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On-line chromatic number of matroids

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Contents

Intruduction

This thesis is mostly devoted to matroids – one of basic combinatorial objects. As we study off-line and on-line variants of several invariants of matroids the content of this thesis naturally split into three parts.

In the first part we study several off-line coloring-type invariants of matroids. Proper colorings of matroids (i.e. colorings of the ground set such that every mono-colored set is independent) were studied by Edmonds [11] and Nash-Williams [33] where they showed a formula for a chromatic number of a matroid (i.e. minimal number of colors that admits proper coloring of a matroid) in terms of its rank function. We slightly generalize his result in Section 2.4 where we give a condition for an existence of a weighted almost covering (i.e. in which every, but a fixed number, element is covered several, according to its weight, times) of a matroid. It turns out that the famous Hall's Marriage Theorem about matchings in bipartite graphs can be rephrased in a similar language and admits a similar generalization as well; this is done in Section 2.5. Of a different nature is our concept of a delay coloring of matroids. Inspired by an article by Wilfong, Haxell and Winkler [59] and their concept of delay colorings of bipartite graphs, we assume that our matroid was properly colored and at each of elements we are given a permutation of colors. We say that the coloring is a delay coloring whenever it remains proper after applying the permutations. We conjecture that for a given matroid there exists a delay coloring whenever we are given one more colors than the chromatic number of the matroid. This generalizes famous Ryser, Brualdi Conjecture, comp. [10], on transversals on Latin squares and Aharoni, Berger Conjecture on rainbow matchings [2]. We give some partial results on the number of colors needed to delay color a matroid – in particular we prove a square upper bound for an arbitrary matroid and a linear upper bound for a graphic matroid in terms of its chromatic number.

The second part is devoted to two player games on matroids in which the matroid itself is known to both players; we prove conditions for an existence of a winning strategy for one of the players. In particular we generalize the coloring game on matroids by Bartnicki, Grytczuk and Kierstead [5] and Lasoń [28] in Section 4.1. In Section 4.2 we generalize the indicated coloring game on matroid by Lasoń. Next we prove that Presenter has a winning strategy in the following game we, with Lasoń, introduced in [29]: in each of the steps Presenter reveals which elements can be colored using one of the colors and Algorithm must immediately decide which elements properly color into the color. A natural question arises: if at each element of the ground set Presenter must reveal at least as many colors as the chromatic number can Algorithm color the whole matroid. We answer to that question in the affirmative in Section 4.3. In the next Section 4.4 we ask for the mean number of colors revealed at each of elements of the ground set in this game that allows Algorithm to color the whole matroid.

The third part of this thesis is devoted to one of the most interesting two player games on matroids, inspired by vertex coloring games on graphs by Kierstead, Penrice and Trotter [25]. That is the game in which one of the players, Presenter, reveals one by one elements of the ground set of a given matroid, and the other player, Algorithm, is supposed to color the revealed part properly. In Chapter 4 we present several examples of such games, where Presenter reveals elements of a ground set together with a representation specific for a given class of matroids. For example by a graphic matroid we understand a family of subsets of edges in a graph such that contain no cycle. In the corresponding game Presenter at each step draws a new edge and ask Algorithm to color it into one of the available colors. We show in Section 5.2 that there is no competitive constant between off-line and on-line number of colors needed by Algorithm to properly color such a matroid. By a transversal matroid we understand a family of subsets of a given ground set, each of which admits a transversal to some number of testing sets. A similar game played on transversal matroids is equivalent to the on-line Hall's condition game: given a bipartite graph Presenter reveals elements of the top set together with all its neighbors. Now Algorithm colors the elements such that each mono colored set admits a full matching in the revealed graph. Once again we prove that there is no competitive constant between number of colors needed by Algorithm in the off-line case and the on-line Hall's condition game.

1. Acknowledgments

I would express my gratitude to advisors Jarosaw Grytczuk and Bartłomiej Bosek for introducing me to matroids and their guidance during my entire Ph.D. studies. I would like to thank all my friends, especially to Michał Lasoń, for inspiring conversations, sharing problems and discussions.

2. Introduction to matroids

Matroids are basic combinatorial structures. They abstract the idea of independence from various areas of mathematics, such as algebra, combinatorics or graph theory. Originally matroids were introduced by Whitney back in 1930's [58]. For a basic concepts in matroid theory we refer the reader to Oxley [34].

In this Chapter we give a definition of matroids, show its equivalent formulations in terms of basis, circuits, rank function or closure operator, we define some basic operations on them and give several exemplary subclasses of them.

We prove Matroid Union Theorem 2.4.2, a basic tool which gives a criterion for an existence of a covering of a ground set of a matroid by independent sets in terms of its rank function. By similar methods we prove a generalized Hall's Marriage Theorem 2.5.4, which gives a criterion for an existence of a matching in a bipartite graph.

2.1. **Definitions and terminology.** Matroids are structures that can be defined in many, non trivially equivalent, ways. In this section we present several approaches to axiomatization of matroids together with a short explanation how to return to the original definition in each of the cases.

Definition 2.1.1. By a matroid $\mathfrak{M}$ on a ground set E we mean a family of subsets of a finite set E, $\mathfrak{M} \subseteq \mathbb{P}(E)$, that satisfies the following conditions:

- $\emptyset \in \mathfrak{M}$;
- (hereditary) if $A \in \mathfrak{M}$ and $B \subseteq A$ then $B \in \mathfrak{M}$;
- (augmentation property) if $A, B \in \mathfrak{M}$ and $\sharp A > \sharp B$ then there exists $a \in A \backslash B$ such that $B \cup \{a\} \in \mathfrak{M}$

Elements of a matroid are called independent sets.

In particular every matroid is a special combinatorial simplicial complex. Note that the augmentation property of matroids implies that every two maximal independent sets in a matroid $\mathfrak{M}$ have the same number of elements. The maximal independent sets are called basis of a matroid. Axioms of a matroid can be reformulated in the language of basis, instead of arbitrary independent sets.

Proposition 2.1.2. *Every matroid $\mathfrak{M}$ on a ground set E is uniquely defined by a family $\mathcal{B}(\mathfrak{M})$ of elements of $\mathfrak{M}$, called basis of $\mathfrak{M}$, that satisfy the following conditions:*

- *$\mathcal{B}(\mathfrak{M})$ is nonempty;*
- (base exchange property) *whenever $B_1, B_2 \in \mathcal{B}(\mathfrak{M})$ and $b \in B_1 \backslash B_2$ then there exists $b' \in B_2 \backslash B_1$ such that $B_1 \backslash \{b\} \cup \{b'\} \in \mathcal{B}$.*

Proof. Given a matroid $\mathfrak{M}$ on a ground set E we define the set of basis $\mathcal{B}(\mathfrak{M})$ as the set of maximal (with respect to inclusion) elements in $\mathfrak{M}$.

Whenever we are given the set of basis $\mathcal{B}(\mathfrak{M})$ on a ground set E then we define $\mathfrak{M} := \{A \subseteq E \mid \exists B \in \mathcal{B} \colon A \subseteq B\}$. □

Instead of independent set of a matroid $\mathfrak{M}$ we may consider dependent ones, i.e. the sets that are not elements of $\mathfrak{M}$. Most important are the minimal dependent sets which are traditionally called circuits. They allow us to give another alternative set of axioms for a matroid.

Proposition 2.1.3. *Every matroid $\mathfrak{M}$ on a ground set E is uniquely defined by a family of subsets $\mathcal{C}(\mathfrak{M})$ of E, called circuits of $\mathfrak{M}$, which satisfies the following conditions:*

- *$\emptyset \notin \mathcal{C}$;*

- *whenever* $C_1, C_2 \in \mathcal{C}(\mathfrak{M})$ *then* $C_1 \subseteq C_2$ *implies* $C_1 = C_2$*;*
- *if* $C_1, C_2 \in \mathcal{C}(\mathfrak{M})$ *and* $c \in C_1 \cap C_2$ *then there exists* $C \in \mathcal{C}(\mathfrak{M})$ *such that* $C \subseteq (C_1 \cup C_2) \backslash \{c\}$.

Proof. Given a matroid $\mathfrak{M}$ on a ground set E we define the set of circuits $\mathcal{C}$ as $\mathcal{C}(\mathfrak{M}) := \{A \subseteq E \mid \forall B \subsetneq A \colon B \in \mathfrak{M}, A \notin \mathfrak{M}\}$.

Given a set of circuits $\mathcal{C}(\mathfrak{M}) \subseteq \mathbb{P}(E)$ on the ground set E we define the matroid $\mathfrak{M} := \{A \subseteq E \mid \forall C \in \mathcal{C} \colon C \nsubseteq A\}$. □

Given a matroid $\mathfrak{M}$ on a ground set E we can associate with every subset $A \subseteq E$ the maximal size of an independent set contained in E. This defines a function

$$r_{\mathfrak{M}} \colon \mathbb{P}(\mathbb{E}) \ni A \mapsto r_{\mathfrak{M}}(A) := max\{\sharp B \mid B \subseteq A, B \in \mathfrak{M}\} \in \mathbb{N}$$

called the rank function of $\mathfrak{M}$. We can now rewrite axioms of a matroid in terms of the rank function.

Proposition 2.1.4. *Every matroid* $\mathfrak{M}$ *on a ground set* E *is uniquely defined by a function* $r_{\mathfrak{M}} \colon \mathbb{P}(E) \to \mathbb{N}$*, called the rank function of* $\mathfrak{M}$*, which satisfies the following conditions:*

- *for every* $A \subseteq E$ *we have* $0 \leqslant r_{\mathfrak{M}}(A) \leqslant \sharp A$*;*
- *whenever* $A \subseteq B \subseteq E$ *then* $r_{\mathfrak{M}}(A) \leqslant r_{\mathfrak{M}}(B)$*;*
- (submodularity) *for every* $A, B \subseteq E$ *we have inequality*

$$r_{\mathfrak{M}}(A) + r_{\mathfrak{M}}(B) \geqslant r_{\mathfrak{M}}(A \cap B) + r_{\mathfrak{M}}(A \cup B).$$

Proof. Given a matroid $\mathfrak{M}$ on a ground set E we define the rank function $r_{\mathfrak{M}} \colon \mathbb{P}(E) \to \mathbb{N}$ for $A \subseteq E$ by the formula $r_{\mathfrak{M}}(A) := max\{\sharp B \mid B \subseteq A, B \in \mathfrak{M}\}$.

Given a rank function $r_{\mathfrak{M}} \colon \mathbb{P}(E) \to \mathbb{N}$ on a ground set E we define a matroid $\mathfrak{M}$ as $\mathfrak{M} := \{A \subseteq E \mid r_{\mathfrak{M}}(A) = \sharp A\}$. □

The last of the proposed methods of defining a matroid uses so called closure operator. Closure of a subset A of a ground set E of a matroid $\mathfrak{M}$ is usually defined as a set theoretic sum of A with all elements in the ground set E that do not increase the rank of A in $\mathfrak{M}$.

Proposition 2.1.5. *Every matroid* $\mathfrak{M}$ *on a ground set* E *is uniquely defined by a function* $cl_{\mathfrak{M}} \colon \mathbb{P}(E) \to \mathbb{P}(E)$*, called the closure operator, which satisfies the following conditions:*

- *whenever* $A \subseteq E$ *then* $A \subseteq cl_{\mathfrak{M}}(A)$*;*
- *if* $A \subseteq B \subseteq E$ *then* $cl_{\mathfrak{M}}(A) \subseteq cl_{\mathfrak{M}}(B)$*;*
- *if* $A \subseteq E$ *then* $cl_{\mathfrak{M}}(cl_{\mathfrak{M}}(A)) = cl_{\mathfrak{M}}(A)$*;*
- (Steinitz exchange property) *whenever* $A \subseteq E$*,* $e \in E$ *and* $f \in cl_{\mathfrak{M}}(A \cup \{e\}) \backslash cl_{\mathfrak{M}}(A)$ *then* $e \in cl_{\mathfrak{M}}(A \cup \{f\})$.

Proof. Given a matroid $\mathfrak{M}$ on a ground set E with rank function $r_{\mathfrak{M}} \colon \mathbb{P}(E) \to \mathbb{N}$ we define the closure operator $cl_{\mathfrak{M}} \colon \mathbb{P}(E) \to \mathbb{P}(E)$ for $A \subseteq E$ as $cl_{\mathfrak{M}}(A) := \{e \in E \mid r_{\mathfrak{M}}(A \cup \{e\}) = r_{\mathfrak{M}}(e)\}$.

Now given the closure operator on a ground set E, i.e. a function $cl_{\mathfrak{M}} \colon \mathbb{P}(E) \to \mathbb{P}(E)$, we define a matroid $\mathfrak{M} := \{A \subseteq E \mid \forall a \in A \colon cl_{\mathfrak{M}}(A \backslash \{a\}) \subsetneq cl_{\mathfrak{M}}(A)\}$. □

2.2. **Operations on matroids.** The class of all matroids admits several internal operations. These include direct sums, products, restrictions, contractions, etc. In this section we define several of these operations and explain how the rank function changes under each of them.

Given a matroid $\mathfrak{M}$ on a ground set E we may consider a dual matroid $\mathfrak{M}^*$ on the same ground set whose basis are completions of basis of $\mathfrak{M}$.

Definition 2.2.1. Let $\mathfrak{M}$ be a matroid on a ground set E. We define a matroid $\mathfrak{M}^*$, called the dual matroid on $\mathfrak{M}$, as $\mathfrak{M}^* := \{A \subseteq E \mid \exists \text{ a base } B \in \mathfrak{M} \colon B \cap A = \emptyset\}$.

If $r_{\mathfrak{M}}$ is a rank function of $\mathfrak{M}$ and $r_{\mathfrak{M}^*}$ is the rank function of $\mathfrak{M}^*$ then for every $A \subseteq E$ we have $r_{\mathfrak{M}^*}(A) = \sharp A - r_{\mathfrak{M}}(E) + r_{\mathfrak{M}}(E \backslash A)$.

One of the simplest operation is taking a restriction of a matroid on a ground set E to a subset $A \subseteq E$.

Definition 2.2.2. Let $\mathfrak{M}$ be a matroid on a ground set E and let $A \subseteq E$. We define a matroid $\mathfrak{M}|_A$ on the ground set A, called restriction of the matroid $\mathfrak{M}$ to the set A, as $\mathfrak{M}|_A := \{X \subseteq A \mid X \in \mathfrak{M}\}$.

Note that whenever $r_{\mathfrak{M}}$ is a rank function of $\mathfrak{M}$ and $r_{\mathfrak{M}|_A}$ is the rank function of $\mathfrak{M}|_A$ then $r_{\mathfrak{M}|_A} = r_{\mathfrak{M}}|_{\mathbb{P}(A)}$.

