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Wojciech Bielas

Uniwersytet Śląski

Subfunctors and sieves in a topos $\mathbf{Set}^{\mathbf{C}}$

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Opiekun pracy: prof. dr hab. Aleksander Błaszczyk

KAPITAŁ LUDZKI
NARODOWA STRATEGIA SPÓJNOŚCI

UNIA EUROPEJSKA
EUROPEJSKI
FUNDUSZ SPOŁECZNY

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Abstract. We investigate the correspondence between subfunctors of certain fixed functor $F : \mathbf{C} \to \mathbf{Set}$ and families of sieves. In the third part of the paper we obtain a characterization of those families of sieves which are determined by some subfunctor. Uniqueness of such subfunctors is extensively used in proving that the category $\mathbf{Set}^{\mathbf{C}}$ has power objects. Finally, we prove that the category $\mathbf{Set}^{\mathbf{C}}$ is a topos. This fact with more or less detailed proofs can be found in [2], [3] and [4].

1 Basic facts and definitions

In this section we give the definitions of basic categorical notions. For the definition of a category, functor natural transformation and monomorphism, the Reader should see for example [1].

Maybe the most intuitive example of a category is a class of all sets with functions as arrows. We denote this category by **Set**. This category has many important properties like having products, coproducts, equalizers etc. which makes possible advanced constructions in the set theory. These properties follows from the ZFC axioms, thus they can be repeated in any model of ZFC. But there are important structures which are formally not models of ZFC, like Boolean-valued models, in which still the most of the set-theoretical constructions can be done. These cases are covered by the concept of *topos*. In order to define this notion we have to recall some basic notions from the category theory.

We say that **C**-object 1 is a *terminal object* in **C** if for every **C**-object A there is exactly one arrow $A \to 1$. Every singleton is a terminal object in the category **Set**.

We say that the diagram

$$\begin{array}{ccc} E & \xrightarrow{k} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array}$$

is a *pullback* if for every **C**-arrows $h' : E' \to A$ and $k' : E' \to B$ such that $g \circ k' = f \circ h'$ there is exactly one **C**-arrow $i : E' \to E$ such that $k \circ i = k'$ and $h \circ i = h'$. We also say that the pair h, k is a pullback of the pair f, g.

Notions of a terminal object and a pullback are important for defining what is the "family of all subsets of a set". Let **C** be a category with a terminal object 1. We say that the category **C** has a *subobject classifier* if there is a **C**-object Ω with an arrow $\top : 1 \to \Omega$ such that

(i) for each monomorphism $m : B \to A$ there is a unique arrow $\chi(m) : A \to \Omega$ such that the diagram

$$\begin{array}{ccc} B & \longrightarrow & 1 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ A & \xrightarrow{\chi(m)} & \Omega \end{array}$$

is a pullback,

(ii) for every $u : A \to \Omega$ there is a pullback of the pair $u : A \to \Omega$, $\top : 1 \to \Omega$.

The category **Set** has a subobject classifier, it can be taken as $\Omega = \{0, 1\}$ with $\top : 1 \to \Omega$, $\top(0) = 1$.

Let **C** be a category with subobject classifier $\Omega, \top$. We say that (PA, e_A) is a *power object* of a **C**-object A if for every **C**-object B and an arrow $f : A \times B \to \Omega$ there exists exactly one arrow $\widehat{f} : B \to PA$ such that the diagram

$$\begin{array}{ccc} A \times B & & \\ {\scriptstyle 1_A \times \widehat{f}}\big\downarrow & \searrow^{f} & \\ A \times PA & \xrightarrow{e_A} & \Omega \end{array}$$

commutes. We say that a category *has power objects* if its every object has a power object.

We say that a category is a *topos* if it has finite products, a subobject classifier and power objects. This is one from a few equivalent definitions, it is used in [2].

Let **C** be a small category, i.e. a category with the class of all **C**-objects being a set. It is easy observation that the class $\mathbf{Set}^{\mathbf{C}}$ of all functors

$\mathbf{C} \to \mathbf{Set}$ with natural transformations as arrows is a category. Since one of our goals is to show that $\mathbf{C} \to \mathbf{Set}$ is a topos we will make a distinction for a special kind of a functor: we observe every $\mathbf{C}$-object A determines a functor $H_A : \mathbf{C} \to \mathbf{Set}$, $H_A B = \mathbf{C}(A, B)$, $(H_A f)g = f \circ g$, where $\mathbf{C}(A, B)$ denotes the set of all $\mathbf{C}$-arrows f such that $\operatorname{dom} f = A$ and $\operatorname{codom} F = B$.

Let $\operatorname{Nat}(F, G)$ denote the class of all natural transformations $F \to G$. Symbols $\operatorname{Ob} \mathbf{C}, \operatorname{Arr} \mathbf{C}$ denote the class of all $\mathbf{C}$-objects and $\mathbf{C}$-arrows, respectively.

We denote the fact that objects A, B are isomorphic in a category by $A \cong B$. In the category $\mathbf{Set}$ two objects A, B are isomorphic if and only if there is a bijection between them.

Below we recall very useful theorem. It says that values of a functor can be taken as sets of natural transformations. This will help us to determine the form of some functors. A proof of this theorem can be found in [2].

Theorem 1.1 (Yoneda lemma) *If $F : \mathbf{C} \to \mathbf{Set}$ is a functor then*

$$\operatorname{Nat}(H_A, F) \cong FA$$

for every $\mathbf{C}$-object A.

2 Category $\mathbf{Set}^{\mathbf{C}}$ has a terminal object and binary products

We recall that since a product of the empty diagram is a terminal object, the category $\mathbf{C}$ has finite products if and only if it has a terminal object and binary products. Thus we should prove the following.

Fact 2.1 *Category $\mathbf{Set}^{\mathbf{C}}$ has a terminal object.*

Proof. Let $F : \mathbf{C} \to \mathbf{Set}$, $FA = \{0\}$, for every $\mathbf{C}$-object A, and let Ff be a unique function $\{0\} \to \{0\}$, for every $\mathbf{C}$-arrow $f : A \to B$. Observe that $\{0\}$ is a terminal object in $\mathbf{Set}$.

Since a unique function $\{0\} \to \{0\}$ is the identity $1_{\{0\}}$ we have $F(1_A) = 1_{FA}$.

