Wojciech Politarczyk

Uniwersytet A. Mickiewicza w Poznaniu

Dirac operators on compact manifolds

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Opiekun pracy: Stefan Jackowski

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Supervisor: prof. Stefan Jackowski

Wojciech Politarczyk
Adam Mickiewicz University in Poznań

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1 Introduction

The aim of this paper is to describe briefly the analytic properties of Dirac operators on compact manifolds. Dirac operator is a partial differential operator whose square is equal to the laplasian modulo the terms of lower order. These operators proved to be very important in mathematics.

Dirac operators appeared for the first time in the work of the physicist P.A.M. Dirac who was looking for an equation describing the behaviour of electrons. In his considerations he was led to the point were he was supposed to find the square root of the Klein-Gordon operator. He managed to solve this problem introducing a new vector valued partial differential operator called the Dirac operator. The vector valued functions which were acted upon by this operator are called spinors. Curiously the spinors are only defined up to a sign and a consistent choice of the sign led in the Diracs theory to the passage from the classical Lorentz group to its double cover.

The ideas of Dirac were adapted to the case of positive definite metrics. M.F. Atiyah and I.M. Singer realized that if the structure group of a manifold can be reduced to the simply connected group $\mathrm{Spin}(n)$ then Diracs construction can be carried out. In this way they obtained elliptic operator canonically associated to the given riemannian metric. Using this operator and its twisted versions they were led to a general formula for the index of arbitrary elliptic operator.

Applications of Dirac operators in geometry and topology are very diverse: group actions on manifolds, existence of positive scalar curvature metrics, immersion problems and computations of holonomy groups just to mention few of them. Many of these topics are described in [3]. What is more the theory of Dirac operators was used by E. Witten to solve a very important conjecture in general relativity namely the positive mass conjecture.

Another very important application of this theory is the Seiberg-Witten theory in low dimensional topology and geometry. Seiberg-Witten theory investigates topological properties of the space of solutions of non-linear perturbation of the Dirac operator on 4-manifold. This approach proved to be very fruitful. The importance of Seiberg-Witten invariants stems from their ability to distinguish different smooth structures on manifolds of dimension 4. These invariants were used to prove that the h-cobordism theorem is false in dimension 4. This discovery revealed the wildness of the world of 4-diemensional manifolds. See [2], [5], [6].

This paper due to its limited volume sketches only the most important parts of the theory of Dirac operators. Only in the fourth section, were we describe properties of Dirac operators, almost every theorem is proved. For a very detailed exposition of this topic see [3].

The paper is organized as follows. In the Section 2 we sketch the theory of Clifford algebras, spin groups and their representations. For more detailed account of this theory see [1], [2], [2]. Section 3 is concerned with applications of the material from Section 2 to geometry. The spin and $\mathrm{spin}^{\mathbb{C}}$structures, spinor bundles and spin connections are defined. Section 4 is the heart of this paper because it is gathers all the information from Section 3 to construct and describe properties of Dirac operators. In this section we used elements of global analysis which are sketched in Appendix A.

2 Clifford algebras, $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ groups.

In this section we gather all necessary information about Clifford algebras, $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ groups. These objects will play essential role in latter geometrical constructions.

2.1 Clifford algebras and their basic properties

Let (V, q) be a K-vector space, where $K = \mathbb{R}, \mathbb{C}$, and $q: V \to K$ be a quadratic form. Let $T(V)$ denote the tensor algebra of V and let I be a two-sided ideal in $T(V)$ generated by elements of the form $v \otimes v + q(v)$.

Definition 2.1.1. The **Clifford algebra** of (V, q) is the quotient algebra

$$\mathrm{C}\ell(V, q) = T(V)/I.$$

There exists a canonical injection

$$V \to T(V) \to T(V)/I = \mathrm{C}\ell(V, q).$$

Clifford algebras are characterized by the following universal property.

Proposition 2.1.2 (Universal property). *Suppose A is an associative K-algebra with unit and we are given a linear map*

$$f: V \to A,$$

with the property that for every $v \in V$ $f(v)^2 = -q(v)1$, then there exists a unique algebra homomorphism

$$\widetilde{f}: \mathrm{C}\ell(V, q) \to A$$

extending f.

