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A construction of the Urysohn spaces

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A CONSTRUCTION OF THE URYSOHN SPACES

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Abstract. We give a construction of universal metric spaces. This is done in two steps. First, we define a hedgehog-like space and prove that iterating its construction gives us a κ-homogeneous space. Observing that κ-homogeneous spaces are complete for $\kappa > \omega$, we need to take the completion for $\kappa = \omega$. The second construction is a slight modification of a given metric space by taking distance of fixed pair of points to be as small as possible. This modification helps us to show that the completion of ω-homogeneous metric space is still ω-homogeneous and universal for separable metric spaces.

1. Introduction

P. S. Urysohn [4] proved that there exists a complete metric space (U, d) which is *universal* for separable metric spaces, i.e. has the property that any separable metric space is isometric to a subspace of U. As usual, isometry of a metric space (X, ρ) into (U, d) is a function $f\colon X \to U$ such that

$$d(f(x), f(y)) = \rho(x, y)$$

for all $x, y \in X$. Therefore, isometry is always one-to-one but is not "onto" in general. If $D = \{x_n : n < \omega\} \subseteq X$ is dense, then it sufficient to embed isometrically D into U and use the fact, that every isometry can be extended over the closure of D since (U, d) is complete. Embedding of D into X can be obtained as an extension of a given isometry $\{x_k : k < n\} \to X$ to an isometry $\{x_k : k < n + 1\} \to X$, for $n < \omega$. This leads us to the notion of *homogeneity.*

If κ is a cardinal, then say that the metric space (X, d) is κ-*homogeneous* if, for any $A \subseteq X$with $|A| < \kappa$ and every isometry $f_0 : A \to f_0[A] \subseteq X$ there exists an isometry f of X onto X such that $f \restriction A = f_0$. We say that the metric space (X, d) is *universal for the class* $\mathcal{K}$ if for any $(Y, \sigma) \in \mathcal{K}$ there exists an isometry $f : Y \to f[Y] \subseteq X$. We say that the metric space is (*strongly*) κ-*universal* if it is universal for the class of all metric spaces of cardinality (of weight) not greater than κ. We also say that a space of weight κ is *Urysohn universal* if it is κ-homogeneous and strongly κ-universal.

Theorems proved here were known in the literature, see e.g. [3]. The purpose of this paper is to present another proofs and in particular to describe a connection between these theorems and some new lemmas to be formulated in the next section.

2. Main lemmas

A metric ρ on X is called rational whenever $\rho(x,y)$ ia a rational number for all $x,y \in X$. The lemmas mentioned above are the following:

Lemma 2.1 *Let $(X,d),(X_s,d_s), s \in S$, be metric spaces such that*

(i) $d \restriction X \cap X_s = d_s \restriction X \cap X_s$, *for all* $s \in S$,

(ii) $X \cap X_s \neq \emptyset$, *for all* $s \in S$,

(iii) $X_s \cap X_t \subseteq X$, *for all* $s \neq t$.

Then there exists a metric ρ on $\bigcup_{s\in S} X_s$ such that $\mathrm{id}_{X_s} : X_s \to \bigcup_{s\in S} X_s$ is an isometry, for all $s \in S$. Moreover, if d and all the metrics d_s are rational and $|X \cap X_s| < \omega$ for every $s \in S$, then the metric ρ is rational as well.

Proof. We define the metric ρ as follows:

$$\rho(a,b) = \inf \{d_s(a,x) + d(x,y) + d_t(y,b) : x \in X \cap X_s, y \in X \cap X_t\}.$$

We will check the triangle inequality $\rho(a,c) \leq \rho(a,b)+\rho(b,c)$ in the following cases:

(a) $a,b,c \in X_s$ or $a,b,c \in X$,

(b) $a,b \in X_s$ and $c \in X \setminus X_s$,

(c) $a \in X_s$ and $b,c \in X$,

(d) $a,b \in X_s$, $c \in X_t \setminus X$ and $s \neq t$,

(e) $a \in X_s$, $b \in X_t$, $c \in X$ and $s \neq t$,

Case (a). If $a,b,c \in X$ or $a,b,c \in X_s$ then the triangle inequality

$$\rho(a,c) \leq \rho(a,b) + \rho(b,c).$$

follows from the fact that d and d_s are metrics.

