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On convergence of sequences of Radon measures

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ON CONVERGENCE OF SEQUENCES OF RADON MEASURES

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Abstract. We give a detailed presentation of an unpublished note [5] of G. Plebanek on an example of a compact space K such that the space $C(K)$ is not Grothendieck while for every compact and separable $L \subseteq K$, the space $C(L)$ is Grothendieck. Next, under assumption of Martin's axiom, we strengthen this theorem to the subspaces of density less than continuum. Finally, we give a relatively simple construction of a first-countable space K such that the space of all probability measures on K contains an isomorphic copy of $\beta\omega$.

1 Definitions and basic facts

In every Banach space X of infinite dimension there are at least two topologies: the norm topology and the weak topology. The latter is induced by X^*, the space of all linear and continuous functionals defined on X. In the space X^* we have the third possibility: the weak* topology, which is induced by the family $\{\Phi_x : x \in X\}$, where $\Phi_x : X^* \to \mathbb{R}$, $\Phi_x(x^*) = x^*(x)$, $x^* \in X^*$. We describe below typical situation when the convergence in weak* topology differs from the weak one.

Let us consider the space $C(K)$, of all continuous functions from a compact space K to $\mathbb{R}$. This space has the usual norm defined by

$$\|f\| = \sup_{x\in K} |f(x)|.$$

Observe that each measure $\nu : \mathrm{Bor}(K) \to [0,\infty)$, where $\mathrm{Bor}(K)$ is the family of all Borel subsets of K, gives us a functional $x^* \in C(K)^*$, such that

$$x^*(f) = \int_K f \,\mathrm{d}\nu, \quad \text{for every } f \in C(K). \tag{1}$$

By the well-known Riesz theorem, for every functional $x^* \in C(K)^*$ such that $x^*(f) \geqslant 0$ for $f \geqslant 0$, there exists exactly one measure $\nu : \mathrm{Bor}(K) \to [0,\infty)$ such that the condition (1) is satisfied and

$$\nu(A) = \sup\{\nu(B) : B \text{ is a compact subset of } A\}$$

for every $A \in \mathrm{Bor}(K)$. If ν satisfies the last equality, then we say that it is a *Radon measure*.

Let us assume that the space K contains sequence $(x_n)_{n<\omega}$ convergent to x, where $x_n \neq x$ for every $n < \omega$. Let $\delta_{x_n} : \mathrm{Bor}(K) \to \{0,1\}$ be the Dirac measure concentrated at the point x_n, i.e. $\delta_{x_n}(A) = 1$ if and only if $x_n \in A$. Then we see that

$$\int_K f \,\mathrm{d}\delta_{x_n} = f(x_n) \to f(x) = \int_K f \,\mathrm{d}\delta_x,$$

for all $f \in \mathrm{C}(K)$; therefore functionals $\int_K \cdot \,\mathrm{d}\delta_{x_n}, n < \omega$, converge weakly* to $\int_K \cdot \,\mathrm{d}\delta_x$.

Given a Borel set A, we can define $\mathrm{pr}_A : \mathrm{C}(K)^* \to \mathbb{R}$ by the formula

$$\mathrm{pr}_A(x^*) = \nu(A)$$

where measure ν is such that $x^* = \int_K \cdot \,\mathrm{d}\nu$. The function pr_A is linear and continuous for every $A \in \mathrm{Bor}(K)$. We observe that

$$\mathrm{pr}_{\{x_n : n<\omega\}}(\delta_{x_m}) = \delta_{x_m}(\{x_n : n < \omega\}) = 1$$

for all $m < \omega$ and $\mathrm{pr}_{\{x_n:n<\omega\}}(\int_K \cdot \,\mathrm{d}\delta_x) = 0$. Thus functionals δ_{x_n} do not converge weakly to δ_x.

The situation described above leads us to the following definition. We say that a Banach space X is a *Grothendieck space* if every sequence $(x_n^*)_{n<\omega}$, in X^*, weak* convergent to 0 is also weak convergent.

In the special case $X = \mathrm{C}(K)$, we also say that the sequence $(\nu_n)_{n<\omega}, \nu_n : \mathrm{Bor}(K) \to [0, \infty)$, of Radon measures is *weak** (*weak*) convergent if the sequence of functionals, given by these measures, is weak* (weak) convergent.

Thus we have showed that if compact space K contains nontrivial convergent sequence then the space $\mathrm{C}(K)$ is not Grothendieck.

On the other hand, each Banach space X which is reflexive, i.e. such that for every $x^{**} \in X^{**}$ there exists $x \in X$ such that $x^{**}(x^*) = x^*(x)$ for all $x^* \in X^*$, is Grothendieck. In order to show another property of Grothendieck spaces, we recall the following notion.

Every linear and continuous operator $T : X \to Y$ between Banach spaces has the adjoint operator $T^* : Y^* \to X^*$ such that for all $y^* \in Y^*$, $x \in X$, we have

$$(T^* y^*)(x) = y^*(Tx).$$

If T is a surjection then T^* is an injection, and if T is an injection onto a closed subspace then T^* is a surjection.

The following facts will be used in the next section.

