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Maps between classifying spaces of unitary groups

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# MAPS BETWEEN CLASSIFYING SPACES OF UNITARY GROUPS

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ABSTRACT. We give an elementary proof of the classification theorem for maps between classifying spaces for unitary groups in certain range of dimensions. We investigate the case of homotopy representation of  $U(p)$  at a prime  $p$ .

A homotopy (complex) representation of a compact Lie group  $L$  at a prime  $p$  is a map from  $BL$  into the  $p$ -completion (in the sense of Bousfield and Kan) of the classifying space of the unitary group  $BU(n)_p^\wedge$ . A family of homotopy representations of  $L$  at  $p$  where  $p$  runs over the set of primes is called a homotopy representation of  $L$  if restriction of every two maps in the family gives the same map up to homotopy on the classifying spaces of the maximal tori.

In this article we give a short proof of the classification theorem which states that homotopy representations of  $U(n)$  of dimensions  $< n^2$  can be obtained by composing Adams operations (as in Jackowski, McClure and Oliver [1] where uniqueness of the operations is proved) and homomorphisms of unitary groups with only one exception – for  $n$  that are not a power of a prime the adjoint representation can be further reduced in the homotopy case. A homotopy representation not induced by the Adams operations and homomorphisms (or compositions of these two) will be called further on *exceptional*. We show that each of these homotopy representations extends to maps  $BU(n) \rightarrow BU(m)$  where  $m < n^2$  is the dimension of the representation and that the list of such mappings is complete. However postpone the proof of existence and extensibility of a reduced adjoint map to [3].

## 1. INTRODUCTION

For the purpose of this article let us call a map  $\phi$  *admissible* from  $G$  to  $G'$  ( $G$  and  $G'$  are compact Lie groups) if it is a homomorphism between the maximal tori  $\phi: T \rightarrow T'$  and for every  $w \in W = N(T)/T'$  there exists

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$w' \in W' = N(T')/T'$  such that  $w' \circ \phi = \phi \circ w$ . The definition is motivated by the following theorem:

**Theorem 1.1** ([2], Theorem 1.1). *For any  $f: BG \rightarrow BG'$  there exists an admissible map  $\phi: T \rightarrow T'$  such that  $B\phi \simeq f|_{BT}$  considered as maps  $BT \rightarrow BG'$ . Furthermore for any other admissible map  $\phi'$  satisfying the theorem we have  $\phi' = w \circ \phi$  for some  $w \in W'$ .*

Since we know the structure of  $\text{hom}(T, T')$  we get that every map  $BG \rightarrow BG'$  can be characterized (not necessarily uniquely) by a weight system  $\mu$  invariant under permutation of roots. The homomorphism corresponding to  $\mu = \{(k_1^i, \dots, k_n^i) : i = 1, \dots, m\}$  is the following:

$$\phi_\mu: T \ni (a_j)_{j=1}^n \mapsto \left( \prod a_j^{k_j^i} \right)_{i=1}^m \in T'.$$

Note that the homomorphism is defined up to a permutation of coordinates in  $T'$ . Moreover we have ([1], Proposition 1.2) that

$$[BG, BG']_\phi \cong \prod_{p \nmid \#W} [BG, BG'_p]_\phi$$

(where  $\phi$  in lower indices means all homotopy classes of maps corresponding to  $\phi$  via theorem 1.1). The most important question is the following: which weight systems define maps between classifying spaces of Lie groups?