Case (b). Let $x, y \in X \cap X_s$ be such that $\rho(a,c)+\varepsilon > d_s(a,x)+d(x,c)$ and $\rho(c,b)+\varepsilon > d(c,y)+d_s(y,b)$. Since $d(x,y) = d_s(x,y)$, we obtain

$$\rho(a,b) = d_s(a,b) \le d_s(a,x) + d(x,y) + d_s(y,b) \le$$

$$\le d_s(a,x) + d(x,c) + d(c,y) + d_s(y,b) < \rho(a,c) + \rho(c,b) + 2\varepsilon$$

and

$$\rho(a,c) \le d_s(a,y) + d(y,c) \le$$
$$\le d_s(a,b) + d_s(b,y) + d(y,c) = \rho(a,b) + \rho(b,c) + \varepsilon.$$

Case (c). Let $\varepsilon > 0$ be fixed and $x, y \in X \cap X_s$ be such that $\rho(a,b)+\varepsilon > d_s(a,x)+d(x,b)$ and $\rho(a,c)+\varepsilon > d_s(a,y)+d(y,c)$. Then

$$\rho(b,c) = d(b,c) \le d(b,x) + d(x,y) + d(y,c) \le$$

$$\le d(b,x) + d_s(x,a) + d_s(a,y) + d(y,c) < \rho(b,a) + \rho(a,c) + 2\varepsilon$$

and

$$\rho(a,c) = d_s(a,y) + d(y,c) \le d_s(a,x) + d(x,c) \le$$
$$\le d_s(a,x) + d(x,b) + d(b,c) < \rho(a,b) + \rho(b,c) + \varepsilon.$$

Case (d). Let $x, y, z, w \in X$ be such that $\rho(a,c)+\varepsilon > d_s(a,x)+d(x,y)+d_t(y,c)$ and $\rho(c,b)+\varepsilon > d_t(c,z)+d(z,w)+d_s(w,b)$. Then, using previous cases,

$$\rho(a,b) \le \rho(a,y) + \rho(y,w) + \rho(w,b) = \rho(a,y) + d_t(y,w) + \rho(w,b) \le$$

$$\le d_s(a,x)+d(x,y)+d_t(y,c)+d_t(c,w)+d(w,z)+d_s(z,b) < \rho(a,c)+\rho(c,b)+2\varepsilon$$

and

$$\rho(a,c) \le d_s(a,z) + d(z,w) + d_t(w,c) \le$$
$$\le d_s(a,b) + d_s(b,z) + d(z,w) + d_t(w,c) \le \rho(a,b) + \rho(b,c) + \varepsilon.$$

Case (e). Let $x, y, z, w \in X$ be such that $\rho(a,b)+\varepsilon > d_s(a,x)+d(x,y)+d_t(y,b)$, $\rho(b,c)+\varepsilon > d_t(b,w)+d(w,c)$ and $\rho(a,c)+\varepsilon > d_s(a,z)+d(z,c)$. Then

$$\rho(a,b) \le d_s(a,z) + d(z,c) + d(c,w) + d_t(w,b) < \rho(a,c) + \rho(c,b) + 2\varepsilon$$

and, using previous cases,

$$\rho(a,c) \le d_s(a,x) + d(x,c) \le$$

$$\le d_s(a,x) + d(x,y) + d_t(y,b) + d_t(b,w) + d(w,c) < \rho(a,b) + \rho(b,c) + 2\varepsilon.$$

□

Lemma 2.2 *Let $\{(Z_\beta, \rho_\beta) : \beta < \alpha\}$ be a family of metric spaces such that $\mathrm{id}_{Z_\beta} : Z_\beta \to Z_\gamma$ is an isometry, for all $\beta < \gamma$. Then there exists metric ρ on the set $\bigcup_{\gamma<\alpha} Z_\gamma$ such that $\mathrm{id}_{Z_\beta} : Z_\beta \to \bigcup_{\gamma<\alpha} Z_\gamma$ is an isometry, for all $\beta < \alpha$. Moreover, if ρ_β is rational for all $\beta < \alpha$, then ρ is also rational.*

