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Measurable colorings of $\mathbb{R}^n$

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Abstract

A well known generalization of Alon's "splitting nacklace theorem" by Longueville and Živaljević states that every k-colored n-dimensional cube can be fairly split using only k cuts in each dimension. Here we prove that for every t there exist a finite coloring (with at least $(t+4)^d - (t+3)^d + (t+2)^d - 2^d + d(t+2) + 3$ different colors) of $\mathbb{R}^n$ such that no n-dimensional cube can be fairly split using at most t cuts in each dimension. In particular there is a finite coloring of $\mathbb{R}^n$ such that no two disjoint n-dimensional cubes have the same measure of each color.

1 Introduction

In 2009 Alon, Grytczuk, Lasoń and Michałek [3] proved that there is a measurable $t+3$-coloring of the real line such that no interval has a fair t-splitting. As one of the corollaries they proved that there is a 5-coloring of the real line such that no two intervals have the same measure of each of the 5 colors. In the present paper we present a multidimensional generalization of their result - there is a $(t+4)^d - (t+3)^d + (t+2)^d - 2^d + d(t+2) + 3$-coloring of the d-dimensional Euclidean space such that no cube has a fair d-dimensional t-splitting.

By a d-dimensional spitting of a cube $[0,1]^d$ of size at most t in each dimension we mean a choice, for $i = 1, \ldots, d$ of t_i, $0 \leqslant t_i \leqslant t$, of nonnegative

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numbers $0 \leqslant z_1^i \leqslant \ldots \leqslant z_{t_i}^i \leqslant 1$ in the unit interval $[0,1]$ such that $\sum t_i \geqslant 1$. The hyperplanes defined by z_j^i (i.e. $I \times \ldots \times \{z_j^i\} \times \ldots \times I \subseteq \mathbb{R}^d$) cut the cube $[0,1]^d$ into at most $\prod(t_i+1)$ smaller cubes. Whenever we speak of a splitting of exact size we mean that the number of t_i for each i is given and we assume that the inequalities between z_j^i's, 0 and 1 are sharp.

By a k-coloring of $A \subseteq \mathbb{R}^d$ we mean a function $\phi\colon A \to \{1,\ldots,k\}$ such that each set $f^{-1}(i)$ is Lebesgue measurable.

Theorem 1.1. *For every $k \geqslant (t+4)^d - (t+3)^d + (t+2)^d - 2^d + d(t+2) + 3$ there is a k-coloring of $\mathbb{R}^d$ such that no d-dimensional cube has a splitting of size at most t.*

The proof is similar in nature to that of [3] and uses Baire category argument applied to the space of all measurable colorings. The same methods with only minor changes in proofs lead to a different version of the result:

Theorem 1.2. *There is a $2^{2d+1} - 2\cdot 3^d + 4d + 4$-coloring of $\mathbb{R}^d$ such that no two (disjoint) cubes have the same measure of each color.*

2 Proof of theorem 1.1

Recall that a set in a metric space is nowhere dense if the interior of its closure is empty. A set is said to be of first category if it can be represented as a countable union of nowhere dense sets. In the proof of theorem 1.1 we apply the Baire category theorem.

Theorem 2.1 (Baire)**.** *If X is a complete metric space and A is a set of first category in X then $X \backslash A$ is dense in X (and in particular is nonempty).*

Our plan is to mimic the argument of [3] and construct a suitable metric space of colorings of $\mathbb{R}^d$ and demonstrate that the subset of bad colorings is of first category.