Let us consider $\mathbf{C}$-arrows $f : A \to B$ and $g : B \to C$. Then

$$F(g \circ f) = 1_{\{0\}} = 1_{\{0\}} \circ 1_{\{0\}} = F(g) \circ F(f).$$

Thus F is a functor. We will show that this functor is a terminal object in $\mathbf{Set}^{\mathbf{C}}$. To see this let us consider functor $G : \mathbf{C} \to \mathbf{Set}$. We define η_A to be a unique function $GA \to \{0\}$. Then for a **C**-arrow $f : A \to B$ we have that the diagram

$$\begin{array}{ccc} GA & \xrightarrow{\eta_A} & FA \\ {\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Ff} \\ GB & \xrightarrow{\eta_B} & FB \end{array}$$

commutes since there is exactly one function $GA \to FB$. Thus

$$\eta = (\eta_A)_{A \in \mathrm{Ob}\,\mathbf{C}}$$

is a natural transformation. This η is unique since for every **C**-object A, the component η_A is unique. Thus F is a terminal object in $\mathbf{Set}^{\mathbf{C}}$. □

Fact 2.2 *Category* $\mathbf{Set}^{\mathbf{C}}$ *has binary products.*

Proof. Let us consider functors $F, G : \mathbf{C} \to \mathbf{Set}$. We define $(F \times G)A = FA \times GA$ for **C**-object A, and $((F \times G)f)(x, y) = ((Ff)(x), (Gf)(y))$ for **C**-arrow $f : A \to B$ and $(x, y) \in FA \times GA$.

Let us consider **C**-arrows $f : A \to B$ and $g : B \to C$. Then

$$((F \times G)1_A)(x, y) = ((F1_A)(x), (G1_A)(y)) = (1_{FA}(x), 1_{GA}(y)) = (x, y).$$

Thus $(F \times G)1_A = 1_{FA \times GA} = 1_{(F \times G)A}$. Next we have

$$((F \times G)(g \circ f))(x, y) = ((F(g \circ f))(x), (G(g \circ f))(y)) =$$
$$((Fg \circ Ff)(x), (Gg \circ Gf)(y)) = ((Fg)((Ff)(x)), (Gg)((Gf)(y))) =$$
$$((F \times G)g)((Ff)(x), (Gf)(y)) = ((F \times G)g)(((F \times G)f)(x, y)) =$$
$$(((F \times G)g) \circ ((F \times G)f))(x, y)$$

which proves that $(F \times G)(g \circ f) = ((F \times G)g) \circ ((F \times G)f)$. Thus $F \times G : \mathbf{C} \to \mathbf{Set}$ is a functor.

For every **C**-object A, let $(\pi_1)_A : (F \times G)A \to FA$, $(\pi_1)_A(x, y) = x$, and $(\pi_2)_A : (F \times G)A \to GA$, $(\pi_2)_A(x, y) = y$, for $(x, y) \in FA \times GA$. Let us consider $f : A \to B$. Then

$$(Ff)((\pi_1)_A(x, y)) = (Ff)x = (\pi_1)_B(((Ff)x, (Gf)y)) =$$
$$(\pi_1)_B(((F \times G)f)(x, y)).$$

Thus the diagram

$$\begin{array}{ccc} FA \times GA & \xrightarrow{(\pi_1)_A} & FA \\ {\scriptstyle (F\times G)f}\downarrow & & \downarrow{\scriptstyle Ff} \\ FB \times GB & \xrightarrow{(\pi_1)_B} & FB \end{array}$$

commutes and $\pi_1 : F \times G \to F$ is a natural transformation. Similarly we show that $\pi_2 : F \times G \to G$ is a natural transformation. Let $K : \mathbf{C} \to \mathbf{Set}$ be a functor and let $\eta : K \to F$, $\xi : K \to G$ be natural transformations.

Suppose that $\alpha : K \to F \times G$ is a natural transformation such that $\pi_1 \circ \alpha = \eta$ and $\pi_2 \circ \alpha = \xi$. Let us consider a $\mathbf{C}$-object A. Then $(\pi_1)_A \circ \alpha_A = \eta_A$ and $(\pi_2)_A \circ \alpha_A = \xi_A$. Thus $\alpha_A(x) = (\eta_A(x), \xi_A(x))$ for every $x \in KA$. This shows that α, if exists, is unique.

Let us define $\alpha_A : KA \to FA \times GA$, $\alpha_A(x) = (\eta_A(x), \xi_A(x))$ for every $x \in KA$ and every $\mathbf{C}$-object A.

Let us consider a $\mathbf{C}$-arrow $f : A \to B$. We see that

$$((F \times G)f)(\alpha_A(x)) = ((F \times G)f)(\eta_A(x), \xi_A(x)) =$$

$$((Ff)(\eta_A(x)), (Gf)(\xi_A(x))) = (\eta_B((Kf)(x)), \xi_B((Kf)(x))) = \alpha_B((Kf)(x))$$

which means that α is a natural transformation.

This shows that the functor $F \times G$ with natural transformations π_1, π_2 is a product of F and G. □

We leave the following fact without a proof as it is an easy computation.

Fact 2.3 *If $F_1, F_2, G_1, G_2 : \mathbf{C} \to \mathbf{Set}$ are functors and $\eta : F_1 \to G_1$, $\xi : F_2 \to G_2$ are natural transformations then $(\eta \times \xi)_A = \eta_A \times \xi_A$ for every $\mathbf{C}$-object A.*

3 Subfunctors and sieves

In order to find a subobject classifier in the category $\mathbf{Set}^{\mathbf{C}}$ we will examine monomorphisms in this category. Let us observe that if $m : A \to B$ is a monomorphism in the category $\mathbf{Set}$ then there exists the set $C \subseteq B$ such that $C \cong A$. It is easy to see that $C = m[A]$. As usual we say that C is a *subset* of A. We will show that the similar situation holds in the category $\mathbf{Set}^{\mathbf{C}}$. To do that we need to know what is a *subfunctor*.

Let $G, F : \mathbf{C} \to \mathbf{Set}$ be functors. We say that G is a *subfunctor of* F if for every $\mathbf{C}$-object A we have $GA \subseteq FA$ as sets in the category $\mathbf{Set}$, and

$(Gf)(x) = (Ff)(x)$ for $x \in GA$ and $\mathbf{C}$-arrow $f : A \to B$. We write it as $G \subseteq F$. We will show that for each monomorphism $m : G \to F$ in $\mathbf{Set}^{\mathbf{C}}$ there is a subfunctor $K \subseteq F$ such that $K \cong G$. First we have to prove that each component of a monomorphism is an injection. This is done by the following Lemma which is an exercise in [3], p. 202.

Lemma 3.1 *If $m : G \to F$ is a monomorphism in the category $\mathbf{Set}^{\mathbf{C}}$ then for every $\mathbf{C}$-object A, component $m_A : GA \to FA$ is an injection.*

Proof. Suppose that there is $\mathbf{C}$-object A and $x, y \in GA$, $x \neq y$ such that $m_A(x) = m_A(y)$. Let $\sigma : GA \to GA$ be a permutation,

$$\sigma(z) = \begin{cases} y, & \text{for } z = x, \\ x, & \text{for } z = y, \\ z, & \text{else.} \end{cases}$$

Let $KB = GB$ for every $\mathbf{C}$-object B. If f is a $\mathbf{C}$-arrow then there are four possibilities: the object A is the domain of f, in this case we define $Kf = (Gf) \circ \sigma$, the object A is the codomain of f, in this case we define $Kf = \sigma \circ (Gf)$, the object A is the domain of f and its codomain, in this case we define $Kf = \sigma \circ (Gf) \circ \sigma$, and for the rest we define $Kf = Gf$. We will show that such defined K is a functor.