*From now on we focus our attention to unitary groups.*

Fix a positive integer  $n$ . Let  $T$  (or  $T_n$  to avoid confusion) be the maximal torus in  $U(n)$  and let  $N_p(T)$  be a  $p$ -normalizer of  $T$ . We call  $f: BN_p(T) \rightarrow BU(m)_p^\wedge$  a  $\mathcal{R}_p(U(n))$ -invariant if for any two subgroups  $P, Q \subseteq N_p(T)$  which are  $p$ -stubborn in a sense of [1] and any homomorphism  $\alpha: P \rightarrow Q$  which is a restriction of a conjugation in  $U(n)$ , the maps  $f|_{BP}$  and  $f|_{BQ} \circ B\alpha$  are homotopic. Every  $\mathcal{R}_p(U(n))$ -invariant representation gives therefore an element of  $\lim_{\mathcal{R}_p(U(n))} [BP, BU(m)_p^\wedge]$ . There is an obvious map:

$$\Theta: [\text{hocolim}_{\mathcal{R}_p(U(N))} EU(n) \times_{U(n)} U(n)/P, BU(m)_p^\wedge] \rightarrow \lim_{\mathcal{R}_p(U(n))} [BP, BU(m)_p^\wedge]$$

Since its range is isomorphic to  $\text{Rep}(P^\infty, U(m))$ , the set of representations of a  $p$ -discrete approximation of  $P$  (comp. for example [5]), and the domain is isomorphic to  $[BU(n), BU(m)_p^\wedge]$  we have an approach to construct maps  $BU(n) \rightarrow BU(m)$  as long as we know when  $\Theta$  is a surjection. Following [9] we address this problem with help of a suitable obstruction theory (this will be done in the last section of this article).

The main result of this paper is the following:

**Theorem 1.2.** *All homotopy classes of maps  $BU(n) \rightarrow BU(m)$  where  $m < n^2$  are induced by homotopy representations of  $U(n)$  of dimensions*

$< n^2$ . Every irreducible homotopy representation of  $U(n)$  of dimensions  $< n^2$  is, up to tensoring with the determinant (or its inverse), equal to one of the following:

- (1) the Adams operation  $\psi^k$  (for  $k$  relatively prime to  $n! = |W(U(n))|$ );
- (2) exterior powers of Adams operations  $\psi^k \wedge \psi^k$  for  $n > 3$  and  $\psi^k \wedge \psi^k \wedge \psi^k$  for  $n = 6, 7, 8$ ;
- (3) representation  $\phi^k$  such that  $\phi^k \oplus \Theta^\alpha \cong \psi^k \circ \text{ad}$  where  $\text{ad}$  is the adjoint representation,  $\alpha \geq 0$  and  $\Theta$  is the 1-dimensional trivial representation.

**Remark 1.3.** Note that the list above contains all representations induced by homomorphisms of unitary groups, namely the identity, exterior powers of it and the adjoint representation.

**Example 1.4.** Let us consider the simplest nontrivial case. For  $n = 6$  we get a family of irreducible representations of  $U(n)$  of dimensions  $< n^2$  of the following weight systems:

- (1)  $\mu_1 = \{(1, 1, 1, 1, 1, 1)\}$ ;
- (2)  $\mu_2 = \{(k, 0, 0, 0, 0, 0) + \text{permutations}\}$ ;
- (3)  $\mu_3 = \{(k, k, 0, 0, 0, 0) + \text{permutations}\}$ ;
- (4)  $\mu_4 = \{(k, -k, 0, 0, 0, 0) + \text{permutations},$   
 $(0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0)\}$ .

where  $k$  is relatively prime to  $6!$ . Of course  $\mu_2$ ,  $\mu_3$  and  $\mu_4$  can be tensored by the determinant (which is induced by  $\mu_1$ ). The only exceptional representation is the one induced by  $\mu_4$ . Let us denote it by  $\phi^k$ . Note that  $\phi^k \oplus \Theta \cong \psi^k \circ \text{ad}$  where  $\psi^k$  is the Adams operation. Later on we will show that a single exceptional representation appears in the considered range of dimensions if and only if  $n$  is not a power of a prime.