2.1 The setting.

Let k be a fixed positive integer and let $\{1,2,\ldots,k\}$ be the set of colors. Let f and g be two measurable colorings of $\mathbb{R}^d$ ($d \geqslant 1$). For a positive integer n we set

$$D_n(f,g) = \{x \in [-n,n]^d\colon f(x) \neq g(x)\}.$$

Clearly $D_n(f,g)$ is Lebesgue measurable so we may define the normalized distance between f and g on $[-n,n]^d$ by

$$d_n(f,g) = \frac{\lambda(D_n(f,g))}{n^d},$$

where λ is the d-dimensional Lebesgue measure. Since $d_n(f,g)$ is bounded from above by 2^d we may define the distance between two measurable colorings f and g by

$$d(f,g) = \sum_{n=1}^{\infty} \frac{d_n(f,g)}{2^{n+1}}.$$

Identifying colorings whose distance is zero gives a metric space $\mathcal{M}$ of equivalence classes of all measurable k-colorings. Note that the splitting properties are preserved by equivalent colorings.

Lemma 2.2. *The space $\mathcal{M}$ is a complete metric.*

We omit the proof since this is a simple generalization (see [5] or [3]) of a result stating that sets of finite measure in any metric space form a complete metric space with symmetric difference as the distance function.

Let $t \geqslant 1$ be a fixed integer. Let D_t be the subspace of $\mathcal{M}$ consisting of those k-colorings that avoid intervals having a d-dimensional splitting of size at most t in each dimension. We will show that D_t is not empty provided that $k \geqslant (t+4)^d - (t+3)^d + (t+2)^d - 2^d + d(t+2) + 3$. By granularity of a splitting we mean the length of the shortest subinterval $[z_j^i, z_{j+1}^i]$ in the splitting. For $n \geqslant 1$ and $r_1, \ldots, r_n$ let $B_n^{(r_i)}$ be the set of those colorings from $\mathcal{M}$ for witch there exists at least one d-dimensional cube in $[-n,n]^d$ having a d-dimensional splitting of size exactly r_i in the i-th dimension for each i and granularity at least $1/n$. Finally let us denote all the bad colorings by

$$B_n(t) = \bigcup_{r_i \leqslant t} B_n^{(r_i)}.$$

Clearly we have

$$D_t = \mathcal{M} \setminus \bigcup_{n=1}^{\infty} B_n(t).$$

Now our aim is to apply Baire category theorem to show that the sets $B_n(t)$ are nowhere dense, provided that $k \geqslant (t+4)^d - (t+3)^d + (t+2)^d - 2^d + d(t+2) + 3$.

2.2 The sets $B_n(t)$.

We show that each set $B_n^{(r_i)}$ is a closed subset of $\mathcal{M}$. Since $B_n(t)$ is a finite union of these sets, it must be closed too.

Theorem 2.3. *The set $B_n^{(r_i)}$ for every $r_i \geqslant 1$ and $n \geqslant 1$ is a closed subset of $\mathcal{M}$.*

Proof. Let $\{f_m\}$ be a sequence of colorings converging in $\mathcal{M}$ to f. For each m let C_m denote a d-dimensional cube in $[-n,n]^d$ of granularity $\geqslant 1/n$ and having a fair splitting into exactly r_i points in the i-th dimension. Let us denote by $\phi_m \colon [r_1] \times \ldots \times [r_d] \to \{1,2\}$ the labeling function defining the two families from the fair splitting of C_m. Since $[-n,n]^d$ is compact we may assume that vertices of the sliced cube C_m converge to vertices of some cube C and since there is finite number of labeling functions we may assume that $\phi_m = \phi$ for every m. Now it is easy to see that ϕ gives a fair splitting for C. □

Next we prove that each $B_n(t)$ has empty interior provided the number of colors k satisfies $k > (t+4)^d - (t+3)^d + (t+2)^d - 2^d + d(t+2) + 2$. For this purpose let us call $f \in \mathcal{M}$ a cube coloring on $[-n,n]^d$ if there is a partition of $[-n,n]^d$ into some number of (half open) d-dimensional cubes of equal size in each dimension, each filled with only one color. Let I_n denote the set of all colorings from $\mathcal{M}$ that are cube colorings on $[-n,n]^d$.

