ssdnm
środowiskowe
studia doktoranckie
z nauk matematycznych

Wanda Niemyska

Uniwersytet Śląski

Laws of Contraposition and Law of Importation for Probabilistic Implications and Probabilistic S-implications

Praca semestralna nr 3

(semestr letni 2012/13)

Opiekun pracy: prof. Przemysław Grzegorzewski

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

University of Silesia

Wanda Niemyska

Laws of Contraposition and Law of Importation for Probabilistic Implications and Probabilistic S-implications

Term paper
(June 30, 2013)

Under the supervision of:
prof. Przemysław Grzegorzewski

1 Introduction

Recently Przemysław Grzegorzewski [2] has introduced new families of fuzzy implications - probabilistic implications and probabilistic S-implications. They are based on conditional copulas and make a bridge between probability theory and fuzzy logic. In the article [2] author gives a motivation to his idea and indicates some interesting connections between new families of implications and the dependence structure of the underlying environment.
A lot of authors investigated various properties of fuzzy implications and it seems proper to make such studies for probabilistic implications and probabilistic S-implications as well. In this paper I have checked if (or when) they satisfy Laws of Contraposition (CP, L-CP, R-CP) and Law of Importation (LI). Let us begin with an introduction of those Laws.

One of the most important tautologies in the classical two-valued logic is the Law of Contraposition:

$$p \to q \equiv \neg q \to \neg p.$$

Since the classical negation satisfies the law of double negation ($\neg(\neg p) \equiv p$), the following laws are also tautologies in the classical logic:

$$\neg p \to q \equiv \neg q \to p$$
$$p \to \neg q \equiv q \to \neg p$$

Natural generalization of those classical tautologies to fuzzy logic is based on fuzzy negations and fuzzy implications and plays an important role in the applications of fuzzy implications.

Definition. Let I be a fuzzy implication and N be a fuzzy negation. I is said to satisfy the

1. *Law of Contraposition* with respect to N, if

$$I(x,y) = I(N(y), N(x)) \quad \forall_{x,y\in[0,1]} \quad \textbf{(CP),(CP(N))}.$$

2. *Law of Left Contraposition* with respect to N, if

$$I(N(x),y) = I(N(y), x) \quad \forall_{x,y\in[0,1]} \quad \textbf{(L-CP),(L-CP(N))}.$$

3. *Law of Right Contraposition* with respect to N, if

$$I(x,N(y)) = I(y, N(x)) \quad \forall_{x,y\in[0,1]} \quad \textbf{(R-CP),(R-CP(N))}.$$

During my studies I use following well-known lemma ([1], p.21):

Lemma 1. *If $I\colon [0,1]^2 \to [0,1]$ is any function and N is a strong negation, then the following equivalence is satisfied*

$$I \textit{ satisfies } (CP(N)) \Leftrightarrow I \textit{ satisfies } (L-CP(N)) \Leftrightarrow I \textit{ satisfies } (R-CP(N)).$$

The equation

$$(p \wedge q) \to r \equiv (p \to (q \to r),$$

known as the *Law of Importation*, is another tautology in classical logic. The general form of the above equivalence is introduced in the following definition

Definition. Let I be a fuzzy implication and T be a t-norm. I is said to satisfy the *Law of Importation* with t-norm T, if

$$I(T(x,y),z) = I(x, I(y,z)) \forall_{x,y,z \in [0,1]} \quad \textbf{(LI)}.$$

Grzegorzewski [2] examined basic properties of families of probabilistic implications and probabilistic S-implications, among them the Left Neutrality Property (NP), the Exchange Principle (EP), the Identity Principle (IP) and the Ordering Property (OP).
My goal was to examine, if those new families of implications satisfy Laws of Contraposition (CP, L-CP, R-CP) and Law of Importation (LI). That's a brief review of my results:

- no probabilistic implication satisfies Laws of Contraposition (CP) and (L-CP). However, every probabilistic implication satisfies (R-CP), but only with respect to the lowest negation - N_{D1};

- if any probabilistic S-implication satisfies any Law of Contraposition with respect to the negation N, then N is specific strong negation - $N_C(u) = 1 - u$.
  Because N_C is a strong negation, if particular probabilistic S-implication satisfies any Law of Contraposition with respect to N_C, then it satisfies all of them wit respect to N_C.
  Some probabilistic S-implications $\tilde{I}_C$ satisfy Laws of Contraposition with pespect to N_C (for instance implications generated from copulas Π, M, W, family $FGM(\Theta)$) and some don't (for instance implications generated from copulas from family $AMH(\Theta)$);

- the Law of Importation (LI) may be satisfied for probabilistic implications only with positive t-norms. Some probabilistic implications satisfy (LI) with all positive t-norms, some only with t-norm T_P and some don't satisfy (LI) with any t-norm;

- for any probabilistic S-implication $\tilde{I}_C$ there exists unique function $T(x,y) = x - C(x, 1-y)$, which is only one candidate to be such t-norm, that (LI) is satisfied for $\tilde{I}_C$ and T:
  $\tilde{I}_C$ and T satisfy (LI) $\rightharpoonup$ ($\Rightarrow$, but $\not\Leftarrow$) $T(x,y) = x - C(x, 1-y)$.