Another operation, which can be considered dual to the restriction, is called contraction.

Definition 2.2.3. Let $\mathfrak{M}$ be a matroid on a ground set E and let $A \subseteq E$. We define a matroid $\mathfrak{M}/A$ on the ground set $E \backslash A$, called contraction of the matroid $\mathfrak{M}$ on the set A, as $\mathfrak{M}/A := \{B \subseteq E \backslash A \mid \forall C \in \mathfrak{M}|_A \colon C \cup B \in \mathfrak{M}\}$

Whenever $r_{\mathfrak{M}}$ is a rank function of $\mathfrak{M}$ and $r_{\mathfrak{M}/A}$ is the rank function of $\mathfrak{M}/A$ then for $B \subseteq E \backslash A$ we have $r_{\mathfrak{M}/A}(B) = r_{\mathfrak{M}}(B \cup A) - r_{\mathfrak{M}}(A)$.

A matroid $\mathfrak{M}'$ is called a minor of a matroid $\mathfrak{M}$ if and only if $\mathfrak{M}'$ can be obtained from $\mathfrak{M}$ by a series of restrictions and contractions.

The following define two ways for extending a ground set of a matroid. Direct sum of matroids and a blow-up of matroid in an element of the ground set.

Definition 2.2.4. For $i = 1, 2, \ldots, m$ let $\mathfrak{M}_i$ be a matroid on a groung set E_i. We define a matroid $\sqcup_{i=1,2,\ldots,m} \mathfrak{M}_i$ on the ground set $E := \sqcup_{i=1,2,\ldots,m} E_i$, called direct sum of matroids E_i, as $\sqcup_{i=1,2,\ldots,m} \mathfrak{M}_i := \{A_1 \cup A_2 \cup \ldots \cup A_m \mid \forall i = 1, 2, \ldots, m \colon A_i \in \mathfrak{M}_i\}$.

If $r_{\mathfrak{M}_i}$ is the rank function of $\mathfrak{M}_i$ and $r_{\sqcup_{i=1,2,\ldots,m} \mathfrak{M}_i}$ is the rank function for $\sqcup_{i=1,2,\ldots,m} \mathfrak{M}_i$ then for every $A \subseteq E$ we have $r_{\sqcup_{i=1,2,\ldots,m} \mathfrak{M}_i}(A) = \Sigma_{i=1,2,\ldots,m} r_{\mathfrak{M}_i}(A \cap E_i)$.

Definition 2.2.5. Let $\mathfrak{M}$ be a matroid on a ground set E, $e \in E$ and $e_1, e_2 \notin E$. We define a matroid $\mathfrak{M} \uparrow e$ on the ground set $E \uparrow e := E \backslash \{e\} \cup \{e_1, e_2\}$, called the blow-up of the matroid $\mathfrak{M}$ at element e, as $\mathfrak{M} \uparrow e := \{A \subseteq E \uparrow e \mid \pi(A) \in \mathfrak{M},\ \sharp(A \cap \{e_1, e_2\}) \leqslant 1\}$ where $\pi \colon E \uparrow e \to E$, $\pi|_{E \backslash \{e\}} = \mathrm{id}_{E \backslash \{e\}}$ and $\pi(e_i) = e$.

When $r_{\mathfrak{M}}$ is a rank function of $\mathfrak{M}$, $r_{\mathfrak{M} \uparrow e}$ is a rank function of $\mathfrak{M} \uparrow e$ then for every $A \subseteq E \uparrow e$ we have $r_{\mathfrak{M} \uparrow e}(A) = r_{\mathfrak{M}}(\pi(A))$.

In each of the cases we omitted proofs that matroids with the presented families of independent sets exist. These can be found in classical books on matroid theory, e.g. [34].

At the end of this section let us define

2.3. **Examples of matroids.** In this section we define several subclasses of matroids we will use in the following chapters . Three of them: graphic, representable and transversal matroids serve as three main sources of examples.

We begin with a trivial example.

Definition 2.3.1 (Unitary matroid)**.** Let $n \geqslant k \geqslant 0$. We denote by $U_{k,n}$ the matroid on the ground set $E_n := \{1, 2, \ldots, n\}$ in which a set $A \subseteq E_n$ is independent if and only if $\sharp A \leqslant k$.

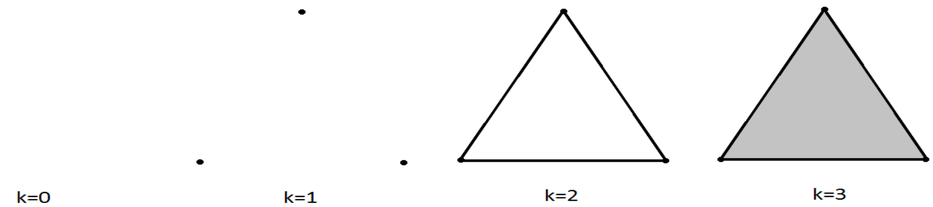

FIGURE 1. The matroid $U_{k,3}$ for various k presented as a simplicial complex.

Note that every circuit in the uniform matroid $U_{k,n}$ is of size $k+1$. If we weaken this condition by allowing sparse circuits of size $k-1$.

Definition 2.3.2 (Sparse paving matroid)**.** Let $\mathfrak{M}$ be a matroid on the ground set E. We call $\mathfrak{M}$ sparse paving matroid of rank k whenever:

- every $(k-1)$-element subset of E is independent and there exists n-element subset of E which is independent;
- every $(k+1)$-element subset of E is dependent;
- if C_1 and C_2 are different circuits in $\mathfrak{M}$ of size k then $\sharp(C_1 \cap C_1) \leqslant k-2$.

In [32] Mayhew, Newman, Welsh and Whittle conjectured that asymptotically almost all matroids are sparse paving matroids. Since their structure is quite simple they serve as the first testing ground for various conjectures and in fact many of them were confirmed for the sparse paving matroids [8].

One of the most important family of matroids comes from graph theory. By a graph we understand an ordered pair $G := (V, E)$ where V are vertexes and E are edges in G, where we allow loops (edges starting and ending at the same vertex) and multiple edges (several edges with similar set of ends).

Definition 2.3.3 (Graphic matroid)**.** Let $G := (V_G, E_G)$ be a graph. We call a matroid $\mathfrak{M}_G$ on the ground set E_G graphic whenever a subset $A \subseteq E_G$ is independent in $\mathfrak{M}$ if and only if A contains no cycle in G.

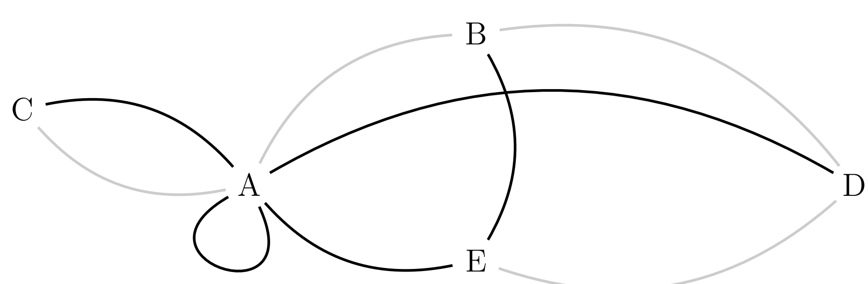

FIGURE 2. A graph with an independent set colored in gray.

A wider class consists of matroids originating from linear algebra.

Definition 2.3.4 (Representable matroid)**.** Let E be a finite set in the linear space V over the field $\mathbb{F}$. We call a matroid $\mathfrak{M}_E$ on the ground set E representable over the field $\mathbb{F}$ whenever $X \subseteq E$ is independent if and only if X is linearly independent in V.

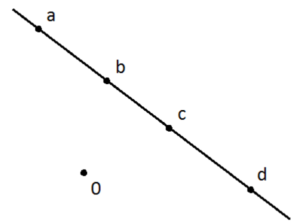

FIGURE 3. The matroid $U_{2,4}$ on the ground set $\{a, b, c, d\}$ presented as a representable matroid over $\mathbb{R}$.

Every graphic matroid is representable over any field $\mathbb{F}$, comp. [50]. The inverse implication is not true (an example of such a matroid is given in Figure 4).

Definition 2.3.5 (Regular matroid)**.** We call a matroid $\mathfrak{M}$ on a ground set E regular whenever for every field $\mathbb{F}$ the matroid $\mathfrak{M}$ is isomorphic to a matroid representable over the field $\mathbb{F}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

FIGURE 4. The matroid R_{10} with its representation as columns of a matrix over any field $\mathbb{F}$.

Of course not every representable matroid is regular. In Figure 5 we present the Fano matroid which is representable over a field $\mathbb{F}$ is and only if the characteristic of $\mathbb{F}$ is 2. On the other hand regular matroids have rather pleasant structure – Seymour [41] showed that they can be constructed by gluing together a graphic, a cographic (i.e. dual to a graphic) and a certain matroid R_{10} on a 10-element ground set presented in Figure 4.

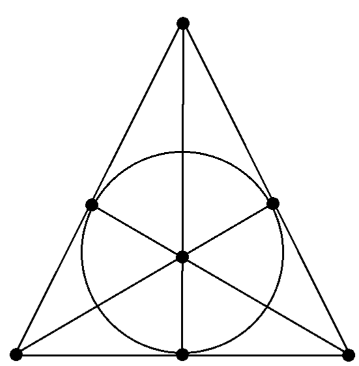

FIGURE 5. The Fano matroid of rank 3 on a 7-element ground set. Three nodes form a base if and only if the do not lie on one line (with the circle considered as one).

A famous conjecture by Rota [36] states that for every finite field $\mathbb{F}$ there is a finite set of matroids $Forb(\mathbb{F})$, called forbidden minors for $\mathbb{F}$, such that a matroid $\mathfrak{M}$ is representable over $\mathbb{F}$ if and only if no element of $Forb(\mathbb{F})$ is a minor of $\mathfrak{M}$. This conjecture has been proved in various cases, for example $Forb(\mathbb{F}_2) = \{U_{2,4}\}$ (comp. Tutte [49]). In 2013 Geelen, Gerards and Whittle reported they had a proof of this conjecture although it has not been published yet.

Definition 2.3.6 (Transversal matroid)**.** Let $\{A_\alpha\}_{\alpha \in A}$ be a family of subsets of a set E. We call a matroid $\mathfrak{M}$ on the ground set E transversal defined by the family $\{A_\alpha\}_{\alpha \in A}$ whenever $X \subseteq E$ is independent in $\mathfrak{M}$ if and only if there exists an injection $\iota_X \colon X \to A$ such that for each $x \in X$ we have $x \in A_{\iota_X(x)}$.

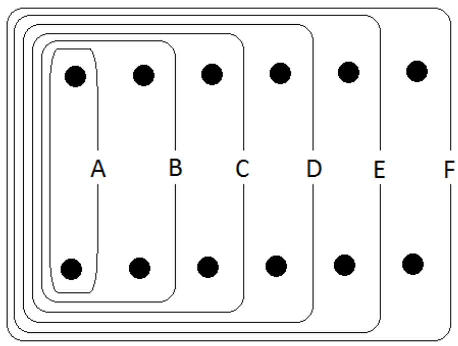

FIGURE 6. A transversal matroid on 12-element ground set. The six sets defining the matroid are denoted by A, B, C, D, E, F.

An example of a transversal matroid was presented in Figure 6. In 5.4 we will also define a subclass of transversal matroids, called lattice path matroids.

2.4. **Matroid union theorem.** The following theorem gives a beautiful criterion for an existence of a covering of a ground set of a matroid with independent set; usually it appears in the literature as Matroid Union Theorem, in this form first proved by Nash-Williams.

Theorem 2.4.1 (Nash-Williams [33])**.** *Let $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_m$ be matroids on the same ground set E. Then the following conditions are equivalent:*

(1) there exists $V_1 \in \mathfrak{M}_1, V_2 \in \mathfrak{M}_2, \ldots, V_m \in \mathfrak{M}_m$ such that $E = V_! \cup V_2 \cup \ldots \cup V_m$;
(2) for every $A \subseteq E$ we have $r_1(A) + r_2(A) + \ldots + r_m(A) \geqslant \sharp A$.

The above result is a simple corollary to its weighted version we will prove below.

Let $\omega \colon E \to \mathbb{N}$ be an arbitrary function called weight assignment for elements of a set E. Now a family $\{V_i\}_{i \in I}$ is said to be an ω-covering of E whenever for every e we have $\sharp\{i \in I \mid e \in V_i\} = \omega(e)$. If $\omega \equiv k$ for some integer k then we call ω covering a k-covering.

For $s \in \mathbb{N}$ we say that $\{V_i\}_{i \in I}$ is a s-almost ω covering of E if and only if for every $e \in E$ we have $\omega(e) \geqslant \sharp\{i \in I \mid e \in V_i\}$ and

$$\sum_{e \in E} \omega(e) - \sharp\{i \in I \mid e \in V_i\} \leqslant s.$$

Prove of the following theorem, for $s = 0$, can be found for example in [28].

Theorem 2.4.2. *Let $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_m$ be matroids on the same ground set E with rank functions denoted by $r_1, r_2, \ldots, r_m$ respectively, $s \in \mathbb{N}$ and let $\omega \colon E \to \mathbb{N}$ be a weight assignment for element of E. Then the following conditions are equivalent:*

(1) there exists $V_1 \in \mathfrak{M}_1, V_2 \in \mathfrak{M}_2, \ldots, V_m \in \mathfrak{M}_m$ *an* s*-almost* ω*-covering of* E*;*
(2) for every $A \subseteq E$ *we have* $r_1(A) + r_2(A) + \ldots + r_m(A) \geqslant \sum_{e \in A} \omega(e) - s$*.*

Proof. Implication (1) ⇒ (2) is immediate so we will show only (2) ⇒ (1).

We prove the implication by induction on the sum $\sharp E + \Sigma_i \Sigma_{A \subseteq E} r_i(A)$. Let

$$f(A) = \sum_i r_i(A) - \sum_{a \in A} \omega(a).$$

From the assumption we get that $f(A) \geqslant -m$.