Corollary 2.1.3. The Clifford algebra $\mathrm{C}\ell(V, q)$ is functorial in (V, q).

The universal property is used to construct two canonical linear automorphisms of the Clifford algebra. First consider an involution on $T(V)$ given by the formula

$$v_1 \otimes \ldots \otimes v_r \to v_r \otimes \ldots \otimes v_1.$$

Using the universal property 2.1.2 and this involution we can construct an anti-automorphism

$$(-)^t: \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q)$$

of the Clifford algebra called the **transpose**.

Let $\beta: V \to \mathrm{C}\ell(V, q)$ be a composition of multiplication by -1 and the canonical inclusion $V \hookrightarrow \mathrm{C}\ell(V, q)$. By the universal property β posses an extension to an involution

$$\alpha: \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q).$$

This involution defines $\mathbb{Z}/2$-grading

$$\mathrm{C}\ell(V,q) = \mathrm{C}\ell^0(V,q) \oplus \mathrm{C}\ell^1(V,q),$$

where $\mathrm{C}\ell^0(V,q)$ and $\mathrm{C}\ell^1(V,q)$ are ±1-eigenspaces of α.

Suppose that (V_1,q_1) and (V_2,q_2) are vector spaces equipped with quadratic forms. The direct sum $V_1 \oplus V_2$ posses a canonical quadratic form $q_1 \oplus q_2$. The next proposition gives the relation between Clifford algebras of (V_1,q_1), (V_2,q_2) and $(V_1 \oplus V_2, q_1 \oplus q_2)$.

Proposition 2.1.4. *There exists an isomorphism*

$$\mathrm{C}\ell(V_1 \oplus V_2, q_1 \oplus q_2) \to \mathrm{C}\ell(V_1,q_1) \widehat{\otimes} \mathrm{C}\ell(V_2,q_2),$$

where $\widehat{\otimes}$ *denotes the* $\mathbb{Z}/2$*-graded tensor product.*

Clifford algebras are closely related to the exterior algebras of the underlying vector space V. If $q = 0$, then $\mathrm{C}\ell(V,q) = \Lambda^* V$. Suppose now that q is a non-zero quadratic form. Let $r \in \mathbb{N}$ and consider a map

$$f_r \colon \prod_r V \to \mathrm{C}\ell(V,q)$$
$$f_r(v_1,\ldots,v_r) = \frac{1}{r!}\sum_{\sigma\in S_r} \mathrm{sign}(\sigma) v_{\sigma(1)} \ldots v_{\sigma(r)}.$$

this map is a well defined, injective and K-linear. Taking the sum of all maps f_r we obtain the following proposition.

Proposition 2.1.5. *For every* (V,q) *there exists a canonical isomorphism of vector spaces*

$$\Lambda^* V \to \mathrm{C}\ell(V,q).$$

Consider a pair $(\mathbb{R}^{r+s}, q_{r,s})$, where

$$q_{r,s} = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2.$$

Let $\mathrm{C}\ell_{r,s} = \mathrm{C}\ell(\mathbb{R}^{r+s}, q_{r,s})$. Theorems of elementary linear algebra imply that every real and non-degenerate quadratic form (V,q) is isomorphic to $(\mathbb{R}^{r+s}, q_{r,s})$ for some r and s, thus we can restrict our attention to considering only those quadratic forms. As a consequence every real Clifford algebra is isomorphic to $\mathrm{C}\ell_{r,s}$ for some r, s. From now on, we will use the following shortened notation $\mathrm{C}\ell_n = \mathrm{C}\ell_{n,0}$ and $\mathrm{C}\ell_n^* = \mathrm{C}\ell_{0,n}$.

In case of quadratic forms over complex numbers there is only one, up to isomorphism, non-degenerate quadratic form of rank n

$$q_n^{\mathbb{C}} = z_1^2 + \ldots + z_n^2.$$

So let us denote $\mathbb{C}\ell_n = \mathrm{C}\ell(\mathbb{C}^n, q_n^{\mathbb{C}})$. The following proposition is an easy consequence of the universal property of Clifford algebras.