Fact 1.1 *Let X, Y be Banach spaces. Let $T : X \to Y$ be continuous and linear surjection. If X is a Grothendieck space then Y is a Grothendieck space.*

Proof. Let us consider a sequence $y_1^*, y_2^*, \ldots \in Y^*$ weak* convergent to 0. Then $T^* y_1^*, T^* y_2^*, \ldots$ converges weakly* to 0 in X^* and, since X is Grothendieck, converges also weakly to 0. Let us consider $y \in Y^{**}$. Since T^{**} is a surjection there exists $x^{**} \in X^{**}$ such that $T^{**} x^{**} = y^{**}$. Then

$$y^{**}(y_n^*) = (T^{**} x^{**})(y_n^*) = x^{**}(T^* y_n^*) \to 0.$$

$\square$

Fact 1.2 *If X, Y are Grothendieck spaces then $X \oplus Y$ is also Grothendieck.*

Proof. Let us consider a sequence $(z_n^*)_{n<\omega}$ in $(X \oplus Y)^*$, weak* convergent to 0. Then we have $(z_n^* \restriction X)_{n<\omega} \in X^*$, $(z_n^* \restriction Y)_{n<\omega} \in Y^*$, weak* convergent to 0, thus weak convergent. Let us consider $z^{**} \in (X \oplus Y)^{**}$. Then

$$z^{**}(z_n^*) = z^{**}(z_n^* \restriction X) + z^{**}(z_n^* \restriction Y) \to 0.$$

$\square$

Let $\mathfrak{C}$ be a Boolean algebra. The symbol $\widehat{c} = \{p \in \mathrm{Ult}\, \mathfrak{C} : c \in p\}$ denotes the basic closed-open set in the Stone space $\mathrm{Ult}\, \mathfrak{C}$ of the algebra $\mathfrak{C}$. Lemma 1.3 says that each measure $\nu : \mathfrak{C} \to [0, \infty)$ can be extended to a Radon measure on the space $\mathrm{Ult}\, \mathfrak{C}$.

Lemma 1.3 *Let $\mathfrak{C}$ be a Boolean algebra with a finitely additive measure $\nu : \mathfrak{C} \to [0, \infty)$. Then there is a Radon measure $\widehat{\nu} : \mathrm{Bor}(\mathrm{Ult}\, \mathfrak{C}) \to [0, \infty)$ such that $\widehat{\nu}(\widehat{c}) = \nu(c)$ for every $c \in \mathfrak{C}$.*

Proof. Let Y be the algebra of simple functions, generated by the family $\{\chi_{\widehat{a}} : a \in \mathfrak{C}\}$. We define $x_0^* : Y \to \mathbb{R}$ such that

$$x_0^*(\sum_i r_i \chi_{\widehat{a_i}}) = \sum_i r_i \nu(a_i)$$

for $a_i \in \mathfrak{C}$. It is routine to check that x_0^* is continuous and linear. From the Hahn–Banach theorem there exists linear and continuous $x^* : \mathrm{C}(K) \to \mathbb{R}$ such that $x^* \restriction Y = x_0^*$.

In order to use the Riesz theorem we have to know that $x^*(f) \geqslant 0$ for $f \geqslant 0$. Let us consider $f \in \mathrm{C}(K)$, $f \geqslant 0$. From the Stone–Weierstrass theorem, f is the limit of a uniformly convergent sequence of functions from Y. Let us consider $\varepsilon > 0$. Let $g = \sum_{i=1}^{n} r_i \chi_{\widehat{a_i}} \in Y$ be such that $|x^*(f) - x^*(g)|, \|f - g\| < \varepsilon$. We can assume that elements $a_1, \dots, a_n$ are disjoint. Then

$$x^*(g) = \sum_{i=1}^{n} r_i \nu(a_i) \geqslant -\varepsilon \cdot \nu(\mathbf{1})$$

where $\mathbf{1}$ is the biggest element in the algebra $\mathfrak{C}$. Thus $x^*(f) \geqslant -\varepsilon(\nu(\mathbf{1}) + 1)$.

From the Riesz theorem there is a unique Radon measure $\widehat{\nu} : \mathrm{Bor}(\mathrm{Ult}\,\mathfrak{C}) \to [0, \infty)$ such that $x^*(f) = \int_K f \; \mathrm{d}\widehat{\nu}$ for every $f \in \mathrm{C}(K)$. Then $\widehat{\nu}(\widehat{c}) = x^*(\chi_{\widehat{c}}) = x_0^*(\chi_{\widehat{c}}) = \nu(c)$. □

Lemma 1.4 *Let $\mathfrak{C}$ be a Boolean algebra and let $\nu_n : \mathrm{Bor}(\mathrm{Ult}\,\mathfrak{C}) \to [0, \infty)$ be a Radon measure for $n < \omega$. If the sequence $(\nu_n(\widehat{c}))_{n<\omega}$ is convergent to $\nu(\widehat{c})$ for each $c \in \mathfrak{C}$ then the sequence $(\nu_n)_{n<\omega}$ converges weakly* to ν.*