If there exists a proper nonempty subset $X \subsetneq E$ such that $f(X) = -m$ then we consider $\mathfrak{M}_1|_X, \ldots, \mathfrak{M}_d|_X$ as matroids on X. The induction hypothesis now gives $W_1, \ldots W_d$ – s-almost ω-covering of X. Note that for matroids $\mathfrak{M}_1/X, \ldots, \mathfrak{M}_d/X$ on $E \backslash X$ (with rank functions $\hat{r_i}$) the function f is nonnegative. Indeed for every $F \subseteq E \backslash X$ we get that

$$f(F) = \sum_i \hat{r_i}(F) - \sum_{a \in F} \omega(a) = \sum_i r_i(F \cup X) - \sum_i r_i(X) - \sum_{a \in F} \omega(a) =$$

$$= \sum_i r_i(F \cup X) - \sum_{a \in F \cup X} \omega(a) + s \geqslant 0.$$

Hence, once again from the induction hypothesis, we get that $E \backslash X$ can be n-covered by $W_1' \in \mathfrak{M}_1/X, \ldots, W_m' \in \mathfrak{M}_m/X$. Now $V_i := W_i \cup W_i'$ gives us the required s-almost ω-covering.

If for every proper nonempty subset $A \subsetneq E$ we get that $f(A) > -m$ then either there exist $e \in E$ and i_0 such that $r(E \backslash e) = r_{i_0}(E)$ and $r_{i_0}(e) = 1$ or each of the matroids $\mathfrak{M}_i$ is focused on their basis, i.e. there exists a basis $B_i \in \mathfrak{M}_i$ for each i such that $r_i(E \backslash B_i) = 0$.

In the former case instead of $\mathfrak{M}_{i_0}$ we consider $\mathfrak{M}_{i_0}/\{e\}$ and for $\neq i_0$ we consider $\mathfrak{M}_i|_{E \backslash \{e\}}$ instead of $\mathfrak{M}_i$. Now for matroids $\mathfrak{M}_1|_{E \backslash \{e\}}, \ldots, \mathfrak{M}_i/\{e\}, \ldots, \mathfrak{M}_d|_{E \backslash \{e\}}$, since the rank function of the i_0-th matroid could decrease by at most by 1, we use the induction hypothesis to get an s-almost ω-covering $V_1, V_2, \ldots, V_m$ of $E \backslash \{e\}$. Of course $V_1, V_2, \ldots, V_{i_0} \cup \{e\}, \ldots, V_m$ is then as s-almost ω-covering of E.

In the latter case the basis $B_1, \ldots, B_d$ form an s-almost ω-covering of E. Indeed let $X := \{x \in E \mid \sharp\{i \mid x \in B_i\} < \omega(x)\}$ then for every $A \subseteq E$ we have

$$\sum_i r_i(A) = \sum_i \sharp(A \cap B_i) \geqslant \sum_{a \in A \backslash X} \omega(a) + \sum_{x \in A \cap X} \sharp\{i \mid x \in B_i\} =$$

$$\sum_{a \in A \backslash X} \omega(a) + \sum_i r_i(A \cap X) \geqslant \sum_{a \in A} \omega(a) - s.$$

Since we can start our induction with 1-element ground set for which the assertion is obviously true, the result follows. □

Another corollary to Theorem 2.4.2 is the following result which covers the case of k-coverings of the ground set E of a matroid with independent sets in it.

Corollary 2.4.3. *Let* $\mathfrak{M}$ *be a matroid on a ground set* E*. Then the following conditions are equivalent:*

(1) there exists a k*-covering of* E *consiting of* m *independent sets in* $\mathfrak{M}$*;*
(2) for every $A \subseteq E$ *we have* $m \cdot r(A) \geqslant k \cdot \sharp A$*.*

2.5. **Hall's Marriage Theorem.** By a bipartite graph $G := (U, V, E)$ we mean a simple graph in which the vertex set can be partitioned into two subsets U and V such that every edge in E connects one vertex in U to one vertex in V.

Definition 2.5.1. Let $G := (U, V, E)$ be a bipartite graph. We say that a set $A \subseteq U$ satisfies the Hall's condition whenever for every $B \subseteq A$ we have $\sharp B \leqslant \sharp \mathcal{N}(B)$ where $\mathcal{N}(B)$ is the neighborhood of B in G.

We say that G satisfies the Hall's condition whenever the set U satisfies the Hall's condition.

Definition 2.5.2. Let $G := (U, V, E)$ be a bipartite graph. We say that there exists a matching in G from $A \subseteq U$ whenever for every $a \in A$ there exists an edge $\{a, v_a\}$ in G such that $v_a \neq v_{a'}$ for different $a, a' \in A$.

We say that there exists a perfect matching in G whenever there is a matching in G from U.

The following is the classical formulation of Hall's Marriage Theorem.

Theorem 2.5.3 (comp. [23])**.** *Let $G := (U, V, E)$ be a bipartite graph. Then the following conditions are equivalent:*

(1) there is a perfect matching in G;
(2) G satisfies the Hall's condition.

Proof. Implication (1) ⇒ (2) follows directly from the definition.

For the opposite implication let us assume that in G there do not exists a perfect matching. Let us choose a matching in G from a maximal $U' \subseteq U$, with edges $\{a, v_a\}$ where $a \in U'$ and $v_a \in V$. Let $b \in U \backslash U'$. We consider all alternating paths (i.e. using alternatively edges in the matching and outside it) starting from b. Set T to be the set of vertexes in U and W the set of vertexes in V connected to b by such paths. Note that no such path can end in a vertex from W as this would allow us to find a matching from $U' \cup \{b\}$. Therefore every element in W is matched bijectively to an element in $T \backslash \{b\}$. On the other hand $\mathcal{N}(T) \subseteq W$ so that $\sharp T - 1 = \sharp W = \sharp \mathcal{N}(T)$. □

Hall's Marriage Theorem is stated in the language of bipartite graphs that do not fit into our model of graphic matroids. Note, however, that bipartite graphs may represent transversal matroids as follows. Whenever a matroid $\mathfrak{M}$ on a ground set E is defined by a (finite) family $\{A_i\}_{i \in I}$ of subsets of E then we define a bipartite graph $G_{\mathfrak{M}} := (E, I, N)$ where an edge $\{e, i\} \in N$ if and only if $e \in A_i$. Note that this process is fully reversible and given a bipartite graph $G := (U, V, E)$ we associate to it a matroid $\mathfrak{M}_G$ on the ground set U defined by the family $\{A_v\}_{v \in V}$, where $A_v := \mathcal{N}(v)$. We will need the following generalization of Hall's Marriage Theorem in Chapter 5 while estimating efficiency of some algorithms on transversal matroids.

Theorem 2.5.4. *Let $G := (U, V, E)$ be a bipartite graph. Then the following conditions are equivalent:*

(1) there exists a covering $U_1, U_2, \ldots, U_m$ of U such that there exists a matching in G from each U_i;
(2) for every $A \subseteq U$ we have $m \cdot \sharp \mathcal{N}(A) \geqslant \sharp A$.

Proof. The implication (1) ⇒ (2) is immediate so we focus on (2) ⇒ (1).

Let $G' := (U, V', E')$ be a bipartite graph where $V' := \sqcup_{i=1,2,\ldots,m} V^{(i)}$, where $V^{(i)}$ is the i-th copy of V and for $u \in U$ and $v \in V^{(i)}$ we have that $\{u, v\} \in E'$ if and only if $\{u, \pi(v)\} \in E$ where $\pi \colon \sqcup_{i=1,2,\ldots,m} V^{(i)} \to V$ is the natural identification map. For

$A \subseteq U$ we denote by $\mathcal{N}'(A)$ the neighborhood of A in G'. Note that we have $\sharp\mathcal{N}(A) \geqslant m \cdot \sharp\mathcal{N}(A) \geqslant \sharp A$. From 2.5.3 for every $a \in U$ we have an edge $\{a, v_a\}$ in G' such that $v_a \neq v_{a'}$ for $a \neq a'$. Now the sets $U_i := \{a \in U \mid v_a \in V^{(i)}\}$ satisfy the assertion. □

Corollary 2.5.5. *Let $G := (U, V, E)$ be a bipartite graph and let $\mathfrak{M}_G$ be the associated matroid on the ground set U with a rank function $r\colon \mathbb{P}(U) \to \mathbb{N}$. The following conditions are equivalent:*

(1) there exists a covering $U_1, U_2, \ldots, U_m$ of U such that there exists a matching in G from each U_i;
(2) there is a covering $U_1, U_2, \ldots, U_m$ of U by independent sets in $\mathfrak{M}_G$;
(3) for every $A \subseteq U$ we have $m \cdot \sharp\mathcal{N}(A) \geqslant \sharp A$;
(4) for every $A \subseteq U$ we have $m \cdot r(A) \geqslant \sharp A$.

Proof. Note that (1) is equivalent to (2) by the construction of the associated matroid to a bipartite graph. Now (1) is equivalent to (3) by 2.5.4 and (2) is equivalent to (4) by 2.4.2. □

3. Colorings of a matroid

This Chapter extends the concept of a proper graph coloring to matroids. A coloring of a matroid is called proper whenever mono colored sets are independent. We recall in 4.3.2 an explicit formula of Edmonds [11] for number of colors needed to find a proper coloring of matroid and we present weighted Seymour's Theorem 3.2.3 on existence of a proper coloring from list.

Next we define delay coloring of a matroid, which is a translation of a concept of delay colorings of matroid by Wilfong, Haxell and Winkler [59]. We state the conjecture the given one more colors than a chromatic number of a matroid we can always delay color the matroid.

This generalize conjectures by Brualdi, Ryser [10] on transversals in Latin squares and Aharoni, Berger [2] on rainbow matchings.

3.1. **Chromatic number.** Let us start this section with our main object.

Definition 3.1.1. Let $\mathfrak{M}$ be a matroid on a ground set E. A coloring of $\mathfrak{M}$ is an assignment of colors (usually positive integers) to elements of E; in other words a coloring is a choice of a function $c\colon E \to \mathbb{N}$.

We say that a coloring $c\colon E \to \mathbb{N}$ is proper whenever for each $i \in \mathbb{N}$ the set $c^{-1}(i)$ is independent in $\mathfrak{M}$.

In this way we can identify proper colorings of $\mathfrak{M}$ with coverings of E by independent sets in $\mathfrak{M}$. By the analogy this allows us to define k-coloring as well as ω-coloring where $\omega\colon E \to \mathbb{N}$ is a weight assignment function.

Definition 3.1.2. Let $\mathfrak{M}$ be a matroid on a ground set E and let $\omega\colon E \to \mathbb{N}$ be a weight assignment function. A function $c\colon E \to \mathbb{P}(\mathbb{N})$ is called a proper ω-coloring of $\mathfrak{M}$ whenever

- for every $e \in E$ we have $\sharp c(e) = \omega(e)$;
- for every $i \in \mathbb{N}$ the set $\{e \in \mathbb{N}\colon i \in c(e)\} \in \mathfrak{M}$.

Whenever $\omega \equiv k$ is a constant function then we call a proper ω-coloring of $\mathfrak{M}$ as a proper k-coloring of $\mathfrak{M}$.

Let $\mathfrak{M}$ be a matroid on a ground set E. We call an element $e \in E$ a loop for $\mathfrak{M}$ whenever $\{e\} \notin \mathfrak{M}$. We say that $\mathfrak{M}$ is loopless whenever there are no loops for $\mathfrak{M}$ in E. Note that a matroid with a loop can never be properly colored.

Definition 3.1.3. Let $\mathfrak{M}$ be a matroid on a ground set E. The chromatic number of $\mathfrak{M}$, denoted by $\chi(\mathfrak{M})$, is the minimal number m such that there exists a proper coloring $c\colon E \to \{1, 2, \ldots, m\}$.

Note that whenever $\mathfrak{M}$ is a graphic matroid then its chromatic number is the minimal number of colors using which we can color the edges of the graph defining the matroid such that each cycle is colored by at least two colors.

If $\mathfrak{M}$ is a transversal matroid then its chromatic number is the minimal number of colors using which we can color the ground set of the matroid such that there is a matching from any set of vertexes colored by the same color in the associated bipartite graph $G_{\mathfrak{M}}$.

Whenever $\mathfrak{M}$ is a representable matroid such that its ground set spans a vector space V then chromatic number is the minimal number of independent sets spanning V, into which we may divide the ground set of the matroid.

Definition 3.1.4. We define fractional chromatic number for a matroid $\mathfrak{M}$ on a ground set E as the infimum of fractions $\frac{a}{b}$ such that there exists a proper a-coloring of $\mathfrak{M}$ using b colors. We denote the fractional chromatic number of $\mathfrak{M}$ by $\chi_f(\mathfrak{M})$.

There is an explicit formula for characteristic number and fractional characteristic number of a matroid by Edmonds [11].

Theorem 3.1.5. *Let $\mathfrak{M}$ be a loopless matroid on a ground set E with the rank function $r\colon E \to \mathbb{N}$. Then*
$$\chi_f(\mathfrak{M}) = \max_{\emptyset \neq A \subseteq E} \tfrac{\sharp A}{r(A)} \text{ and } \chi(\mathfrak{M}) = \max_{\emptyset \neq A \subseteq E} \lceil \tfrac{\sharp A}{r(A)} \rceil$$

Proof. Let us set $R(\mathfrak{M}) := \max_{\emptyset \neq A \subseteq E} \frac{\sharp A}{r(A)}$. Whenever for $\mathfrak{M}$ exists a proper a-coloring $c\colon E \to \{1, 2, \dots, b\}$ then 2.4.3 for any $A \subseteq E$ implies that $b \cdot r(A) \geqslant a \cdot \sharp A$. This implies immediately that $\chi(\mathfrak{M}) \geqslant \chi_f(\mathfrak{M}) \geqslant R(\mathfrak{M})$.

For the inverse inequalities let $R(\mathfrak{M}) = \frac{\sharp B}{r(B)}$ for some $B \subseteq E$. Again from 2.4.3 we know that for every $A \subseteq E$ we have $\sharp B \cdot r(A) \geqslant r(B) \cdot \sharp A$ and that there exists a proper $r(B)$-coloring $c\colon E \to \{1, 2, \dots, \sharp B\}$. This implies that $\chi_f(\mathfrak{M}) \leqslant R(\mathfrak{M})$.

Now for every $A \subseteq E$ we have $\lceil \frac{\sharp B}{r(B)} \rceil \cdot r(A) \geqslant \frac{\sharp B}{r(B)} \cdot r(A) \geqslant \frac{\sharp A}{r(A)} \cdot r(A) = \sharp A$ and that there exists a proper 1-coloring $c\colon E \to \{1, 2, \dots, \lceil \frac{\sharp B}{r(B)} \rceil\}$. This implies that $\chi(\mathfrak{M}) \leqslant \lceil R(\mathfrak{M}) \rceil$.

This ends the proof. □

Note that this implies that the chromatic number for matroids is less than one bigger than its fractional chromatic number.