## 2. $\mathcal{R}_p(U(p))$ INVARIANT REPRESENTATIONS

Every weight system  $\mu$  induces a class function  $\chi(\mu): N_p(T) \rightarrow \mathbb{C}$  in the following way: every element of the  $p$ -normalizer of torus  $N_p(T)$  is in  $U(n)$  conjugated to a unique element of the maximal tori, where the class function is given by  $\phi_\mu$ . We have

**Theorem 2.1.** *A class function induced by a weight system  $\mu$  which is closed under permutation of roots is a character of a  $\mathcal{R}_p(U(n))$ -invariant representation if and only if*

$$\forall \psi \in \text{IrrRep}(N_p(T)): \langle \chi(\mu), \chi(\psi) \rangle \in \mathbb{Z}_+ \cup \{0\}.$$

*Proof.* Since every morphism in the category of  $p$ -stubborn subgroups is induced by a conjugation in  $U(m)$  and the weight system is itself invariant under the action of the Weyl group the result follows easily from the character theory.  $\square$

What remains to check is the uniqueness of such a representation – which in our case is a direct consequence of

**Theorem 2.2** ([2], Corollary 1.14). *Every weight system induces at most one  $\mathcal{R}_p(U(n))$ -invariant representation.*

Now let

$$\begin{aligned}\mu &= \{(k, \dots, k)\} \\ \mu' &= \{(k_1, \dots, k_p) + \text{permutations}\}\end{aligned}$$

be systems of weights for  $U(p)$ ; we assume that  $k_1, \dots, k_p$  is not a constant sequence. For clearence we denote by

$$\mathcal{N} = \{\sigma \in \Sigma_p : (k_1, \dots, k_p) = (k_{\sigma(1)}, \dots, k_{\sigma(p)})\}$$

hence  $\mu' = \{(k_{\sigma(1)}, \dots, k_{\sigma(p)}) : \sigma \in \Sigma_p / \mathcal{N}\}$

According to the previous theorem it is enough to compute the scalar product of the class functions induced by  $\mu$  and  $\mu'$  and characters of irreducible representations of  $N_p(T)$  and check whenever it is a nonnegative integer.

Irreducible representations of  $N_p(T)$  are of the following form:

- $\rho_{k, \dots, k}^\epsilon$  where  $(k, \dots, k)$  denotes the roots of the representation and  $\epsilon$  is a  $p$ -root of unity and describes the action of a generator of  $\pi_0(N_p(T)) \cong C_p$  (the cyclic group of order  $p$ );
- $\text{ind}_T^{N_p(T)} \rho_{k_1, \dots, k_p}$ , where  $k_1, \dots, k_p$  denotes the roots of the representation (and cannot be constant).

To simplify the notation we denote by  $e(x) := \exp(2\pi i x)$  and by

$$\langle k_1, \dots, k_p \rangle : T \rightarrow \mathbb{C}$$

the function which assigns to a diagonal matrix its weighted power, i.e.

$$\langle k_1, \dots, k_p \rangle \left( \begin{bmatrix} e(x_1) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & e(x_p) \end{bmatrix} \right) = e\left(\sum k_i x_i\right).$$

Moreover let  $t$  be any generator of  $\pi_0(N_p(T))$  then let  $\delta_{t^s}$  be a function equal to 1 on the connected component of  $t^s$  and 0 elsewhere. We get

$$\chi(\rho_{k, \dots, k}^\epsilon) = \sum_{i=0}^{p-1} \delta_{t^i} \epsilon^i \langle k, \dots, k \rangle,$$

$$\chi(\text{ind}_T^{N_p(T)} \rho_{k_1, \dots, k_p}) = \delta_1(\langle k_1, \dots, k_p \rangle + \langle k_2, \dots, k_p, k_1 \rangle + \dots + \langle k_p, k_1, \dots, k_{p-1} \rangle).$$

Similar calculations show that

$$\chi(\mu) = \left( \sum_{i=0}^{p-1} \delta_{t^i} \right) \langle k, \dots, k \rangle,$$

$$\chi(\mu') = \sum_{\sigma \in \Sigma_p/\mathcal{N}} \delta_1 \langle k_{\sigma(1)}, \dots, k_{\sigma(p)} \rangle + \left( \sum_{i=1}^{p-1} \delta_{t^i} \right) C_{k_{\sigma(1)}, \dots, k_{\sigma(p)}} \langle 1/p \sum k_i, \dots, 1/p \sum k_i \rangle$$

where  $C_{k_{\sigma(1)}, \dots, k_{\sigma(p)}} = e(1/p \cdot (\sum_i (i-1)k_{\sigma(i)}))$ .