In the next section I recall some definitions and examples connected to fuzzy implications, probabilistic implications and probabilistic S-implications. Sections 3,4,5 and 6 contain precise discussions about four statements which are pointed above .
There are still some questions, if it's possible to generalize somehow results, which are introduced in this paper.

2 Preliminaries

At first let us recall general definitions of fuzzy operators, which will be exploited henceforth - fuzzy implications, t-norms and fuzzy negations.

Definition. A function $I\colon [0,1]^2 \to [0,1]$ is called a ***fuzzy implication*** if it satisfies the following conditions for all $x, y \in [0,1]$

1. $I(\cdot, y)$ is decreasing,
2. $I(x, \cdot)$ is increasing,
3. $I(0,0) = 1$,
4. $I(1,1) = 1$,
5. $I(1,0) = 0$.

Definition. A function $T\colon [0,1]^2 \to [0,1]$ is called a ***triangular norm (t-norm)*** if it satisfies, for all $x, y, z \in [0,1]$, the following conditions

1. $T(x,y) = T(y,x)$,
2. $T(x, T(y,z)) = T(T(x,y), z)$,
3. $T(x, \cdot)$ is decreasing,
4. $T(x,1) = x$.

Definition. A function $N\colon [0,1] \to [0,1]$ is called a ***fuzzy negations*** if it satisfies the following conditions

1. $N(0) = 1, \quad N(1) = 0$,
2. N is decreasing.

New families of fuzzy implications - probabilistic implications and probabilistic S-implications - are based on conditional copulas:

Definition. A ***copula*** (specifically a 2-copula) is a function $C\colon [0,1]^2 \to [0,1]$ which satisfies the following conditions

(a) $C(u,0) = C(0,v) = 0 \quad \forall_{u,v\in[0,1]}$,

(b) $C(u,1) = u \quad \forall_{u\in[0,1]}$,

(c) $C(1,v) = v \quad \forall_{v\in[0,1]}$,

(d) $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geqslant 0 \quad \forall_{u_1,u_2,v_1,v_2\in[0,1]: u_1\leqslant u_2, v_1\leqslant v_2}$.

Example 1. Following copulas I've examined henceforth:

1. $C(u,v) = M(u,v) = \min(u,v)$

2. $C(u,v) = W(u,v) = \max(u+v-1,0)$
3. $C(u,v) = \Pi(u,v) = u \cdot v$
4. Farlie-Gumbel-Morgenstern's family, $FGM(\Theta)$, where $\Theta \in [-1,1]$:

$$C_\Theta(u,v) = u \cdot v + \Theta u \cdot v(1-u)(1-v)$$

5. Ali-Mikhail-Haq's family, $AMH(\Theta)$, where $\Theta \in [-1,1]$:

$$C_\Theta(u,v) = \frac{u \cdot v}{1 - \Theta(1-u)(1-v)}$$

It can be shown that every copula is bounded by the so-called $Fréchet - Hoeffding$ bounds, i.e. for any copula C and for all $u, v \in [0,1]$

$$W(u,v) \leqslant C(u,v) \leqslant M(u,v).$$

Definition. A function $I_C \colon [0,1]^2 \to [0,1]$ given by

$$I_C(u,v) = \begin{cases} 1, & u = 0 \\ \frac{C(u,v)}{u}, & u > 0 \end{cases},$$

where C is a copula, is called a ***probabilistic implication*** (based on copula C).

Definition. A function $\tilde{I}_C : [0,1]^2 \to [0,1]$ given by

$$\tilde{I}_C(u,v) = C(u,v) - u + 1,$$

where C is a copula, is called a ***probabilistic S-implication*** (based on copula C).

Remark. Not every probabilistic implication is also a fuzzy implication - we need to add a condition, that $I_C(\cdot, v)$ is decreasing (the rest of conditions from fuzzy implication's definition are satisfied for any probabilistic implication).
Among implications based on discussed above examples of copulas, I_M and I_Π are fuzzy implications, but I_W is not. Probabilistic implications based on copulas from family $FGM(\Theta)$ are fuzzy implications only for $\Theta \geqslant 0$, whereas probabilistic implications based on copulas from family $AMH(\Theta)$ are fuzzy implications for all $\Theta \in [-1,1]$.
However any probabilistic S-implication is a fuzzy implication.

3 The Laws of Contraposition for Probabilistic Implications

Lemma 2. *No probabilistic implication satisfies the Laws of Contraposition (CP) and (L-CP).*

Dowód. .