Proof. Let us consider $\varepsilon > 0$, $K = \mathrm{Ult}\,\mathfrak{C}$. As in the proof of Lemma 1.3, there exists $g = \sum_{i=1}^{m} r_i \chi_{\widehat{a_i}} \in Y$ such that $\|f - g\| < \varepsilon$. Then

$$|\int_K f \; \mathrm{d}\nu_n - \int_K f \; \mathrm{d}\nu| \leqslant |\int_K f \; \mathrm{d}\nu_n - \int_K g \; \mathrm{d}\nu_n| + |\int_K g \; \mathrm{d}\nu_n - \int_K g \; \mathrm{d}\nu| +$$

$$|\int_K g \; \mathrm{d}\nu - \int_K f \; \mathrm{d}\nu| \leqslant \varepsilon(\nu_n(K) + \nu(K)) + \sum_{i=1}^{m} r_i(\nu(a_i) - \nu_n(a_i)) \longrightarrow 2\varepsilon\nu(K).$$

□

2 A question of Piotr Koszmider

The following question was posed by Piotr Koszmider:

> Is there a compact space K such that $\mathrm{C}(K)$ is not Grothendieck while $\mathrm{C}(L)$ is Grothendieck for every separable compact $L \subseteq K$?

In his unpublished note [5] G. Plebanek answered this question positively. We will present his solution complemented by necessary details. In the next section, under the assumption of Martin's axiom, we will extend this result for the subspaces L of density strictly less than continuum.

Let $\mathfrak{A}$ be the algebra of the Lebesgue measure λ on the interval $[0,1]$. If $[A],[B] \in \mathfrak{A}$, $[A]=[B]$, then $\lambda(A \triangle B)=0$ and $\lambda(A)=\lambda(B)$. This shows that the function $\overline{\lambda}:\mathfrak{A}\to[0,1]$, $\overline{\lambda}([A])=\lambda(A)$, is correctly defined. Thus we will not distinguish between these two measures.

The following theorem can be found for example in [1].

Theorem 2.1 $\mathrm{C}(\mathrm{Ult}\,\mathfrak{A})$ *is a Grothendieck space.*

The theorem above is a special case of a more general fact that for every compact and extremally disconnected space K, the space $\mathrm{C}(K)$ is Grothendieck. Recall that the Stone space $\mathrm{Ult}\,\mathfrak{C}$ is extremally disconnected for complete Boolean algebra $\mathfrak{C}$. The completeness of a Boolean algebra $\mathfrak{C}$ can be replaced by the *subsequential completeness property* (SCP) and in this case we also obtain Grothendieck space $\mathrm{C}(\mathrm{Ult}\,\mathfrak{C})$. For the definition of SCP and proof of the above fact, see [4].

Let $s(\mathfrak{A})$ be the set of all decreasing sequences $(a_k)_{k<\omega}\in{}^{\omega}\mathfrak{A}$ with the property $\lim_{k\to\infty}\lambda(a_k)=0$. Let us fix $(s_k)_{k<\omega}\in s(\mathfrak{A})$ such that $\lambda(s_k)>2\lambda(s_{k+1})$ for every $k<\omega$.

Throughout the paper an element $a\in\mathfrak{A}$ will be called *big* if

$$\lim_{k\to\infty}\frac{\lambda(a\cdot s_k)}{\lambda(s_k)}=1,$$

and *small* if the limit above is 0. It is easy to see that an element a is small if and only if $-a$ is big.

Lemma 2.2 *Suppose that* $(a_k^n)_{k<\omega}\in s(\mathfrak{A})$ *and* a_0^n *is small, for every* $n<\omega$. *Then there is* $g:\omega\to\omega$ *such that*

$$a_g=\sum_{n<\omega}a_{g(n)}^n$$

is small.

Proof. Let us consider $\varepsilon>0$. We claim that for every n there exists $g(n)$ such that for every k,

$$\lambda(a_{g(n)}^n\cdot s_k)\leqslant\lambda(s_k)\frac{\varepsilon}{2^{n+1}}. \tag{2}$$

In order to prove (2), let us fix $n<\omega$. Let $m<\omega$ be such that for every $k\geqslant m$,

$$\frac{\lambda(a_0^n\cdot s_k)}{\lambda(s_k)}<\frac{\varepsilon}{2^{n+1}}.$$

Let $g(n)$ be a natural number such that for every $k < m$,

$$\lambda(a^n_{g(n)} \cdot s_k) < \frac{\varepsilon}{2^{n+1}} \lambda(s_k).$$

If $k \geqslant m$, then

$$\lambda(a^n_{g(n)} \cdot s_k) \leqslant \lambda(a^n_0 \cdot s_k) < \frac{\varepsilon}{2^{n+1}} \lambda(s_k),$$

which gives the condition (2).

Thus $a_g = \sum_n a^n_{g(n)}$ satisfies the inequality

$$\lambda(a_g \cdot s_k) \leqslant \sum_n \lambda(a^n_{g(n)} \cdot s_k) \leqslant \sum_n \frac{\varepsilon}{2^{n+1}} \lambda(s_k) \leqslant \lambda(s_k)\varepsilon$$

for all k.