3.2. **List chromatic number.** In this section we consider colorings of matroids with constrains on available colors on each of the elements of the ground set.

Let $\mathfrak{M}$ be a matroid on a ground set E. We call an arbitrary function $L\colon E \to \mathbb{P}(\mathbb{N})$ such that $\sharp L(e) = l(e)$ for $l\colon E \to \mathbb{N}$ a list assignment of size l.

Definition 3.2.1. Let $\mathfrak{M}$ be a matroid on a ground set E, $L\colon E \to \mathbb{P}(\mathbb{N})$ a list assignment of size $l\colon E \to \mathbb{N}$ and let $\omega\colon E \to \mathbb{N}$ be a weight assignment function.

We call a function $c\colon E \to \mathbb{P}(\mathbb{N})$ a proper ω-coloring of $\mathfrak{M}$ from lists L of size l whenever

- c is a proper ω-coloring of $\mathfrak{M}$;
- for every $e \in E$ we have $c(e) \subseteq L(e)$.

The following remarkable result by Seymour [42] shows that if we assure the lists are of size equal to the chromatic number of a matroid then there exists a proper coloring from the lists.

Theorem 3.2.2. *Let $\mathfrak{M}$ be a matroid on a ground set E with rank function $r\colon \mathbb{P}(\mathbb{N}) \to \mathbb{N}$. Then the following conditions are equivalent*

(1) $\chi(\mathfrak{M}) \leqslant m$;
(2) there exists a proper coloring of $\mathfrak{M}$ from any list of a constant size $l \equiv m$.

Proof. Since we may consider constant list $L\colon E \ni e \mapsto L(e) := \{1, 2, \dots, m\}$ we see that (2) implies (1).

Now let us assume that (1) holds and let us consider and arbitrary list assignment $L\colon E \to \mathbb{P}(\mathbb{N})$ of size $\chi(\mathfrak{M})$. For every $i \in \mathbb{N}$ we set $E_i := \{e \in E \mid i \in L(e)\}$, $k := \max \bigcup_{i \in I} L(i)$, $\mathfrak{M}_i := \mathfrak{M}|_{E_i}$ with rank functions $r_i\colon E_i \to \mathbb{N}$. Note that we may trivially extend matroids $\mathfrak{M}_i$ to the ground set E. Now for every $A \subseteq E$ we have $m \cdot r(A) \geqslant \sharp A$ and
$$\sum_{i \in \mathbb{N}_{\leqslant k}} r_i(A) = \sum_{i \in \mathbb{N}_{\leqslant k}} r(A \cap E_i) \geqslant \sum_{i \in \mathbb{N}_{\leqslant k}} \tfrac{\sharp A \cap E_i}{m} = \tfrac{m \cdot \sharp A}{m} = \sharp A$$
which implies, by from 2.4.2, existence of a coloring from the lists L. □

A natural extension of Theorem 3.2.2 to the case of weighted colorings is the following result.

Theorem 3.2.3 (comp. [28]). *Let $\mathfrak{M}$ be a matroid on a ground set E with rank function $r\colon \mathbb{P}(\mathbb{N}) \to \mathbb{N}$, $\omega\colon E \to \mathbb{N}$ be a weight assignment function for $\mathfrak{M}$ and let $l\colon E \to \mathbb{N}$. Then*

(1) there exists a proper ω-coloring of $\mathfrak{M}$ from any lists of size l;

(2) there exists a proper ω-coloring of $\mathfrak{M}$ from the lists

$$L_l\colon E \ni e \mapsto L_l(e) := \{1, 2, \dots, l(e)\} \in \mathbb{P}(\mathbb{N});$$

(3) for every $A \subseteq E$ we have $\sum_{i\in\mathbb{N}_{>0}} r(\{e \in A\colon l(e) \geqslant i\}) \geqslant \sum_{e\in A} \omega(a)$.

Proof. Let $E'_i := \{e \in E \mid i \in L_l(e)\}$, $k' := \max \bigcup_{e\in E} L_l(e)$. For $i \leqslant k'$ let $\mathfrak{M}_i := \mathfrak{M}|_{E'_i}$ be matroids with rank functions $r_i\colon E'_i \to \mathbb{N}$. Note that we may trivially extend $\mathfrak{M}_i$ to the ground set E such that for every $A \subseteq E$ we have $r_i(A) = r_i(A \cap E'_i)$.

Let $A \subseteq E$. Since for every $i \in \mathbb{N}$ we have $\{e \in A\colon l(e) \geqslant i\} = A \cap E'_i$ then

$$\sum_{i\in\mathbb{N}_{\leqslant k'}} r(\{e \in A\colon l(e) \geqslant i\}) = \sum_{i\in\mathbb{N}_{\leqslant k'}} r_i(A)$$

so that equivalence of (2) and (3) follows directly from 2.4.2.

Note that (2) trivially implies (1). To prove that (1) implies (2) let $L\colon E \to \mathbb{P}(\mathbb{N})$ be any list assignment of size l. We set $E_i := \{e \in E \mid i \in L(e)\}$, $k := \max \bigcup_{e\in E} L(e)$. It is enough to show that

$$\sum_{i\in\mathbb{N}_{\leqslant k}} r(A \cap E_i) \geqslant \sum_{i\in\mathbb{N}_{\leqslant k'}} r(A \cap E'_i)$$

but this follows directly from submodularity of the rank function as the family $\{E'_i\}_{i\leqslant k'}$ forms a descending chain. □

As a simple corollary to Theorem 3.2.3 we present the following multiple exchange property for independent sets.

Corollary 3.2.4. *Let I_1 and I_2 be two independent sets of a matroid $\mathfrak{M}$ on a ground set E. Then for every $X \subseteq I_1$ there exists $Y \subseteq I_2$ such that both sets, $(I_1 \setminus X) \cup Y$ and $(I_2 \setminus Y) \cup X$, are independent.*

Proof. Let $I = I_1 \cap I_2$. We can restrict to the case where $I = \emptyset$. Indeed, if $I \neq \emptyset$, then consider matroid $\mathfrak{M}$ with contracted set I and two independent sets $I_1 \setminus I$, and $I_2 \setminus I$. For $X \setminus I_2$ we get Y, which is also good in the previous case.

Now let $I_1 \cap I_2 = \emptyset$. Let $\mathfrak{M}_1$ be matroid $\mathfrak{M}$ restricted to the set $X \cup I_2$, and let $\mathfrak{M}_2$ be matroid $\mathfrak{M}$ restricted to the set $(I_1 \setminus X) \cup I_2$. Let $I_1 \cup I_2$ be their common ground set, and denote their rank functions by r_1, r_2 respectively. Observe that for each $A \subseteq I_1 \cup I_2$ we have:

$$\begin{aligned} r_1(A) + r_2(A) = r(A \cap (X \cup I_2)) + r(A \cap ((I_1 \setminus X) \cup I_2)) \geq \\ \geq r(A \cap (I_1 \cup I_2)) + r(A \cap I_2) \geq |A \cap I_1| + |A \cap I_2| = |A|, \end{aligned}$$

where the first inequality is just a submodularity of a rank function. From 3.2.3 it follows that $I_1 \cup I_2$ can be covered by sets I'_1, I'_2 independent in $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively, so also in $\mathfrak{M}$. Now $Y = I_2 \cap I'_2$ is a good choice, since $(I_1 \setminus X) \cup Y = I'_2$ and $(I_2 \setminus Y) \cup X = I'_1$. □

As a special case we get the multiple basis exchange property.

Corollary 3.2.5. (Multiple basis exchange property) *Let B_1 and B_2 be two bases of a matroid $\mathfrak{M}$. Then for every $X \subseteq B_1$ there exists $Y \subseteq B_2$, such that $(B_1 \setminus X) \cup Y$ and $(B_2 \setminus Y) \cup X$ are also bases.*

3.3. **Delay chromatic number.** Inspired by an article by Wilfong, Haxell and Winkler [59] on delay colorings of bipartite graphs we define in this Section a delay coloring of a matroid and prove bounds for the number of colors needed to delay color a matroid.

Let $\mathfrak{M}$ be a matroid on a ground set E with rank function $r\colon \mathbb{P}(\mathbb{N}) \to \mathbb{N}$. We assume that there is a function, called a delay function, $\sigma\colon E \ni e \mapsto \sigma_e \in \Sigma_k$ where Σ_k is the symmetric group on k-elements.

Recall that a proper k-coloring of $\mathfrak{M}$ is a function $c\colon E \to \{1, 2 \ldots, k\}$ such that $U_i(c) := c^{-1}(i) \in \mathfrak{M}$ for every i.

Definition 3.3.1. We say that a proper coloring $c\colon E \to \{1, 2, \ldots, k\}$ is a k-delay coloring of $\mathfrak{M}$ for $\sigma\colon E \to \Sigma_k$ whenever $\sigma \circ c\colon E \ni e \mapsto \sigma_e(c(e)) \in \{1, 2, \ldots, k\}$ is a proper coloring of $\mathfrak{M}$.

We denote the smallest k such that for every $\sigma\colon E \to \Sigma_k$ there exists a k-delay coloring $\mathfrak{M}$ for σ by $D\chi(\mathfrak{M})$.

We propose the following far reaching conjecture.

Conjecture 3.3.2. *For every matroid $\mathfrak{M}$ we have that $D\chi(\mathfrak{M}) \leqslant \chi(\mathfrak{M}) + 1$.*

Whenever $\mathfrak{M}$ is a loopless matroid on a ground set E of rank 1 then the above conjecture is equivalent to a conjecture by Aharoni and Berger [2] (comp. Figure 7).

Conjecture. *Let G be a bipartite graph. Every n matchings from sets of size n have a rainbow matching of size $n - 1$.*

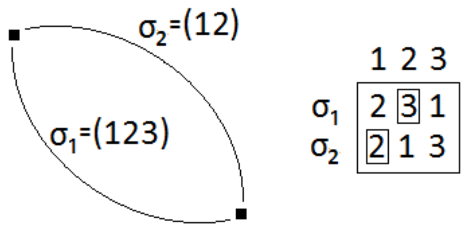

FIGURE 7. Matroid $U_{1,2}$ together with a delay function. Delay coloring was depicted in a table as a full transversal.

Of course when $\chi(\mathfrak{M}) = 1$ then $D\chi(\mathfrak{M}) = 1$. It is easy to find examples of matroids for which $D\chi(\mathfrak{M}) > \chi(\mathfrak{M}) \geqslant 2$.

In the remaining part of this section we will show several partial results to Conjecture 3.3.2. We start with a general upper bound for the delay chromatic number in terms of chromatic number.

Proposition 3.3.3. *Let $\mathfrak{M}$ be a matroid on a ground set E. If $\chi(\mathfrak{M}) = n$ then $D_\chi(\mathfrak{M}) \leqslant \frac{1}{2}n(n+1)$.*

Proof. Let $\sigma\sigma\colon E \to \Sigma_{\frac{1}{2}n(n+1)}$ be a delay function and let $B_1, B_2, \ldots, B_n$ form a covering of E by n independent sets in $\mathfrak{M}$. We define a lists size $l\colon E \to \mathbb{N}$ for $e \in E$ as $l(e) = i$ whenever $e \in B_i$.

Clearly there exists 1-coloring of $\mathfrak{M}$ from the lists L_l (comp. 3.2.3). We define lists

$$L(e) = \begin{cases} \{\sigma_e^{-1}(1)\}, & e \in B_1, \\ \{\sigma_e^{-1}(2), \sigma_e^{-1}(3)\} & e \in B_2, \\ \ldots & \\ \{\sigma_e^{-1}(\frac{1}{2}(n-1)n+1), \sigma_e^{-1}(\frac{1}{2}(n-1)n+2), \ldots, \sigma_e^{-1}(\frac{1}{2}n(n+1))\} & e \in B_n \end{cases}$$

Note that 3.2.3 implies that there exists a 1-coloring of $\mathfrak{M}$ from L, namely $c\colon E \to \{1, 2, \dots, n\}$ – and this is also a n-delay coloring of $\mathfrak{M}$ for σ. $\square$

Let B, B' be two disjoint basis in $\mathfrak{M}$ and let $A \cup C = B$ be a disjoint covering. Then there exists $E \cup F = B'$ such that $A \cup F \in \mathfrak{M}$ and $C \cup E \in \mathfrak{M}$.

Proposition 3.3.4. *Let $\mathfrak{M}$ be a matroid on a ground set E. If $\chi(\mathfrak{M}) = 3$ then $D_\chi(\mathfrak{M}) \leqslant 5$.*

Proof. Let $\sigma\colon E \to \Sigma_5$ be a delay function and let B_1, B_2 and B_3 form a covering of E by three independent sets in $\mathfrak{M}$. We define a lists size

$$l\colon E \ni e \mapsto \begin{cases} 1, & e \in B_1 \\ 2, & e \in B_2 \\ 4, & e \in B_3 \end{cases}$$

and a weight assignment function

$$\omega\colon E \ni e \mapsto \begin{cases} 1, & e \in B_1 \\ 1, & e \in B_2 \\ 2, & e \in B_3 \end{cases}.$$

Clearly there exists an ω-coloring of $\mathfrak{M}$ from L_l (comp. 3.3.2). Now we define the list

$$L(e) = \begin{cases} \{\sigma_e^{-1}(1)\}, & e \in B_1 \\ \{\sigma_e^{-1}(2), \sigma_e^{-1}(3)\} & e \in B_2 \\ \{\sigma_e^{-1}(2), \sigma_e^{-1}(3), \sigma_e^{-1}(4), \sigma_e^{-1}(5)\} & e \in B_3 \end{cases}.$$

Now 3.2.3 implies that there exists an ω-coloring of $\mathfrak{M}$ from L, namely $c\colon E \to \{1, 2, 3, 4, 5\}$. We set $U_i := c^{-1}(i)$.

Let $A := \{e \in B_2\colon e \in U_{\sigma_e^{-1}(2)}\}$ and $C := \{e \in B_2\colon e \in U_{\sigma_e^{-1}(3)}\}$ then $A \cup C = B_2$ and from the Remark we get $E \cup F = B_3$ such that $A \cup F, C \cup E \in \mathfrak{M}$. Now we define

$$V_i := U_i \cap (B_1 \cup B_2 \cup (F \cap \{e \in B_3\colon e \in U_{\sigma_e^{-1}(2)}\} \cup E \cap \{e \in B_3\colon e \in U_{\sigma_e^{-1}(3)}\}))$$

Now V_i defines a coloring $\tilde{c}\colon E \to \{1, 2, 3, 4, 5\}$, such that $\tilde{c}(e) = i$ if and only if $e \in V_i$. It is easy to see that it is a 5-delay coloring of $\mathfrak{M}$ for σ. $\square$

Theorem 3.3.5. *Let $\mathfrak{M}$ be a graphic matroid on a ground set E such that $\chi(\mathfrak{M}) \leqslant k$. Then $D_\chi(\mathfrak{M}) \leqslant 4k - 1$.*

Proof. We will prove the statement by induction on the number of elements in the ground set E. For $\sharp E = 1$ it is trivial so let us assume $\sharp E > 1$. Let G be the graph defining $\mathfrak{M}$ and $\sigma\colon E \to \Sigma_{4k-3}$ be a delay function.