(1) **(CP)** $I_C(x,y) = I_C(N(y),N(x))$
Let $x = 1$. Then

$$L = \frac{C(x,y)}{x} = \frac{C(1,y)}{1} = y.$$

Whereas

$$R = \begin{cases} 1, & N(y) = 0 \\ \frac{C(N(y),N(1))}{N(y)}, & N(y) > 0 \end{cases} = \begin{cases} 1, & N(y) = 0 \\ \frac{C(N(y),0)}{N(y)}, & N(y) > 0 \end{cases} = \begin{cases} 1, & N(y) = 0 \\ 0, & N(y) > 0 \end{cases}$$

Therefore for $y \in (0,1)$ we get $L \neq R$.

(2) **(L-CP)** $I_C(N(x),y) = I_C(N(y),x)$
The argumentation is analogous to the (CP)'s case. Let $x = 0$. Then

$$L = \frac{C(N(x),y)}{N(x)} = \frac{C(N(0),y)}{N(0)} = \frac{C(1,y)}{1} = y.$$

Whereas

$$R = \begin{cases} 1, & N(y) = 0 \\ \frac{C(N(y),x)}{N(y)}, & N(y) > 0 \end{cases} = \begin{cases} 1, & N(y) = 0 \\ \frac{C(N(y),0)}{N(y)}, & N(y) > 0 \end{cases} = \begin{cases} 1, & N(y) = 0 \\ 0, & N(y) > 0 \end{cases}$$

Therefore for $y \in (0,1)$ we get $L \neq R$.

□

Lemma 3. *Every probabilistic implication satisfies $(R-CP)$, but only with respect to the lowest negation* $N_{D1}(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$.

Dowód. **(R-CP)** $I_C(x,N(y)) = I_C(y,N(x))$
First of all let us check the conditions which functions I_C and N must satisfy, if we want the Law $(R-CP)$ to be satisfied on the endpoints of the unit square $[0,1]^2$:

- $x = 0$

  $$L = I_C(0,N(y)) = 1; \quad R = I_C(y,N(0)) = \begin{cases} 1, & y = 0 \\ \frac{C(y,1)}{y}, & y > 0 \end{cases} = \begin{cases} 1, & y = 0 \\ \frac{y}{y}, & y > 0 \end{cases} = 1$$

  Thus $L = R \quad \forall_{y \in [0,1]}$.

- $x = 1$

  $$L = I_C(1,N(y)) = \frac{C(1,N(y))}{1} = N(y); \quad R = I_C(y,N(1)) = \begin{cases} 1, & y = 0 \\ \frac{C(y,0)}{y}, & y > 0 \end{cases} = \begin{cases} 1, & y = 0 \\ 0, & y > 0 \end{cases}.$$

  Thus $L = R \quad \forall_{y \in [0,1]} \quad \iff \quad N(y) = \begin{cases} 1, & y = 0 \\ 0, & y > 0 \end{cases}.$

- $y = 0$

$$L = I_C(x, N(0)) = \begin{cases} 1, & x = 0 \\ \frac{C(x,1)}{x}, & x > 0 \end{cases} = \begin{cases} 1, & x = 0 \\ \frac{x}{x}, & x > 0 \end{cases} = 1; \quad R = I_C(0, N(x)) = 1.$$

Thus $L = R \quad \forall_{x \in [0,1]}$.

- $y = 1$

$$L = I_C(x, N(1)) = \begin{cases} 1, & x = 0 \\ \frac{C(x,0)}{x}, & x > 0 \end{cases} = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}; \quad R = I_C(1, N(x)) = \frac{C(1, N(x))}{1} = N(x).$$

Thus $L = R \quad \forall_{x \in [0,1]} \iff N(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$.

Therefore there exists exactly one fuzzy negation, which any probabilistic implication may satisfy $(R - CP)$ with respect to. And any probabilistic implication satisfies $(R - CP)$ with respect to this negation on the endpoints of the square $[0, 1]^2$. This negation is the lowest one:

$$N_{D1}(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$$

Now let us check, which probabilistic implications satisfy $(R-CP)$ with respect to N_{D1} on the whole unit square $[0, 1]^2$.
Let $x, y \in (0, 1)$. Then $L = I_C(x, N(y)) = I_C(x, 0) = \frac{C(x,0)}{x} = 0$, and $R = I_C(y, N(x)) = I_C(y, 0) = \frac{C(y,0)}{y} = 0$.
Thus for any copula C the equality $L = R$ is held for all $x, y \in (0, 1)$. This ends the proof.

