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A generalization of Seymour's theorem

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A generalization of Seymour's Theorem

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Abstract. We call a coloring of a matroid *proper* if vertices of the same color form an independent set. Seymour theorem says that when a matroid is properly colorable using k colors, then it can be properly colored from any lists of size k. We generalize this theorem by giving a weigted version, with arbitrary sizes of lists, and a stronger assertion which says that a matroid can be properly on-line colored from lists of appropriate sizes. As corollaries we get that for matroids there is an equality between on-line list chromatic number and list chromatic number and also between fractional on-line list chromatic number and fractional list chromatic number. The main tool we use is the multiple basis exchange property, we give a simple proof of it.

1. Introduction

We call a coloring of a matroid *proper* if vertices of the same color form an independent set. The *chromatic number* of a matroid is the least possible number of colors that suffice to color properly all its vertices. The *list chromatic number* of a matroid M is the least possible number k, such that it is possible to color all vertices of M from any assignment of k-element lists of its vertices (each vertex has to be colored with a color from its list).

Let M be a matroid on a ground set E with an assignment $\mathbf{W} : E \to \mathbb{N}$ of weights of elements of E, and let $\mathbf{l} : E \to \mathbb{N}$. We say that a list assignment $L : E \to \mathcal{P}(\mathbb{N})$ *is of size* $\mathbf{l}$, if for each $e \in E$ we have $|L(e)| = \mathbf{l}(e)$. We say that M is *properly* $\mathbf{W}$*-colorable from lists* L, if it is possible to properly color each $e \in E$ with $\mathbf{W}(e)$ colors from its list $L(e)$.

The *fractional chromatic number* of a matroid M is the infimum of fractions $\frac{b}{a}$, such that M is properly $\mathbf{W} \equiv a$-colorable using b colors. The *fractional list chromatic number* of a matroid M is the infimum of fractions $\frac{b}{a}$, such that M is properly $\mathbf{W} \equiv a$-colorable from any lists of size $\mathbf{l} \equiv b$.

The *on-line* $\mathbf{l}$*-list* $\mathbf{W}$*-coloring game* on M is a two person game in which colors in the lists appear and are used sequentially. In the i-th round Mr. Paint shows nonempty set W_i of vertices which have i on its lists, and Mrs. Correct has to color its independent subset I_i using color i. The only restriction for Mr. Paint is that for all $e \in E$ always $|\{i : e \in W_i\}| \leq \mathbf{l}(e)$, and the game ends when there is an equality everywhere. Mrs. Correct wins if each vertex e has been colored with $W(e)$ colors.

Definition 1. *If Mrs. Correct has a winning strategy then we say that* matroid M is properly on-line $\mathbf{W}$-colorable from lists of size $\mathbf{l}$.

Definition 2. *The* on-line list chromatic number *of a matroid M is the least possible number k, such that M is properly on-line $\mathbf{W} \equiv 1$-colorable from lists of size $\mathbf{l} \equiv k$.*

Of course on-line list chromatic number of a matroid M is greater or equal to its list chromatic number, which can be considered as its "off-line" version.

Definition 3. *The* fractional on-line list chromatic number *of a matroid M is the infimum of fractions $\frac{b}{a}$, such that M is properly on-line $\mathbf{W} \equiv a$-colorable from lists of size $\mathbf{l} \equiv b$.*

All above parameters were originally defined for graphs. Definitions for graphs are the same, we change only the meaning of proper coloring - vertices of the same color have to form an independent set in a graph.

For graphs list chromatic number does not have to be equal to the chromatic number, it is not even bounded by any function of chromatic number. Seymour's theorem says that for matroids they are equal.

For graphs on-line list chromatic number does not have to be equal to the list chromatic number, however it is bounded by an exponential function of list chromatic number. Definition of the on-line list chromatic number was first introduced by Schauz in [**7**]. For graphs this topic has been intensively studied in the papers [**2**], [**8**]. Also for graphs fractional chromatic number equals to the fractional list chromatic number and by [**1**] they are equal to the fractional on-line list chromatic number. However infimum from the definition may not be attained.