If the graph G is not connected then the matroid $\mathfrak{M}$ can be presented as a direct sum of matroids with smaller ground sets and we can use induction on each of them to obtain our assertion.

Let us assume that G is connected. We know that G can be decomposed into at most k forests, with the biggest forest built up from c edges. Now in G there are at least $c + 1$ vertexes and at most kc edges. Therefore there is a vertex v of degree $\deg(v) \leqslant \frac{2kc}{c+1} < 2k$. Let $e \in E$ be any edge adjacent to v. Note that e can belong to at most $2k - 2$, disjoint outside e, cycles in G so to at most $2k - 2$ subsets $V_1, V_2 \dots, V_{2k-2}$ not belonging to $\mathfrak{M}$. Now, by induction, we find a $(4k - 3)$-delay coloring of $E \backslash \{e\}$, that is $c\colon E\backslash\{e\} \to \{1, 2, \dots, 4k - 3\}$. It follows that $\{i \mid c^{-1}(i) \cup \{e\} \notin \mathfrak{M}\} \leqslant 2k - 2$ and $\{i \mid (\sigma \circ c)^{-1}(i) \cup \{e\} \notin \mathfrak{M}\} \leqslant 2k - 2$ which follows that there is a color we may use to extend c to a $(4k - 3)$-delay coloring of E. $\square$

4. On-line games on a matroid

In on-line colorings one of the players try to get an optimal coloring of a combinatorial structure (graph, poset, matroid, etc.) whenever the other strives to prevent it from doing so. The basic scheme of games presented in this chapter is the following. The two players are given, known to both of them, a matroid. Now one of the players put some restrictions on possible colorings of elements of its ground sets (such as colors of some elements, colors in lists of some elements) and the other player tries to properly color the whole matroid.

These games include the Coloring Game introduced on graphs by Brams [9] and Bartnicki, Grytczuk, Kierstead [5]. We show a generalization of it and present a winning strategy for one of the players in Theorem 4.1.1, extending results of Lasoń [28].

The Indicated Coloring Game introduced for graphs by Grytczuk [18]. In Theorem 4.2.2 we construct a strategy for one of the players, extending results of Lasoń [30].

The third game, On-line List Coloring Game, was introduced for graphs by Schauz [37]. We prove a strategy for one of the players in Theorem 4.3.1, extending previously mentioned Seymour Theorem 3.2.2.

4.1. **Generalized coloring game.** We consider the following game.

Game 1. *Assume that we are given matroids $\mathfrak{M}_1, \dots, \mathfrak{M}_d$ on a ground set E visible to Presenter and Algorithm, we are given a set $\{1, 2, \dots, d\}$ of available colors and positive integers s.*

At each of the steps Algorithm colors up to s elements of E, previously colored with less available colors than their weights, using one of the unused available colors. Then, symmetrically, Presenter colors up to s elements of E, previously colored with less available colors than their weights, using one of the unused available colors. At any point of the game points colored by the i'th color must form an independent set in $\mathfrak{M}_i$.

Presenter wins the game if all elements of E are 1-colored; Algorithm wins if at some point there is an element $e \in E$ not colored and that cannot be colored using any of the available colors.

Theorem 4.1.1. *Let s be a positive integer and $\mathfrak{M}_1, \dots, \mathfrak{M}_d$ be matroids on the same ground set E. Suppose there are sets $V_1, \dots, V_d$ such that V_i is independent in $\mathfrak{M}_i$ and $\{V_i\}_{i=1,\dots,d}$ is a $s+1$-covering of E. Then Presenter has a winning strategy in Game 1.*

Proof. The proof goes by induction on $\sharp E$. Whenever $\sharp E \leqslant d$ the assertion is obviously true. Let us assume then that $\sharp E > d$. Let $r_1, \dots, r_d$ denote the rank functions of the matroids $\mathfrak{M}_1, \dots, \mathfrak{M}_d$. For each $\emptyset \neq A \subseteq E$ we have the following inequality:

$$r_1(A) + \ldots + r_d(A) \geqslant (s+1) \cdot \sharp A.$$

In his step Algorithm colors s elements $e_1, e_2, \dots, e_{s'}$, $s' \leqslant s$, of the ground set E. Let us assume for $e_1 \in E$ it used color $i_1 \in \{1, 2, \dots, d\}$. Now instead of $\mathfrak{M}_1, \dots, \mathfrak{M}_d$ we consider $\mathfrak{M}_1, \dots, \mathfrak{M}'_{i_1}, \dots, \mathfrak{M}_d$ where $\mathfrak{M}'_{i_1} = \mathfrak{M}_{i_1}/\{e_1\}$ and we consider the weigh assignment $\omega' \colon E \to \mathbb{N}$ where $\omega'(e) = \omega(e)$ for $e \neq e_1$ and $\omega'(e_1) = \omega(e_1) - 1$. We repeat this operation for all elements of E colored by Algorithm. We obtain matroids $\mathfrak{N}_1, \mathfrak{N}_2, \dots, \mathfrak{N}_d$ with rank functions $r'_1, r'_2, \dots, r'_d$ such that

$$r'_1(A) + r'_2(A) + \ldots + r'_d(A) \geqslant (s+1) \cdot \sharp A - s.$$

From 2.4.2 we know that there is an s-almost $(s+1)$-covering of $E \backslash \{e_1, e_2, \dots, e_{s'}\}$ by sets $V'_1, \dots, V'_d$ such that $V'_i \in \mathfrak{N}_i$. Let $y_1, y_2, \dots, y_l$ for $l \leqslant s$ be the elements that belong to $< s+1$ sets of the covering. Now Presenter colors these elements with a color equal to index of a set of the covering it belongs to (note that there is always at least one such). □

4.2. **Generalized indicated coloring game.** We consider the following game.

Game 2. *Assume that we are given matroids $\mathfrak{M}_1, \ldots, \mathfrak{M}_d$ on a ground set E visible to Presenter and Algorithm, we are given a set $\{1, 2, \ldots, d\}$ of available colors and a positive integer k.*

At each of the steps Presenter indicates an element $e \in E$, previously colored with colors $C_e = \{x_1, x_2, \ldots, x_l\} \subseteq \{1, 2, \ldots, d\}$ for $l < k$ colors, and Algorithm adds to colors at e an available color $c \in \{1, 2, \ldots, d\} \backslash C_e$. At any point of the game points colored by the i'th color must form an independent set in $\mathfrak{M}_i$.

Presenter wins the game if all elements of E are k-colored; Algorithm wins if at some point an indicated element of E cannot be colored at all.

Before proving the main result of this section we need the following simple lemma.

Lemma 4.2.1. *Let $\mathfrak{M}$ be a matroid on a ground set $E := A \sqcup B$ (disjoint union of sets) and let U be a base of $\mathfrak{M}|A$ then $r(A \cup B) = r(U \cup B)$.*

Proof. Each independent set, e.g. U, extends to a base V of $\mathcal{M}$. Note that $V \cap B = U$ (otherwise there would be bigger then U independent set in $\mathfrak{M}|B$). □

Theorem 4.2.2. *Let $\mathfrak{M}_1, \ldots, \mathfrak{M}_d$ be matroids on the same ground set E. Suppose there are sets $V_1, \ldots, V_d$ such that V_i is independent in $\mathfrak{M}_i$ and $\{V_i\}_{i=1,\ldots,d}$ is a k-covering of E. Then Presenter has a winning strategy in Game 2.*

Proof. The proof goes by induction on the number of elements of E. For a singleton it clearly holds. For $\sharp E > 1$ let $r_1, \ldots, r_d$ denote the rank functions of the matroids $\mathfrak{M}_1, \ldots, \mathfrak{M}_d$. For each $\emptyset \neq A \subseteq E$ we have the following inequality:

$$r_1(A) + \ldots + r_d(A) \geqslant k \cdot \sharp A.$$

Consider two separate cases.

Case 1: There is a proper subset $\emptyset \neq A \subsetneq E$ with equality

$$r_1(A) + \ldots + r_d(A) = k \cdot \sharp A.$$

Now Presenter plays the game on A with matroids $\mathfrak{M}_1|A, \ldots, \mathfrak{M}_k|A$. Let U_i denote the subset of A colored by Algorithm with i-th color. Note that, since the set colored by the i-th color is supposed to be independent we have

$$r_1(U_1) + \ldots + r_k(U_k) = \sharp A.$$

Now using the assumption about A we obtain that each U_i is a base for $\mathfrak{M}_i|_A$ (i.e. that $r_i(U_i) = r_i(A)$) (*). We have

$$r_1(A \cup B) - r_1(A) + \ldots + r_k(A \cup B) - r_k(A) \geqslant \sharp A \cup B - \sharp A = \sharp B.$$

Translating this inequality using 4.2.1 and the (*) we obtain

$$r_1(U_1 \cup B) - r_1(U_1) + \ldots + r_k(U_k \cup B) - r_k(U_k) \geqslant \sharp A \cup B - \sharp A = \sharp B.$$

This proves that the collection of matroids $(\mathfrak{M}_1/U_1)|A, \ldots, (\mathfrak{M}_k/U_k)|A$ satisfies assumptions of the theorem. Hence there is a winning strategy for Presenter.

Case 2: For all $\emptyset \neq A \subsetneq E$ holds

$$r_1(A) + \ldots + r_k(A) > \sharp A.$$

Then Presenter indicates an arbitrary vertex $e \in E$. Let us assume that Algorithm colored it into the i_0'th color. We set $\mathfrak{M}'_i = \mathfrak{M}_i$ for $i \neq i_0$ and M'_{i_0} equal to $\mathfrak{M}_i/\{e\}$ trivially extended to E. □

4.3. **On-line list coloring game.** Consider the following game.

Game 3. *Assume we are given a matroid $\mathfrak{M}$ on a ground set E and let k be a fixed positive integer (both are known to Presenter and Algorithm).*

In the first round Presenter chooses arbitrary non-empty subset $B_1 \subseteq E$ and inserts color 1 to the lists of all elements of B_1. Then Algorithm chooses some independent set $A_1 \subseteq B_1$ and colors its elements by color 1. In the second round Presenter picks arbitrarily a non-empty subset $B_2 \subseteq E$ and inserts color 2 to the lists of all elements of B_2. Then Algorithm chooses an independent subset $A_2 \subseteq B_2 \setminus A_1$ and colors its elements with color 2. And so on, until all lists will have exactly k elements.

If at the end of the play the whole ground set E is colored, then Algorithm is the winner. In the opposite case, Presenter is the winner.

We denote by $\widetilde{\chi}(\mathfrak{M})$ the minimum number k that guarantees existence of a winning strategy for Algorithm.

Theorem 4.3.1. *Every matroid $\mathfrak{M}$ satisfies $\widetilde{\chi}(\mathfrak{M}) = \chi(\mathfrak{M})$.*

We may consider a generalized game on a matroid $\mathfrak{M}$ with given weight assignment function $\omega\colon E \to \mathbb{N}$ and lists size function $l\colon E \to \mathbb{N}$, which goes in the same way as described in Game 4 except that the goal of Algorithm is a ω-coloring of $\mathfrak{M}$ from lists of size l.

Game 4. *Assume we are given a matroid $\mathfrak{M}$ on a ground set E, weight assignment function $\omega\colon E \to \mathbb{N}$, lists size function $l\colon E \to \mathbb{N}$. All information is known to Presenter and Algorithm.*

In the first round Presenter chooses arbitrary non-empty subset $B_1 \subseteq E$ and inserts color 1 to the lists of all elements of B_1. Then Algorithm chooses some independent set $A_1 \subseteq B_1$ and colors its elements by color 1. In the second round Presenter picks arbitrarily a non-empty subset $B_2 \subseteq E$ and inserts color 2 to the lists of all elements of B_2. Then Algorithm chooses an independent subset $A_2 \subseteq B_2 \setminus A_1$ and colors its elements with color 2. And so on, until lists at every $e \in E$ will have exactly $l(e)$ elements.

If at the end of the play the ground set E is ω-colored, then Algorithm is the winner. In the opposite case, Presenter is the winner.

If Algorithm has a winning strategy, then we say that $\mathfrak{M}$ is *on-line (ω, l)-colorable.*

For a given subset $U \subseteq E$, let $\mathbf{c}_U$ denote the characteristic function of U, that is, $\mathbf{c}_U(e) = 1$ if $e \in U$ and $\mathbf{c}_U(e) = 0$, otherwise. Now we prove the following inductive step lemma.

Lemma 4.3.2. *Let $I_1, \dots, I_k$ be a collection of independent sets in a matroid $\mathfrak{M}$ forming a ω-covering of its ground set E. Then for every set $V \subseteq E$ there exists an independent set $I \subseteq V$ and independent sets $I'_1, \dots, I'_k$ satisfying the following conditions.*

1. *The sets $I'_1, \dots, I'_k$ form a $(\omega - \mathbf{c}_I)$-covering of E.*
2. *For each $e \in E$, if $e \in I'_s$ then $e \in I_t$ for some $t \geq s + \mathbf{c}_V(e)$.*

Proof. Let $X_1 = (V \cap I_1) \setminus (I_1 \cap I_2)$. By Lemma 3.2.4 there exists $Y_2 \subseteq I_2$ such that $I'_1 = (I_1 \setminus X_1) \cup Y_2$ and $I''_2 = (I_2 \setminus Y_2) \cup X_1$ are independent. In general let $X_i = (V \cap I''_i) \setminus (I''_i \cap I_{i+1})$. So again by Lemma 3.2.4 there exists $Y_{i+1} \subseteq I_{i+1}$, such that $I'_i = (I''_i \setminus X_i) \cup Y_{i+1}$ and $I''_{i+1} := (I_{i+1} \setminus Y_{i+1}) \cup X_i$ are independent. Let $I = X_k$. It is easy to see that conditions (1) and (2) are satisfied. □

We are ready to prove the following generalization of Theorem 4.3.1.