Proposition 2.1.6. *For every $r, s \in \mathbb{N}$ there is an isomorphism*

$$\mathrm{C}\ell_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\ell_{r+s}.$$

Clifford algebras can also be described in terms of their generators and relations. The easiest way to give such a description is to choose the orthonormal basis and write down all relations coming from the generators of the ideal I.

Proposition 2.1.7. *If $e_1, \dots, e_n$ is an orthonormal basis of $\mathbb{R}^n$, then $\mathrm{C}\ell_n$ is generated as an algebra by these elements subject to the following relations*

$$e_i^2 = -1, \qquad e_i \cdot e_j = -e_j \cdot e_i, \text{ for } i \neq j, \quad i, j = 1, \dots, n.$$

The proposition 2.1.5 states that there is a canonical vector space isomorphism $\Lambda^*\mathbb{R}^n \to \mathrm{C}\ell_n$. Using this isomorphism we can write the formula for Clifford multiplication in terms of linear operations on the exterior algebra. First, we have to define the **interior product** in $\Lambda^*\mathbb{R}^n$. Let $v \in \mathbb{R}^n$ then the interior product by v is a linear map

$$v\llcorner \colon \Lambda^p\mathbb{R}^n \to \Lambda^{p-1}\mathbb{R}^n$$

given by the formula

$$v \llcorner (v_1 \wedge \dots \wedge v_p) = \sum_{i=1}^{p} (-1)^{i+1} \langle v, v_i \rangle v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_p,$$

where $\widehat{v_i}$ means that we omit this term in the wedge product.

Proposition 2.1.8. *For $v \in \mathbb{R}^n$ and $x \in \Lambda^*\mathbb{R}^n$ the Clifford multiplication by v is given by the following formula*

$$v \cdot x = v \wedge x - v \llcorner x.$$

2.2 Clifford algebras as matrix algebras

In the previous section it was proved that every Clifford algebra is isomorphic to $\mathrm{C}\ell_{r,s}$ for some positive integers r and s. This section is devoted to showing that Clifford algebras are isomorphic to matrix algebras. Proofs are omitted in this section. Reader can find them in [3].

First it is easy to check that there are isomorphisms.

$$\begin{array}{lll} \mathrm{C}\ell_1 \cong \mathbb{C}, & \mathrm{C}\ell_1^* \cong \mathbb{R} \oplus \mathbb{R}, & \mathbb{C}\ell_1 \cong \mathbb{C} \oplus \mathbb{C}, \\ \mathrm{C}\ell_2 \cong \mathbb{H}, & \mathrm{C}\ell_2^* \cong \mathbb{R}(2), & \mathbb{C}\ell_2 \cong \mathbb{C}(2), \end{array}$$

where $K(n)$ denotes the algebra on $n \times n$ matrices over the ring K.

Theorem 2.2.1. *For all $n \geq 1$ there are isomorphisms*

$$\mathrm{C}\ell_n \otimes \mathrm{C}\ell_2^* \cong \mathrm{C}\ell_{n+2}^*,$$
$$\mathrm{C}\ell_n^* \otimes \mathrm{C}\ell_n \cong \mathrm{C}\ell_{n+2}.$$

The theorem 2.2.1 enables us to describe $\mathrm{C}\ell_n$ and $\mathbb{C}\ell_n$ for every $n \geq 0$ as a matrix algebra. Aside from the aforementioned isomorphisms there is another very important consequence of this theorem. Namely there are certain periodicity isomorphisms between Clifford algebras. These isomorphisms are directly related to the Bott periodicity in real and complex K-theory.