Using the above construction for $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ we obtain functions $g_1, g_2, \ldots$ satisfying inequalities

$$\lambda(a_{g_n} \cdot s_k) \leqslant \frac{1}{n} \lambda(s_k)$$

for all k. We can assume that $g_1 \leqslant g_2 \leqslant \ldots$

Let g be a function dominating the family $\{g_n : n \in \mathbb{N}\}$, with respect to $\leqslant^*$. Let us consider $m < \omega$. Let ℓ be such that $g_m(n) \leqslant g(n)$ for $n > \ell$. Then

$$\frac{\lambda(a_g \cdot s_k)}{\lambda(s_k)} \leqslant \frac{\lambda(\sum_{n=0}^{\ell} a^n_{g(n)} \cdot s_k)}{\lambda(s_k)} + \frac{\lambda(\sum_{n \geqslant \ell+1} a^n_{g_m(n)} \cdot s_k)}{\lambda(s_k)} \leqslant$$

$$\leqslant \frac{\lambda(\sum_{n=0}^{\ell} a^n_{g(n)} \cdot s_k)}{\lambda(s_k)} + \frac{1}{m} \stackrel{k \to \infty}{\longrightarrow} \frac{1}{m}.$$

□

Let $\mathfrak{B} = \{a \in \mathfrak{A} : a \text{ is big or small}\}$. It is easy to see that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$.

Lemma 2.3 *Family $p = \{a \in \mathfrak{B} : a \text{ is big}\}$ is an ultrafilter in $\mathfrak{B}$.*

Proof. Since the union of two small elements is small, we see that if $a, b \in \mathfrak{B}$, $a + b \in p$ then at least one of them is big. □

Theorem 2.4 *The space* $\mathrm{C}(\mathrm{Ult}\,\mathfrak{B})$ *is not Grothendieck, but for all compact and separable* $L \subseteq \mathrm{Ult}\,\mathfrak{B}$*, the space* $\mathrm{C}(L)$ *is Grothendieck.*

Proof. Let us assume $p = \{a \in \mathfrak{B} : a \text{ is big}\}$ and $K = \mathrm{Ult}\,\mathfrak{B}$. Then $p \in K$. We define $\nu_k : \mathfrak{B} \to [0, 1]$,

$$\nu_k(a) = \frac{1}{\lambda(s_k)}\lambda(s_k \cdot a), \quad \text{for } a \in \mathfrak{B}.$$

Then ν_k is a probability measure on the algebra $\mathfrak{B}$. From Lemma 1.3, the measure ν_k defines a Radon measure $\widehat{\nu_k} : \mathrm{Bor}(K) \to \mathbb{R}$ such that $\widehat{\nu_k}(\widehat{b}) = \nu_k(b)$ for every $b \in \mathfrak{B}$.

Let δ_p be the Dirac measure concentrated at the point p. We will show that $\widehat{\nu_k}$ converges weakly* to δ_p. Let us consider $b \in \mathfrak{B}$. If b is small then $\widehat{\nu_k}(\widehat{b}) = \nu_k(b)$ which converges to 0. Then also $b \notin p$, thus $\delta_p(b) = 0$. If b is big then $\nu_k(b)$ converges to 1, and $\delta_p(b) = 1$.

From Lemma 1.4, sequence $(\widehat{\nu_k})_{k<\omega}$ is weak* convergent to δ_p.

Let $\mathrm{pr} : \mathrm{C}(\mathrm{Ult}\,\mathfrak{B})^* \to \mathbb{R}$, $\mathrm{pr}(\nu) = \nu(\widehat{s_0} \setminus \{p\})$. The function pr is linear and continuous. Since $\lambda(s_k) > 2\lambda(s_{k+1})$ we have $\lambda(s_k - s_{k+1}) > \lambda(s_k)/2$. Then

$$\widehat{\nu_k}(\widehat{s_0} \setminus \{p\}) \geqslant \widehat{\nu_k}(\widehat{s_k - s_{k+1}}) = \frac{\lambda((s_k - s_{k+1}) \cdot s_k)}{\lambda(s_k)} \geqslant \frac{1}{2}$$

hence $\liminf_{k\to\infty} \widehat{\nu_k}(\widehat{s_0} \setminus \{p\}) \geqslant 1/2$.

On the other hand $\delta_p(\widehat{s_0} \setminus \{p\}) = 0$. Therefore $(\mathrm{pr}(\widehat{\nu_k}))_{k<\omega}$ does not converge to $\mathrm{pr}(\delta_p)$ and thus the sequence $(\widehat{\nu_k})_{k<\omega}$ is not weak convergent to δ_p.

Let $L \subseteq \mathrm{Ult}\,\mathfrak{B}$ be a separable and closed subspace. Consider the case $p \notin L$. Let $\{q_n : n < \omega\}$ be a dense subset of L. Since $q_n \neq p$ for every n, every ultrafilter q_n contains some small a_0^n. Using the properties of measure algebra we obtain sequence $(a_k^n)_{k<\omega} \in s(\mathfrak{A})$ such that $a_k^n \in q_n$ for every $k < \omega$. From Lemma 2.2 there exists $g : \omega \to \omega$ such that $a_g = \sum_n a_{g(n)}^n$ is small. Obviously $a_g \in \mathfrak{B}$, moreover $\mathfrak{A}\restriction a_g = \mathfrak{B}\restriction a_g$.