Theorem 4.3.3. *Let $\mathfrak{M}$ be a matroid on a ground set E and let $: E \to \mathbb{N}$ be a list size function. If there exists a ω-coloring of $\mathfrak{M}$ from lists $L_l(e) = \{1, 2, \ldots, l(e)\}$, $e \in E$, then $\mathfrak{M}$ is on-line (ω, l)-colorable.*

Proof. We prove it by the induction on the number $\omega(E) = \sum_{e\in E} \omega(e)$. If $\omega(E) = 0$, then ω is the zero vector and the assertion holds trivially. Suppose now that $\omega(E) \geq 1$ and the assertion of the theorem holds for all ω' with $\omega'(E) < \omega(E)$. Let $V \subseteq E$ be the set of elements picked by Presenter in the first round of the game. So, all elements of V have color 1 in their lists. Let $I_1, \ldots, I_k$ be a ω-coloring of $\mathfrak{M}$ which exists by the assumption. By Lemma 4.3.2, there exist independent sets $I \subseteq V$ and $I'_1, \ldots, I'_k$, such that $I'_1, \ldots, I'_k$ is a $(\omega - \mathbf{c}_I)$-cover of E. Now Algorithm colors all elements from I with color 1. By condition (2) of Lemma 4.3.2, matroid $\mathfrak{M}$ is $(\omega - \mathbf{c}_I)$-colorable from lists $L'(e) = \{1, 2, \ldots, l(e) - \mathbf{c}_V(e)\}$. The assertion of the theorem follows by induction. □

4.4. **Strong on-line list coloring game.** We consider the following game.

Game 5. *Assume that we are given a matroid $\mathfrak{M}$ on a ground set E visible to Presenter and Algorithm and empty lists assignment $L_0 \colon E \ni e \mapsto \emptyset \in \mathbb{P}(\mathbb{N})$.*

Given $L_{i-1} \colon E \to \mathbb{P}(\mathbb{N})$ and a coloring $c \colon E_{i-1} \to \mathbb{N}$ of some subset $E_{i-1} \subseteq E$. At the i'th steps Presenter chooses an element $e \in E \backslash E_{i-1}$, a natural number $k_i \notin L_{i-1}(e)$, and defines $L_i \colon E \to \mathbb{P}(\mathbb{N})$ as $L_i|_{E\backslash\{e\}} = L_{i-1}$ and $L_i(e) = L_{i-1}(e) \cup \{k_i\}$. Algorithm now either sets $E_i := E_{i-1}$ and $c_i := c_{i-1}$ or colors the element e using k_i, i.e. $E_i := E_{i-1} \cup \{e\}$ and $c \colon E_i \to \mathbb{N}$ is defined by $c_i|_{E_{i-1}} = c_i$ and $c_i(e) = k_i$; the coloring c_i must be a proper coloring of E_i.

Algorithm wins the game if the whole E is colored at some point of the game and $\sum_{e\in E} \sharp L(e) \leqslant \sharp E \cdot (\chi(\mathfrak{M}) - 1)$; if at some point of the game $\sum_{e\in E} \sharp L(e) \geqslant \sharp E \cdot (\chi(\mathfrak{M}) - 1)$ then Presenter wins.

Theorem 4.4.1. *Let $\mathfrak{M}$ be a matroid on a ground set E. Then Algorithm has a winning strategy in Game 5.*

Proof. Let us describe the strategy for Algorithm.

Let $d := \chi(\mathfrak{M})$. There is a 1-covering $\{V_1, V_2, \ldots, V_d\}$ of E by independent sets in $\mathfrak{M}$.

We assume that before the i'th step we are given a list assignment function L_{i-1} and a coloring c_{i-1} of E_{i-1}. We assume also that Algorithm constructed a list assignment function $F_{i-1} \colon E \to \mathbb{P}(\mathbb{N})$ (called forbidden color lists) such that for each color $j \in \mathbb{N}$ and $1 \leqslant s \leqslant d$ we have $c_{i-1}^{-1}(j) \cup W_{i-1}^{j,s} \in \mathfrak{M}$, $W_{i-1}^{j,s} := V_s \cap \{e \in E \mid j \notin F_{i-1}(e)\}$; where we start with $E_0 := \emptyset$, $c_0 := \emptyset$, $L_0 := \emptyset$, $F_0 := \emptyset$.

When at the i'th step Presenter chooses $e \in E \backslash E_{i-1}$, a color $k_i \notin L_{i-1}(e)$ and defines L_i then Algorithm checks if $k_i \in F_{i-1}(e)$. If so then it defines $E_i := E_{i-1}$, $c_i := c_{i-1}$, $F_i := F_{i-1}$. If this is not the case then $E_i := E_{i-1} \cup \{e\}$, $c_i \colon E_i \to \mathbb{N}$ is such that $c_i|_{E_{i-1}} = c_{i-1}$, $c_i(e) = k_i$. Note that the conditions on F_{i-1} assures this is a proper coloring. Now if $e \in V_l$ then for every $1 \leqslant s \leqslant d$, $s \neq l$, there exists $e_s \in W_{i-1}^{k_i,s}$ (unless the set $W_{i-1}^{k_i,s}$ is empty) such that $c_i^{-1}(k_i) \cup W_{i-1}^{k_i,s} \backslash \{e_s\} \in \mathfrak{M}$, as follows from the augmentation property. Now we define $F_i \colon E \to \mathbb{P}(\mathbb{N})$ as $F_i(e) := F_{i-1}(e)$ if $e \neq e_s$ and $F_i(e_s) := F_{i-1}(e_s) \cup \{k_i\}$.

Since coloring of one element of the ground set E forces Algorithm to enlarge the lists F_i by at most $\chi(\mathfrak{M}) - 1$ elements, the result follows. □

5. Matroids on-line

In this chapter we present several result concerning the following game.

Assume that we are given an infinite countable set E'. Presenter is given a matroid on a ground set $E \subseteq E'$ not known to Algorithm.

At each step Presenter reveals one of the elements of E together with the matroids structure on the revealed part. Now Algorithm (if it is possible) colors the element into one of the available colors such that the whole revealed part was colored properly.

Algorithm wins whenever at each step he can perform his move until the whole ground set E will be revealed. In the other case Presenter wins.

The game is inspired by a similar vertex coloring game on graphs by Kierstead, Penrice and Trotter [25]. We try to find competitive constant between chromatic number of the matroid and number of colors that Algorithm needs to properly color the whole ground set in several variations of the game. In each case we assume that Algorithm knows that the matroid is in a specific class (i.e. is a sparse paving matroid, transversal matroid, graphic matroid, regular matroid etc.) and at each step Presenter reveals not only the matroid structure on the revealed part but also the corresponding part of its representation as a member of one of the classes.

5.1. Sparse paving matroid.

Recall the definition of sparse paving matroids.

Definition 5.1.1 (Sparse paving matroid)**.** Let $\mathfrak{M}$ be a matroid on the ground set E. We call $\mathfrak{M}$ sparse paving matroid of rank n whenever:

- every $(n-1)$-element subset of E is independent and there exists n-element subset of E which is independent;
- every $(n+1)$-element subset of E is dependent;
- if C_1 and C_2 are different circuits in $\mathfrak{M}$ of size n then $\sharp(C_1 \cap C_1) \leqslant n-2$.

We play the following two-person game on sparse paving matroids.

Game 6. *Assume that we are given an infinite countable set E'. Presenter is given a sparse paving matroid $\mathfrak{M}$ on the ground set $E \subseteq E'$ not known to Algorithm.*

At each step Presenter reveals one of the elements that remained hidden from Algorithm together with all circuits of $\mathfrak{M}$ contained in the revealed part of E. Now Algorithm (if it is possible) colors the element into one of the available colors such that the whole revealed part was colored properly in $\mathfrak{M}$.

Algorithm wins whenever at each step he can perform his move until the whole ground set E will be revealed. In the other case Presenter wins.

Definition 5.1.2. The smallest number of colors m that assures Algorithm a winning strategy in the above game is called on-line chromatic number of a sparse paving matroid $\mathfrak{M}$ and is denoted by $\chi_{sp}^{ol}(\mathfrak{M})$.

Given an on-line variant of chromatic number we compare it with the standard off-line version.

Definition 5.1.3. We set

$$H_{sp}^{ol}(n) := \max\{\chi_{sp}^{ol}(\mathfrak{M})\colon \chi(\mathfrak{M}) = n,\ \mathfrak{M} \text{ is sparse paving}\}$$

and call it on-line chromatic ratio for sparse paving matroids.

Theorem 5.1.4. *There are equalities*

- $H^{ol}_{sp}(1) = 1$;
- $H^{ol}_{sp}(n) = n + 1$ *for* $n > 1$.

Proof. The equality $H^{ol}_{sp}(1) = 1$ is obvious. Let us then assume that $n > 1$. At i'th step, given a sparse paving matroid $\mathfrak{M}_{i-1}$ on the ground set E_{i-1} such that $\sharp E_{i-1} = i - 1$ and $\chi(M_{i-1}) = n$, Presenter reveals an element $e \in E \backslash E_{i-1}$ and a sparse paving matroid $\mathfrak{M}_i$ on $E_i := E_{i-1} \cup \{e\}$ such that $\mathfrak{M}_i|_{E_{i-1}} = \mathfrak{M}_{i-1}$, $\chi(\mathfrak{M}_i) = n$.

Since the number $H^{ol}_{sp}(n)$ is obtained as a maximum of $\chi^{ol}_{sp}(\mathfrak{M})$ over all sparse paving matroids $\mathfrak{M}$ satisfying $\chi(\mathfrak{M}) = n$, we will assume that at the i'th step Presenter constructs matroid $\mathfrak{M}_i$ on the ground set E_i such that $\chi(\mathfrak{M}_i) = n$ and $\mathfrak{M}_i|_{E_{i-1}}\mathfrak{M}_{i-1}$ depending on the color choices made by the Algorithm.

For $n \geqslant 2$ let us assume that $H^{ol}_{sp}(n) = n$. We denote colors available to Algorithm by natural numbers $C := \{1, 2, \ldots, n\}$. The last, n'th, of the colors will be called black.

Let us assume that at the end of the $(2n - 1)$'st step Presenter revealed matroid $\mathfrak{M}_{2n-1} := U_{2,2n-1}$ defined on the ground set $E_{2n-1} := \{1, 2, \ldots, 2n - 1\}$. Either Presenter already won or Algorithm used each but one of the colors twice. Let us assume that the color used once was black and element $\{e\} \in E_{2n-1}$ was already colored black. Now let $\mathfrak{M}_{2n}$ be a matroid in which every three element set is dependent and $\{e, e_{2n}\}$ is the only 2-element circuit. Of course $\mathfrak{M}_{2n}$ is sparse paving, $\mathfrak{M}_{2n}|_{E_{2n-1}} = \mathfrak{M}_{2n-1}$ and $\chi(\mathfrak{M}_{2n}) = n$. On the other hand Algorithm cannot use any of the colors in C to color e_{2n}.

Now let us show that $H^{ol}_{sp}(n) \leqslant n + 1$. We denote colors available to Algorithm by natural numbers $\{1, 2, \ldots, n + 1\}$. Winning strategy for Algorithm is the following.

Algorithm colors revealed elements into the first color until for some i the set E_i contains a circuit C. We have two possible cases $\sharp C = i - 1$ or $C = E_1$. In the former case rank r of the matroid $\mathfrak{M}$ equals to $i - 1$. In the latter we color e_i into the second color and wait for the $(i + 1)$'st step. If $E_{i-1} \cup \{e_{i+1}\}$ is a circuit then rank of our matroid is $i - 1$ and we color e_{i+1} into the second color (the case of a matroid of rank 1 is trivial) and if not then the rank equals to i and we color e_{i+1} into the first color.

Now if the Algorithm used r times colors $\{1, 2, \ldots, k\}$ and at most once color $k + 1$'st then it colors revealed elements into the $k + 1$ color until some e_i forms with the set colored into $k + 1$'st color a dependent set. It the set is of order $r + 1$ then we follow by induction and if r then we color element e_i into $k + 2$'nd color and e_{i+1} into $k + 1$'st and in this case induction ends the proof as well. □

5.2. **Graphic matroid.** Let us briefly recall the definition of graphic matroid.

Definition 5.2.1 (Graphic matroid)**.** Let $G := (V_G, E_G)$ be a graph. We call a matroid $\mathfrak{M}_G$ on the ground set E_G graphic whenever a subset $A \subseteq E_G$ is independent in $\mathfrak{M}$ if and only if A contains no cycle in G.

We play the following two person game on a graphic matroid $\mathfrak{M}_G$:

Game 7. *Assume we are given an infinite set of vertices V. Presenter is given a finite graph $G := (V_G, E_G)$ such that $V_G \subseteq V$. It defines a graphic matroid $\mathfrak{M}_G$. Neither G nor $\mathfrak{M}_G$ are known to Algorithm.*

At the i'th step Presenter reveals one of the edges of the graph G. Now Algorithm (if it is possible) colors the edge into one of the available colors such that the revealed part of G was colored properly in $\mathfrak{M}_G$.

Algorithm wins whenever at each step he can perform him move until the whole graph G will be revealed. In the other case Presenter wins.

Definition 5.2.2. The smallest number of colors m that assures Algorithm a winning strategy in the above game is called on-line chromatic number of a graphic matroid $\mathfrak{M}_G$ and is denoted by $\chi_g^{ol}(\mathfrak{M}_G)$

Given an on-line variant of chromatic number we compare it with the standard off-line version.

Definition 5.2.3. We set

$$H_g^{ol}(n) := \max\{\chi_g^{ol}(\mathfrak{M}_G) \colon \chi(\mathfrak{M}_G) = n\}$$

and call it on-line chromatic ratio for graphic matroids.

Theorem 5.2.4. *We have the following equalities:*

- $H_g^{ol}(1) = 1$*;*
- $H_g^{ol}(m) = \infty$ *for* $m > 1$*.*

We postpone proof of this theorem to the end of this section.

We play the following two person game.

Game 8. *Assume we are given a forest* F *(not knows to Algorithm) and an infinite number of vertexes and that Algorithm is given a finite set of available colors.*

At the i'th step Presenter draws an edge between two of the vertexes such that the revealed graph was a forest. Now Algorithm colors the edge into one of the available colors (without any constraints).

Presenter wins if, after some finite number of steps, there is a mono-colored forest isomorphic to F*. In the other case Algorithm wins.*

To prove Theorem 5.2.4 we need the following result which, in a slightly different context, first appeared in [19].

Lemma 5.2.5. *Presenter has always a winning strategy in Game 8.*

Proof. We prove the statement by induction on the number of colors n_c available to Algorithm and the number of edges $n_e(F)$ in the forest F. Let $\mathcal{S}(n_c, n_e(F))$ be a winning strategy for Presenter (if any).