□

4 The Laws of Contraposition for Probabilistic S-Implications

Lemma 4. *Let $\tilde{I}_C$ be any probabilistic S-implication. If $\tilde{I}_C$ satisfies the Law (CP) with respect to the negation N, then N is a strong negation $N_C(x) = 1-x$.*

Dowód. (CP) $\quad \tilde{I}_C(x, y) \stackrel{?}{=} \tilde{I}_C(N(y), N(x)) \iff$

$$C(x, y) - x + 1 = C(N(y), N(x)) - N(y) + 1 \tag{1}$$

Let $y = 0$. Then equation (1) takes the form

$$C(x, 0) - x + 1 = C(N(0), N(x)) - N(0) + 1 \iff$$

$$0 - x + 1 = N(x) - 1 + 1 \iff N(x) = 1 - x.$$

□

Remark. The equation (1) is held for N_C and any copula C also for $x = 0, x = 1$ or $y = 1$.

Corollary 2. *Therefore probabilistic S-implication $\tilde{I}_C$ satisfies (CP) (with respect to N_C) if and only if a copula C satisfies following equation for all $x, y \in (0,1)$*

$$C(x,y) - x + 1 = C(1-y, 1-x) + y \tag{2}$$

Remark. Some probabilistic S-implications $\tilde{I}_C$ satisfy the Law of Contraposition (CP) with respect to N_C (for instance implications based on copulas Π, M, W or copulas from family $FGM(\Theta)$), and some do not (for instance implications based on copulas from family $AMH(\Theta)$).

Dowód. Because of the last corollary we can consider only $x, y \in (0,1)$, without endpoints of the interval.

1. $C = \Pi$ **OK**
   The equation (2) takes the form:

$$xy - x + 1 \stackrel{?}{=} (1-x)(1-y) + y$$

$$R = (1 - x - y + xy) + y = xy - x + 1 = L$$

2. $C = M$ **OK**
   The equation (2) takes the form:

$$\min(x,y) - x + 1 \stackrel{?}{=} \min(1-x, 1-y) + y$$

   We consider two cases:
   1). $x \leqslant y$ $\quad (\Rightarrow 1 - y \leqslant 1 - x)$ $\qquad L = 1, R = 1;$
   2). $x > y$ $\quad (\Rightarrow 1 - y > 1 - x)$ $\qquad L = y - x + 1, R = 1 - x + y.$

3. $C = W$ **OK**
   The equation (2) takes the form:

$$\max(x+y-1, 0) - x + 1 \stackrel{?}{=} \max(1-x+1-y-1, 0) + y = \max(-(x+y-1), 0) + y$$

   We consider two cases:
   1). $x + y - 1 \geqslant 0$ $\qquad L = y, R = y;$
   2). $x + y - 1 < 0$ $\qquad L = -x + 1, R = -(x + y - 1) + y = -x + 1.$

4. $C \in FGM(\Theta)$ **OK**
   The equation (2) takes the form:

$$xy + \Theta xy(1-x)(1-y) - x + 1 \stackrel{?}{=} (1-x)(1-y) + \Theta(1-x)(1-y)xy + y$$

$$\iff \quad xy - x + 1 \stackrel{?}{=} (1-x)(1-y) + y$$

   $L = R$, as in the case $C = \Pi$.

   We might suppose that any probabilistic S-implication satisfies (CP) with respect to N_C after above four examples. However the next example proves that it is not true.

5. $C \in AMH(\Theta)$ **X**
The equation (2) takes the form:

$$\frac{xy}{1-\Theta(1-x)(1-y)} - x + 1 \overset{?}{=} \frac{(1-x)(1-y)}{1-\Theta xy} + y$$

This equation is satisfied not for every $\Theta \in [-1,1]$ and $x, y \in [0,1]$. For instance let $\Theta = 1, x = y = \frac{1}{4}$. Then:

$$L = R \quad \Leftrightarrow \quad \frac{\frac{1}{16}}{1-\frac{9}{16}} + \frac{3}{4} = \frac{\frac{9}{16}}{1-\frac{1}{16}} + \frac{1}{4} \quad \Leftrightarrow \quad \frac{1}{7} + \frac{3}{4} = \frac{9}{15} + \frac{1}{4} \quad \Leftrightarrow \quad \frac{25}{28} = \frac{17}{20}$$

Contradiction.
(We can get the result, that the equation (2) is satisfied $\Leftrightarrow (\Theta = 0 \vee x \in \{0,1\} \vee y \in \{0,1\} \vee x + y = 1$*).*

□

Lemma 5. *Let $\tilde{I}_C$ be any probabilistic S-implication. If $\tilde{I}_C$ satisfies (L-CP) or (R-CP) with respect to N, then N is a strong negation $N_C(x) = 1 - x$. Furthermore, some probabilistic S-implications $\tilde{I}_C$ satisfy (L-CP) and (R-CP) with respect to N_C (for instance implications based on copulas Π, M, W or copulas from family $FGM(\Theta)$), and some do not (for instance implications based on copulas from family $AMH(\Theta)$).*

Dowód. The proof of the first thesis is analogous to the proof of Lemma 4. Next to prove that specific probabilistic S-implications satisfy (L-CP) and (R-CP) or do not satisfy them, we use Lemma 1 and the last Remark. □

The next Remark points new big groups of probabilistic S-implications which appropriately satisfy (CP) and do not satisfy (CP).