For every $n < \omega$, the ultrafilter q_n can be uniquely extended to $q_n' \in \mathrm{Ult}\,\mathfrak{A}$. Let $L' = \mathrm{cl}\,\{q_n' : n < \omega\}$. It is not difficult to see that $L' \subseteq \widehat{a_g}$ in $\mathrm{Ult}\,\mathfrak{A}$. We will show that L and L' are homeomorphic. Observe that for every small element a, we have

$$\widehat{a}^{\mathfrak{A}} \cap \{q_n' : n < \omega\} = \widehat{a}^{\mathfrak{B}} \cap \{q_n : n < \omega\},$$

where $\widehat{a}^{\mathfrak{A}} = \{q \in \mathrm{Ult}\,\mathfrak{A} : a \in q\}$, $\widehat{a}^{\mathfrak{B}} = \{q \in \mathrm{Ult}\,\mathfrak{B} : a \in q\}$. Thus the function $f : \{q_n : n < \omega\} \to \{q_n' : n < \omega\}$, $f(q_n) = q_n'$, is a continuous bijection hence it has unique continuous extension to the bijection $g : L \to L'$. Since L, L' are compact, g is homeomorphism.

Since $\widehat{a_g}$ is homeomorphic to $\mathrm{Ult}\,\mathfrak{A}$, spaces $\mathrm{C}(\widehat{a_g})$ and $\mathrm{C}(\mathrm{Ult}\,\mathfrak{A})$ are isometric and from Theorem 2.1 we have that $\mathrm{C}(\widehat{a_g})$ is a Grothendieck space. Then $\mathrm{C}(L')$ is Grothendieck as continuous and linear image of $\mathrm{C}(\widehat{a_g})$ (see Fact 1.1). Once again, since L is homeomorphic to L' we have that $\mathrm{C}(L)$ is Grothendieck.

Let us consider the case $p \in L$. Let $\{q_n : n < \omega\} \subseteq L$ be a dense subset of L. Suppose that $q_n \neq p$ for every $n < \omega$. Then, as in the previous case, we obtain $\mathrm{cl}\,\{q_n : n < \omega\} \subseteq \widehat{a_g}$ and $p \notin \widehat{a_g}$; a contradiction. Thus we can assume that $q_n \neq p$ for $n \geqslant 1$. Similarly, for some a_g we have $\mathrm{cl}\,\{q_n : n \geqslant 1\} \subseteq \widehat{a_g}$ and $p \notin \widehat{a_g}$, thus p is an isolated point in L. As in the previous case, since $L \setminus \{p\}$ is closed and separable, the space $\mathrm{C}(L \setminus \{p\})$ is Grothendieck. Then $\mathrm{C}(L) = \mathrm{C}(L \setminus \{p\}) \oplus \mathbb{R}$. From Fact 1.2 $\mathrm{C}(L)$ is a Grothendieck space.$\square$

3 Using MA

In this section, under the assumption of Martin's axiom, we will strengthen Theorem 2.4 obtained in the previous section, by claiming that for every closed subspace $L \subseteq K = \mathrm{Ult}\,\mathfrak{B}$ of density less than continuum, the space $\mathrm{C}(L)$ is Grothendieck.

Let us recall that the formula $\rho(a, b) = \lambda(a \bigtriangleup b)$ defines a metric for $a, b \in \mathfrak{A}$. Let $Q = \{(p, q) : p, q \in \mathbb{Q}\}$. For every $\varepsilon > 0$ and measurable subset $A \subseteq [0, 1]$ there exist $I_1, \dots, I_n \in Q$ such that $\lambda(A \bigtriangleup \bigcup_{k=1}^{n} I_k) < \varepsilon$. This shows that the metric space $(\mathfrak{A}, \rho)$ is separable.

The following theorem is due to Fremlin, see [3].

Theorem 3.1 (MA) *Let $\kappa < \mathfrak{c}$. For every $\xi < \kappa$, let $(a_k^\xi)_{k<\omega} \in {}^\omega\mathfrak{A}$ be a decreasing sequence such that $\lim_{k\to\infty} \lambda(a_k^\xi) = 0$. Then for every $\varepsilon > 0$ there exists $a \in \mathfrak{A}$ such that $\lambda(a) < \varepsilon$ and for every $\xi < \kappa$, there is k_ξ such that $a_{k_\xi}^\xi \leqslant a$*

It is known that under the assumption of Martin's axiom the bounding number is equal to continuum, $\mathfrak{b} = \mathfrak{c}$.

Lemma 3.2 (MA) *Let $g_\alpha : \omega \to (0, \infty)$ be a sequence convergent to 0, for every $\alpha < \kappa < \mathfrak{c}$. Then there exists $h : \omega \to (0, \infty)$ such that $\lim_{n\to\infty} h(n) = 0$ and for every $\alpha < \kappa$,*

$$\lim_{n\to\infty} \frac{g_\alpha(n)}{h(n)} = 0.$$

Proof. For $\alpha < \kappa$ let $\varphi_\alpha : \omega \to \omega$ be an increasing function such that if $n \geqslant \varphi_\alpha(i)$ then $g_\alpha(n) < 1/i^2$. Since $\mathfrak{b} = \mathfrak{c}$, there exists increasing function $\varphi : \omega \to \omega$ such that $\varphi_\alpha \leqslant^* \varphi$ for every $\alpha < \kappa$. We define $h : \omega \to \omega$, putting $h(n) = 1/i$ for $\varphi(i) \leqslant n < \varphi(i+1)$.