We see that $K(1_A) = \sigma \circ (G(1_A)) \circ \sigma = \sigma \circ 1_{GA} \circ \sigma = \sigma \circ \sigma = 1_{GA} = 1_{KA}$. If $B \neq A$ is a $\mathbf{C}$-object then $K(1_B) = G(1_B) = 1_{GB} = 1_{KB}$. Let us consider $\mathbf{C}$-objects B, C, D different from A. If $f : B \to C$, $g : C \to D$ are $\mathbf{C}$-arrows then we have

$$K(g \circ f) = G(g \circ f) = (Gg) \circ (Gf) = (Kg) \circ (Kf).$$

Functoriality of K in the rest of cases goes as following:

(i) for $f : A \to B$, $g : B \to C$, we have

$$K(g \circ f) = (G(g \circ f)) \circ \sigma = (Gg) \circ (Gf) \circ \sigma = (Kg) \circ (Kf),$$

(ii) for $f : B \to A$, $g : A \to C$, we have

$$K(g \circ f) = G(g \circ f) = (Gg) \circ \sigma \circ \sigma \circ (Gf) = (Kg) \circ (Kf),$$

(iii) for $f : B \to C$, $g : C \to A$, we have

$$K(g \circ f) = \sigma \circ (G(g \circ f)) = \sigma \circ (Gg) \circ (Gf) = (Kg) \circ (Kf),$$

(iv) for $f : A \to A$, $g : A \to B$, we have

$$K(g \circ f) = (G(g \circ f)) \circ \sigma = (Gg) \circ (Gf) \circ \sigma =$$

$$(Gg) \circ \sigma \circ \sigma \circ (Gf) \circ \sigma = (Kg) \circ (Kf),$$

(v) for $f : A \to B$, $g : B \to A$, we have

$$K(g \circ f) = \sigma \circ (G(g \circ f)) \circ \sigma = \sigma \circ (Gg) \circ (Gf) \circ \sigma = (Kg) \circ (Kf),$$

(vi) for $f : B \to A$, $g : A \to A$, we have

$$K(g \circ f) = \sigma \circ (G(g \circ f)) = \sigma \circ (Gg) \circ (Gf) =$$

$$\sigma \circ (Gg) \circ \sigma \circ \sigma \circ (Gf) = (Kg) \circ (Kf),$$

(vii) for $f : A \to A$, $g : A \to A$, we have

$$K(g \circ f) = \sigma \circ (G(g \circ f)) \circ \sigma =$$

$$\sigma \circ (Gg) \circ \sigma \circ \sigma \circ (Gf) \circ \sigma = (Kg) \circ (Kf).$$

We will show that if $\eta_B = 1_{KB} = 1_{GB}$ for every **C**-object $B \neq A$, and $\eta_A = \sigma$ then $\eta : K \to G$ is a natural transformation. Let us consider $f : A \to A$. Then $Kf = \sigma \circ (Gf) \circ \sigma$ thus

$$(Gf) \circ \eta_A = (Gf) \circ \sigma = \sigma \circ (Kf) = \eta_A \circ (Kf).$$

In case $f : A \to B$ and $B \neq A$ we have $\eta_B \circ (Kf) = Kf = (Gf) \circ \sigma = (Gf) \circ \eta_A$. In case $f : B \to A$ and $B \neq A$ we have $(Kf) \circ \eta_B = Kf = \sigma \circ (Gf) = \eta_A \circ (Gf)$. If $f : B \to C$ and $C \neq A$ in a **C**-object then $(Kf) \circ \eta_B = Kf = \eta_C \circ (Kf)$.

Since the category $\mathbf{Set}^{\mathbf{C}}$ has binary products there exists $G \times K : \mathbf{C} \to \mathbf{Set}$. Let $\pi_1 : G \times K \to G$, $\pi_2 : G \times K \to K$ be projections. We define $\xi = \eta \circ \pi_2$. Then $\pi_1, \xi : G \times K \to G$.

Let us consider **C**-object $B \neq A$, and $(z, t) \in GB \times KB$. We see that

$$(m \circ \pi_1)_B(z, t) = m_B(z) = m_B(\eta_B(z)) =$$

$$m_B(\eta_B((\pi_2)_B(z, t))) = (m \circ \eta \circ \pi_2)_B(z, t)$$

and, since $m_A(x) = m_A(y)$,

$$(m \circ \pi_1)_A(z, t) = m_A(z) = m_A(\eta_A(z)) =$$

$$m_A(\eta_A((\pi_2)_A(z,t))) = (m \circ \eta \circ \pi_2)_A(z,t).$$

Thus $m \circ \pi_1 = m \circ \eta \circ \pi_2$. Moreover

$$(\pi_1)_A(x,x) = x \neq y = \sigma(x) = \eta_A(x) = \eta_A((\pi_2)_A(x,x))$$

hence $\pi_1 \neq \eta \circ \pi_2 = \xi$. This is a contradiction since m is a monomorphism.□

Lemma 3.2 *If G is a subfunctor of F then $m : G \to F$ given by the formula $m_A(x) = x$ is a monomorphism in the category $\mathbf{Set}^{\mathbf{C}}$.*

Proof. The naturality of m follows directly from the definition of a subfunctor. Let us consider a functor $K : \mathbf{C} \to \mathbf{Set}$ and natural transformations $\alpha, \beta : K \to G$ such that $m \circ \alpha = m \circ \beta$. Let us consider a $\mathbf{C}$-object A and $x \in KA$. Then $\alpha_A(x) = m_A(\alpha_A(x)) = m_A(\beta_A(x)) = \beta_A(x)$. This shows that $\alpha = \beta$. □

Lemma 3.3 *If $m : G \to F$ is a monomorphism in the category $\mathbf{Set}^{\mathbf{C}}$ then there exists a subfunctor $K \subseteq F$, naturally isomorphic to G.*

Proof. Let us consider $x \in GA$ and $\mathbf{C}$-arrow $f : A \to B$. Then

$$(Ff)(m_A(x)) = m_B((Gf)(x)) \in m_B[GB].$$

Thus the function $K : \mathbf{C} \to \mathbf{Set}$, $KA = m_A[GA]$, $(Kf)(x) : m_A[GA] \to m_B[GB]$, $(Kf)(x) = (Ff)(x)$ for every $x \in m_A[GA]$ and $\mathbf{C}$-arrow $f : A \to B$, is defined correctly.