Proof. Since family $\{(Z_\beta, \rho_\beta) : \beta < \alpha\}$ is a chain, function $\rho(x, y) = \rho_\beta(x, y)$, for $x, z \in Z_\beta$, is defined correctly. For all $x, y, z \in \bigcup_{\gamma<\alpha} Z_\gamma$, there exists $\beta < \alpha$ such that $x, y, z \in Z_\beta$ and ρ_β is a metric, thus ρ is also a metric. □

Let us observe that in fact we have $\rho = \bigcup_{\beta<\alpha} \rho_\beta$. In the situation described in the above lemma instead of $(\bigcup_{\beta<\alpha} Z_\beta, \bigcup_{\beta<\alpha} \rho_\beta)$ we shall use the symbol

$$\bigcup_{\beta<\alpha}(Z_\beta, \rho_\beta).$$

The following lemma can be found in [1] and thus we an omit the proof.

Lemma 2.3 *If (X, d) is a finite metric space, $x_0 \in X$, $\varepsilon > 0$ and $y \notin X$, then there exists a metric ρ on the set $X \cup \{y\}$ such that*

(1) $\rho \restriction X \times X = d$,

(2) $\rho(x_0, y) < \varepsilon$,

(3) $\rho(x, y) \in \mathbb{Q}$ *for all* $x \in X$.

In the next lemma we shall change the metric in two points.

Lemma 2.4 *If (X, ρ) is a metric space, $|X| \geq 3$ and $a, b \in X$ are fixed, then the function $\sigma: X \times X \to \mathbb{R}$ given by the formula*

$$\sigma(a, b) = \sup\{|\rho(z, a) - \rho(z, b)| : z \in X \setminus \{a, b\}\},$$

and $\sigma(x, y) = \rho(x, y)$, for $\{x, y\} \neq \{a, b\}$, is a metric on X, which is different from ρ at most in $\{a, b\}$.

Proof. It suffices to check the triangle inequality for the triple a, b, x where x is not in $\{a, b\}$. Then we have

$$\rho(a, x) - \rho(b, x) \leq \sigma(a, b)$$

and thus $\sigma(a, x) = \rho(a, x) \leq \sigma(a, b) + \rho(b, x) = \sigma(a, b) + \sigma(b, x)$. Since

$$\rho(a, z) \leq \rho(a, x) + \rho(x, b) + \rho(b, z),$$

we see that $\rho(a, z) - \rho(b, z) \leq \rho(a, x) + \rho(x, b)$, for all $z \in X \setminus \{a, b\}$, which implies $\sigma(a, b) \leq \rho(a, x) + \rho(x, b) = \sigma(a, x) + \sigma(x, b)$ and completes the proof. □

Lemma 2.5 *Let (X,d) be a metric space and $\kappa \geq \omega$ be a cardinal number and the following condition holds true:*

($*$) *if (Y,σ) is a (rational, if $\kappa=\omega$) metric space, $|Y|<\kappa$ and $y\in Y$ and $f_0 : Y\setminus\{y\} \to X$ is an isometry, then there exists an isometry $f : Y \to X$ such that $f\restriction(Y\setminus\{y\}) = f_0$.*

Then (X,d) is κ-homogeneous and κ-universal.

Proof. Let $Y \subseteq X$, $|Y|<\kappa$ and $f_0 : Y \to f_0[Y] \subseteq X$ be an isometry. Let $X\setminus Y = \{x_\alpha : \alpha<\kappa\}$ and $X\setminus f_0[Y] = \{y_\alpha : \alpha<\kappa\}$. Assume that there exists an isometry f_α such that $Y\cup\{x_\beta : \beta<\alpha\} \subseteq \operatorname{dom} f_\alpha \subseteq X$ and $f_0[Y]\cup\{y_\beta : \beta<\alpha\} \subseteq \operatorname{rng} f_\alpha \subseteq X$. If $x_\alpha \in \operatorname{dom} f_\alpha$ and $y_\alpha \in \operatorname{rng} f_\alpha$, then we set $f_{\alpha+1} = f_\alpha$. If $x_\alpha \notin \operatorname{dom} f_\alpha$, then by the assumption, there exists $y \in X\setminus \operatorname{rng} f_\alpha$ such that