Results of calculating the scalar product are the following:

$$(2.1) \quad \langle \chi(\mu), \chi(\rho_{k, \dots, k}^\epsilon) \rangle = 1/p(1 + \epsilon + \dots + \epsilon^{p-1}) = \begin{cases} 1 & \epsilon \neq 1 \\ 0 & \epsilon = 1 \end{cases}$$

$$(2.2) \quad \langle \chi(\mu), \chi(\text{ind}_T^{N_p(T)} \rho_{k_1, \dots, k_p}) \rangle = 0$$

(2.3)

$$\langle \chi(\mu'), \chi(\rho_{1/p \sum k_i, \dots, 1/p \sum k_i}^\epsilon) \rangle = 1/p(\epsilon + \dots + \epsilon^{p-1}) \left( \sum_{\sigma \in \Sigma_p/\mathcal{N}} C_{k_{\sigma(1)}, \dots, k_{\sigma(p)}} \right)$$

$$(2.4) \quad \langle \chi(\mu'), \chi(\text{ind}_T^{N_p(T)} \rho_{k_1, \dots, k_p}) \rangle = 1$$

and any other scalar products where the roots do not match the above is 0. It follows that invariance depends on 2.3.

Note that whenever  $\sum k_i = r \neq 0$  in  $\mathbb{Z}/p$  then, since cyclic permutation acts freely on  $\Sigma_p/\mathcal{N}$  and it changes the value of  $C_{k_{\sigma(1)}, \dots, k_{\sigma(p)}}$  by  $r$ , then  $\sum_{\sigma \in \Sigma_p/\mathcal{N}} C_{k_{\sigma(1)}, \dots, k_{\sigma(p)}} = 0$ . Therefore for that sequence  $\chi(\text{ind}_T^{N_p(T)} \rho_{k_1, \dots, k_p})$  is a character of a  $\mathcal{R}_p(U(p))$ -invariant representation. Hence we showed the first part of the following:

**Theorem 2.3.** *If  $\sum k_i \neq 0$  in  $\mathbb{Z}/p$  then  $\chi(\mu')$  is a character of a  $\mathcal{R}_p(U(p))$ -invariant representation. If  $\sum k_i = 0$  then this is never the case.*

Proof of the remaining part of this theorem will be given in the next section.

### 3. WEIGHTED SUMS

For a prime  $p$  let us consider a finite sequence  $k_0, \dots, k_{p-1}$  of elements of  $\mathbb{Z}/p$  (we treat the indices also as elements of  $\mathbb{Z}/p$ ) such that  $\sum k_i = 0$ ; note that for computational reasons the indexes are shifted by one with respect

to the previous chapret. We set:

$$(3.1) \quad \mathcal{A}(k_i) = \{\sigma \in \Sigma_p : \sum ik_{\sigma(i)} = 0\}$$

$$(3.2) \quad \mathcal{B}_r(k_i) = \{\sigma \in \Sigma_p : \sum ik_{\sigma(i)} = r\}$$

$$(3.3) \quad \mathcal{B}(k_i) = \bigcup_{0 < r < p} \mathcal{B}_r$$

Moreover we define additional maps on  $\mathcal{C} = \{(k_1, \dots, k_p) \in (\mathbb{Z}/p)^{\oplus p} : \sum k_i = 0\}$  as follows:

$$\tau : \mathcal{C} \ni (a_i) \mapsto (a_{i+1}) \in \mathcal{C}$$

$$\sigma_j : \mathcal{C} \ni (a_i) \mapsto (a_{i+ij}) \in \mathcal{C} \text{ for } j = 0, 1, \dots, p-2$$