We see that

$$(K(1_A))(x) = (F(1_A))(x) = 1_{FA}(x) = x = 1_{KA}(x)$$

for $x \in KA$, thus $K(1_A) = 1_{KA}$.

Let us consider $\mathbf{C}$-arrows $f : A \to B$, $g : B \to C$, and $x \in KA$. Then

$$(K(g \circ f))(x) = (F(g \circ f))(x) = (Fg)((Ff)(x)) = (Kg)(Kf)(x)$$

hence $K(g \circ f) = (Kg) \circ (Kf)$. This proves that K is a subfunctor of F.

From the Lemma 3.1 we know that $m_A : GA \to KA$ is an isomorphism for every $\mathbf{C}$-object A. With codomain changed from FA to KA, m becomes a natural isomorphism of G and K. $\Box$

Let us assume that G is a subfunctor of F. Then $GA \subseteq FA$ for every $\mathbf{C}$-object A. Thus we have a restriction of the set FA to GA. There is also a natural restriction of the set of $\mathbf{C}$-arrows: every pair A, x where $x \in FA$ and A is a $\mathbf{C}$-object, determines some special subset $S(G, A, x)$ of $\mathbf{C}$-arrows,

$$S(G, A, x) = \{f \in \operatorname{Arr} \mathbf{C} : \operatorname{dom} f = A, (Ff)(x) \in G(\operatorname{codom} f)\}.$$

This subset has one particular property: if $f \in S(G, A, x)$ and $\operatorname{dom} g = \operatorname{codom} f$ then $g \circ f \in S(G, A, x)$. Any set $S \subseteq \operatorname{Arr} \mathbf{C}$ such that

(1) $\operatorname{dom} f = A$ for all $f \in S$,

(2) if $f \in S$ and $\operatorname{dom} g = \operatorname{codom} f$ then $g \circ f \in S$

is called a *sieve* on A. Thus every subfunctor $G \subseteq F$ determines the family

$$\mathcal{F}_G = \{S(G, A, x) : A \in \operatorname{Ob} \mathbf{C}, x \in FA\}$$

of sieves. For each $\mathbf{C}$-object A we have the function $S(G, A, \cdot) : FA \to \Omega A$, where ΩA is the set of all sieves on A. This notation suggests that $S(G, A, \cdot)$ is a component of a natural transformation, but at this moment Ω is not a functor. We need to define Ωf for a $\mathbf{C}$-arrow f. If we want $S(G, A, \cdot)$ to be a component of a natural transformation $F \to \Omega$ then there should be $(\Omega f)(S(G, A, x)) = S(G, B, (Ff)(x))$ for $f : A \to B$. If $g \in S(G, B, (Ff)(x))$ then $\operatorname{dom} g = B = \operatorname{codom} f$ and

$$(F(g \circ f))(x) = (Fg)((Ff)(x)) \in G(\operatorname{codom} g) = G(\operatorname{codom} g \circ f).$$

Thus $g \circ f \in S(G, A, x)$ and

$$S(G, B, (Ff)(x)) = \{g \in \operatorname{Arr} \mathbf{C} : g \circ f \in S(G, A, x)\}.$$

We can define this operation for any sieve S on A, taking

$$S_f = \{g \in \operatorname{Arr} \mathbf{C} : g \circ f \in S\}.$$

Then

$$(\Omega f)(S(G, A, x)) = S(G, B, (Ff)(x)) = S(G, A, x)_f.$$

Observe that if $g : B \to C$ is an element of S_f and $h : C \to D$ is a $\mathbf{C}$-arrow then $h \circ g \circ f \in S$ since $g \in S_f$ and S is a sieve. Thus $h \circ g \in S_f$ hence S_f is a sieve on B. Now we can define $\Omega : \mathbf{C} \to \mathbf{Set}$ as follows:

$$\Omega A = \{S \subseteq \operatorname{Arr} \mathbf{C} : S \text{ is a sieve on } A\},$$

$(\Omega f)(S) = S_f$ for $S \in \Omega A$ and $f : A \to B$.

Corollary 3.4 *If $\Omega : \mathbf{C} \to \mathbf{Set}$ is defined as above and G is a subfunctor of F then $(S(G, A, \cdot))_{A \in \mathrm{Ob}\,\mathbf{C}}$ is a natural transformation $F \to \Omega$.*

Knowing that each subfunctor of F determines the family of sieves on each $\mathbf{C}$-object, we can expect that Ω, as a place of all sieves of all $\mathbf{C}$-objects, will be a subobject classifier. Using the Yoneda lemma we will find even more natural way to the subobject classifier. Now we are interested in the operation $S \mapsto S_f$ for sieves. The Theorem 3.6 below shows that the statement about a subfunctor generating a family of sieves, can be inverted. In order to establish this theorem we observe simple but very useful remark about sieves.

Remark 3.5 *Let $f : A \to B$ a $\mathbf{C}$-arrow and let S be a sieve on A. Then we have the following equivalences:*

$$f \in S \Leftrightarrow 1_B \in S_f \Leftrightarrow S_f = \max_B .$$

Theorem 3.6 *Let us assume that $F : \mathbf{C} \to \mathbf{Set}$ is a functor and $\{S(A, x) : A \in \mathrm{Ob}\,\mathbf{C}, x \in FA\}$ is a family of sieves. Then there exist subfunctors $G^-, G^+ \subseteq F$ such that*

(1) *$S(G^-, A, x) \subseteq S(A, x) \subseteq S(G^+, A, x)$ for every $\mathbf{C}$-object A and $x \in FA$,*

(2) *if G', K' are subfunctors of F such that*

$$S(G', A, x) \subseteq S(A, x) \subseteq S(K', A, x)$$

for every $\mathbf{C}$-object A and $x \in FA$, then $G' \subseteq G^-$ and $G^+ \subseteq K'$.

Moreover the following conditions are equivalent:

(a) *$S(A, x)_f = S(B, (Ff)(x))$ for every $\mathbf{C}$-arrow $f : A \to B$ and $x \in FA$,*

(b) *$S(A, x) = S(G^+, A, x)$ for every $\mathbf{C}$-object A and $x \in FA$,*

(c) *there exists subfunctor $G \subseteq F$ such that $S(A, x) = S(G, A, x)$ for every $\mathbf{C}$-object A and $x \in FA$,*

(d) *$S(G^-, A, x) = S(G^+, A, x)$, for every $\mathbf{C}$-object A and $x \in FA$,*

(e) *$G^- = G^+$,*

(f) *there exists a unique subfunctor* $G \subseteq F$ *such that* $S(A,x) = S(G,A,x)$ *for every* **C***-object* A *and* $x \in FA$.