Remark. I have showed that for any copula C such that $C = \Theta C_1 + (1-\Theta)C_2$, where C_1 and C_2 are copulas, $\Theta \in [0,1]$, and $\tilde{I}_{C_1}$ or $\tilde{I}_{C_2}$ satisfy (CP) with respect to N_C, the following equivalence is true:

$\tilde{I}_C$ satisfies (CP) with respect to N_C $\iff$ both $\tilde{I}_{C_1}$ and $\tilde{I}_{C_2}$ satisfy (CP) with respect to N_C.

5 The Law of Importation (LI) for probabilistic implications

$$(LI) \quad I_C(T(x,y),z) = I_C(x, I_C(y,z))$$

Lemma 6. *If any probabilistic implication I_C and t-norm T satisfy the Law of Importation (LI), then T is positive.*
(t-norm T is positive $\iff \neg\exists_{x,y\neq 0} T(x,y) = 0$).

Dowód. For $x = 0 \vee y = 0$ it is easy to check, that $L = R$ for any copula C and t-norm T.
Let us assume, that $x, y \neq 0$ and $z = 0$. Then:

$$R = I_C(x, I_C(y,0)) = I_C(x,0) = 0$$

and:

$$L = I_C(T(x,y),0) = \begin{cases} 1, & T(x,y) = 0 \\ 0, & T(x,y) > 0 \end{cases}$$

Therefore if there exist $x, y \neq 0$ such, that $T(x,y) = 0$, then $L \neq R$. □

Now we will study five examples of copulas and corresponding to them probabilistic implications, looking for positive t-norms, which those specific implications satisfy (LI) with. We can assume that $x, y, z \neq 0$. In this case (LI) is equivalent to

$$\frac{C(T(x,y),z)}{T(x,y)} = \frac{C(x, \frac{C(y,z)}{y})}{x}. \tag{3}$$

Example 3. $C = \Pi$ *—with any positive t-norm*

$$\text{The equation (3)} \;\Leftrightarrow\; \frac{T(x,y)\cdot z}{T(x,y)} = \frac{x \cdot \frac{y \cdot z}{y}}{x} \;\Leftrightarrow\; z = z.$$

Therefore probabilistic implication I_Π satisfies (LI) any positive t-norm T.

Example 4. $C = M$ *— only with T_P*

$$\text{The equation (3)} \;\Leftrightarrow\; \frac{\min(T(x,y),z)}{T(x,y)} = \frac{\min(x, \frac{\min(y,z)}{y})}{x}$$

Let us consider two cases:

1). $y \leqslant z$.

$$R = \frac{\min(x, \frac{y}{y})}{x} = \frac{x}{x} = 1 \text{ and } L = \frac{T(x,y)}{T(x,y)} = 1,$$

because $T(x,y) \leqslant T(1,y) = y \leqslant z$.
Thus in this case for any positive t-norm T equation $L = R$ is satisfied.

2). $y > z$.

$$(3) \;\Leftrightarrow\; \frac{\min(T(x,y),z)}{T(x,y)} = \frac{\min(x, \frac{\min(y,z)}{y})}{x} \;\Leftrightarrow\; \frac{\min(T(x,y),z)}{T(x,y)} = \frac{\min(x, \frac{z}{y})}{x}$$

$$\Leftrightarrow \quad \min(1, \frac{z}{T(x,y)}) = \min(1, \frac{z}{xy})$$

Thus for any $x, y, z \in (0,1] : \frac{z}{xy} < 1$ the equation $T(x,y) = xy$ must be true. But for any pair of $(x,y) \in (0,1]^2$ we can choose such $z \in (0,1]$, that $z < xy$. From this we have

$$T(x,y) = xy \quad \forall_{x,y \in (0,1]}$$

We've proved, that probabilistic implication I_M with t-norm T satisfies $(LI) \Leftrightarrow T = T_P$.

Example 5. $C = W$ *— only with T_P*

The equation (3) $\Leftrightarrow \quad \dfrac{\max(T(x,y)+z-1,0)}{T(x,y)} = \dfrac{\max(x+\frac{\max(y+z-1,0)}{y}-1,0)}{x}$

$$\Leftrightarrow \quad \max(1+\frac{z-1}{T(x,y)},0) = \max(1+\frac{\max(y+z-1,0)}{xy}-\frac{1}{x},0)$$

We can simplify the form of the right side of the above equation:

$$R = \begin{cases} \max(1+\frac{y+z-1}{xy}-\frac{1}{x},0), & y+z \geqslant 1 \\ \max(1+\frac{0}{xy}-\frac{1}{x},0), & y+z<1 \end{cases} = \begin{cases} \max(1+\frac{z-1}{xy},0), & y+z \geqslant 1 \\ 0, & y+z<1 \end{cases} = \max(1+\frac{z-1}{xy},0)$$

The last equation results from the fact, that $1+\frac{z-1}{xy} < 0 \quad \Leftrightarrow \quad xy+z-1 < 0 \quad \Leftrightarrow \quad xy+z<1$, and for $y+z<1$ we have $xy+z < 1\cdot y+z<1$, thus $\max(1+\frac{z-1}{xy},0)=0$ for $y+z<1$.