Theorem 2.2.2. *There are isomorphisms*

$$\mathrm{C}\ell_n \otimes \mathrm{C}\ell_8 \cong \mathrm{C}\ell_{n+8},$$
$$\mathbb{C}\ell_n \otimes \mathbb{C}\ell_2 \cong \mathbb{C}\ell_{n+2}.$$

Corollary 2.2.3. For every $n \geq 0$ there are isomorphisms.

$$\begin{array}{ll}
\mathrm{C}\ell_{8n} \cong \mathbb{R}(16^n), & \mathrm{C}\ell_{8n+1} \cong \mathbb{C}(16^n), \\
\mathrm{C}\ell_{8n+2} \cong \mathbb{H}(16^n), & \mathrm{C}\ell_{8n+3} \cong \mathbb{H}(16^n) \oplus \mathbb{H}(16^n), \\
\mathrm{C}\ell_{8n+4} \cong \mathbb{H}(2 \cdot 16^n), & \mathrm{C}\ell_{8n+5} \cong \mathbb{C}(4 \cdot 16^n), \\
\mathrm{C}\ell_{8n+6} \cong \mathbb{R}(8 \cdot 16^n), & \mathrm{C}\ell_{8n+7} \cong \mathbb{R}(8 \cdot 16^n) \oplus \mathbb{R}(8 \cdot 16^n), \\
\mathbb{C}\ell_{2n} \cong \mathbb{C}(2^n), & \mathbb{C}\ell_{2n+1} \cong \mathbb{C}(2^n) \oplus \mathbb{C}(2^n).
\end{array}$$

Remark 2.2.4. Notice that Clifford algebras are semi-simple algebras i.e. they can be decomposed as a direct product of their minimal left ideals. This is the consequence of the fact that matrix algebras are simple and semi-simple. Moreover $\mathrm{C}\ell_n$ is simple in every dimension except for $\neq 8n+3, 8n+7$. The $\mathbb{C}\ell_n$ is simple for n even. These facts are essential in representation theory of Clifford algebras.

2.3 Representations of Clifford algebras

The aim of this paper is to describe the construction of Dirac operator on a compact manifold X. This operator acts on the set of sections of the so called spinor bundle over the manifold. Every fibre of the spinor bundle is equipped with a structure of a Clifford module over the Clifford algebra of the tangent space at the corresponding point. In order to perform all these constructions it is necessary to describe representations of Clifford algebras.

Definition 2.3.1. Suppose we are given a representation

$$\rho\colon \mathrm{C}\ell_n \to \mathrm{End}_{\mathbb{R}}(W),$$

i.e. ρ is a homomorphism of unital algebras. Then W is called a **Clifford module** and ρ is called **Clifford multiplication**. We say that W is **simple** Clifford module (ρ is an **irreducible representation**) if W does not contain any proper Clifford submodule (proper subrepresentation).

As was pointed out in remark 2.2.4 Clifford algebras are semi-simple, thus any Clifford module is semi-simple. In other words every Clifford module splits as a direct sum of simple Clifford modules. Consequently in order to obtain full classification of representations of Clifford algebra we need to describe irreducible representations.

Theorem 2.3.2. *Let K be a real division algebra. Then the only irreducible representation of $K(n)$ is the representation given by the identity homomorphism. The algebra $K(n) \oplus K(n)$ has two inequivalent irreducible representations given by the projection on the first and second factor.*

Let $\nu(\mathrm{C}\ell_n)$ denote the number of real and irreducible representations of $\mathrm{C}\ell_n$, $d(\mathrm{C}\ell_n)$ denote the real dimension of one of the real irreducible representation of $\mathrm{C}\ell_n$. Note that by the previous theorem and the classification of Clifford algebras the function d is well defined.

Theorem 2.3.3. *Let $n = 8k + r$, where $0 \le r < 8$. Then*

$$\nu(C\ell_n) = \begin{cases} 2, & r = 3, 7, \\ 1, & r = 0, 1, 2, 4, 5, 6. \end{cases} \qquad d(C\ell_n) = \begin{cases} 16^k, & r = 0, \\ 2 \cdot 16^k, & r = 1, \\ 4 \cdot 16^k, & r = 2, 3, \\ 8 \cdot 16^k, & r = 4, 5, 6, 7. \end{cases}$$

Definition 2.3.4. The representation ρ is **complex** if there exists a complex structure $J \in \mathrm{End}_{\mathbb{R}}(W)$, which commutes with ρ. The algebra $\mathrm{span}_{\mathbb{R}}(Id, J)$ is called a **commuting subalgebra** of ρ. Analogously, ρ is **quaternionic** if there exists a quaternionic structure $I, J, K \in \mathrm{End}_{\mathbb{R}}(W)$, which commutes with ρ. The commuting algebra is $\mathrm{span}_{\mathbb{R}}(Id, I, J, K)$.

