_0, z) \le \frac{1}{2} \log(-\log \min_{j=k+2,\dots,n} |z_j|) + C,$$

$$(z', z_{k+1}, z_{k+2}, \dots, z_{j-1}, 0, z_{j+1}, \dots, z_n) \in \partial D,\ j = k+2, \dots, n,$$

$$d_D(z) \le \min_{j=k+2,\dots,n} |z_j|$$

or in the case $(n-k-1).2$

$$(z', z_{k+1}, \dots, z_{j-1}, 0, z_{j+1}, \dots, z_n) \in \partial D,\ j = k+1, \dots, n.$$

This property let us estimate $d_D(z)$ from above by $\min_{j=k+1,\dots,n} |z_j|$ and use (4) to finish the proof. $\square$

Proof of Theorem 3. The proof has two main parts; in the first the claim is proved for $\zeta_0 \in \partial D \cap \mathbb{C}_*^n$ thanks to the effective formulas for the Kobayashi distance in special domains and in the second part the remaining case is amounted to the lower-dimensional situation with a boundary point having all non-zero coordinates.

Let $\zeta_0 \in \partial D \cap \mathbb{C}_*^n$ and consider $z \in D \cap \mathbb{C}_*^n$ close to ζ_0. From the convexity of the set $\log D$ there exist $\alpha \in \mathbb{R}^n$ and $c > 0$ such that the hyperplane

$$\{x \in \mathbb{R}^n : \langle \alpha, x \rangle_{\mathbb{R}^n} = \log c\}$$

contains point $\log |\zeta_0|$ and $\log D$ lies on the one side of this hyperplane. Assume, without loss of generality, that this side is $\{x \in \mathbb{R}^n : \langle \alpha, x \rangle_{\mathbb{R}^n} < \log c\}$ since in the case of $\log D \subset \{x \in \mathbb{R}^n : \langle \alpha', x \rangle_{\mathbb{R}^n} > \log c'\}$ it suffices to define

$$\alpha := -\alpha' \text{ and } c := 1/c'.$$

Therefore

$$\{(e^{x_1}, \dots, e^{x_n}) : x \in \log D\} \subset \{w \in \mathbb{C}^n : |w|^\alpha < c\} =: D_{\alpha,c}{}^{1},$$

where by a point satisfying the condition $|w|^\alpha < c$ we mean such point w whose coordinate w_j is non-zero when $\alpha_j < 0$ (and satisfies $|w|^\alpha < c$ in the usual sense). To affirm that $D \subset D_{\alpha,c}$, we have to check that the above restriction for points w does not remove from D points with some zero coordinates. Indeed, if there is no such inclusion, we can assume that the order of zero coordinates of point $w \in D$ and negative terms of the sequence α is as follows:

$$w_1, \dots, w_k \neq 0,\ w_{k+1}, \dots, w_n = 0$$

$$\alpha_{k+1}, \dots, \alpha_l \geq 0,\ \alpha_{l+1}, \dots, \alpha_n < 0,$$

where $1 \leq k \leq l < n$. In some neighbourhood of the point w contained in D there exist points $v \in \mathbb{C}_*^n$ with coordinates v_j such that

$$|v_1|, \dots, |v_l| > \varepsilon > 0$$

and $|v_{l+1}|, \dots, |v_n|$ arbitrarily close to zero (i.e. moved from w in a direction of subspace $\{0\}^l \times \mathbb{C}^{n-l}$ and next moved from it by a constant vector in the direction $\mathbb{C}^l \times \{0\}^{n-l}$). Then there exist points $u \in \log D$ whose coordinates u_j satisfy

$$u_1, \dots, u_l > \log \varepsilon > -\infty$$

however $u_{l+1}, \dots, u_n$ are arbitrarily close to $-\infty$. But it contradicts a fact that values of the expression

$$\sum_{j=l+1}^{n} \alpha_j u_j$$

are for these points u bounded from above by a constant $\log c - \sum_{j=1}^{l} \alpha_j \log \varepsilon$.

We will use effective formulas for the Kobayashi distance in domains $D_{\alpha,c}$ [9]. Define

$$l := \#\{j = 1, \dots, n : \alpha_j < 0\}$$

and

$$\widetilde{\alpha} := \min\{\alpha_j : \alpha_j > 0\} \ \text{ if } \ l < n.$$

We first consider a situation $l < n$. The formula in this case gives

$$k_D(z_0, z) \geq k_{D_{\alpha,c}}(z_0, z) \geq p\left(0, \frac{|z|^{\alpha/\widetilde{\alpha}}}{c^{1/\widetilde{\alpha}}}\right) + C.$$