$$f(x) = \begin{cases} f_\alpha(x), & \text{if } x \in \operatorname{dom} f_\alpha, \\ y, & \text{if } x = x_\alpha, \end{cases}$$

is an isometry. If $y_\alpha \in \operatorname{rng} f$, then we set $f_{\alpha+1} = f$. If $y_\alpha \notin \operatorname{rng} f$, then there exists $z \in X\setminus \operatorname{dom} f$ such that

$$f_{\alpha+1}(x) = \begin{cases} f(x), & \text{if } x \in \operatorname{dom} f, \\ y_\alpha, & \text{if } x = z, \end{cases}$$

is an isometry. If α is a limit ordinal, then we define $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Thus f_κ is an isometry of X into X.

Let (Y,σ) be a (rational, if $\kappa=\omega$) metric space such that $Y=\{y_\alpha : \alpha<\kappa\}$. Let $f(y_0)\in X$ be chosen arbitrarily. If f is defined on the set $\{y_\beta : \beta<\alpha\}$ then we set $f(y_\alpha)$ to be such that $f : \{y_\beta : \beta\leq\alpha\} \to \{f(y_\beta) : \beta\leq\alpha\}$ is an isometry. □

3. A construction

In this section we give a general construction of the Urysohn universal spaces of weight κ, if $\omega<\kappa=\kappa^{<\kappa}$, and some special space for $\kappa=\omega$. The case $\kappa=\omega$ needs special treatment since weight of the Urysohn universal space in this case is strictly less than 2^ω, the cardinality of all possible values of metric. Thus we should considered values of metric restricted to some countable field, e.g. $\mathbb{Q}$. Such a modification leads us to the space which is rational, countable, ω-homogeneous and ω-universal *for rational metric spaces.*

It can be shown (cf. [3]) that if $\kappa < \kappa^{<\kappa}$, then there is no Urysohn universal space of weight κ. However, if $\kappa = \kappa^{<\kappa}$, then such a space exists.

If κ is a cardinal number, then we write $[A]^\kappa = \{B \subseteq A : |B| = \kappa\}$ and $[A]^{<\kappa} = \{B \subseteq A : |B| < \kappa\}$.

Let $\kappa = \kappa^{<\kappa}$ be a cardinal and let

$$\mathcal{M} = \{(X,d) : (X,d)\text{is a metric space, } |X| \leq \kappa\}.$$

Let us fix $(X,d) \in \mathcal{M}$ (if $\kappa = \omega$ then we assume additionally that $\mathcal{M}$ consists of the rational metric spaces) and a cardinal $\lambda < \kappa$. We also fix a set A of cardinality κ, disjoint with X, and a partition $\{A_\alpha : \alpha < \kappa\} \subseteq [X]^\kappa$ of the set A. Since $|X| \leq \kappa$ we have $|[X]^{<\kappa}| \leq \kappa^{<\kappa} = \kappa$. Let us enumerate

$$[X]^{<\kappa} = \{X_\alpha : \alpha < \kappa\}\text{and } A_\alpha = \{a_{\alpha,\beta} : \beta < \kappa\}.$$

In case $\omega < \kappa$ the family of all ($\mathbb{R}$-valued) metrics on a set of cardinality λ is of cardinality at most $|2^\omega|^\lambda \leq \kappa^\lambda \leq \kappa$ and in case $\kappa = \omega$ the family of all rational metrics on a set of cardinality λ is of cardinality $|\omega|^\lambda = \omega$. Thus for all $\beta < \kappa$ there exist metrics $d_{\alpha,\beta}$ on the sets $X_\alpha \cup \{a_{\alpha,\beta}\}$, such that if (Y,σ) is a metric space (rational metric space, in case $\kappa = \omega$) and if $f_0 : Y \setminus \{y\} \to X_\alpha$ is an isometry, then there exists an isometry $f : Y \to X_\alpha \cup \{a_{\alpha,\beta}\}$ such that $f \restriction (Y \setminus \{y\}) = f_0$.