Proof. We define

$$G^+ = \{(Ff)(x) : f \in \mathbf{C}(C,A) \cap S(C,x), x \in FC, C \in \mathrm{Ob}\,\mathbf{C}\}.$$

Let us consider $f : A \to B$ and $x \in G^+A$. We will show that $(Ff)(x) \in G^+B$. Since $x \in G^+A$ there exist **C**-object C, $y \in FC$ and $g \in \mathbf{C}(C,A) \cap S(C,y)$ such that $(Fg)(y) = x$. Then $(Ff)(x) = (F(f \circ g))(y)$ and since $f \circ g \in S(C,y)$ we obtain $(Ff)(x) \in G^+B$. Thus the function $G^+f : G^+A \to G^+B$, $(G^+f)(x) = (Ff)(x)$, is correctly defined.

Let us consider **C**-object A, $x \in FA$ and $f \in S(A,x)$, $f : A \to B$. Then $(Ff)(x) \in G^+B$ hence $f \in S(G^+,A,x)$. This shows that $S(A,x) \subseteq S(G^+,A,x)$.

Now we define

$$G^-A = \{(Ff)(x) : f \in \mathbf{C}(C,A) \cap S(C,x), x \in FC, C \in \mathrm{Ob}\,\mathbf{C},$$

$$\forall_{g:B\to D}\forall_{h:A\to D}\forall_{y\in FB}((F(h \circ f))(x) = (Fg)(y) \to g \in S(B,y))\}.$$

Let us consider $f : A \to E$ and $x \in G^-A$. We will show that $(Ff)(x) \in G^-E$. Since $x \in G^-A$ there exist **C**-object C, $y \in FC$ and $g \in \mathbf{C}(C,A) \cap S(C,y)$ such that $(Fg)(y) = x$ and for all $k : B \to D$, $h : A \to D$ and $z \in FB$, if $(F(h \circ g))(x) = (Fk)(z)$ then $k \in S(B,z)$. Of course $f \circ g \in S(C,y)$ and $(Ff)(x) = (F(f \circ g))(y)$. Let us consider $k : B \to D$, $h : E \to D$ and $z \in FB$ such that $(F(h \circ f \circ g))(y) = (Fk)(z)$. From the property of g we have $k \in S(B,z)$. Thus $(Ff)(x) \in G^-B$ and the function $G^-f : G^-A \to G^-B$, $(G^-f)(x) = (Ff)(x)$, is correctly defined.

Let us consider $f \in S(G^-,A,x)$, $f : A \to E$. This means that $(Ff)(x) \in G^-E$. Thus there exists $g : C \to E$, $g \in S(C,y)$ such that $(Fg)(y) = (Ff)(x)$ and for all $k : B \to D$, $h : E \to D$ and $z \in FB$, if $(F(h \circ g))(x) = (Fk)(z)$ then $k \in S(B,z)$. Taking $h = 1_E : E \to E$ we see that $(Ff)(x) = (F(1_E \circ g))(y)$ thus $f \in S(A,x)$. This shows that $S(G^-,A,x) \subseteq S(A,x)$.

Let us assume that G', K' are subfunctors of F such that $S(G',A,x) \subseteq S(A,x) \subseteq S(K',A,x)$ for every **C**-object A and $x \in FA$. Let us consider $f \in S(G,A,x)$, $f : A \to B$. This means that $(Ff)(x) \in GB$. It suffices to show that $(Ff)(x) \in G^-B$. Let us consider $g : B \to D$, $h : C \to D$ and $y \in FC$ such that $(F(g \circ f))(x) = (Fh)(y)$. Since $(Ff)(x) \in GB$ we have $(Fh)(y) = (F(g \circ f))(x) \in GD$, hence $h \in S(G,C,y) \subseteq S(C,y)$. This shows that f has the property asserting that $(Ff)(x) \in G^-B$. Thus $f \in S(G^-,A,x)$ and $S(G,A,x) \subseteq S(G^-,A,x)$.

Now let us consider $f \in S(G^+, A, x)$, $f : A \to B$. This means that $(Ff)(x) \in G^+B$. Then there exists $g : C \to B$ and $y \in FC$ such that $g \in S(C, y)$ and $(Ff)(x) = (Fg)(y)$. Since $S(C, y) \subseteq S(K, A, x)$ we have $g \in S(K, A, x)$. Therefore $(Ff)(x) = (Fg)(y) \in KB$ and $f \in S(K, A, x)$. This shows that $S(G^+, A, x) \subseteq S(K, A, x)$.

Now we turn to the second part of the theorem.

(a)⇒(b). Let us assume that $S(A, x)_f = S(B, (Ff)(x))$ for every **C**-arrow $f : A \to B$ and $x \in FA$. Since $S(A, x) \subseteq S(G^+, A, x)$ it suffices to show that $S(G^+, A, x) \subseteq S(A, x)$. Let us consider $f \in S(G^+, A, x)$, $f : A \to B$. This means that $(Ff)(x) \in G^+B$. Thus there exist $g : C \to B$ and $y \in FC$ such that $(Fg)(y) = (Ff)(x)$ and $g \in S(C, y)$. Since $g \in S(C, y)$, we have from our assumption and the Remark 3.5 that

$$\max_B = S(C, y)_g = S(B, (Fg)(y)) = S(B, (Ff)(x)) = S(A, x)_f$$

hence $f \in S(A, x)$.

(b)⇒(c). This implication is obvious.

(c)⇒(d). From the point (2) of the Theorem 3.6, we have that

$$S(G, A, x) \subseteq S(G^-, A, x) \subseteq S(G^+, A, x) \subseteq S(G, A, x)$$

thus $S(G^-, A, x) = S(G^+, A, x)$.

(d)⇒(e). We see that

$$x \in G^-A \Leftrightarrow 1_A \in S(G^-, A, x) \Leftrightarrow 1_A \in S(G^+, A, x) \Leftrightarrow x \in G^+A.$$

(e)⇒(a). In this case $S(G^-, A, x) = S(G^+, A, x)$ hence $S(G^+, A, x) = S(A, x)$. Let us consider $f : A \to B$. Then

$$S(A, x)_f = S(G^+, A, x)_f = S(G^+, B, (Ff)(x)) = S(B, (Ff)(x)).$$

The point (f) follows from the fact that if $S(G, A, x) = S(K, A, x)$ for every **C**-object A, $x \in FA$, and $y \in GA \setminus KA$ then $1_A \in S(G, A, x) \setminus S(K, A, x)$ which is a contradiction. □

If a family $\mathcal{F}$ of sieves satisfies the condition (a) in the Theorem 3.6 then we call $\mathcal{F}$ an Ω-*matching family*, see [2].