Finally

$$L = R \quad \Leftrightarrow \quad \max(1+\frac{z-1}{T(x,y)},0) = \max(1+\frac{z-1}{xy},0)$$

Thus for all $x,y,z \in (0,1] : 1+\frac{z-1}{xy} > 0$ the equation $T(x,y)=xy$ must be satisfied. And for $z=1$, and for any $x,y \in (0,1]$ an inequality $1+\frac{z-1}{xy}>0$ is true. From this we have

$$T(x,y) = xy \quad \forall_{x,y\in(0,1]}$$

We've proved, that probabilistic implication I_W with t-norm T satisfies $(LI) \Leftrightarrow T = T_P$.

After above three examples we might suppose that product t-norm T_P satisfies (LI) with any probabilistic implication. However the next example proves that it is not true.

Example 6. $C \in FGM(\Theta)$ *— for no T, if $\Theta \neq 0$*

$$\text{The equation (3)} \Leftrightarrow \quad \frac{C(T(x,y),z)}{T(x,y)} = \frac{C(x,\frac{C(y,z)}{y})}{x}$$

$$L = \frac{T(x,y)z+\Theta T(x,y)z(1-T(x,y))(1-z)}{T(x,y)} = z+\Theta z(1-T(x,y))(1-z)$$

$$R = \frac{\cancel{x}\cdot\frac{C(y,z)}{y}+\Theta\cancel{x}\cdot\frac{C(y,z)}{y}(1-x)(1-\frac{C(y,z)}{y})}{\cancel{x}}$$
$$= \underbrace{[z+\Theta z(1-y)(1-z)]}_{\frac{C(y,z)}{y}} + \Theta\cdot[z+\Theta z(1-y)(1-z)]\cdot(1-x)\cdot(1-[z+\Theta z(1-y)(1-z)])$$
$$= z+\Theta z(1-y)(1-z)+\Theta z(1+\Theta(1-y)(1-z))(1-x)(1-z)(1-\Theta z(1-y))$$
$$= z+\Theta z(1-z)\cdot[(1-y)+(1+\Theta(1-y)(1-z))(1-x)(1-\Theta z(1-y))]$$

Thus for $z \notin \{0,1\}$ and $\Theta \neq 0$ we have

$$\begin{aligned} L = R \quad &\iff \quad 1 - T(x,y) = [(1-y) + (1+\Theta(1-y)(1-z))(1-x)(1-\Theta z(1-y))] \\ &\iff \quad T(x,y) = y - (1+\Theta(1-y)(1-z))(1-x)(1-\Theta z(1-y)) \end{aligned}$$

We've obtained the formula for t-norm T, which is dependent not only on x and y, but also on z. Therefore to make it sensible, it should take the same value for all $z \in (0,1)$ for any specific pair $(x,y) \in (0,1]^2$. It is easy to check that it is not true.

This way we've proved that probabilistic implications generated from copulas from family $FGM(\Theta)$ do not satisfy (LI) with any t-norm T (except the case when $\Theta = 0$, but then we deal with product copula Π we've studied before).

Example 7. $C \in AGH(\Theta)$ *— if $\Theta \neq 0$, then the only option is $\Theta = 1$ and $T = T_P$*

$$\text{The equation (3)} \ \Leftrightarrow \ \frac{C(T(x,y),z)}{T(x,y)} = \frac{C(x, \frac{C(y,z)}{y})}{x}$$

Thus for $C(x,y) = \frac{xy}{1-\Theta(1-x)(1-y)}$ we have:

$$L = \frac{T(x,y)\cdot z}{T(x,y)\cdot[1-\Theta(1-T(x,y))(1-z)]} = \frac{z}{1-\Theta(1-T(x,y))(1-z)}$$

$$\begin{aligned} R &= \frac{\cancel{x}\cdot\frac{C(y,z)}{y}}{\cancel{x}\cdot[1-\Theta(1-x)(1-\frac{C(y,z)}{y})]} = \frac{z}{1-\Theta(1-y)(1-z)}\cdot\frac{1}{1-\Theta(1-x)(1-\frac{z}{1-\Theta(1-y)(1-z)})} \\ &= \frac{z}{1-\Theta(1-y)(1-z)-\Theta(1-x)[1-\Theta(1-y)(1-z)-\frac{z\cdot(1-\Theta(1-y)(1-z))}{1-\Theta(1-y)(1-z)}]} \\ &= \frac{z}{1-\Theta(1-y)(1-z)-\Theta(1-x)(1-z)(1-\Theta(1-y))} \\ &= \frac{z}{1-\Theta(1-z)[1-y+(1-x)(1-\Theta(1-y))]} \end{aligned}$$