Using lemma 2.1, for spaces $(X_\alpha \cup \{a_{\alpha,\beta}\}, d_{\alpha,\beta}$ with $\beta < \kappa$, we obtain metric space $(X_\alpha \cup A_\alpha, \rho_\alpha) \in \mathcal{M}$. Using lemma 1.1 for spaces $(X_\alpha \cup A_\alpha, \rho_\alpha)$, with $\alpha < \kappa$, we obtain a metric space $(X \cup A, \rho) \in \mathcal{M}$ which has the following property:

(2) if (Y,σ) is a (rational, if $\kappa = \omega$) metric space, $|Y| < \kappa$ and $f_0 : Y \setminus \{y\} \to X$ is an isometry, then there exists an isometry $f : Y \to X \cup A$ such that $f \restriction (Y \setminus \{y\}) = f_0$.

Let $F((X,d)) = (X \cup A, \rho)$, where $(X \cup A, \rho)$ is as above. Observe that in the case $\kappa = \omega$ the space $F((X,d))$ is rational.

4. A construction of the Urysohn rational space

In this section we assume that $\kappa = \omega$. A metric space (X,d) is called a *Urysohn rational space* if it is countable, rational, ω-homogeneous and universal for the class of all countable and rational metric spaces.

Let $F^1 = F$, $F^{n+1} = F \circ F^n$, for $n < \omega$, where F is the function defined in the previous section. The defined is correct since $F((X,d)) \in \mathcal{M}$, for $(X,d) \in \mathcal{M}$.

Theorem 4.6 *For any countable rational metric space* (X,d)*, the metric space* $(\mathbb{U}_0, \mathrm{d}_0) = \bigcup_{n<\omega} F^n((X,d))$ *is the Urysohn rational space.*

Proof. Since every $F^n((X,d))$ is countable, the union

$$\bigcup_{n<\omega} F^n((X,d))$$

is countable.

Let $Y \subseteq X$, where (X,d) is a countable rational metric space, be a finite set and let $f_0 : Y \setminus \{y\} \to \mathbb{U}_0$ be an isometry. Since $f_0[Y \setminus \{y\}]$ is finite we have $f_0[Y \setminus \{y\}] \subseteq F^n((X,d))$ for some $n < \omega$. From the definition of F we obtain an isometry $f : Y \to f[Y] \subseteq F(F^n((X,d))) \subseteq \mathbb{U}_0$ such that $f \restriction (Y \setminus \{y\}) = f_0$. From the lemma 2.5 the space $(\mathbb{U}_0, \mathrm{d}_0)$ is ω-homogeneous and ω-universal. □

5. A construction of the universal Urysohn space of weight ω

We will prove that taking $(\mathbb{U}, \mathrm{d})$ to be the completion of $(\mathbb{U}_0, \mathrm{d}_0)$ we obtain Urysohn space.

Theorem 5.7 $(\mathbb{U}, \mathrm{d})$ *is the universal Urysohn space of weight* ω*.*

Proof. We will check condition $(*)$ from lemma 2.5. Let us assume that (X, ρ) is a metric space such that $X \setminus \{x_0\} = \{x_1, \ldots, x_n\} \subseteq \mathbb{U}$. Assume that there exists $Z_k = \{z^i_\ell : \ell < k, 1 \leq i \leq n\} \cup \{y_\ell : \ell < k\} \subseteq \mathbb{U}_0$ and a metric ρ_k on the set $Z_k \cup X$ such that

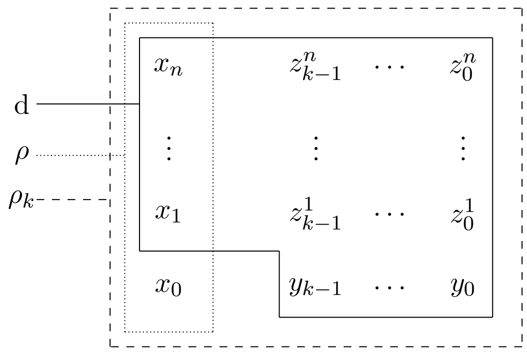
(i) $\mathrm{d}(x_i, z^i_\ell) < 1/(\ell+1)$ for $i \leq n$,

(ii) ρ_k is equal to d on the set $Z_k \cup X \setminus \{x_0\}$ and to ρ on the set X,