Lemma 2.4 (comp. [3])**.** *Let $f \in \mathcal{M}$ be a k-coloring. Then for every $\epsilon > 0$ and $n \in \mathbb{N}$ there exists a coloring $g \in I_n$ such that $d(f,g) < \epsilon$.*

Proof. Let $C_i = f^{-1}(i) \cap [-n,n]^d$ and let $C_i^* \subseteq [-n,n]^d$ be a finite union of intervals such that

$$\lambda((C_i^* \backslash C_i) \cup (C_i \backslash C_i^*)) < \frac{\epsilon}{2k^2}$$

for each $i = 1,2,\ldots,k$. Define coloring h so that for each $i = 1,2,\ldots,k$ the set $C_i^* \backslash (C_1^* \cup \ldots \cup C_{i-1}^*)$ is filled with color i, the rest of the cube $[-n,n]^d$ is filled with any of these colors. Moreover we set h to be equal f outside $[-n,n]^d$. Note that $d(f,h) < \epsilon/2$ and $h^{-1}(i) \cap [-n,n]^d$ is a finite union of cubes. Let $A_1, A_2, \ldots, A_N$ be the whole family of these cubes. Now split the cube $[-n,n]^d$ into M^d cubes $B_1, \ldots B_{M^d}$ equally spaced in $[-n,n]^d$. We define g to be equal h on A_i whenever $A_i \subseteq B_j$ for some j and $g(A_i)$ is of any color otherwise. Note that g differs from h on a set of d-dimensional measure at most $t((2n+4n/M)^d-(2n)^d)$ so that for sufficiently large M $d(g,h)<\epsilon/2$ and we get $d(f,g)<\epsilon$. □

In order to state the next lemma we will use the following notation:

$$D(d) := \sum_{i=1}^{d} \binom{d}{i} (t+2)^i (2^{d-i}-1) = (t+4)^d + 1 - (t+3)^d - 2^d$$

Lemma 2.5. *If $k > (t+2)^d + d(t+2) + 1 + D(d)$ then each $B_n(t)$ has empty interior.*

Proof. Let $f \in B_n(t)$ be any bad coloring. Let $U(f,\epsilon)$ be the open ϵ-neighborhood of f in the space $\mathcal{M}$. Assume the assertion of the lemma is false: there is some $\epsilon>0$ for witch $U(f,\epsilon)\subseteq B_n(t)$. By lemma 2.4 there is a coloring $g\in I_n$ such that $d(f,g)<\epsilon/2$, so that $U(g,\epsilon/2)\subseteq B_n(t)$. The idea is to modify slightly the cube coloring g so that the new coloring will still be close to g, but there will be no cube in $[-n,n]^d$ possessing a fair splitting of size at most t and granularity at least $1/n$. Without loss of generality we may assume that there are equally spaced cubes $C_{i_1,\ldots,i_d}$ for $i_1,\ldots,i_d\in\{1,2,3,\ldots,N\}$ in $[-n,n]^d$ such that $1>6n^2/N$ each cube is filled with a unique color in the cube coloring g. Let $\delta>0$ be a real number satisfying

$$\delta < min\{\frac{\sqrt[d]{\epsilon}}{2N}, \frac{2n}{N^2}\}.$$

Choose a color (which we will call from now on "white").

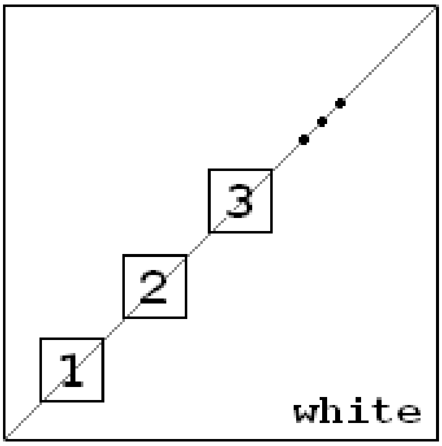
Let $W'_{i_1,\ldots,i_d}$ where $i_1,\ldots,i_d\in\{1,2,\ldots,N\}$ be a cube $[0,2\delta]^d$ colored as follows: choose a countable set