Corollary 2.3.5. Let $n = 8k+r$, where $0 \le r < 8$. For $r = 2, 3, 4$ every irreducible representation of $\mathrm{C}\ell_n$ has natural quaternionic structure. For $r = 1, 5$ every irreducible Clifford module has a natural complex structure.

Representations of Clifford algebras are closely related to the representations of the **Clifford group**. The Clifford group $F_n \subset C\ell_n$ is the group generated by a fixed orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$. Let $x = e_1^2 \in F_n$. This element lies in the center of F_n. There is a canonical isomorphism.

$$\mathrm{C}\ell_n = \mathbb{R}F_n/(1 + x),$$

where $(1 + x)$ denotes the two-sided ideal generated by $1 + x$. In other words representations of $C\ell_n$ are exactly those representations

$$\rho \colon F_n \to \mathrm{GL}(\mathbb{R}^N)$$

such that $\rho(x) = \pm Id$. These considerations yields the corollary.

Corollary 2.3.6. If $\rho \colon \mathrm{C}\ell_n \to \mathrm{End}_{\mathbb{R}}(W)$ is a representation, and

$$L \subset \mathrm{End}_{\mathbb{R}}(W)$$

is the maximal commuting subalgebra, then there exists an L-invariant inner product $g \colon W \times W \to \mathbb{R}$ such that if $e \in \mathbb{R}^n$ and $|e| = 1$, then the Clifford multiplication by e is a g-orthogonal transformation. Furthermore for every $v \in \mathbb{R}^n$ the Clifford multiplication by v is skew-adjoint transformation with respect to g.

Proof. This is the consequence of the fact that every $\mathbb{R}, \mathbb{C}, \mathbb{H}$-representation of finite group is equivalent to orthogonal, unitary, symplectic representation. The skew-adjointness follows from the fact that the Clifford multiplication by elements of the basis is skew-adjoint. □

Let $\nu^{\mathbb{C}}(\mathbb{C}\ell_n)$ denote the number of irreducible complex representations of $\mathbb{C}\ell_n$ and let $d^{\mathbb{C}}(\mathbb{C}\ell_n)$ denote the complex dimension of the irreducible $\mathbb{C}\ell_n$-module.

Theorem 2.3.7. *Let $n = 2k + r$, where $0 \leq r \leq 1$. Then*

$$\nu^{\mathbb{C}}(\mathbb{C}\ell_n) = \begin{cases} 1, & r = 0, \\ 2, & r = 1, \end{cases} \qquad d^{\mathbb{C}}(\mathbb{C}\ell_n) = 2^k.$$

Now consider an element $\omega_n = e_1 \cdot \ldots \cdot e_n \in \mathrm{C}\ell_n$ called the **volume element**. For $n \equiv 0, 3 \pmod 4$ the following equality holds $\omega_n^2 = 1$. In these dimensions the volume element yields canonical decomposition of every Clifford module. Next propositions describe this decomposition.

Proposition 2.3.8. *Suppose that $n \equiv 0 \pmod 4$. Let V be a $\mathrm{C}\ell_n$-module. Then there is a decomposition $V = (1+\omega_n)V \oplus (1-\omega_n)V$ into vector subspaces. Furthermore if $v \in \mathbb{R}^n$ with $q(v) \neq 0$ then the Clifford multiplication by v interchanges summands in the decomposition.*

Proof. This is the consequence of the fact that $\omega_n^2 = 1$ and for every $v \in \mathbb{R}^n$ we have $v \cdot \omega_n = -\omega_n \cdot v$. □