**Lemma 3.1.** *For every prime  $p \geq 3$  and  $(a), (b) \in \mathcal{C}$  we have*

- 1)  $\sigma_j \circ \tau = \tau^{j+1} \circ \sigma_j$ ;
- 2)  $\sum(\tau(a)) = \sum(a)$ ;
- 3)  $(j+1) \sum(\sigma_{j'}(a)) = \sum(a)$ ;
- 4) if  $\sum(\sigma_j(a)) = \sum(\sigma_{j'}(a))$ ,  $j \neq j'$  then  $\sum(a) = 0$ ;
- 5) if  $\sigma_j(a) = \sigma_{j'}(b)$  then there exists  $k$  such that  $(a) = \sigma_k(b)$ .

*Proof.* 1).  $\sigma_j(\tau(a_i)) = \sigma_j(a_{i+1}) = a_{i+1+ij+j} = \tau^{j+1}(a_{i+ij}) = \tau^{j+1}(\sigma_j(a))$ .

2) is obvious.

3). Note that numbers  $j+1$  and  $p$  are relatively prime numbers so that the assignment  $i \mapsto i+ij$  is 1 to 1 in  $\mathbb{Z}/p$ . Hence

$$\begin{aligned} \sum(a_i) - \sum(\sigma_j(a_i)) &= \sum_i i \cdot a_i - \sum_i i \cdot a_{i+ij} = \sum_i (i+ij) \cdot a_{i+ij} - \sum_i i \cdot a_{i+ij} \\ &= \sum_i ij \cdot a_{i+ij} = j. \end{aligned}$$

4). If  $\sum(\sigma_j(a)) = \sum(\sigma_{j'}(a))$  then according to point 3) there is an equality

$$(j'+1) \sum(a) = (j+1) \sum(a)$$

therefore  $0 = (j-j') \sum(a)$ . Since  $j-j' \neq 0$  we get that  $\sum(a) = 0$ .

5). Note that for  $0 \leq j \leq p-2$  we can find  $t$  such that  $(t+1)(j+1) = 1$ . Therefore we get that  $(a) = \sigma_0(a) = \sigma_t \circ \sigma_j(a) = \sigma_t \circ \sigma_{j'}(b) = \sigma_{j'+t+j't}(a)$  as long as  $0 \leq j'+t+j't \leq p-2$ . Note however that the case  $j'+t+j't = p-1 \Leftrightarrow (j'+1)(t+1) = p$  implies that either  $j' = 0$  or  $t = 0$  which ends the proof.  $\square$

Note that the above lemma shows that a relation defined as follows:

$$(a) \sim (b) \Leftrightarrow \exists_k : \sigma_k(a) = (b)$$

is an equivalence relation. Point 1) implies that the relation is invariant under the action of cyclic permutations (and the action is free as long as the sequence is not constant). Note that all sequences from  $\mathcal{B}$  have the following property

**Lemma 3.2.** *Every  $(a) \in \mathcal{C}$  such that  $\sum(a) \neq 0$  satisfies  $\{\sum(\sigma_i(a))\}_{j=0}^{p-2} = \{1, 2, \dots, p-1\}$ .*

*Proof.* This is a direct consequence of point 4) of the previous lemma.  $\square$

Note that this lemma implies that whenever  $(a) \in \mathcal{B}$  then the set  $[(a)]_{\sim}$  has exactly  $p-1$  elements. Moreover – since the relation is invariant under the action of cyclic permutations then every  $\#\mathcal{B}_r$  is divided by  $p$  as well as  $\#\mathcal{A}$  (for nonconstant sequences).