$$\{m^j_{i_1,\ldots,i_d}\}_{j=1,\ldots,k;\; i_1,\ldots,i_d\in\{1,2,3\ldots N\}}$$

of linearly independent over $\mathbb{Q}$ real numbers such that $0<m^j_{i_1,\ldots,i_d}<(\delta/k)^d$. We color $W'_{i_1,\ldots,i_d}$ white except for small cubes

$$V^\eta_{i_1,\ldots,i_d} = (\frac{2\eta-1}{k}\delta,\ldots,\frac{2\eta-1}{k}\delta) + \prod_{j=1}^{d}[-\sqrt[d]{m^j_{i_1,\ldots,i_d}}, \sqrt[d]{m^j_{i_1,\ldots,i_d}}]$$

colored using color η for $\eta=1,2,\ldots k$ (compare Pic. 1 for the 2-dimensional case). Note that the d-dimensional Lebesgue measure of $V^\eta_{i_1,\ldots,i_d}$ is equal $2^d m^j_{i_1,\ldots,i_d}$ hence measures of these cubes are linearly independent over $\mathbb{Q}$.



Pic. 1

Now modify the coloring g to get a coloring h outside $B_n(t)$. The coloring h is equal to g outside $[-n,n]^d$. Inside $C_{i_1,\dots,i_d}$ the coloring h is equal g except in

$$W_{i_1,\dots,i_d} = ((i_1 - \frac{1}{2} - \delta)\frac{2n}{N} - n, \dots, (i_d - \frac{1}{2} - \delta)\frac{2n}{N} - n) + W'_{i_1,\dots,i_d}$$

where h is defined by the coloring of $W'_{i_1,\dots,i_d}$.

Note that $d(g,h) < \epsilon/2$ so that there exists a d-dimensional cube C in $[-n,n]^d$ with granularity at least $1/n$ such that there is a fair splitting of size at least t. The fair splitting divides C into at most $(t+1)^d$ cubes hence we obtain a d-dimensional cell complex in $[-n,n]^d$ (which we will also denote by C).

Let us denote by A the measure of $C_{i_1,\dots,i_d} \backslash W_{i_1,\dots,i_d}$ (note that A does not depend on the set of indexes chosen and we may assume it is linearly independent with the $m^j_{i_1,\dots,i_d}$ chosen before).

By the determinant of $C_{i_1,\dots,i_d}$ in C (den. by $\det_C C_{i_1,\dots,i_d}$) we mean the lowest dimension of cells C that intersect $C_{i_1,\dots,i_d}$ (there is only one cell reaching the minimum – denoted by $d_C(C_{i_1,\dots,i_d})$). If $C_{i_1,\dots,i_d}$ lays outside C we set $\det_C C_{i_1,\dots,i_d} = d$. Note that cells of C divide each cube $C_{i_1,\dots,i_d}$ into $2^{codimdet_C C_{i_1,\dots,i_d}}$ cubes of measures

$$\alpha_1(d_C(C_{i_1,\dots,i_d})), \alpha_2(d_C(C_{i_1,\dots,i_d})), \dots, \alpha_{2^{codimd_C C_{i_1,\dots,i_d}}}(d_C(C_{i_1,\dots,i_d}))$$

and their sum is equal to A. In fact (up to indexing) $\alpha_i(d_C(C_{i_1,\dots,i_d}))$ does not depend on $d_C(C_{i_1,\dots,i_d})$ but on the $det_C(C_{i_1,\dots,i_d})$-dimensional subspace of

$\mathbb{R}^d$ spanned by it. The subspace can be identified by a suitable choice of $codimdet_C C_{i_1,\ldots,i_d}$ slices (or ends) of C on some of the dimensions. Hence we get that $\alpha_i(d_C(C_{i_1,\ldots,i_d})) = \alpha_i(t_1,\ldots,t_s)$ for $t_1,\ldots,t_s \in \{0,1,2,\ldots,t+1\}$ and $s = 0,1,2,\ldots,d$. Of course $\alpha_1(\emptyset) = A$.