If $n_c = 1$ or $n_e(F) = 1$ then the winning strategy is obvious.

For $n_c > 1$ and $n_e(F) > 1$ we construct winning strategy for Presenter as follows. Let F' be the forest F without one leaf e adjacent to a node $v(F')$ in F', i.e. $F' := F \backslash \{e\}$.

From the induction assumption we know winning strategies for Presenter $\mathcal{S}(n_c, n_e(F'))$ as well as $\mathcal{S}(n_c - 1, n_e(F))$.

Assume that the strategy $\mathcal{S}(n_c - 1, n_e(F))$ uses N nodes.

Now Presenter repeats $N(n_c - 1) + 1$ times the strategy $\mathcal{S}(n_c, n_e(F'))$, each time on a completely new set of nodes. It follows that there is a color, let us assume it is the n_c'th one, in which Algorithm colored the forest $G := \sqcup_{i=1,2,\ldots,N} F'_i$ in which $F'_i \cong F'_j \cong F'$ and each copy of such F' lies in a different connected component of the graph constructed in the process. Now for every i the forest F'_i contains a node $v(F'_i)$.

Now Presenter uses the strategy $\mathcal{S}(n_c - 1, n_e(F))$ using the $v(F'_1), \ldots, v(F'_N)$ as nodes. If at any point Algorithm uses the n_c'th color then Presenter automatically wins. If this is not the case then the strategy $\mathcal{S}(n_c - 1, n_e(F))$ assures the existence of the graph F mono colored into one of the colors $1, 2, \ldots, n_c - 1$. □

Proof of Theorem 5.2.4. Let $N(n, c)$ be the number of nodes needed to construct a mono-colored path of n edges when Algorithm is allowed to use c different colors in Lemma 5.2.5.

We set recursively

$$M(n) := \begin{cases} 2 & \text{for } n = 1; \\ N(2M(n-1), n) & \text{for } n > 0. \end{cases}.$$

Now let us assume that Algorithm is given $n_c > 0$ element set of available colors. Presenter uses $M(n_c)$ nodes and building forest F_1 on them forces Algorithm to create a mono-colored path P_1 of $2M(n_c - 1)$ edges. Now Presenter takes every second node on the path and plays further on on the $M(n_c - 1)$ nodes. Note that the color used by Algorithm on P_1 cannot be used any more (as it results in a mono-colored cycle) so that it has $n_c - 1$ available colors.

At the i'th step, given $M(n_c - i + 1)$ nodes and $n_c - i + 1$ colors available to Algorithm, Presenter constructs a forest F_i and forces Algorithm to create a mono-colored path P_i of length $2M(n_c - i)$. He choses every second node on the path ending with $M(n_c - i)$ nodes and $n_c - i$ available colors for Algorithm for the future game.

Continuing in this fashion we end up with a forest F_{n_c} equal to a path P_{n_c} of 2 edges in the last of the colors.

Now Presenter draws a path P_{n_c+1} between the ends of the path P_{n_c} which cannot be colored by Algorithm. Presenter wins if we show that $F := \cup_{i=1,2,\dots,n_c+1} F_i$ is a graph covered by two forests.

For every i we order edges of P_i along the path (we choose any of the two orders) and consider a forest A_i containing even edges in P_i. We set

$$X := \bigcup_{i=1,\dots,n_c+1} G_i \backslash A_i \text{ and } Y := \bigcup_{i=1,\dots,n_c+1} A_i.$$

We claim these are the forests that cover F.

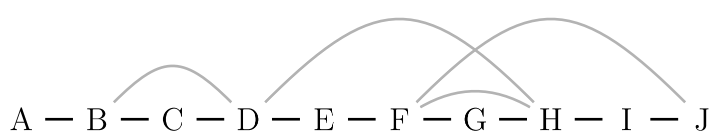

FIGURE 8. Part of a graph constructed by Presenter with paths P_i (black), P_{i+1} (gray).

Indeed if C is a cycle in X or Y then let

$$i_0 := \min\{i : C \cap F_i \neq \emptyset\}.$$

Note that $C \cap F_i$ must contain two consecutive edges on the path P_i which contradicts the choice of A_i. □

5.3. Transversal matroid.

Definition 5.3.1 (Transversal matroid)**.** Let $\{A_\alpha\}_{\alpha \in A}$ be a family of subsets of a set E. We call a matroid $\mathfrak{M}$ on the ground set E transversal defined by the family $\{A_\alpha\}_{\alpha \in A}$ whenever $X \subseteq E$ is independent in $\mathfrak{M}$ if and only if there exists an injection $\iota_X : X \to A$ such that for each $x \in X$ we have $x \in A_{\iota_X(x)}$.

We play the following two person game on a transversal matroid:

Game 9. *Assume we are given an infinite set E'. Presenter is given a transversal matroid $\mathfrak{M}$ defined by $\{A_\alpha\}_{\alpha \in A}$ (we assume the set A to be countable, we do not assume the sets A_α to be not empty) on a ground set $E \subseteq E'$.*

At the i'th step Presenter reveals one of the elements e of the ground set E together with all the indexes $A(e) \subseteq A$ such that $e \in A_\alpha$ for every $\alpha \in A(e)$. Now Algorithm colors properly the element e such that the revealed part was colored properly in $\mathfrak{M}$.

Algorithm wins whenever he can properly color the whole matroid $\mathfrak{M}$. In the other case Presenter wins.

Definition 5.3.2. The smallest number of colors m that assures Algorithm a winning strategy in the above game is called on-line chromatic number of a transversal matroid $\mathfrak{M}$ defined by $\{A_\alpha\}_{\alpha\in A}$ and is denoted by $\chi^{ol}_{tr}(\mathfrak{M})$.

Given an on-line variant of chromatic number we compare it with the standard off-line version.

Definition 5.3.3. We set

$$H^{ol}_{tr}(n) := \max\{\chi^{ol}_{tr}(\mathfrak{M})\colon \chi(\mathfrak{M}) = n,\ \mathfrak{M} \text{ is a transversal matroid defined by } \{A_\alpha\}_{\alpha\in A}\}$$

and call it the on-line chromatic ratio for transversal matroids.

Theorem 5.3.4. *We have the following equalities:*

- $H^{ol}_{tr}(1) = 1$*;*
- $H^{ol}_{tr}(m) = \infty$ *for* $m > 1$*.*

We postpone proof of this theorem to the end of this section.

We play the following two person game which can be understood as an *on-line Hall's condition.*

Game 10. *Assume we are given two infinite sets of vertexes: the upper U' and the lower V' and a bipartite graph $G := (U, V, E)$ where $U \subseteq U'$, $V \subseteq V'$ (not knows to Algorithm) and that Algorithm is given a finite set of available colors.*

At each of the steps Presenter reveals one of the vertexes $v \in V$ and draws all the edges from v to U' that exists in G. Now Algorithm colors the vertex v into one of the available colors such that mono-colored sets satisfy the Hall's condition.

Algorithm wins if at each step he can perform his move until the whole graph G will be revealed. In the other case Algorithm wins.

Proposition 5.3.5. *A winning strategy for Algorithm in Game 9 exists if and only if a winning strategy for Algorithm exists in Game 10.*

Proof. It is enough to note that every transversal $\mathfrak{M}$ matroid defined by $\mathcal{A} := \{A_\alpha\}_{\alpha\in A}$ corresponds to a bipartite graph $G := (U, A, E)$ in which U is a underlying set of the matroid $\mathfrak{M}$, A is the indexing set of the family $\mathcal{A}$ and $(u, \alpha) \in E$ if $u \in A_\alpha$.

Under this correspondence revealing elements of the ground set of $\mathfrak{M}$ by Presenter is equivalent to revealing elements of U together with all edges form them into A.

Now the condition for a subset $X \subseteq U$ to be independent corresponds exactly to the existence of a matching from X into A which is equivalent to the Hall's condition. □

Proof of Theorem 5.3.4. We will construct a strategy for Algorithm in Game 10 instead of Game 9.

Let us assume that Presenter reveals elements of the ground set of a matroid $\mathfrak{M}$ such that $\chi(\mathfrak{M}) = 2$ and Algorithm has n_c available colors. We will use the translation of the problem proposed in Proposition 5.3.5.

Let us assume that we are given a bipartite graph $G := (U_1, V_1, E_1)$ colored properly by Algorithm using $c_G\colon U_1 \to \{1, 2, \ldots, n_c\}$, off-line properly colorable using two colors and where $U_1 \subseteq U'$, $V_1 \subseteq V'$. For subsets $X \subseteq U'\backslash U_1$ and $Y \subseteq V'$ we denote by $G(X,Y)$ the bipartite graph in which every $x \in X$ is connected to every element of Y.

For $Y \subseteq V'$ we define degree of V in G as

$$\deg_G Y := \max\{\sharp X\colon X \subseteq U'\backslash U_1,\ G \cup G(X,Y) \text{ is colorable into two colors}\}$$

and on-line degree of V in G as

$$\deg^{ol}_G Y := \max\{\sharp X\colon X \subseteq U'\backslash U_1,\\ \exists \tilde{c}_G\colon U_1 \cup X \to \{1,2,\ldots,n_c\} \text{ a proper coloring and } \tilde{c}_G|_{U_1} = c_G\}.$$

Of course whenever at some point of our game there exists $Y \subseteq V'$ such that $\deg_G Y > 0$ and $\deg^{ol}_G Y < 0$ then there is a winning strategy for Presenter.

Since Algorithm is given a finite number of colors we may assume that whenever Presenter forced Algorithm to color an arbitrary finite number of copies $G_0, G_1, \ldots, G_N$ of a given bipartite graph $G := (U_G, V_G, E_G)$ with an ordered set of distinguished vertexes (note the ordering) $\mathcal{V} := (v_1, \ldots, v_{l_G})$ identically. We denote that distinguished elements in G_i by $(v^i_1, \ldots, v^i_{l_G})$ For any $i > 0$ we reveal elements $u^i_1, \ldots, u^i_{l_G}$ such that for any j there are edges $\{v^0_j, u^i_j\}$ and $\{v^i_j, u^i_j\}$. Once again from the finiteness of the number of available colors we may assume that for every $1 \leqslant j \leqslant l_G$ Algorithm colored u^1_j, u^2_j into the same color. We denote the resulting graph by $m(G)$ with the set of distinguished elements $m(\mathcal{V}) := (v^1_1, \ldots, v^1_{l_G}, v^2_1, \ldots, v^2_{l_G})$. Note that $\deg_{m(G)}(m(\mathcal{V})) = 2\deg_G(\mathcal{V})$ and that $\deg^{ol}_{m(G)}(m(\mathcal{V})) = 2\deg^{ol}_G(\mathcal{V}) - l_G$.

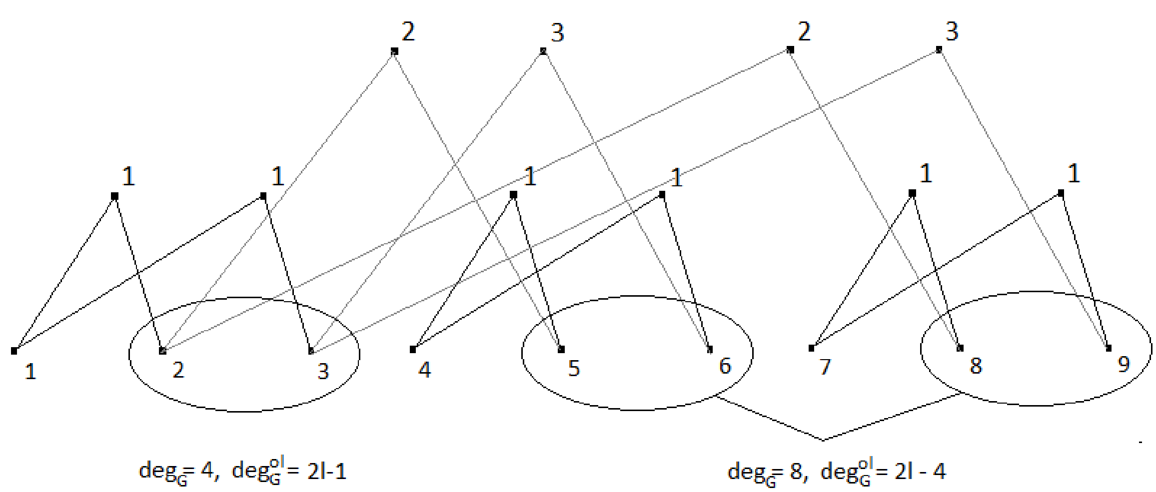

FIGURE 9. The first two steps of the game; black edges were revealed by Presenter in the first step of the game, gray edges were revealed by Presenter in the second step of the game. Captions of the top vertexes denote exemplary colors used by Algorithm.

Now the strategy for Presenter goes as follows (comp. Figure 9). We start with $G_0 := (\emptyset, \{v_0\}, \emptyset)$ with distinguishes set of elements $\mathcal{V}_0 := (v_0)$. Then $\deg_{G_0}(\mathcal{V}_0) = 2$ and $\deg^{ol}_{G_0}(\mathcal{V}_0) = n_c$. Now given a bipartite graph G_i with set of distinguished elements $\mathcal{V}_i$ of size 2^i then we define $G_{i+1} := m(G_i)$ and $\mathcal{V}_{i+1} := m(\mathcal{V}_i)$. Then $\deg_{G_i}(\mathcal{V}_i) = 2^{i+1}$ and

$\deg^{ol}_{G_i}(\mathcal{V}_i) = 2^i n_c - i2^{i-1}$. Note that after a finite number of steps $\deg^{ol}_{G_i}(\mathcal{V}_i) < 0$ which proves the assertion. □

5.4. **Lattice path matroids.** We start by defining lattice path matroids, which create a subclass of transversal matroids.

Definition 5.4.1. Let $P := p_1p_2\dots p_m$ be a path in the lattice $\mathbb{Z}\times\mathbb{Z}$ starting at $(0,0)$. We call P nondecreasing whenever it uses only North and East steps (i.e. $(0,1)$ and $(1,0)$ respectively).

Definition 5.4.2. For a path $P := p_1p_2\dots p_m$ we define its restriction to $Z \subseteq \{1,2,\dots,m\}$ as $P|_Z := \{p_{z_1}p_{z_2}\dots p_{z_k}\}$ where $Z = \{z_1, z_2, \dots, z_k\}$ and $z_i < z_{i+1}$ for $i = 1,2,\dots,k-1$.