**Theorem 3.3.** *There is no equality  $(p-1)\#\mathcal{A}(k_i) = \#\mathcal{B}(k_i)$ .*

*Proof.* First note that the equality does not hold when the sequence is constant. Since  $\#\mathcal{A}(k_i) + \#\mathcal{B}(k_i) = p!$  then from the equality we would get that  $\#\mathcal{A}(k_i) = (p-1)!$ . Hence for a nonconstant sequence we would get that  $p$  divides  $(p-1)!$ .  $\square$

*Proof of theorem 2.3.* Let us consider only the case when  $\sum k_i = 0$ . Note that  $\chi(\mu')$  is a character of a  $\mathcal{R}_p(U(p))$ -invariant representation if and only if  $\sum_{\sigma \in \Sigma_p/\mathcal{N}} C_{k_{\sigma(1)}, \dots, k_{\sigma(p)}} = 0$ . Note that  $\#\mathcal{B}_1 = \dots = \#\mathcal{B}_{p-1}$  and

$$\sum_{\sigma \in \mathcal{B}/\mathcal{N}} C_{k_{\sigma(1)}, \dots, k_{\sigma(p)}} = -\#\mathcal{B}_1/\mathcal{N}.$$

Since  $\sum_{\sigma \in \mathcal{A}/\mathcal{N}} C_{k_{\sigma(1)}, \dots, k_{\sigma(p)}} = \#\mathcal{A}/\mathcal{N}$  we get that

$$\sum_{\sigma \in \Sigma_p/\mathcal{N}} C_{k_{\sigma(1)}, \dots, k_{\sigma(p)}} = \#\mathcal{A}/\mathcal{N} - \#\mathcal{B}_1/\mathcal{N}$$

and this is nonzero from theorem 3.3.  $\square$

**Example 3.4.** 1) for  $(k_0, \dots, k_{p-1}) = (k, \dots, k)$  we have  $\#\mathcal{A}(k_i) = p!$  and  $\#\mathcal{B}(k_i) = 0$ ;

2) for  $(k_0, \dots, k_{p-1}) = (s, \dots, s, k, \dots, k)$  we have  $\#\mathcal{A}(k_i) = p!$  and  $\#\mathcal{B}(k_i) = 0$ ;

3) for  $(k_0, \dots, k_{p-1}) = (k, k', 0, \dots, 0)$  we have  $\#\mathcal{A}(k_i) = 0$  and  $\#\mathcal{B}(k_i) = p!$ .

**Corollary 3.5.** *If  $(k_1, \dots, k_p) \in \mu$  where  $\mu$  is a weight system inducing a  $\mathcal{R}_p(U(n))$ -invariant representation then there exists  $(k'_1, \dots, k'_p) \in \mu$  such that  $\sum k_i = \sum k'_i$  and  $\#\mathcal{B}(k'_i) \neq 0$ .*



4. HOMOTOPY REPRESENTATIONS OF  $U(n)$  OF DIMENSIONS  $< n^2$ 

In this section we prove the following (theorem 1.2):

**Theorem 4.1.** *Every irreducible homotopy representation of  $U(n)$  of dimensions  $< n^2$  is, up to tensoring with the determinant (or its inverse), an extension of the following weight system:*

- (1) the Adams operation  $\psi^k$  (for  $k$  relatively prime to  $n! = |W(U(n))|$ );
- (2) exterior powers of Adams operations  $\psi^k \wedge \psi^k$  for  $n > 3$  and  $\psi^k \wedge \psi^k \wedge \psi^k$  for  $n = 6, 7, 8$ ;
- (3) representation  $\phi^k$  such that  $\phi^k \oplus \Theta^\alpha \cong \psi^k \circ \text{ad}$  where  $\text{ad}$  is the adjoint representation,  $\alpha \geq 0$  and  $\Theta$  is the 1-dimensional trivial representation.

In [3] we list all homotopy representations of  $U(2)$  and  $U(3)$  so we can prove the above theorem for  $n \geq 4$ . Note also that for every  $n$  we have a homomorphism  $U(n) \rightarrow U(1)$ , namely the determinant. Of course it induces a one dimensional homotopy representation of  $U(n)$ . Moreover it is invertible hence tensoring by it helps to simplify the notation – we assume that the generic root of every weight is equal to 0.