Assuming that $z \neq 0$ and $\Theta \neq 0$ we have:

$$\begin{aligned} L = P \quad &\iff \quad \frac{z}{1-\Theta(1-T(x,y))(1-z)} = \frac{z}{1-\Theta(1-z)[1-y+(1-x)(1-\Theta(1-y))]} \\ &\iff \quad 1-T(x,y) = 1-y+(1-x)(1-\Theta(1-y)) \\ &\iff \quad T(x,y) = x+y-1+\Theta(1-x)(1-y) \end{aligned}$$

We've got unique function T. There remains the question if this function is a t-norm? It turns out, that only for $\Theta = 1$ the set of values of T is $[0,1]$:

- $\Theta = 1 \quad \Rightarrow \quad T(x,y) = xy$
- $\Theta < 1 \quad \Rightarrow \quad$ for $x = y = \frac{1}{n}$ we have:

  $$T(\frac{1}{n},\frac{1}{n}) = \frac{2}{n} - 1 + \Theta(\frac{n-1}{n})^2 \geqslant 0 \quad \Leftrightarrow \quad n^2(\Theta-1) + n2(1-\Theta) + \Theta \geqslant 0$$

  A coefficient of n^2 is negative ($\Theta - 1 < 0$), thus on the left side of the last inequality there is a concave quadratic function, which can't have its whole plot above x-axis. Therefore for some values n we have $T(\frac{1}{n},\frac{1}{n}) < 0$, which implies that function T is not a t-norm.

Finally we can conclude, that among probabilistic implications generated from copulas from family $AMH(\Theta)$ for $\Theta \neq 0$, only an implication for $\Theta = 1$, with product t-norm T_P, satisfies (LI).

Let us notice one more fact:

Remark. Probabilistic implications generated from Π and M satisfy (LI) with t-norm T_P, however no implication generated from copulas $C = \Theta\Pi + (1-\Theta)M$ for $\Theta \in (0,1)$ satisfy (LI) (with T_P and any other t-norm T).

6 The Law of Importation (LI) for probabilistic S-implications

$$\begin{aligned} (LI) \quad & \tilde{I}_C(T(x,y),z) = \tilde{I}_C(x, \tilde{I}_C(y,z)) \\ \iff \quad & C(T(x,y),z) - T(x,y) + 1 = C(x, C(y,z) - y + 1) - x + 1 \end{aligned}$$

Lemma 7. *If any probabilistic S-implication $\tilde{I}_C$ satisfies (LI) with any t-norm T, then T must be of the form $T(x,y) = x - C(x, 1-y)$.*

Dowód. Let $z = 0$. Then

$$\begin{aligned} & C(T(x,y),z) - T(x,y) + 1 = C(x, C(y,z) - y + 1) - x + 1 \\ \iff \quad & 0 - T(x,y) + 1 = C(x, 0 - y + 1) - x + 1 \\ \iff \quad & T(x,y) = x - C(x, 1-y) \end{aligned}$$

□

Remark. On the other endpoints of x, y, z's domains ($x = 0$, $y = 0$, $x = 1$, $y = 1$ and $z = 1$) the equation (LI) is satisfied for any copula C and t-norm $T(x,y) = x - C(x, 1-y)$.
What we have to do now is to check for every probabilistic S-implication $\tilde{I}_C$ if function $T(x,y) = x - C(x,1-y)$ satisfies (LI) on the rest of its domain, i.e. for $x, y, z \in (0,1)$, and if such defined T is a t-norm at all.

The following table includes the list of functions which are the only candidates for t-norms which particular probabilistic S-implications may satisfy (LI) with:

C_Π	$T_P(x,y) = xy$
C_M	$T_{LK}(x,y) = \max(x+y-1, 0)$
C_W	$T_M(x,y) = \min(x,y)$
$C \in FGM(\Theta)$	$T(x,y) = xy - \Theta xy(1-x)(1-y)$
$C \in AMH(\Theta)$	$T(x,y) = xy\frac{1-\Theta(1-x)}{1-\Theta(1-x)y}$

Remark. The pairs of functions $(\tilde{I}_{C_\Pi}, T_P), (\tilde{I}_{C_M}, T_{LK}), (\tilde{I}_{C_W}, T_M)$ satisfy (LI), whereas probabilistic S-implications generated from copulas from families $FGM(\Theta)$ or $AMH(\Theta)$ do not satisfy (LI) with functions listed in above table.

Dowód. Let us check if those pairs of functions satisfy considered equation:

$$C(T(x,y),z) - T(x,y) = C(x, C(y,z) - y + 1) - x \qquad (LI')$$

for $x, y, z \in (0,1)$.