---

[1] These sets are called *elementary Reinhardt domains.*

But

$$z = \zeta_{D_{\alpha,c}}(z) - d_{D_{\alpha,c}}(z)\nu_{D_{\alpha,c}}(z)$$

and hence

$$|z|^{\alpha/\tilde{\alpha}} = \prod_{j=1}^{n} |\zeta_{D_{\alpha,c}}(z)_j - d_{D_{\alpha,c}}(z)\nu_{D_{\alpha,c}}(z)_j|^{\alpha_j/\tilde{\alpha}} = c^{1/\tilde{\alpha}} - \rho(z)d_{D_{\alpha,c}}(z)$$

for some bounded positive function ρ. Thus

$$p\left(0, \frac{|z|^{\alpha/\tilde{\alpha}}}{c^{1/\tilde{\alpha}}}\right) = p\left(0, 1 - \frac{\rho(z)}{c^{1/\tilde{\alpha}}}d_{D_{\alpha,c}}(z)\right) \geq -\frac{1}{2}\log\left(\frac{\rho(z)}{c^{1/\tilde{\alpha}}}d_{D_{\alpha,c}}(z)\right) \geq$$

$$\geq -\frac{1}{2}\log d_{D_{\alpha,c}}(z) + C.$$

We will show that

$$d_{D_{\alpha,c}}(z) \approx d_D(z) \text{ as } z \longrightarrow \zeta_0 \text{ non-tangentially.}$$

By the definition there exists a cone $\mathcal{A}$ with a vertex ζ_0 and a semi-axis $-\nu_{D_{\alpha,c}}(\zeta_0)$ which contains considered points z. Thanks to the $\mathcal{C}^1$-smoothness of D we have a cone $\mathcal{B}$ with the vertex ζ_0 and the semi-axis $-\nu_{D_{\alpha,c}}(\zeta_0)$, whose intersection with some neighbourhood of the point ζ_0 is contained in D and contains in its interior the cone $\mathcal{A}$. Therefore

$$1 \geq \frac{d_D(z)}{d_{D_{\alpha,c}}(z)} = \frac{\|z - \zeta_D(z)\|}{\|z - \zeta_{D_{\alpha,c}}(z)\|} \geq \frac{\|z - \zeta_D(z)\|}{\|z - \zeta_0\|} \geq \frac{\|z - \zeta_{\mathcal{B}}(z)\|}{\|z - \zeta_0\|} =$$

$$= \sin\angle(z, \zeta_0, \zeta_{\mathcal{B}}(z)) \geq \sin\theta,$$

where $\angle(X, Y, Z)$ is an angle with vertex Y, whose arms contain points X, Z and θ is an angle between these generatrices of cones $\mathcal{A}$, $\mathcal{B}$ which lie in one plane with the axis of both cones.[2]

The second case $l = n$ gives

$$k_D(z_0, z) \geq k_{D_{\alpha,c}}(z_0, z) \geq p\left(0, \frac{|z|^{\alpha}}{c}\right) + C.$$

Similarly as before

$$|z|^{\alpha} = \prod_{j=1}^{n} |\zeta_{D_{\alpha,c}}(z)_j - d_{D_{\alpha,c}}(z)\nu_{D_{\alpha,c}}(z)_j|^{\alpha_j} = c - \sigma(z)d_{D_{\alpha,c}}(z)$$

with a bounded positive function σ. Hence

$$p\left(0, \frac{|z|^{\alpha}}{c}\right) \geq -\frac{1}{2}\log d_{D_{\alpha,c}}(z) + C \geq -\frac{1}{2}\log d_D(z) + C.$$

Now, take $\zeta_0 \in \partial D \setminus \mathbb{C}^n_*$. We may assume that the first k coordinates of ζ_0 are non-zero and the last $n - k$ are zero, where $0 \leq k \leq n - 1$. Notice that $k \neq 0$. Indeed, the assumption $k = 0$ is equivalent to $0 \in \partial D$. Using Facts 1 and 2 we see that the $\mathcal{C}^1$-smoothness of D implies, thanks to the Fu condition, $D \cap V_j^n \neq \emptyset$ for $j = 1, \ldots, n$. Hence D is complete i.e. $0 \in D$ — a contradiction. Finally, point ζ_0 has a form

$$\zeta_0 = (\zeta_{01}, \ldots, \zeta_{0k}, 0, \ldots, 0),\ \zeta_{0j} \neq 0,\ 1 \leq j \leq k \leq n - 1.$$

[2]In other words, θ is a difference of angles appearing in the definitions of the cones $\mathcal{B}$, $\mathcal{A}$.