Let us consider $\alpha < \kappa$ and $\varepsilon > 0$. Let m be such that $1/m < \varepsilon$ and for all $n \geqslant m$, $\varphi_\alpha(n) \leqslant \varphi(n)$. Let us consider $n \geqslant \varphi_\alpha(m)$. Let i be such that $\varphi_\alpha(i) \leqslant n < \varphi_\alpha(i+1)$. Then $i \geqslant m$, $g_\alpha(n) < 1/i^2$ and $n < \varphi(i+1)$. Thus $h(n) \geqslant 1/i$ and

$$\frac{g_\alpha(n)}{h(n)} \leqslant \frac{1}{i} \leqslant \frac{1}{m} < \varepsilon.$$

□

Let $(s_k)_{k<\omega}$ be a fixed decreasing sequence of nonzero elements of $\mathfrak{A}$ as in the previous section.

Theorem 3.3 (MA) *For every $\xi < \kappa < \mathfrak{c}$, let $(a_n^\xi)_{n<\omega}$ be decreasing sequence of elements of $\mathfrak{A}$ such that a_0^ξ is small. Then there exists small $b > \mathbf{0}$ such that for every $\xi < \kappa$ there is $n < \omega$ such that $a_n^\xi \leqslant b$.*

Proof. For every $\xi < \kappa$, let $g_\xi(n) = \lambda(a_0^\xi \cdot s_n)/\lambda(s_n)$ for $n < \omega$. We see that $\lim_{n\to\infty} g_\xi(n) = 0$. From Lemma 3.2 there is function $h : \omega \to (0,\infty)$ such that $\lim_{n\to\infty} h(n) = 0$ and for every $\xi < \kappa$, $\lim_{n\to\infty} g_\xi(n)/h(n) = 0$.

Additionally we define for every $b \in \mathfrak{A}$ a function $h_b : \omega \to (0,\infty)$,

$$h_b(n) = \frac{\lambda(b \cdot s_n)}{\lambda(s_n)}.$$

Let

$$P = \{b > \mathbf{0} : \exists_{t<1} h_b \leqslant t \cdot h, \lim_{n\to\infty} \frac{h_b(n)}{h(n)} = 0\}.$$

Then we can ordered the set P taking $a \preceq b$ if $a \geqslant b$ in the Boolean algebra $\mathfrak{A}$. Let

$$D_\xi = \{b \in P : \text{there exists } n \text{ such that } a_n^\xi \leqslant b\}.$$

We will show that D_ξ is dense in P. Let us consider $b \in P$ and let $t < 1$ be such that $h_b \leqslant t \cdot h$. Since $\lim_{n\to\infty} g_\xi(n)/h(n) = 0$ there exists $m < \omega$ such that for every $n \geqslant m$ and $k < \omega$,

$$\frac{\lambda(a_k^\xi \cdot s_n)}{\lambda(s_n)} \leqslant \frac{\lambda(a_0^\xi \cdot s_n)}{\lambda(s_n)} < \frac{1-t}{2} h(n).$$

Let k be such that

$$\frac{\lambda(a_k^\xi \cdot s_n)}{\lambda(s_n)} < \frac{1-t}{2} h(n)$$

for every $n < m$. Then

$$h_{b+a_k^\xi}(n) \leqslant \frac{\lambda(b \cdot s_n)}{\lambda(s_n)} + \frac{\lambda(a_k^\xi \cdot s_n)}{\lambda(s_n)} < th(n) + \frac{1-t}{2} h(n) < \frac{1+t}{2} h(n)$$

for every $n < \omega$. Moreover

$$\frac{h_{b+a_k^\xi}(n)}{h(n)} \leqslant \frac{h_b}{h(n)} + \frac{g_\xi(n)}{h(n)} \longrightarrow 0$$

thus $b + a_k^\xi \in P$. Finally, $b + a_k^\xi \in D_\xi$ and $b + a_k^\xi \preceq b$.

We will show that $(P, \preceq)$ satisfies ccc. Suppose that there is a family $\{b_\alpha : \alpha < \aleph_1\}$ such that $b_\alpha \perp b_\beta$ for $\alpha < \beta < \aleph_1$. For each $\alpha < \aleph_1$ there is $t_\alpha < 1$ such that $h_{b_\alpha} < t \cdot h$. Thus, taking suitable subfamily, we can assume that for all $\alpha < \aleph_1$, $t_\alpha \leqslant t < 1$. Since metric space $\{b_\alpha : \alpha < \aleph_1\}$ is separable, it contains an accumulation point b_β. Let $m < \omega$ be such that for every $n \geqslant m$, $h_{b_\beta}(n)/h(n) < (1-t)/2$. Since b_β is an accumulation point there exists $\alpha < \aleph_1$, $\alpha \neq \beta$ such that for every $n < m$,

$$\frac{\lambda((b_\beta - b_\alpha) \cdot s_n)}{\lambda(s_n)} < \frac{(1-t)}{2} h(n).$$

If $n \geqslant m$ then

$$\frac{\lambda((b_\alpha + b_\beta) \cdot s_n)}{\lambda(s_n)} < th(n) + \frac{1-t}{2} h(n) = \frac{1+t}{2} h(n).$$

If $n < m$ then

$$\frac{\lambda((b_\alpha + b_\beta) \cdot s_n)}{\lambda(s_n)} < th(n) + \frac{\lambda((b_\beta - b_\alpha) \cdot s_n)}{\lambda(s_n)} < th(n) + \frac{(1-t)}{2} h(n).$$

Moreover

$$\frac{\lambda((b_\alpha + b_\beta) \cdot s_n)}{\lambda(s_n)} \leqslant \frac{\lambda(b_\alpha \cdot s_n)}{\lambda(s_n)} + \frac{\lambda(b_\beta \cdot s_n)}{\lambda(s_n)} \longrightarrow 0.$$

Thus $b_\beta + b_\alpha \in P$ and $b_\beta + b_\alpha \preceq b_\beta, b_\alpha$; a contradiction with $b_\beta \perp b_\alpha$.