Proposition 2.3.9. *Let W be an irreducible $\mathrm{C}\ell_n$-module, where $n \equiv 0 \pmod 4$. Then there is a decomposition $W = (1+\omega_n)W \oplus (1-\omega_n)W$. These subspaces are invariant under the action of $\mathrm{C}\ell_n^0 \cong \mathrm{C}\ell_{n-1}$ and correspond to the two distinct irreducible representations of $\mathrm{C}\ell_{n-1}$.*

Proposition 2.3.10. *Suppose that $n \equiv 3 \pmod 4$. Then there exists a decomposition $\mathrm{C}\ell_n = \mathrm{C}\ell_n^+ \oplus \mathrm{C}\ell_n^-$, where $\mathrm{C}\ell_n^\pm = (1 \pm \omega_n)\,\mathrm{C}\ell_n$ are two-sided ideals. Furthermore $\mathrm{C}\ell_n^+$ and $\mathrm{C}\ell_n^-$ are isomorphic subalgebras and $\alpha(\mathrm{C}\ell_n^\pm) = \mathrm{C}\ell_n^\mp$.*

Proof. For n odd we have that ω_n is central, so the eigenspaces of the multiplication by ω_n are ideals in $\mathrm{C}\ell_n$, and direct summands by semi-simplicity. Furthermore $\alpha(\omega_n) = -\omega_n$, so α interchanges summands of the direst sum. □

Analogously we can define the **complex volume element**

$$\omega_n^{\mathbb{C}} = i^{\left[\frac{n+1}{2}\right]} e_1 \cdot \ldots \cdot e_n.$$

For odd n the $\omega_n^{\mathbb{C}}$ is central thus we obtain decomposition

$$\mathbb{C}\ell_n = \mathbb{C}\ell_n^+ \oplus \mathbb{C}\ell_n^-$$

as in proposition 2.3.10.

Remark 2.3.11. From the classification of Clifford algebras we know that for $n \equiv 3 \pmod 4$ algebra $C\ell_n$ decomposes into a direct sum of two isomorphic two-sided ideals. In fact this decomposition is exactly the decomposition from proposition 2.3.10. Analogously in case of complex Clifford algebras, the decomposition of $\mathbb{C}\ell_n$ for odd n is the decomposition given by the complex volume element.

2.4 Spin groups and their representations

Let $C\ell_n^\times$ denote the group of units of the Clifford algebra. The $C\ell_n^\times$ acts on the Clifford algebra. This action gives rise to the **adjoint representation**

$$\mathrm{Ad}\colon C\ell_n^\times \to \mathrm{Aut}(C\ell_n),$$

where $Ad(x)y = xyx^{-1}$. Now define the **twisted adjoint representation**

$$\begin{aligned} \widetilde{\mathrm{Ad}}\colon C\ell_n^\times &\to \mathrm{Aut}(C\ell_n) \\ \widetilde{\mathrm{Ad}}(x)y &= \alpha(x)yx^{-1}, \end{aligned}$$

where α is the involution defined in subsection 2.1.

Proposition 2.4.1. *Let $v \in \mathbb{R}^n \subset C\ell_n$. Then*

$$\widetilde{\mathrm{Ad}}(v)\mathbb{R}^n \subset \mathbb{R}^n.$$

Furthermore $\widetilde{\mathrm{Ad}}(v)|_{\mathbb{R}^n}$ is a reflection across $v^\perp$.

Proof. Let $e_1 = \frac{v}{\sqrt{q(v)}}$. We can choose vectors $e_2, \dots, e_n \in \mathbb{R}^n$ such that $e_1, \dots, e_n$ is an orthonormal basis. Then

$$\begin{aligned} \widetilde{\mathrm{Ad}}(v)e_1 &= -e_1, \\ \widetilde{\mathrm{Ad}}(v)e_i &= e_i, \qquad i \geq 2. \end{aligned}$$

□

Definition 2.4.2. The $\mathrm{Pin}(n)$ group is the subgroup of $C\ell_n^\times$ generated by the elements $v \in \mathbb{R}^n$ such that $q(v) = 1$. The $\mathrm{Spin}(n)$ group is defined as $\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap C\ell_n^0$. These groups can be described as

$$\begin{aligned} \mathrm{Pin}(n) &= \{v_1 \cdot \ldots \cdot v_l \colon q(v_1) = \ldots = g(v_l) = 1\} \\ \mathrm{Spin}(n) &= \{v_1 \cdot \ldots \cdot v_{2k} \colon q(v_1) = \ldots = q(v_{2k}) = 1\}. \end{aligned}$$