We prove a generalization of Seymour's theorem - Theorem 3. It gives an easy necessary and sufficient condition for a matroid to be properly on-line $\mathbf{W}$-colorable from lists of a fixed size $\mathbf{l}$. As a main tool we use the multiple basis exchange property - Theorem 2, for which we give a simple proof. As corollaries we get that for matroids there is an equality between on-line list chromatic number and list chromatic number and also between fractional on-line list chromatic number and fractional list chromatic number, additionally infimums from the definition are attained.

2. Preliminaries

The only preliminary we need is the following exchange property.

Theorem 1. *Let I_1 and I_2 be independent sets of a matroid M. Then for every $V \subset I_1$ there exists $U \subset I_2$ such that $(I_1 \setminus V) \cup U$ and $(I_2 \setminus U) \cup V$ are also independent.*

Proof. Let $I = I_1 \cap I_2$. We can restrict to the case then $I = \emptyset$, indeed if $I \neq \emptyset$, then consider matroid M with contracted set I, and two independent sets $I_1 \setminus I_2$, and $I_2 \setminus I_1$. For $V \setminus I_2$ we get U, which is also good in the previous case.

Now let $I_1 \cap I_2 = \emptyset$. Let M_1 be matroid M restricted to the set $V \cup I_2$, and let M_2 be matroid M restricted to the set $(I_1 \setminus V) \cup I_2$. Let $I_1 \cup I_2$ be their common ground set, and denote their rank functions by r_1, r_2 respectively. Observe that for each $A \subset I_1 \cup I_2$ we have:

$$r_1(A) + r_2(A) = r(A \cap (V \cup I_2)) + r(A \cap ((I_1 \setminus V) \cup I_2)) \geq$$

$$\geq r(A \cap (I_1 \cup I_2)) + r(A \cap I_2) \geq |A \cap I_1| + |A \cap I_2| = |A|,$$

where the first inequality is just a submodularity of a rank function. From the matroid union theorem (see [**5**]) it follows that $I_1 \cup I_2$ can be covered by sets I'_1, I'_2 independent in M_1 and M_2 respectively, so also in M. Now $U = I_2 \cap I'_2$ is a good choice, since $(I_1 \setminus V) \cup U = I'_2$ and $(I_2 \setminus U) \cup V = I'_1$. □

As a corollary we get the multiple base exchange property (see [**3**]).

THEOREM 2. (Multiple base exchange property) *Let B_1 and B_2 be basis of a matroid M. Then for every $V \subset B_1$ there exists $U \subset B_2$, such that $(B_1 \setminus V) \cup U$ and $(B_2 \setminus U) \cup V$ are also basis.*

PROOF. If I_1, I_2 are basis of M, then $(I_1 \setminus V) \cup U$ and $(I_2 \setminus U) \cup V$ have to be of equal size and so they are also basis. □

3. The main result

We say that sets $I_1, \dots, I_k$ form a **W**-cover of a set E if for each $e \in E$ we have $|\{i : e \in I_i\}| = W(e)$. We introduce an inductive step lemma.

LEMMA 1. *Let $I_1, \dots, I_k$ be independent sets in a matroid M which **W**-cover its ground set E. Let $W \subset E$ be an arbitrary set (not necessarily independent), then there exists $I \subset W$ and independent sets $I'_1, \dots, I'_k$, such that:*

(1) *I is an independent set*
(2) *sets $I'_1, \dots, I'_k$ form a $(\mathbf{W} - \mathbf{I})$-cover of E (for each $e \in E$ $(\mathbf{W} - \mathbf{I})(e) := \mathbf{W}(e) - \chi_I(e)$)*
(3) *for each $e \in E$ if $e \in I'_s$, then $e \in I_t$ for some $t \geq s + \chi_W(e)$ ($\mathbf{l} : E \ni e \to \max\{s : e \in I_s\}$ is supposed to play the role of size of lists assignment).*