*Proof.* The proof follows by case by case check of all possible weight systems  $\mu$  that induce *irreducible* homotopy representation. Frequently we will use the fact that  $\mu$  has to be invariant under permutation of roots. We order all weights such that  $(k_1, \dots, k_n) > (k'_1, \dots, k'_n)$  whenever

$$\max\{s: k_{i_1} = \dots = k_{i_s}\} > \max\{s: k'_{i_1} = \dots = k'_{i_s}\}$$

or inductively when

$$\max\{s: k_{i_1} = \dots = k_{i_s}\} = \max\{s: k'_{i_1} = \dots = k'_{i_s}\} = s_0$$

and  $(l_1, \dots, l_{n-s_0}) < (l'_1, \dots, l'_{n-s_0})$  where  $(l_1, \dots, l_{n-s_0})$  and  $(l'_1, \dots, l'_{n-s_0})$  are obtained from  $(k_1, \dots, k_n)$  and  $(k'_1, \dots, k'_n)$  by deleting  $k_{i_1}, \dots, k_{i_{s_0}}$  and respectively  $k'_{i_1}, \dots, k'_{i_{s_0}}$  used in the equality of maxima.

Each time we consider smaller, bigger or equivalent weights we mean smaller, bigger or equal weights with respect to this order. Usually we will choose  $p$  and check invariance of a suitable representation of  $\mathcal{R}_p(U(n))$ . The list of all  $p$ -stubborn subgroups of  $U(n)$  as well as all morphisms in their category of orbits can be find in [8]. We will frequently use the notation used there. Note also that we will use (without mentioning) the uniqueness theorem recalled in 2.2.

- 1)  $\mu$  contains only weights equivalent to  $(0, \dots, 0)$ . Then it is the determinant itself.

- 2)  $\mu$  contains weights of the form  $(k, 0, \dots, 0)$  where  $k \neq 0$  and smaller.
- a) if the numbers  $k$  and  $n!$  are relatively prime then we obtain the Adams operations  $\psi^k$ ;
  - b) if  $(k, n!) = m > 1$  and  $p|m$  is the biggest prime factor,  $p \geq 2$ , let us consider a  $p$ -stubborn subgroup of the form  $\Gamma_1^U \wr E_p \times G_0$ . Since the representation restricted to them is a product of representations we will focus our attention to the first factor. From theorem 2.3 we get that a weight system  $\mu' = \{(k, 0, \dots, 0) + \text{permutations}\}$  never gives a  $\mathcal{R}_p(U(p))$ -invariant representation. According to corollary 3.5 there exists  $(k_1, \dots, k_p, 0, \dots, 0) \in \mu$  such that  $\sum k_i = k$  and  $\sharp\mathcal{B}(k_i) \neq 0$ . In particular  $\mu$  must contain a weight bigger than  $(k, 0, \dots, 0)$ .
- 3)  $\mu$  contains weights of the form  $(k, k, 0, \dots, 0)$  and smaller.
- a) if the numbers  $k$  and  $n!$  are relatively prime then we obtain a representation of the form  $\psi^k \wedge \psi^k$ ;
  - b) if  $(k, n!) = m > 1$  and  $p|m$  is the biggest prime factor  $p \geq 2$  let us once again consider a  $p$ -stubborn subgroup of the form  $\Gamma_1^U \wr E_p \times G_0$ . Restricting our attention to the first factor we get that, according to theorem 2.3,  $\mu$  contains a weight of the form  $(k_1, \dots, k_p, 0, \dots, 0)$  such that  $\sum k_i = k$  and  $\sharp\mathcal{B}(k_i) \neq 0$ . In particular  $\mu$  must once again contain a weight bigger than  $(k, k, 0, \dots, 0)$  (corollary 3.5).
- 4)  $\mu$  contains weights of the form  $(k', k'', 0, \dots, 0)$  where  $k' \neq k''$  and smaller.
- (a) if  $(k' + k'', n!) = m > 1$  and  $p|m$  is the biggest prime factor then once again from the theorem 2.3 we get that  $\mu$  contains a weight of the form  $(k_1, \dots, k_p, 0, \dots, 0)$  such that  $\sum k_i = k$  and  $\sharp\mathcal{B}(k_i) \neq 0$ . However  $\sharp\{(k', k'', 0, \dots, 0) + \text{permutations}\} = n^2 - n$  so that  $\mu$  may contain only weights smaller than  $(k', k'', 0, \dots, 0)$ . According to corollary 3.5 and the dimension restriction this can be done only if  $k' + k'' = 0$ . Moreover if any  $p \leq n$  divides  $k'$  (hence also  $k''$ ) then the corollary gives us a not smaller weight in  $\mu$  different from  $(k', k'', 0, \dots, 0)$  so that the dimension restriction cannot be satisfied. Therefore the only possible irreducible homotopy representation can be induced by the weight system of the form