Using the proposition 2.4.1 we obtain representations of $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$

$$\begin{aligned} \widetilde{\mathrm{Ad}}\colon \mathrm{Pin}(n) &\to O(n) \\ \widetilde{\mathrm{Ad}}\colon \mathrm{Spin}(n) &\to SO(n). \end{aligned}$$

The image of $\mathrm{Spin}(n)$ is contained in $SO(n)$, because every element of $\mathrm{Spin}(n)$ is mapped to product of even number of reflections.

Proposition 2.4.3. *Both reprentations* $\widetilde{\mathrm{Ad}}|_{\mathrm{Pin}(n)}$ *and* $\widetilde{\mathrm{Ad}}|_{\mathrm{Spin}(n)}$ *are surjective.*

Proof. Let $A \in O(n)$ then there is an orthonormal basis $e_1, \ldots, e_{2k}, e_{2k+1}, \ldots, e_r$ such that A rotates the linear subspace generated by e_{2i-1}, e_{2i}, through an angle θ_i for $i \leq k$ and $Ae_j = \pm e_j$ for $j > 2k$. Since every rotation of $\mathbb{R}^2$ is a product of two reflections, thus we see that $O(n)$ is generated by reflections, which yields the desired result. □

Proposition 2.4.4. *There are short exact sequences*

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Pin}(n) \xrightarrow{\widetilde{\mathrm{Ad}}} O(n) \longrightarrow 1$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\widetilde{\mathrm{Ad}}} SO(n) \longrightarrow 1$$

Furthermore $\mathrm{Spin}(n)$ *is connected for* $n \geq 2$ *and simply-connected for* $n \geq 3$.

Proof. The exactness of the above sequences follows from the fact that

$$\ker \widetilde{\mathrm{Ad}} = \mathbb{R}^\times \subset \mathbb{R}^n \subset \mathrm{C}\ell_n,$$

see [3, Prop. 2.4], because $\mathrm{Pin}(n) \cap \mathbb{R}^n = \mathrm{Spin}(n) \cap \mathbb{R}^n = \{1, -1\}$.

To prove that $\mathrm{Spin}(n)$ is connected it is sufficient to prove that there exists a path connecting 1 and -1. Let $e_1, e_2 \in \mathbb{R}^n$ be two orthogonal vectors of norm 1. Then

$$\gamma(t) = (e_1 \cos(t) + e_2 \sin(t))(e_1 \cos(t) - e_2 \sin(t))$$

is the desired path.

For $n \geq 3$ we have $\pi_1(SO(n)) = \mathbb{Z}/2$ and $\widetilde{\mathrm{Ad}}$ is nontrivial double covering of $SO(n)$ with connected covering space, thus it has to be the universal covering. □

Definition 2.4.5. Let S_n be the irreducible $\mathrm{C}\ell_n$-module. Using the inclusion $\mathrm{Spin}(n) \subset \mathrm{C}\ell_n$ we obtain the **real spinor representation**

$$\Delta_n \colon \mathrm{Spin}(n) \to \mathrm{GL}(S_n).$$

Proposition 2.4.6. *For* $n \equiv 3 \pmod 4$ *the definition of spinor representation does not depend on the chosen irreducible module. For* $n \equiv 3, 5, 6, 7 \pmod 8$ *the representation* Δ_n *is irreducible. For* $n \equiv 1, 2 \pmod 8$ *the representation* Δ_n *splits into a direct sum of two equivalent irreducible representations.*

Proof. Let $n \equiv 3 \pmod 4$. From the proposition 2.3.10 we know that α interchanges the summands of the decomposition. It is easy to check that $\mathrm{C}\ell_n^0 = \{x + \alpha(x) \colon x \in \mathrm{C}\ell_n\}$. This implies that

$$\mathrm{C}\ell_n^0 = \{(x, \alpha(x)) \in \mathrm{C}\ell_n^+ \oplus \mathrm{C}\ell_n^- \colon x \in \mathrm{C}\ell_n^+\}.$$

Thus α is an isomorphism between two $\mathrm{C}\ell_n^0$-modules $\mathrm{C}\ell_n^+$ and $\mathrm{C}\ell_n^-$. In consequence α is also an isomorphism between two representations of $\mathrm{Spin}(n)$.