Corollary 3.7 *If G, K are subfunctors of $F : \mathbf{C} \to \mathbf{Set}$ and $S(G, A, x) = S(K, A, x)$ for every $\mathbf{C}$-object A and $x \in FA$ then $G = K$.*

4 Subobject classifier in the category $\mathbf{Set}^{\mathbf{C}}$

We will investigate the form of a subobject classifier in $\mathbf{Set}^{\mathbf{C}}$. In order to do this we assume for the moment that $\Omega : \mathbf{C} \to \mathbf{Set}$ is a subobject classifier. From the Yoneda lemma we know that $\Omega A \cong \mathrm{Nat}(H_A, \Omega)$. Since there is a bijection between arrows $H_A \to \Omega$ and subfunctors $G \subseteq H_A$ we see that in order to find ΩA one has to find all subfunctors of H_A. In the previous section we have proved that each subfunctor $G \subseteq F$ determines the family $\{S(G, A, x) : A \in \mathrm{Ob}\,\mathbf{C}, x \in FA\}$ of sieves such that $G^+ = G^- = G$ where G^+, G^- are as in the Theorem 3.6. This situation becomes simpler in the case of subfunctor $G \subseteq H_A$. We have a sieve on A,

$$S(G, A, 1_A) = \{f \in \mathrm{Arr}\,\mathbf{C} : \mathrm{dom}\, f = A, f \in G(\mathrm{codom}\, f)\} = \bigcup_{B \in \mathrm{Ob}\,\mathbf{C}} GB.$$

Then $GB = \{f \in S(G, A, 1_A) : \mathrm{codom}\, f = B\}$ for every $\mathbf{C}$-object B. Thus G is determined by just one sieve on A. This is another suggestion to define Ω as in the previous section.

Fact 4.1 *Ω is a functor.*

Proof. Let us consider $\mathbf{C}$-arrows $f : A \to B$, $g : B \to C$, and $S \in \Omega A$. Then

$$(\Omega(1_A))(S) = S_{1_A} = \{g \in \mathrm{Arr}\,\mathbf{C} : g \circ 1_A \in S\} = S = 1_{\Omega A}(S),$$

thus $\Omega(1_A) = 1_{\Omega A}$.

Furthermore,

$$(\Omega(g \circ f))S = S_{g \circ f} = \{h \in \mathrm{Arr}\,\mathbf{C} : h \circ g \circ f \in S\} = \{h \in \mathrm{Arr}\,\mathbf{C} : h \circ g \in S_f\} =$$

$$(S_f)_g = (\Omega g)(S_f) = (\Omega g)((\Omega f)(S)),$$

thus $\Omega(g \circ f) = (\Omega g) \circ (\Omega f)$. □

Each $\mathbf{C}$-object A has its maximal sieve $\max_A$, the set of all $\mathbf{C}$-arrows f such that $\mathrm{dom}\, f = A$. We define $\top : 1 \to \Omega$, $\top_A(0) = \max_A$. Let us consider $\mathbf{C}$-arrow $f : A \to B$. Then

$$(\Omega f)(\top_A(0)) = (\Omega f)(\max_A) = (\max_A)_f =$$

$$\{g \in \mathrm{Arr}\,\mathbf{C} : \mathrm{dom}\, g = B, g \circ f \in \max_A\} = \max_B = \top_B(0) = \top_B((1f)(0)).$$

Thus $\top$ is a natural transformation.

We have to show that for each monomorphism $m : G \to F$ there is a unique $\chi(m) : F \to \Omega$ such that the diagram

$$(1) \qquad \begin{array}{ccc} G & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \top} \\ F & \xrightarrow{\chi(m)} & \Omega \end{array}$$

is a pullback. We will do this for a subfunctor and then use the Lemma 3.3.

Let us consider subfunctor $G \subseteq F$ and let $m : G \to F$ be its monomorphism from the Lemma 3.2. Assume that $\chi(m)$ is a natural transformation such that the diagram (1) is a pullback. We will investigate the form of $\chi(m)$, proving also that such $\chi(m)$ is unique.

Since the diagram (1) commutes we see that $\chi(m)_A(x) = \max_A$ for every $x \in GA$. Thus $GA \subseteq \chi(m)_A^{-1}[\{\max_A\}]$. The opposite inclusion uses universal property of pullbacks.

Lemma 4.2 *For every* **C***-object* A*,* $\chi(m)_A^{-1}[\{\max_A\}] = GA$*.*

Proof. Suppose that there is **C**-object A and $x \in FA \setminus GA$ such that $\chi(m)_A(x) = \max_A$.

Let $K : \mathbf{C} \to \mathbf{Set}$, $KB = GB \cup \{(Ff)(x) : f \in \mathbf{C}(A, B)\}$. Observe that if $y \in KB$ and $g : B \to C$, then there exists $f : A \to B$ such that $y = (Ff)(x)$. Thus $(Fg)(y) = (Fg)((Ff)(x)) = F(g \circ f)(x) \in KC$. Thus the function $Kf : KB \to KC$, $(Kf)(x) = (Ff)(x)$, for every **C**-arrow $f : B \to C$, is defined correctly.

We will show that $K : \mathbf{C} \to \mathbf{Set}$ is a functor. Let us consider **C**-arrows $g : B \to C$, $h : C \to D$ and $y = (Ff)(x) \in KB$ for some $f : A \to B$. Then $K(1_B)(y) = F(1_B)(y) = y = 1_{KB}(y)$. Furthermore,

$$K(g \circ f)(y) = F(g \circ f)(y) = F(g)((Ff)(y)) = (Kg)((Kf)(y)).$$

Since $KB \subseteq FB$ for every **C**-object B and $(Kf)(y) = (Ff)(y)$, K is a subfunctor of F. Thus $\alpha_B : KB \to FB$, $\alpha_B(y) = y$, is a component of natural transformation $\alpha : K \to F$. Moreover we see that for $y = (Ff)(x) \in KB$,

$$\chi(m)_B(\alpha_B(y)) = \chi(m)_B((Ff)(x)) = (\Omega f)(\chi(m)_A(x)) =$$
$$(\Omega f)(\max_A) = \max_B = \top_B(0).$$

Thus the diagram

$$(2) \qquad \begin{array}{ccc} K & \longrightarrow & 1 \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \top} \\ F & \xrightarrow{\chi(m)} & \Omega \end{array}$$

commutes. Since the diagram (1) is a pullback there exists exactly one natural transformation $\beta : K \to G$ such that $m \circ \beta = \alpha$. Since $x \in KA$ we have $x = \alpha_A(x) = m_A(\beta_A(x)) = \beta_A(x) \in GA$; a contradiction with the fact that $x \in FA \setminus GA$. $\square$

Let us consider $f : A \to B$ and $x \in FA$. Using the Remark 3.5 we obtain

$$f \in \chi(m)_A(x) \Leftrightarrow (\Omega f)(\chi(m)_A(x)) = \max_B .$$

But $\chi(m)$ is a natural transformation hence

$$\chi(m)_B((Ff)(x)) = (\Omega f)(\chi(m)_A(x)).$$

Thus

$$f \in \chi(m)_A(x) \Leftrightarrow \chi(m)_B((Ff)(x)) = \max_B .$$

From the Lemma 4.2 we have

$$\chi(m)_A(x) = \{f \in \max_A : (Ff)(x) \in G(\operatorname{codom} f)\}$$

which proves that $\chi(m)$ is unique. Moreover $\chi(m)_A(x) = S(G, A, x)$ which under the Corollary 3.4 means that $\chi(m)$ is a natural transformation.