(iii) $\rho_k(x_0, y_{k-1}) < 1/2^{k+1}$,

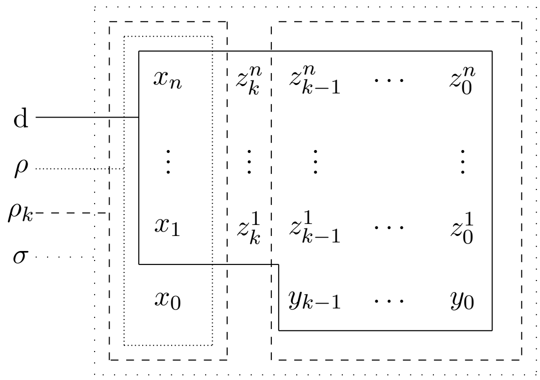
(iv) $\mathrm{d}(y_{k-1}, y_{k-2}) < 1/2^{k-1}$.

Let us illustrate the idea of the proof by the diagram

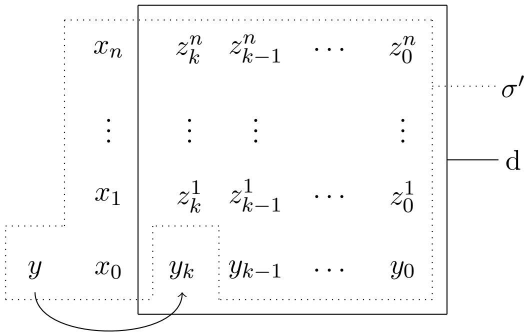


where the solid line is for metric d, the dotted line is for metric ρ and the dashed line is for metric ρ_k. These metrics are compatible on the intersections of their domains.

Let $z^i_k \in \mathbb{U}_0$, $i \le n$, be such that $\mathrm{d}(z^i_k, x_i) < 1/2^{k+3}$. By lemma 2.1 there exists a metric σ on the set $Z_k \cup \{z^i_k : i \le n\} \cup X$ which is equal to the metric d on the set $Z_k \cup \{z^i_k : i \le n\} \cup X \setminus \{x_0\}$ and to the metric ρ_k on the set $Z_k \cup X$:



Using lemma 2.3 we obtain $y \notin \mathbb{U}$ and metric σ' such that σ' extends σ, $\sigma'(x_0, y) < 1/2^{k+3}$ and $\sigma'(x, y) \in \mathbb{Q}$ for all $x \in Z_k \cup \{z^i_k : i \le n\}$. Thus $(Z_k \cup \{z^i_k : i \le n\} \cup \{y\}, \sigma')$ is a rational metric space and there exists $y_k \in \mathbb{U}_0$ such that $(Z_k \cup \{z^i_k : i \le n\} \cup \{y\}, \sigma')$ is isometric to (Z_{k+1}, d) where $Z_{k+1} = Z_k \cup \{z^i_k : i \le n\} \cup \{y_k\}$. Now, the solid line is for the metric d and the dotted line is for the metric σ':



Using lemma 2.1 we obtain the metric ρ_{k+1} which is equal to the metric d on $Z_{k+1} \cup X \setminus \{x_0\}$ and to the metric ρ on X, but still we have to modify value $\rho_{k+1}(x_0, y_k)$ to ensure it is small. From the lemma 2.4 we can assume that

$$\rho_{k+1}(x_0, y_k) = \max\{|\rho_{k+1}(y_k, a) - \rho_{k+1}(x_0, a)| : a \in (X \cup Z_{k+1}) \setminus \{x_0, y_k\}\}.$$

In the case $a = x_i$ we have

$$\rho_{k+1}(y_k, x_i) - \rho_{k+1}(x_0, x_i) \leq \rho_{k+1}(y_k, z_k^i) + \rho_{k+1}(z_k^i, x_i) - \rho_{k+1}(x_0, x_i) <$$

$$\sigma'(y, z_k^i) + 1/2^{k+3} - \rho(x_0, x_i) \leq \sigma'(y, x_0) + \sigma'(x_0, x_i) + 1/2^{k+3} - \rho(x_0, x_i) \leq$$
$$\leq 1/2^{k+3} + 1/2^{k+3} + \rho(x_0, x_i) - \rho(x_0, x_i) = 1/2^{k+2}$$