Consider the projection $\pi_k : \mathbb{C}^n \longrightarrow \mathbb{C}^k$ i.e.

$$\pi_k(z) = (z_1, \dots, z_k).$$

We will show that $D_k := \pi_k(D)$ is a $\mathcal{C}^1$-smooth pseudoconvex Reinhardt domain. A Reinhardt property is clear for D_k. To affirm the pseudoconvexity of D_k it suffices to show that

$$D_k \times \{0\}^{n-k} = D \cap (\mathbb{C}^k \times \{0\}^{n-k}).$$

Inclusion

$$D_k \times \{0\}^{n-k} \supset D \cap (\mathbb{C}^k \times \{0\}^{n-k})$$

is obvious. To prove the opposite inclusion we will use Facts 1 and 2 again. We have $D \cap V_j^n \neq \emptyset$ for $j = k+1, \dots, n$, so D is complete in j-th direction for $j = k+1, \dots, n$. Take some $z \in D_k \times \{0\}^{n-k}$. Then $z = (z_1, \dots, z_k, 0 \dots, 0)$ and $(z_1, \dots, z_k, \widetilde{z}_{k+1}, \dots, \widetilde{z}_n) \in D$ for some $\widetilde{z}_{k+1}, \dots, \widetilde{z}_n \in \mathbb{C}$. Thus $(z_1, \dots, z_k, 0 \dots, 0) \in D$ i.e. $z \in D \cap (\mathbb{C}^k \times \{0\}^{n-k})$.

The local defining function for D_k at point $\zeta \in \partial D_k$ is

$$\widetilde{\rho}(z_1, \dots, z_k) := \rho(z_1, \dots, z_k, 0, \dots, 0),\ (z_1, \dots, z_k) \in \pi_k(U) \cap D_k,$$

where $\rho : U \longrightarrow \mathbb{R}$ is the local defining function for D at point $(\zeta, 0, \dots, 0)$. Indeed, $\nabla\widetilde{\rho} \neq 0$ since

- $\nabla\rho \neq 0$;
- $\frac{\partial\widetilde{\rho}}{\partial\overline{z}_j} = \frac{\partial\rho}{\partial\overline{z}_j}$ for $j = 1, \dots, k$;
- $\frac{\partial\rho}{\partial\overline{z}_j} = 0$ for $j = k+1, \dots, n$,

however the two remaining conditions for a defining function follow easy from the definition of $\widetilde{\rho}$.

If z tends to ζ_0 non-tangentially in a cone $\mathcal{A} \subset \mathbb{C}^n$ then $\pi_k(z)$ tends to $\pi_k(\zeta_0) \in \mathbb{C}_*^k$ non-tangentially in a cone $\pi_k(\mathcal{A}) \subset \mathbb{C}^k$. From the previous case

$$k_D(z_0, z) \geq k_{D_k}(\pi_k(z_0), \pi_k(z)) \geq -\frac{1}{2}\log d_{D_k}(\pi_k(z)) + C.$$

Hence to finish the proof it suffices to show that

$$d_{D_k}(\pi_k(z)) \lesssim d_D(z).$$

Consider a cone $\mathcal{B}$ with vertex ζ_0 and semi-axis $-\nu_{D_\alpha}(\zeta_0)$ whose intersection with some neighbourhood of the point ζ_0 is contained in D and contains in its interior the cone $\mathcal{A}$. Then

$$1 \geq \frac{d_{\mathcal{B}}(z)}{d_D(z)} = \frac{\|z - \zeta_{\mathcal{B}}(z)\|}{\|z - \zeta_D(z)\|} \geq \frac{\|z - \zeta_{\mathcal{B}}(z)\|}{\|z - \zeta_0\|} = \sin\angle(z, \zeta_0, \zeta_{\mathcal{B}}(z)) \geq \sin\theta,$$

where θ is an angle between these generatrices of the cones $\mathcal{A}$, $\mathcal{B}$ which lie in one plane with the axis of both cones. Analogously

$$1 \geq \frac{d_{\pi_k(\mathcal{B})}(\pi_k(z))}{d_{D_k}(\pi_k(z))} \geq \sin\theta',$$

where θ' depends only on $\mathcal{B}$. Therefore

$$\frac{d_{D_k}(\pi_k(z))}{d_D(z)} \approx \frac{d_{\pi_k(\mathcal{B})}(\pi_k(z))}{d_{\mathcal{B}}(z)},$$

however

$$\frac{d_{\pi_k(\mathcal{B})}(\pi_k(z))}{d_{\mathcal{B}}(z)} = \frac{\|\pi_k(z) - \zeta_{\pi_k(\mathcal{B})}(\pi_k(z))\|}{\|z - \zeta_{\mathcal{B}}(z)\|} =$$

$$= \frac{\|\pi_k(z) - \pi_k(\zeta_0)\| \sin \angle(\pi_k(z), \pi_k(\zeta_0), \zeta_{\pi_k(\mathcal{B})}(\pi_k(z)))}{\|z - \zeta_0\| \sin \angle(z, \zeta_0, \zeta_{\mathcal{B}}(z))} \leq$$

$$\leq \frac{\|\pi_k(z) - \pi_k(\zeta_0)\|}{\|z - \zeta_0\| \sin \theta} \leq \frac{1}{\sin \theta}. \qquad \square$$

REMARK. The estimate from below by $-\frac{1}{2} \log d_D + C$ for the Carathéodory (pseudo) distance c_D is not true even for smooth bounded complete pseudoconvex Reinhardt domain D and its boundary point $\zeta_0 \in \mathbb{C}_*^n$.