Let us consider $x \in GA$. Then $(Ff)(x) \in G(\operatorname{codom} f)$ for any $f \in \max_A$. Thus $\chi(m)_A(x) = \max_A = \top_A(0)$ and the diagram (1) commutes.

Lemma 4.3 *The diagram* (1) *is a pullback.*

Proof. Assume that there is natural transformation $\alpha : K \to F$ such that the diagram

$$(3) \qquad \begin{array}{ccc} K & \longrightarrow & 1 \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \top} \\ F & \xrightarrow{\chi(m)} & \Omega \end{array}$$

commutes. Then for every **C**-object A also the diagram

$$(3_A) \qquad \begin{array}{ccc} KA & \longrightarrow & 1 \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \top_A} \\ FA & \xrightarrow{\chi(m)_A} & \Omega A \end{array}$$

commutes. Suppose that $\beta_A : KA \to GA$ is such that $\alpha_A = m_A \circ \beta_A$. Since m_A is an insertion we have $\beta_A(x) = \alpha_A(x)$ for every $x \in KA$. Thus β_A, if exists, is unique. In order to define β_A in this way we have to know that $\alpha_A(x) \in GA$ for every $x \in KA$. But for $x \in KA$, we have $1_A \in \max_A = \top_A(0) = \chi(m)_A(x)$. Thus $x = (F(1_A))(x) \in G(\operatorname{codom} 1_A) = GA$.

Thus β_A is unique and correctly defined. Let us consider $f : A \to B$ and $x \in KA$. Then

$$(Gf)(\beta_A(x)) = (Ff)(\alpha_A(x)) = \alpha_B((Kf)(x)) = \beta_B((Kf)(x))$$

which means that $\beta : K \to G$ is a natural transformation. From the previous part of the proof we have also that $\alpha = m \circ \beta$ and such a β is unique.□

Fact 4.4 *For any natural transformation $u : F \to \Omega$ there exists a functor G and monomorphism $m : G \to F$ such that the pair $m : G \to F$, $G \to 1$ is a pullback of the pair $u, \top$.*

Proof. Let us consider a natural transformation $u : F \to \Omega$. Let $GA = u_A^{-1}[\{\max_A\}] \subseteq FA$ for every **C**-object A. Let us assume that $x \in GA$ and $f : A \to B$ is a **C**-arrow. Then

$$u_B((Ff)(x)) = (\Omega f)(u_A(x)) = (\Omega f)(\max_A) = \max_B .$$

Thus the function $Gf : GA \to GB$, $(Gf)(x) = (Ff)(x)$, is properly defined. This means that G is a subfunctor of F. From the Lemma 3.2 we know that $m : G \to F$ given by the formula $m_A(x) = x$ is a monomorphism. Observe that $u_A(m_A(x)) = u_A(x) = \max_A$, thus the diagram

$$\begin{array}{ccc} G & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \top} \\ F & \xrightarrow{\;u\;} & \Omega \end{array} \tag{3}$$

commutes. Let us consider a functor K and a natural transformation $\alpha : K \to F$ such that $u_A(\alpha_A(x)) = \max_A$ for every $x \in KA$. We have to show that there is a unique natural transformation $\beta : K \to G$ such that $m \circ \beta = \alpha$. Any such transformation satisfies the formula $\beta_A(x) = m_A(\beta_A(x)) = \alpha_A(x)$ which proves its uniqueness. Since $u_A(\alpha_A(x)) = \max_A$ we see that $\alpha_A(x) \in GA$, hence we can define β by the formula $\beta_A(x) = \alpha_A(x)$. If $f : A \to B$ is a **C**-arrow then

$$\beta_B((Gf)(x)) = \alpha_B((Ff)(x)) = (Kf)(\alpha_A(x)) = (Kf)(\beta_A(x))$$

which shows that β is a natural transformation. □

5 Power objects in the category $\mathbf{Set}^{\mathbf{C}}$

If $PF : \mathbf{C} \to \mathbf{Set}$ is a functor then from the Yoneda lemma there is a bijection between $(PF)A$ and $\mathrm{Nat}(H_A, PF)$. If PF is a power object of F then there is a bijection between $\mathrm{Nat}(H_A, PF)$ and $\mathrm{Nat}(F \times H_A, \Omega)$. In the previous section we have showed that for each natural transformation $u : F \times H_A \to \Omega$ there is a subfunctor $G \subseteq F \times H_A$ with monomorphism $m : G \to F \times H_A$ such that the pair $m : G \to F \times H_A$, $G \to 1$ is a pullback of the pair $u, \top$. Thus we should try to define $(PF)A$ as the set of all subfunctors of $F \times H_A$. We also need to define $(PF)f$ for a $\mathbf{C}$-arrow $f : A \to B$. This will be done via families of sieves.

Let G be a subfunctor of $F \times H_A$. Then we have the family

$$\{S(G, C, (x, g)) : C \in \mathrm{Ob}\,\mathbf{C}, (x, g) \in (F \times H_A)C\}.$$

We want $((PF)f)G$ to be a subfunctor of $F \times H_B$. Thus it suffices to find a sieve for every $\mathbf{C}$-object C and $(x, h) \in (F \times H_B)C = FC \times H_B C$. This is done by the formula

$$S(C, (x, h)) = S(G, C, (x, h \circ f)).$$

Let us consider $g : C \to D$. Then

$$k \in S(C, (x, h))_g \Leftrightarrow k \circ g \in S(C, (x, h)) \Leftrightarrow k \circ g \in S(G, C, (x, h \circ f)) \Leftrightarrow$$

$$((F \times H_A)(k \circ g))(x, h \circ f) \in G(\mathrm{codom}\,(k \circ g)) \Leftrightarrow$$

$$((F(k \circ g))(x), (H_A(k \circ g))(h \circ f)) \in G(\mathrm{codom}\,k) \Leftrightarrow$$

$$((Fk)((Fg)(x)), k \circ g \circ h \circ f) \in G(\mathrm{codom}\,k) \Leftrightarrow$$

$$((F \times H_A)k)((Fg)(x), g \circ h \circ f) \in G(\mathrm{codom}\,k) \Leftrightarrow$$