1. (C_Π, T_P) OK

$$L = xyz - xy;$$
$$R = x(yz - y + 1) - x = xyz - xy = L.$$

2. (C_M, T_{LK}) OK

$$L = \min(\max(x+y-1,0), z) - \max(x+y-1,0);$$
$$R = \min(x, \min(y,z) - y + 1) - x.$$

Let us consider such cases:

- $x + y - 1 \leqslant 0 \quad \Longrightarrow \quad L = \min(0,z) - 0 = 0;$
  A). $y \leqslant z \quad \Rightarrow \quad R = \min(x, y - y + 1) - x = x - x = 0 = L;$
  B). $y > z \quad \Rightarrow \quad R = \min(x, z - y + 1) - x;$
  We have $x \leqslant -y + 1 < z - y + 1$, thus $R = x - x = 0 = L$.
- $0 < x + y - 1 \leqslant z \quad \Longrightarrow \quad L = x + y - 1 - (x + y - 1) = 0;$
  A). $y \leqslant z \quad \Rightarrow \quad R = \min(x, y - y + 1) - x = x - x = 0 = L;$
  B). $y > z \quad \Rightarrow \quad R = \min(x, z - y + 1) - x = x - x = 0 = L.$
- $z < x + y - 1$ ($\Rightarrow z < y$):
  $L = z - (x + y - 1);$
  $R = \min(x, z - y + 1) - x = z - y + 1 - x = L.$

3. (C_W, T_M) OK

$$L = \max(\min(x,y) + z - 1, 0) - \min(x,y);$$
$$R = \max(\max(y + z - 1, 0) + x - y, 0) - x.$$

Let us consider such cases:

- $x \leqslant y$
  A). $y + z - 1 \leqslant 0$ ($\Rightarrow x + z - 1 \leqslant 0$)

$$L = \max(x + z - 1, 0) - x = -x;$$
$$R = \max(0 + x - y, 0) - x = -x = L.$$

  B). $y + z - 1 > 0$

$$L = \max(\min(x + z - 1, 0) - x;$$
$$R = \max(z - 1 + x, 0) - x = L.$$

- $x > y$
  A). $y + z - 1 \leqslant 0$

$$L = \max(y + z - 1, 0) - y = -y;$$
$$R = \max(0 + x - y, 0) - x = x - y - x = -y = L.$$

  B). $y + z - 1 > 0$ ($\Rightarrow x + z - 1 > 0$)

$$L = \max(y + z - 1, 0) - y = z - 1;$$
$$R = \max(z - 1 + x, 0) - x = z - 1 = L.$$

4. $(C_{FGM(\Theta)}, T)$ X

$$C_\Theta(x, y) = xy + \Theta xy(1 - x)(1 - y);$$
$$T(x, y) = xy - \Theta xy(1 - x)(1 - y).$$

For instance for $x = y = \frac{1}{2}$ and $z = \frac{1}{4}$ after some tedious calculations we get

$$L = R \iff \Theta = -16 \notin [-1, 1]$$

That is how we have proved that considered pair $(C_{FGM(\Theta)}, T)$ does not satisfy (LI') for any $\Theta \in [-1, 1]$.

5. $(C_{AMH(\Theta)}, T)$ X

$$C(x, y)_\Theta = \frac{xy}{1 - \Theta(1 - x)(1 - y)};$$
$$T(x, y) = xy\frac{1 - \Theta(1 - x)}{1 - \Theta(1 - x)y}.$$

For instance for $x = y = \frac{1}{2}$ and $z = \frac{1}{4}$ after some next tedious calculations we get

$$L = R \iff \Theta_1 = 2, \Theta_2 = 6 \notin [-1, 1]$$

Therefore the pair $(C_{AMH(\Theta)}, T)$ can't satisfies (LI') for any $\Theta \in [-1, 1]$.

□

7 Conclusion

We have examined particular properties of probabilistic implications and probabilistic S-implications, i.e. (CP), (L-CP), (R-CP) and (LI), in this paper. There are still open questions if it's possible to make some conclusions from this paper more general. For instance if there is any particular family of copulas which satisfy the equation (2) from Corollary 2? Or if the formula $T(x, y) = x - C(x, 1 - y)$ from Lemma 7 expresses any special kind of relation between functions T and C (some "duality")?
We may draw hypothesis that if any probabilistic implication satisfies (LI) with any t-norm T, then for sure it satisfies (LI) with product t-norm T_P. It remains to be checked.
The families of probabilistic implications and probabilistic S-implications are still new and there are a lot of other properties which should be examined in their cases.

Literatura

[1] M. Baczyński, B. Yayaram, Fuzzy Implications, Springer 2008, Berlin.

[2] P. Grzegorzewski, Probabilistic Implications, Fuzzy Sets and Systems 226 (2013), 53-66

[3] R.B. Nelsen, An Introduction to Copulas, Springer 1999, New York.