$$\mu = \{(k, -k, 0, \dots, 0) + \text{permutations}, (0, \dots, 0) \text{ added at most } n \text{ times}\}.$$

Hence the representation must be a nontrivial summand of a homotopy representation induced by  $\psi^k \circ \text{ad}$  (in fact we will show in [3] that the homotopy representation can be further reduced if  $n$  is not a power of a prime).

- (b) if  $(k' + k'', n! = 1)$  then the sum  $k' + k''$  must be odd. Hence we may assume that  $k' = 2k$ . Let us consider a 2-stubborn subgroup of the form  $\Gamma_1^U \wr E_2 \times G_0$ . Restricting our attention to the first factor and using the fact that  $n \geq 3$  we may consider the following weight system  $\mu' = \{(k', k''), (k'', k')\}$ . Using classification of irreducible homotopy representations of  $U(2)$  we get that either  $(s, s, 0, \dots, 0) \in \mu$  or  $(s', s'', 0, \dots, 0)$  where  $s, s', s''$  are odd. Nevertheless in each case, using the fact that  $\mu$  is closed under permutation of roots, the dimension of the representation is  $\geq n^2$ .
- 5)  $\mu$  contains weights of the form  $(k, k, k, 0, \dots, 0)$  for  $8 \geq n \geq 6$  and smaller.
- (a) if the numbers  $k$  and  $n!$  are relatively prime then we obtain a representation of the form  $\psi^k \wedge \psi^k \wedge \psi^k$ ;
- (b) if  $(k, n!) = m > 1$  and  $p|m$  is the biggest prime factor  $p \in \{2, 3, 5, 7\}$  we obtain in the very same way as before that there are no such irreducible representations.

To postpone

□

## REFERENCES

- [1] S. Jackowski, J. McClure, B. Oliver, *Homotopy classification of self-maps of BG via G-actions*, Ann Math 135 (1992), 183-226.
- [2] S. Jackowski, J. McClure, B. Oliver, *Maps between classifying spaces revisited*, Contemp Math 181 (1993), 263-298.
- [3] W. Jackowski, K. Ziemiaski, *Examples of maps between classifying spaces of unitary groups*, manuscript.
- [4] K. Ziemiański, *Homotopy representation of SO(7) and Spin(7) at prime 2*, J Pure Appl Algebra 212 (2008), 1525-1541.
- [5] K. Ziemiański, *A faithful unitary representation of the 2-compact group DI(4)*, J Pure Appl Algebra 213 (2009), 1239-1253.
- [6] D. Notbohm, *Classifying spaces for compact Lie groups*, Handbook of Algebraic Topology, North-Holland, 1995, 1049-1094.
- [7] A.K. Bousfield, D.M. Kan, *Homotopy Limits, Completions and Localizations*, LNM, vol. 304, Springer, New York, 1972.
- [8] B. Oliver, *p-stubborn subgroups of classical compact Lie groups*, J Pure Appl Algebra 92 (1994), 55-78.
- [9] Z. Wojtkowiak, *On maps from hocolim F to Z*, LNM, vol. 1298, Springer-Verlag, 1987, 227-236.
- [10] J. Grodal, *Higher limits via subgroup complexes*, Ann Math 155 (2002), 405-457.

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