$$k \in S(G, D, ((Fg)(x), g \circ h \circ f)) \Leftrightarrow$$

$$k \in S(D, ((Fg)(x), g \circ h)) \Leftrightarrow$$

$$k \in S(D, ((Fg)(x), (H_B g)(h))) \Leftrightarrow$$

$$k \in S(D, ((F \times H_B)g)(x, h))$$

thus $S(C,(x,h))_g = S(D,((F\times H_B)g)(x,h))$ and

$$\{S(C,(x,h)) : C \in \mathrm{Ob}\,\mathbf{C}, (x,h) \in FC \times H_B C\}$$

is an Ω-matching family. From the Theorem 3.6 there exists a unique subfunctor $K \subseteq F \times H_B$ such that $S(C,(x,h)) = S(K,C,(x,h))$ for every $\mathbf{C}$-object C and $(x,h) \in FC \times H_B C$. We define $((PF)f)G = K$ and obtain the formula

$$S(((PF)f)G, C, (x,h)) = S(G, C, (x, h\circ f)).$$

In the case $f = 1_A$ we obtain

$$S(((PF)1_A)G, C, (x,h)) = S(G, C, (x, h\circ 1_A)) = S(G, C, (x,h)),$$

thus from Corollary 3.7 we have $((PF)1_A)G = G$ and $(PF)1_A = 1_{(PF)A}$.

Let us consider $f : A \to B$ and $g : B \to D$. Then

$$S(((PF)(g\circ f))G, C, (x,h)) = S(G, C, (x, h\circ g\circ f)) =$$

$$S(((PF)f)G, C, (x, h\circ g)) = S(((PF)g)(((PF)f)G), C, (x,h))$$

for every $\mathbf{C}$-object C and $(x,h) \in FC \times H_D C$. From Corollary 3.7 we obtain

$$((PF)(g\circ f))G = ((PF)g)(((PF)f)G)$$

which proves that $(PF)(g\circ f) = ((PF)g)\circ((PF)f)$. Thus $PF : \mathbf{C} \to \mathbf{Set}$ is a functor.

The last thing to be defined is the *evaluation arrow* $e_F : F \times PF \to \Omega$. If A is a $\mathbf{C}$-object then we define

$$(e_F)_A : FA \times (PF)A \to \Omega A, \quad (e_F)_A(x,G) = S(G, A, (x, 1_A)).$$

Fact 5.1 *e_F is a natural transformation.*

Proof. Let us consider $f : A \to B$. Then

$$(\Omega f)((e_F)_A(x,G)) = S(G, A, (x, 1_A))_f = S(G, B, ((F\times H_A)f)(x, 1_A)) =$$

$$S(G, B, ((Ff)(x), f)) = S(((PF)f)G, B, ((Ff)(x), 1_B)) =$$

$$(e_F)_B((Ff)(x), ((PF)f)G) = (e_F)_B((F\times PF)(x,G)).$$

□

We have to show that for each $\eta : F \times K \to \Omega$ there exists a unique $\widehat{\eta} : K \to PF$ such that $\eta = e_F \circ (1_F \times \widehat{\eta})$. Let us assume that such $\widehat{\eta}$ exists. Then

$$\eta_A(x, y) = (e_F)_A((1_F \times \widehat{\eta})_A(x, y)) = (e_F)_A(x, \widehat{\eta}_A(y)) = S(\widehat{\eta}_A(y), A, (x, 1_A)).$$

Let us consider **C**-object A and $y \in KA$. We define

$$S(C, (x, g)) = \eta_C(x, (Kg)(y))$$

for $x \in FA$ and $g : A \to C$. For $f : C \to D$ we compute

$$S(C, (x, g))_f = (\Omega f)(S(C, (x, g)) = (\Omega f)(\eta_C(x, (Kg)(y))) =$$
$$\eta_D(((F \times K)f)(x, (Kg)(y))) = \eta_D((Ff)(x), (Kf)((Kg)(y))) =$$
$$\eta_D((Ff)(x), (K(f \circ g))(y)) = S(D, ((Ff)(x), f \circ g)) =$$
$$S(D, ((Ff)(x), (H_A f)g)) = S(D, ((F \times H_A)f)(x, g))$$

which proves that $\{S(C, (x, g)) : C \in \mathrm{Ob}\,\mathbf{C}, (x, g) \in FC \times H_A C\}$ is an Ω-matching family. From the Theorem 3.6 there exists a unique subfunctor $G \subseteq F \times H_A$ such that for every **C**-object C and $x \in FC$, $g : A \to C$,

$$S(G, C, (x, g)) = S(C, (x, g)) = \eta_C(x, (Kg)(y)) =$$
$$S(\widehat{\eta}_C((Kg)(y)), C, (x, 1_C)) = S(((PF)g)(\widehat{\eta}_A(y)), C, (x, 1_C))) =$$
$$S(\widehat{\eta}_A(y), C, (x, 1_C \circ g)) = S(\widehat{\eta}_A(y), C, (x, g)).$$

From the Theorem 3.6 we obtain $G = \widehat{\eta}_A(y)$ thus $\widehat{\eta}$ is unique.

Fact 5.2 *$\widehat{\eta}$ is a natural transformation.*

Proof. Let us consider $f : A \to B$, $y \in KA$ and **C**-object C. Since $\widehat{\eta}_B((Kf)(y))$ is a subfunctor of $F \times H_B$ we consider $(x, g) \in FC \times H_B C$. Then

$$S(\widehat{\eta}_B((Kf)(y)), C, (x, g)) = \eta_C(x, (Kg)((Kf)(y))) = \eta_C(x, (K(g \circ f))(y)) =$$
$$S(\widehat{\eta}_A(y), C, (x, g \circ f)) = S(((PF)f)(\widehat{\eta}_A(y)), C, (x, g)).$$

From the Theorem 3.6 we obtain $\widehat{\eta}_B((Kf)(y)) = ((PF)f)(\widehat{\eta}_A(y))$. □

We see that

$$(e_F)_A((1_F \times \widehat{\eta})_A(x, y)) = (e_F)_A(x, \widehat{\eta}_A(y)) = S(\widehat{\eta}_A(y), A, (x, 1_A)) =$$
$$\eta_A(x, (K1_A)(y)) = \eta_A(x, y)$$

hence $\eta = e_F \circ (1_F \times \widehat{\eta})$. This shows that the category $\mathbf{Set}^{\mathbf{C}}$ has power objects.

Finally, we have proved that the category $\mathbf{Set}^{\mathbf{C}}$ is a topos.

References

[1] S. Awodey, *Category Theory*, Oxford University Press, New York, 2006.

[2] J. L. Bell, *Toposes and Local Set Theories: An Introduction*, Oxford University Press, New York, 1988.

[3] R. Goldblatt, *Topoi, the categorial analysis of logic*, Elsevier Science Publishers B.V., Amsterdam, 1984.

[4] S. Mac Lane, I. Moerdijk, *Sheaves in Geometry and Logic*, Springer-Verlag, New York, 1992.