and

$$\rho_{k+1}(x_0, x_i) - \rho_{k+1}(y_k, x_i) = \sigma'(x_0, x_i) - \rho_{k+1}(y_k, x_i) \leq$$
$$\leq \sigma'(x_0, y) + \sigma'(y, z_k^i) + \sigma'(z_k^i, x_i) - \rho_{k+1}(y_k, x_i) \leq$$
$$\leq 1/2^{k+3} + \rho_{k+1}(y_k, z_k^i) + 1/2^{k+3} - \rho_{k+1}(y_k, x_i) \leq$$
$$\leq 2/2^{k+3} + \rho_{k+1}(z_k^i, x_i) < 1/2^{k+2}.$$

In the case $a = z_\ell^i$ for $\ell \leq k$ or $a = y_\ell$ for $\ell < k$ we get

$$|\rho_{k+1}(y_k, a) - \rho_{k+1}(x_0, a)| = |\sigma'(y, a) - \sigma'(x_0, a)| \leq \sigma'(y, x_0) < 1/2^{k+2}.$$

Moreover,

$$\mathrm{d}(y_k, y_{k-1}) = \sigma'(y, y_{k-1}) \leq \sigma'(y, x_0) + \sigma'(x_0, y_{k-1}) =$$

$$= \sigma'(y, x_0) + \rho_k(x_0, y_{k-1}) < 1/2^{k+3} + 1/2^{k+1} < 1/2^k.$$

Hence $\{y_k : k < \omega\}$ is a Cauchy sequence in $(\mathbb{U}, \mathrm{d})$ and thus it is convergent to an element $t \in \mathbb{U}$. We see that

$$\mathrm{d}(x_i, t) = \lim \mathrm{d}(x_i, y_k) = \lim \mathrm{d}(z_k^i, y_k) = \lim \rho_{k+1}(z_k^i, y_k) \leq$$

$$\leq \lim \rho_{k+1}(z_k^i, x_i) + \rho_{k+1}(x_i, x_0) + \lim \rho_{k+1}(x_0, y_k) = \rho(x_i, x_0)$$

and

$$\rho(x_i, x_0) \leq \lim \rho_{k+1}(x_i, z_k^i) + \lim \rho_{k+1}(z_k^i, y_k) + \lim \rho_{k+1}(y_k, x_0) = \mathrm{d}(x_i, t).$$

Thus taking $f(x_0) = t$ we get the desired isometry.

From the lemma 2.5 we conclude that the space $(\mathbb{U}, \mathrm{d})$ is ω-homogeneous and ω-universal. □

5. Urysohn universal spaces of higher cardinalities

In this section we assume that $\omega < \kappa = \kappa^{<\kappa}$. Since $\kappa < \kappa^{\mathrm{cf}\,\kappa}$, we know that κ is regular. Moreover, $2^\omega \leq \kappa^\omega \leq \kappa$.

Let $F^1 = F$ and $F^{\beta+1} = F \circ F^\beta$. By lemma 1.2 for limit ordinal α we can set

$$F^\alpha((X,d)) = \bigcup_{\beta<\alpha} F^\beta((X,d)).$$

Theorem 6.8 *For a metric space (X,d) such that $|X| \leq \kappa$, the space $F^\kappa((X,d))$ is a universal Urysohn space of weight κ.*

Proof. Let $(U,\rho) = F^\kappa((X,d))$. Observe that

$$|F^\kappa((X,d))| \leq \sum_{\beta<\kappa} |F^\beta((X,d))| \leq \kappa \cdot \kappa = \kappa.$$

Thus (U,ρ) is of weight at most κ. Let $A \subseteq U$ with $|A| < \kappa$ and let $f_0 : A \setminus \{y\} \to U$ be an isometry. From regularity of κ and the fact that $|f_0[A \setminus \{y\}]| < \kappa$ we get $f_0[A \setminus \{y\}] \subseteq F^\gamma((X,d))$, for some $\gamma < \kappa$. Then there exists an isometry $f : A \to f[A] \subseteq F(F^\gamma((X,d))) \subseteq U$ such that $f \restriction (A \setminus \{y\}) = f_0$. From the lemma 2.5 we conclude that $F^\kappa((X,d))$ is κ-homogeneous and κ-universal. □

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