From Martin's Axiom there exists a filter G such that $G \cap D_\xi \neq \emptyset$ for every $\xi < \kappa$. We will show that $\lambda(\sum G) \leqslant \varepsilon$. This will be done by choosing countable $G' \subseteq G$ such that $\sum G' = \sum G$. Suppose that there is no such G'. Then for all countable $G' \subseteq G$ we have $\sum G - \sum G' \neq \mathbf{0}$, in fact there is $a \in G$ such that $a - \sum G' \neq \mathbf{0}$. Thus we can recursively define a sequence $\{a_\beta : \beta < \aleph_1\} \subseteq G$ such that $b_\beta = a_\beta - \sum_{\gamma<\beta} a_\gamma \neq \mathbf{0}$. The sequence $\{b_\beta : \beta < \aleph_1\}$ is disjoint and $\lambda(b_\beta) > 0$ since $b_\beta \neq \mathbf{0}$. Thus there exists $n > 0$ such that $|\{\beta < \aleph_1 : \lambda(b_\beta) > 1/n\}| = \aleph_1$. Let $A \subseteq \{\beta < \aleph_1 : \lambda(b_\beta) > 1/n\}$ be infinite and countable. Then $\lambda(\sum_{\beta\in A} b_\beta) = \infty$; a contradiction. Thus there exists countable $G' \subseteq G$ such that $\sum G' = \sum G$. Then

$$\lambda(\sum G) = \lambda(\sum G') = \sup\{\lambda(\sum G'') : G'' \subseteq G', |G''| < \aleph_0\}.$$

Since G is centred we have $\lambda(\sum G'') < \varepsilon$ for all $G'' \subseteq G$, $|G''| < \aleph_0$. Finally, $\lambda(\sum G) \leqslant \varepsilon$.

Let $a = \sum G$. For every $\xi < \kappa$, since $G \cap D_\xi \neq \emptyset$, we have $a^\xi_k \leqslant g \leqslant \sum G$ for some $g \in G$ and $k < \omega$. Thus a is the desired element. □

Now we are ready to prove the stronger version of Theorem 2.4.

Theorem 3.4 (MA) *The space* $\mathrm{C}(\mathrm{Ult}\,\mathfrak{B})$ *is not Grothendieck, but for all compact subspaces* $L \subseteq \mathrm{Ult}\,\mathfrak{B}$ *of density less than* $\mathfrak{c}$*, the space* $\mathrm{C}(L)$ *is Grothendieck.*

Proof. The fact that $\mathrm{C}(\mathrm{Ult}\,\mathfrak{B})$ is not a Grothendieck space is already proved by Theorem 2.4.

Let $L \subseteq \mathrm{Ult}\,\mathfrak{B}$ be closed subspace of density $\kappa < \mathfrak{c}$, $L = \mathrm{cl}\,\{q_\xi : \xi < \kappa\}$. Consider the case $p \notin L$. Since $q_\xi \neq p$ for every $\xi < \kappa$, every ultrafilter q_ξ contains some small a^ξ_0. Using the properties of measure algebra we obtain sequence $(a^\xi_k)_{k<\omega} \in s(\mathfrak{A})$ such that $a^\xi_k \in q_\xi$ for every $\xi < \kappa$. From Theorem 3.3 there exists small $b > \mathbf{0}$ such that for every $\xi < \kappa$ there exists $n < \omega$ such that $a^\xi_n \leqslant b$.

Once we have the element b, the rest of the proof is analogous to that of Theorem 2.4. □

4 Embedding $\beta\omega$ in the space of probability measures

The space $\mathrm{C}(\mathrm{Ult}\,\mathfrak{A})$ is Grothendieck, so in particular the Stone space $\mathrm{Ult}\,\mathfrak{A}$ cannot contain any nontrivial convergent sequence; indeed every such sequence would lead to a weak*, but not weak, convergent sequence in the space $C(\mathrm{Ult}\,\mathfrak{A})^*$. We will show in this section, under the assumption of continuum hypothesis, that even if a compact space K is first-countable, the space $P(K)$, of all probability measures on K equipped with the weak* topology, may contain a homeomorphic copy of $\beta\omega$, therefore a sequence which has no weak* convergent subsequences.

Theorem 4.1 (CH) *There is a first-countable compact space K such that the space $P(K)$ (with the weak* topology) contains a homeomorphic copy of $\beta\omega$.*

Proof. Symbol 2^ω denotes the Cantor set. Let $\mathcal{A}_0$ be the subalgebra of $\mathcal{P}(2^\omega \times \omega)$ generated by the family

$$\{U \times A : U \in \mathrm{Clopen}(2^\omega), A \subseteq \omega \text{ is finite or cofinite}\}.$$

An easy observation gives us that every element of $\mathcal{A}_0$ is a finite union of some elements from the family above.