PROOF. Consider a domain

$$D := \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < R_1,\ |z_2| < R_2,\ |z_1||z_2|^\alpha < R_3\right\},$$

where $R_1, R_2, R_3 > 0$, $\alpha \in (\mathbb{R} \setminus \mathbb{Q})_+$ and $R_1 R_2^\alpha > R_3$. Fix $\zeta_0 \in \partial D$ such that $|\zeta_{01}| < R_1, |\zeta_{02}| < R_2$. This domain is not smooth. Since

$$\log D = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 < \log R_1,\ x_2 < \log R_2,\ x_1 + \alpha x_2 < \log R_3\right\},$$

it is easy to construct smooth bounded convex domain $E \subset \mathbb{R}^2$ such that $\log D \subset E$ and ∂E contains the skew segment

$$(\partial \log D) \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + \alpha x_2 = \log R_3\}.$$

Let $\widetilde{D} \subset \mathbb{C}^2$ be a complete Reinhardt domain such that $\log \widetilde{D} = E$. Then $\widetilde{D}$ is bounded, smooth and, thanks to Fact 1, pseudoconvex. Moreover, $D \subset \widetilde{D}$ and D, $\widetilde{D}$ are identic in the neighbourhood of their common boundary point ζ_0.

We have from the Proposition 4.3.2 in [9]

$$a_\lambda := \frac{g_D(\lambda\zeta_0, 0)}{\log|\lambda|} \to \infty \text{ as } \lambda \to \partial\mathbb{D},$$

where g_D is the pluricomplex Green function (general properties of the Carathéodory (pseudo)distance and the pluricomplex Green function one can find e.g. in [3] and [9]). Certainly

$$d_{\widetilde{D}}(\lambda\zeta_0) = d_D(\lambda\zeta_0) \approx 1 - |\lambda| \text{ as } |\lambda| \to 1$$

and

$$c_D(\lambda\zeta_0, 0) \leq \tanh^{-1} \exp g_D(\lambda\zeta_0, 0).$$

Therefore, if there exists a constant $C > 0$ such that

$$c_{\widetilde{D}}(\lambda\zeta_0, 0) \geq -\frac{1}{2} \log d_{\widetilde{D}}(\lambda\zeta_0) + C,\ |\lambda| \to 1$$

then for $|\lambda| \to 1$

$$c_D(\lambda\zeta_0, 0) \geq -\frac{1}{2} \log d_D(\lambda\zeta_0) + C,$$

$$-\frac{1}{2} \log(1 - |\lambda|) + C \leq \tanh^{-1} |\lambda|^{a_\lambda},$$

$$\frac{1}{1 - |\lambda|} \leq \frac{C'}{1 - |\lambda|^{a_\lambda}}$$

with a constant $C' > 0$. For $|\lambda|$ sufficiently close to 1 we have $a_\lambda \geq C' + 1$. Therefore

$$\frac{1}{1 - |\lambda|} \leq \frac{C'}{1 - |\lambda|^{C'+1}}$$

or equivalently

$$\frac{1-|\lambda|^{C'+1}}{1-|\lambda|} \le C'.$$

The left-hand side tends to $C'+1$ as $|\lambda| \to 1$. □

4. Open problems

(1) Can we improve the estimate from Theorem 1 to $-\frac{1}{2}\log d_D(z) + C$?

(2) Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain and $\zeta_0 \in \partial D \cap \mathbb{C}^n_*$. Does it implies that for some constant C the inequality

$$k_D(z_0, z) \ge -\frac{1}{2}\log d_D(z) + C$$

holds if $z \in D$ tends to ζ_0?

(3) Is it true for pseudoconvex Reinhardt domains $D \subset \mathbb{C}^n$ that if

$$\#\{j : \zeta_{0j} = 0 \text{ and } D \cap V_j^n = \emptyset\} = 0$$

then

$$k_D(z_0, z) \ge -\frac{1}{2}\log d_D(z) + C$$

and in the opposite case

$$k_D(z_0, z) \ge \frac{1}{2}\log(-\log d_D(z)) + C$$

for $z \in D$ near $\zeta_0 \in \partial D$?

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