Let us now consider two nondecreasing paths $P := p_1p_2\dots p_m$ and $Q := q_1q_2\dots q_m$ in the lattice $\mathbb{Z}\times\mathbb{Z}$ starting at $(0,0)$. We assume that P never goes above Q. Let $\{p_{u_1}p_{u_2}\dots p_{u_r}\}$ be the set of North steps in P and $\{q_{l_1}q_{l_2}\dots q_{l_s}\}$ be the set of North steps in Q. Then $s \geqslant r$ and for $i = 1,2,\dots,s$ we define a subset $N_i^{P,Q} \subseteq N$ as

$$N_i^{P,Q} := \begin{cases} [l_i, u_i] & \text{if } i \leqslant r \\ [l_i, m] & \text{if } i > r \end{cases}.$$

Definition 5.4.3 (Lattice path matroid)**.** We say that a matroid $\mathfrak{M}_{P,Q}$ is a path lattice matroid defined by the paths P and Q whenever it is a transversal matroid on the ground set $\{1,2,\dots,s\}$ defined by the family $\{N_i^{P,Q}\}_{i=1,2,\dots,s}$.

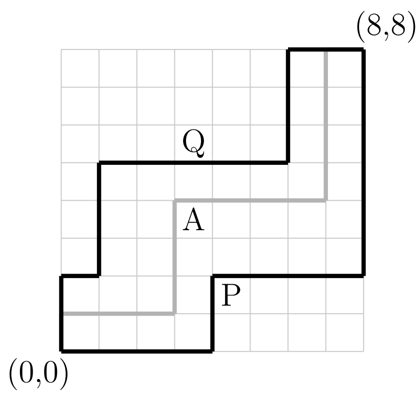

FIGURE 10. An example of a path lattice matroid defined by paths P and Q. Each path using North and East steps from $(0,0)$ to $(8,8)$ between P and Q defines a base of the matroid by taking indexes of its North steps. In particular the path A corresponds to base $\{1,5,6,7,12,13,14,15\}$.

We play the following two person game on a path lattice matroid:

Game 11. *Assume that Presenter is given a path lattice matroid* $\mathfrak{M}$ *on the ground set* $E := \{1,2,\dots,m\}$ *defined by two nondecreasing paths* P *and* Q *starting at* $(0,0)$ *and that Algorithm is given a finite set of available colors.*

At the i'th step Presenter reveals one of the elements e_i *of the ground set* E *together with two paths* $P|_{E_i}$ *and* $Q|_{E_i}$ *where* $E_i := \{e_1, e_2, \dots, e_i\}$*. Now Algorithm colors properly the element* e_i *such that the revealed part was colored properly in* $\mathfrak{M}^{P,Q}|_{E_i}$*.*

Algorithm wins whenever he can properly color the whole matroid $\mathfrak{M}^{P,Q}$*. In the other case Presenter wins.*

Definition 5.4.4. The smallest number of colors m that assures Algorithm a winning strategy in Game 11 is called an on-line chromatic number of a path lattice matroid $\mathfrak{M}^{P,Q}$ and is denoted by $\chi_{pl}^{ol}(\mathfrak{M}^{P,Q})$.

Given an on-line variant of chromatic number we compare it with the standard off-line version.

Definition 5.4.5. We set

$$H_{pl}^{ol}(n) := \max\{\chi_{pl}^{ol}(\mathfrak{M}^{P,Q}) \colon \chi(\mathfrak{M}^{P,Q}) = n\}$$

and call it the on-line chromatic ratio for path lattice matroids.

Theorem 5.4.6. *We have the following equalities:*

- $H_{pl}^{ol}(1) = 1$*;*
- $H_{pl}^{ol}(m) \leqslant 2m - 1$ *for* $m > 1$*;*
- $H_{pl}^{ol}(m) > m$ *for* $m > 1$*.*

We postpone proof of this theorem to the end of this section.

We play the following two person game.

Game 12. *Assume we are given two infinite countable sets of vertexes: the upper* U' *and the lower* V' *and a bipartite graph* $G := (U, V, E)$ *(not known to Algorithm) where* $U \subseteq U'$*,* $V \subseteq V'$*. We assume that there is an ordering*

$$\sigma \colon \{1, 2, \ldots, m\} \to U' = \{u_1, u_2, \ldots, u_m\}$$

such that if we set $\mathcal{N}(u_i) = [l_i, r_i] \subset$ *then*

$$l_{\sigma(1)} \leqslant l_{\sigma(2)} \leqslant \ldots \leqslant l_{\sigma(m)} \text{ and } r_{\sigma(1)} \leqslant r_{\sigma(2)} \leqslant \ldots \leqslant r_{\sigma(m)}.$$

We assume also that Algorithm is given a finite set of available colors.

At each of the steps Presenter reveals one of the vertexes $u \in U$ *and draws all the edges from* u *to* V' *that exists in* G*. Now Algorithm colors the vertex* v *into one of the available colors such that mono-colored sets satisfy the Hall's condition.*

Algorithm wins if at each step he can perform his move until the whole graph G *will be revealed. In the other case Algorithm wins.*

Note that exactly as in Proposition 5.3.5 we have.

Proposition 5.4.7. *A winning strategy for Algorithm in Game 11 exists if and only if a winning strategy for Algorithm exists in Game 12.*

Proof of Theorem 5.4.6. We will construct a strategy for Algorithm in Game 12 instead of Game 11.

Let us prove first the upper bound. We assume that during the game Presented reveals elements of the ground set of a matroid $\mathfrak{M}$ such that $\chi(\mathfrak{M}) = m > 1$.

The following describes a strategy for Algorithm whenever number of colors n_c available to it satisfies $n_c \geqslant 2m - 1$. We will use the description introduced by Proposition 5.4.7.

When in the i'th step Presenter reveals a top vertex v_i of the bipartite graph $G = (U, V, E)$ then it defines an interval $\mathcal{N}(v_i) := [l_i, r_i]$. We have a bijection $\sigma_i \colon \{1, 2, \ldots, i\} \to \{1, 2, \ldots, i\}$ (it is unique up to permutation of equal intervals, which order are set by Algorithm) such that $l_{\sigma_i(1)} \leqslant l_{\sigma_i(2)} \leqslant \ldots \leqslant l_{\sigma_i(i)}$ and $r_{\sigma_i(1)} \leqslant r_{\sigma_i(2)} \leqslant \ldots \leqslant r_{\sigma_i(i)}$ such that $\sigma_i|_{\{1,2,\ldots,i-1\}} = \sigma_{i-1}$. We assume that vertexes $v_1, v_2, \ldots, v_{i-1}$ are already colored by Algorithm. Now Algorithm checks colors of v_i for $m - 1$ indexes to the left and $m - 1$ to the right in the above order and chooses one not used in that range. Therefore every color appears at most every m'th element in the order.

Let us now assume that there is a color for which for the first time in the game, at the i_0'th step, the Hall's condition is not satisfied. In other word there exists the smallest $A := \{a_1, a_2, \dots, a_k\} \subseteq U_{i_0} := \{v_1, v_2, \dots, v_{i_0}\} \subseteq U$ colored by Algorithm into one color such that $k = \sharp A > \sharp\mathcal{N}(A)$ where $\mathcal{N}(A)$ is the neighborhood of A in G. Since A is the smallest such, then $\mathcal{N}(A)$ is an interval $[l_A, r_A]$. We assume moreover that the set A is indexed according to the order σ_{i_0}. Let $X := \{v \in U_{i_0} \colon a_1 \leqslant v \leqslant a_k\}$ then $\sharp X \geqslant m(k-1)+1$. We know that X is off-line properly colorable using m colors; it follows that there is a color using which $Y := \{y_1, y_2, \dots, y_k\} \subseteq X$ are colored. Now $\sharp Y = k = \sharp A > \sharp\mathcal{N}(A) \geqslant \sharp\mathcal{N}(Y)$ which gives a contradiction.

To show that m colors assures Presenter has a winning strategy we set $k := m-1$. Now Presenter reveals the following bipartite graph (in arbitrary order)

$$G := (\{v_1, v_2, \dots, v_k, w_1, w_2, \dots, w_k\}, \{1,2,3\}, \{\{v_i, 1\}, \{v_i, 2\}, \{w_i, 2\}, \{w_i, 3\}\}_{i=1,2,\dots,k}).$$

Algorithm chooses a coloring $c \colon \{v_1, v_2, \dots, v_k, w_1, w_2, \dots, w_k\} \to \{1, 2, \dots, m\}$. Depending on the choices made by Algorithm now Presenter do the following:

If Algorithm used one of the colors twice on the vertexes v_i (on vertexes w_i it goes similar) then Presenter reveals vertexes $u_1, u_2, \dots, u_m$. The resulting graph cannot be properly colored by Algorithm and is off-line properly colorable using m colors.

If $\operatorname{Im} c = \{1, 2, \dots, m\}$ (i.e. Algorithm already used all available colors) then a vertex a and an edge $\{a, 2\}$. We assume that Algorithm colored a into one of the colors he used on $\{v_1, v_2, \dots, v_k\}$. Presenter then reveals $u_1, u_2, \dots, u_m$ with edges $\{u_i, 1\}$. Now Algorithm cannot color all u_i properly and whereas the graph is properly colorable using k colors.

If $\operatorname{Im} c$ consists of k colors and none of the colors were used twice on the vertexes v_i nor on w_i then the vertexes v_i were colored using same k colors as w_i. Now Presenter reveals vertexes a_1, a_2 and edges $\{a_1, 1\}$, $\{a_2, 1\}$ and vertexes $w_1, w_2, \dots, w_m$ with edges $\{w_i, 3\}$. The resulting graph is properly colorable using k colors but cannot be properly colored by Algorithm. □

5.5. **Representable matroids.**

Definition 5.5.1. Let E be a finite set in the linear space V over the field $\mathbb{F}$. We call a matroid $\mathfrak{M}_E$ on the ground set E representable over the field $\mathbb{F}$ whenever $X \subseteq E$ is independent if and only if X is linearly independent in V.

We play the following two person game on a matroid representable over the field $\mathbb{F}$:

Game 13. *Assume we are given a vector space V over $\mathbb{F}$. Presenter is given a finite subset $E \subseteq V$ and the matroid $\mathfrak{M}_E$. We Assume Algorithm is given a set of available colors.*

At the i'th step Presenter reveals one of the elements e of the ground set $E \subseteq V$. Now Algorithm colors properly the element e such that the revealed part was colored properly in $\mathfrak{M}$.

Algorithm wins whenever he can properly color the whole matroid $\mathfrak{M}$. In the other case Presenter wins.

Definition 5.5.2. The smallest number of colors m that assures Algorithm a winning strategy in the above game is called on-line chromatic number of a representable matroid $\mathfrak{M}_E$ for a finite $E \subseteq V$ and is denoted by $\chi^{ol}_{rep}(\mathfrak{M})$.

Given an on-line variant of chromatic number we compare it with the standard off-line version.

Definition 5.5.3. We set

$H^{ol}_{rep}(n) := \max\{\chi^{ol}_{rep}(\mathfrak{M}_E)\colon \chi(\mathfrak{M}_E) = n,\ E$ is a finite subset of a vector space V over the field $\mathbb{F}\}$ and call it the on-line chromatic ratio for matroids representable over $\mathbb{F}$.

Theorem 5.5.4. *We have the following equalities:*

- $H^{ol}_{rep}(1) = 1$*;*
- $H^{ol}_{rep}(m) = \infty$ *for* $m > 1$*.*

Proof. Let us assume that V is a countable dimensional vector space with basis $\{e_i\}_{i\in\mathbb{N}}$ and that Algorithm has n_c available colors whereas at each step Presenter reveals elements of a matroid $\mathfrak{M}$ such that $\chi(\mathfrak{M}) = 2$.

We set $A^0_i := e_i$. We will describe the strategy $\mathcal{S}(n_c)$ for Presenter depending on the number of the available colors.

At the first step Presenter reveals vectors $A^0_1 + A^0_i$ until Algorithm colors two of them, lets say $B^0_{1,1} := A^0_1 + A^0_2$ and $B^0_{1,2} := A^0_1 + A^0_3$, into one of the available colors. Using the fact that these is a finite number of available colors and that Presenter can play on a seperate set of A^0_i's we may assume that Algorithm colored $\mathcal{B}^0 := \{B^0_{i,1}, B^0_{i,2}\}_{i=1,2,\dots,M_0}$ (all of them linearly independent in V) into one and the same color. Up to reindexing we may assume that $A^1_i := B^0_{i,1} - B^0_{i,2} = A^0_{3i+2} - A^0_{3i+3}$.

$$
\begin{array}{rl}
B^0_{1,1} = & (1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \dots)\\
B^0_{1,2} = & (1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \dots)\\
B^0_{2,1} = & (0,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ \dots)\\
B^0_{2,2} = & (0,\ 0,\ 0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ \dots)\\
B^0_{3,1} = & (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 0,\ \dots)\\
B^0_{3,2} = & (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,\ \dots)\\
B^1_{1,1} = & (0,\ 1,\ -1,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0,\ \dots)\\
B^1_{1,2} = & (0,\ 1,\ -1,\ 0,\ 0,\ 0,\ 0,\ 1,\ -1,\ \dots)
\end{array}
$$

FIGURE 11. First eight elements used in the strategy.

Now given a family $\{A^j_i\}_{j=1,2,\dots,N_j}$ Presenter reveals vectors $A^j_1 + A^j_i$ until Algorithm colors two of them, lets say $B^j_{1,1} := A^j_1 + A^j_2$ and $B^j_{1,2} := A^j_1 + A^j_3$, into the same of the available colors. Using the fact that these is a finite number of available colors and that we may assume that $N_j >> 0$ we may assume that Algorithm colored $\mathcal{B}^j := \{B^j_{i,1}, B^j_{i,2}\}_{i=1,2,\dots,M_j}$ into one and the same color. Up to reindexing we may assume that $A^{j+1}_i := B^j_{i,1} - B^j_{i,2} = A^j_{3i+2} - A^j_{3i+3}$. First eight elements used in the strategy were presented schematically in Figure 11.

Note that whenever Presenter knows the number n_c of available colors it may choose N_1 sa that n_c+1 steps of the above procedure was possible to perform. On the other hand for $j_0 > 0$ Algorithm colors the revealed vectors such that $\mathcal{B}^{j_0} := \{B^{j_0}_{i,1}, B^{j_0}_{i,2}\}_{i=1,2,\dots,M_{j_0}}$ is colored with one and the same color and it cannot use this color to color any of the vectors $B^j_{i,k}$ for $j > j_0$ as they are linearly dependent with the set $\mathcal{B}^{j_0}$. This implies that Algorithm needs $n_c + 1$'st color to color the set $\mathcal{B}^{n_c+1}$ which proves the statement. □

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