PROOF. Let $V_1 = (W \cap I_1) \setminus (I_1 \cap I_2)$, by Theorem 1 there exists $U_2 \subset I_2$, such that $I'_1 := (I_1 \setminus V_1) \cup U_2$ and $I''_2 := (I_2 \setminus U_2) \cup V_1$ are independent. In general let $V_i = (W \cap I''_i) \setminus (I''_i \cap I_{i+1})$, so again by Theorem 1 there exists $U_{i+1} \subset I_{i+1}$, such that $I'_i := (I''_i \setminus V_i) \cup U_{i+1}$ and $I''_{i+1} := (I_{i+1} \setminus U_{i+1}) \cup V_i$ are independent. Let $I = V_k$. It is easy to see that conditions $(1) - (3)$ are satisfied. □

We are ready to prove a generalization of Seymour's Theorem.

THEOREM 3. (Generalization of Seymour's Theorem) *Suppose matroid M is properly **W**-colorable from lists $\mathbf{L}(e) = \{1, \dots, \mathbf{l}(e)\}$. Then it is also properly on-line **W**-colorable from lists of size $\mathbf{l}$.*

PROOF. We prove it by the induction on the $\mathbf{W}(E) = \sum_{e \in E} \mathbf{W}(e)$. If $\mathbf{W}(E) = 0$, then $\mathbf{W} \equiv 0$, and the assertion is clearly true. When $\mathbf{W} > 0$, let $W \subset E$ be a set of vertices which have 1 on its list. Let $I_1, \dots, I_k$ be now a **W**-coloring which exists from the assumption. Apply Lemma 1, there exist independent sets $I \subset W$ and $I'_1, \dots, I'_k$, such that $I'_1, \dots, I'_k$ is a $(\mathbf{W} - \mathbf{I})$-covering of E. Now Alice colors vertices from I with 1. Due to condition (3) from Lemma 1 matroid M is properly $(\mathbf{W} - \mathbf{I})$-colorable from lists $\mathbf{L}(e) = \{1, \dots, \mathbf{l}(e) - \chi_W(e)\}$. We conclude by the inductive assumption. □

Observe that the condition from the assumption of Theorem 3 is not only sufficient, but also a necessary one for a matroid to be properly on-line **W**-colorable from lists of size $\mathbf{l}$.

Taking $\mathbf{W} \equiv 1$ and $\mathbf{l} \equiv k$ we get:

Corollary 1. *For matroids the on-line list chromatic number equals to the chromatic number.*

The remarkable Theorem of Seymour [**6**] says that for matroids the list chromatic number equals to the chromatic number, which is the "off-line" version of the above corollary.

Corollary 2. *For matroids the fractional on-line list chromatic number equals to the fractional chromatic number, moreover both infimums are attained.*

Proof. If there exists a $\mathbf{W} \equiv a$-coloring using b colors, when by Theorem 3 matroid is also on-line $\mathbf{W} \equiv a$-colorable from lists of size $\mathbf{l} \equiv b$. This shows that the fractional on-line list chromatic number is less or equal to the fractional chromatic number. The opposite inequality is obvious.

Moreover let us denote $\Delta(M) = \min_{\emptyset \neq A \subset E} \frac{|A|}{r(A)}$, and suppose that the minimum is realized by the set C. It is known that fractional chromatic number of M equals to $\Delta(M)$, and it is attained for $a = r(C)$, $b = |C|$. Now fractional on-line chromatic number is attained for the same values of a and b. $\square$

Another application of Theorem 3 is then we take $\mathbf{W} \equiv 1$ and look at off-line version.

Corollary 3. *If matroid M is properly colorable from lists $\mathbf{L}(e) = \{1, \ldots, \mathbf{l}(e)\}$, then it is also properly colorable from any lists of size $\mathbf{l}$.*

Theorem 3 generalizes also another variation of Seymour's theorem - Theorem 2 from [**4**], which is its off-line version, and which has slightly different proof. The latter theorem shows possible applications to matroid games.

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