Let $\mathcal{P}(\omega) = \{N_\alpha : \alpha < \omega_1\}$, $2^\omega = \{t_\alpha : \alpha < \omega_1\}$. Let λ be the product measure on 2^ω.

We recursively define sequence $(F_\alpha)_{\alpha<\omega_1}$ of closed subsets of 2^ω such that for every $\alpha < \omega_1$,

$$F_\alpha \subseteq 2^\omega \setminus \{t_\beta : \beta < \alpha\} \quad \text{and} \quad \lambda(F_\alpha) \geqslant \frac{1}{2}.$$

Now, let $\mathcal{A}$ be the subalgebra of $\mathcal{P}(2^\omega \times \omega)$ generated by the family $\mathcal{A}_0 \cup \{F_\alpha \times N_\alpha : \alpha < \omega_1\}$. We will show that the space $K = \mathrm{Ult}\,\mathcal{A}$ is the desired one.

Let us consider $\mathcal{F} \in K$. Since $\mathcal{F}$ is centred, there exists an element $t \in \bigcap\{C \in \mathrm{Clopen}(2^\omega) : C \times \omega \in \mathcal{F}\}$. Suppose that there exists $t' \neq t$ with the same property. Let $C, C' \in \mathrm{Clopen}(2^\omega)$ be disjoint sets such that $t \in C$, $t' \in C'$. Then $C \times \omega, C' \times \omega$ are disjoint elements of filter $\mathcal{F}$; a contradiction.

Let $\alpha < \omega_1$ be such that $t = t_\alpha$. We will show that the family $\{\widehat{A} : A \in \mathcal{A}_\alpha\}$, where $\mathcal{A}_\alpha$ is the countable subalgebra generated by the family $\mathcal{A}_0 \cup \{F_\beta \times N_\beta : \beta \leqslant \alpha\}$, contains a base for $\mathcal{F}$. Let us consider $A \in \mathcal{F}$. Then there exist $A_1, \ldots, A_n \in \mathcal{A}_0$, $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_k$ such that

$$A \supseteq A_1 \cap \ldots \cap A_n \cap (F_{\alpha_1} \times N_{\alpha_1}) \cap \ldots \cap (F_{\alpha_m} \times N_{\alpha_m}) \cap$$
$$\cap (F_{\beta_1} \times N_{\beta_1})^{\mathrm{c}} \cap \ldots \cap (F_{\beta_k} \times N_{\beta_k})^{\mathrm{c}} \in \mathcal{F}$$

where D^{c} denotes the complement of D. We see that $t \in F_{\alpha_1} \cap \ldots \cap F_{\alpha_n}$, hence $\alpha_1, \ldots, \alpha_n \leqslant \alpha$. If $\beta_i > \alpha$ then $t \notin F_{\beta_i}$ and there exists $C_i \in \mathrm{Clopen}(2^\omega)$

such that $t \in C_i$ and $C_i \cap F_{\beta_i} = \emptyset$. We do this for every $1 \leqslant i \leqslant k$ such that $\beta_i > \alpha$. Taking intersection of these C_i's we obtain $C \in \mathrm{Clopen}(2^\omega)$ such that $t \in C$ and $C \cap F_{\beta_i} = \emptyset$ for $\beta_i > \alpha$. Assuming that $\beta_i \leqslant \alpha$ for $i \leqslant r$ and $\beta_i > \alpha$ for $i > r$ we have

$$A \supseteq A' = A_1 \cap \ldots \cap A_n \cap (F_{\alpha_1} \times N_{\alpha_1}) \cap \ldots \cap (F_{\alpha_m} \times N_{\alpha_m}) \cap$$

$$\cap (F_{\beta_1} \times N_{\beta_1})^{\mathrm{c}} \cap \ldots \cap (F_{\beta_r} \times N_{\beta_r})^{\mathrm{c}} \cap (C \times \omega) \in \mathcal{F}$$

and $A' \in \mathcal{A}_\alpha$. Since the family $\{\widehat{A} : A \in \mathcal{A}_\alpha\}$ is countable, the space K is first-countable.

Now we define $\mu_n : \mathcal{A} \to \mathbb{R}$ by the formula $\mu_n(A) = \lambda(A^n)$ where $A^n = \{t \in 2^\omega : (t, n) \in A\}$. Then, from Lemma 1.3, for every $n < \omega$ there exists $\widehat{\mu_n} \in P(K)$ such that $\widehat{\mu_n}(\widehat{A}) = \mu_n(A)$ for every $A \in \mathcal{A}$.

Let us consider $N \subseteq \omega$. Let $\alpha < \omega_1$ be such that $N = N_\alpha$. Then for every $n \in N$, $\mu_n(\widehat{F_\alpha \times N_\alpha}) = \lambda(F_\alpha) \geqslant 1/2$, and for every $n \notin N$, $\mu_n(\widehat{F_\alpha \times N_\alpha}) = \lambda(\emptyset) = 0$. Thus we have showed that

$$\mathrm{cl}\,\{\widehat{\mu_n} : n \in N\} \cap \mathrm{cl}\,\{\widehat{\mu_n} : n \in \omega \setminus N\} = \emptyset.$$

Since N was chosen arbitrarily, the subspace $\mathrm{cl}\,\{\widehat{\mu_n} : n < \omega\}$ is homeomorphic to $\beta\omega$, as it satisfies one of its topological characterisations, see [2]. □

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