Note that the dimension of the space spanned by $\alpha_i(t_1,\ldots,t_s)$ where $s > 0$ is no greater than $D(d)$.

Now note that all the vertices of C are colored at most by $(t+2)^d$ colors. Moreover cells of dimensions $d-1$ of C intersect at most one of the cubes $V^\eta_{i_1,\ldots,i_d} \subseteq C_{i_1,\ldots,i_d}$ and two such cell intersect the cubes of the same color if they span the same subspace of $\mathbb{R}^d$. Since there are at most $d(t+2)$ different subspaces of $\mathbb{R}^d$ obtained in such a way then if C intersects one of $V^\eta_{i_1,\ldots,i_d} \subseteq C_{i_1,\ldots,i_d}$ then $d-1$ of C also does and it has one of $d(t+2)$ colors.

Summing up let us consider a color c different from white and the $(t+2)^d + d(t+2)$ colors mentioned before. Since our splitting is fair d-dimensional cells colored partially by c can be divided into two families having equal measure of c. Hence the measure satisfies equality of the form:

$$T(0)A + \sum \epsilon(0)^j_{i_1,\ldots,i_d} 2^d m^j_{i_1,\ldots,i_d} + \sum S(0)^i_{t_1,\ldots,t_s} \alpha_i(t_1,\ldots,t_s)$$
$$- T(0)A - \sum \epsilon(0)^j_{i_1,\ldots,i_d} 2^d m^j_{i_1,\ldots,i_d} - \sum S(0)^i_{t_1,\ldots,t_s} \alpha_i(t_1,\ldots,t_s) = 0$$

where $T(0), T(1) \in \mathbb{N}$, $\epsilon(0)^j_{i_1,\ldots,i_d}, \epsilon(1)^j_{i_1,\ldots,i_d} \in \{0,1\}$, $S(0), S(1) \in \mathbb{N}$ and not all $\epsilon(0)^j_{i_1,\ldots,i_d}$ are equal to 0. Note that for each color the numbers

$$T(0)A \ + \ \sum \epsilon(0)^j_{i_1,\ldots,i_d} 2^d m^j_{i_1,\ldots,i_d} \ - \ T(0)A \ - \ \sum \epsilon(0)^j_{i_1,\ldots,i_d} 2^d m^j_{i_1,\ldots,i_d}$$

are independent over $\mathbb{Q}$. On the other hand they can be generated over $\mathbb{Q}$ by $\alpha_i(t_1,\ldots,t_s)$ so they lie in $D(d)$-dimensional space. Since the number of remaining colors is greater than $D(d)$ we get a contradiction that ends the proof. $\square$

3 Generalizations and open problems

Note that when $t = 1$ and $d = 1$ we get that for $k > 7$ there is a k-coloring of $\mathbb{R}$ such that no consecutive intervals have the same measure of every color. This result is weaker than the one obtained in [3] where it was proved for

$k = 4$, but our approach has a higher-dimensional generalizations. Moreover, even in our approach, the number of colors can be easily reduced to 4 in the case of the real line and one slice (or to $t + 3$ in the case of the real line and t slices).

Nevertheless the difference between the positive answer, i.e. k-colored d-dimensional cube can be fairly split using kd cuts (comp. [4]), and our result is surprisingly big. Hence we hope that our result can be further improved.

Note also that a small modification of our result gives the following

Theorem 3.1. *There is a $2^{2d+1} - 2 \cdot 3^d + 4d + 4$-coloring of $\mathbb{R}^d$ such that no two (disjoint) cubes have the same measure of each color.*

Once again our approach can be improved in the case of the real line to match the result of [3] – there is a 5-coloring of the real line such that no two disjoint intervals have the same measure of each color.

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