For $n \equiv 3, 5, 6, 7 \pmod 8$ Δ_n is irreducible, because if W is irreducible $C\ell_n$-module, then it is also irreducible as an $C\ell^0_{n-1}$-module due to dimensional reasons.

For $n \equiv 1, 2 \pmod 8$ dimension counting shows, that any $C\ell^0_n$ module splits as a direct summand of two isomorphic and irreucible $C\ell^0_n$-modules. □

Definition 2.4.7. Let $S^{\mathbb{C}}_n$ be an irreducible $\mathbb{C}\ell_n$-module. Then the **complex spinor representation**

$$\Delta^{\mathbb{C}}_n : \mathrm{Spin}(n) \to \mathrm{GL}(S^{\mathbb{C}}_n)$$

is a representation obtained by

$$\mathrm{Spin}(n) \subset C\ell^0_n \subset \mathbb{C}\ell_n.$$

Proposition 2.4.8. *For n odd $\Delta^{\mathbb{C}}_n$ does not depend on which irreducible $\mathbb{C}\ell_n$ module we choose. For n even $\Delta^{\mathbb{C}}_n$ splits into a direct sum*

$$\Delta^{\mathbb{C}}_n = \Delta^{\mathbb{C}^+}_n \oplus \Delta^{\mathbb{C}^-}_n.$$

of two inequivalent irreducible complex representations of $\mathrm{Spin}(n)$.

The $\mathrm{Spin}(n)$ group is a compact Lie group, which is a Lie subgroup of $C\ell^\times_n$. The Lie algebra of $C\ell^\times_n$ is just $C\ell_n$ with the standard Lie bracket. The Lie algebra of $\mathrm{Spin}(n)$ is a Lie subalgebra of $C\ell_n$. The next proposition describes this Lie algebra.

Proposition 2.4.9. *Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ and let $\underline{\mathrm{spin}}(n)$ be a Lie algebra of* $\mathrm{Spin}(n)$. *Then*

$$\underline{\mathrm{spin}}(n) = \mathrm{span}\{e_i \cdot e_j : 1 \le i < j \le n\} = \Lambda^2\mathbb{R}^n,$$

where we identify $\Lambda^2\mathbb{R}^n$ with its image by the canonical isomorphism

$$\Lambda^*\mathbb{R}^n \to C\ell_n$$

from prop. 2.1.5.

Proof. Consider a curve

$$y(t) = (e_i \cos t + e_j \sin t)(-e_i \cos t + e_j \sin t) = \cos 2t + e_i \cdot e_j \sin 2t.$$

We have $y(t) \in \mathrm{Spin}(n)$ for all t, $y(t) = 1$ and $y'(t) = 2e_i \cdot e_j$. Thus

$$\Lambda^2\mathbb{R}^n \subset \underline{\mathrm{spin}}(n).$$

Comparing the dimensions of $\mathrm{Spin}(n)$ (which is equal to the dimension of $SO(n)$) and $\underline{\mathrm{spin}}(n)$ we obtain the equality. □

The Lie algebra $\underline{\mathrm{so}}(n)$ of $SO(n)$ consists of skew-symmetric matrices. There exists a natural isomorphism $\Lambda^2\mathbb{R}^n \to \underline{\mathrm{so}}(n)$, where for $v, w \in \mathbb{R}^n$ the linear map $v \wedge w$ is defined by the following formula.

$$v \wedge w(x) = \langle v, x\rangle w - \langle w, x\rangle v, \tag{1}$$

here $\langle\cdot,\cdot\rangle$ denotes the scalar product on $\mathbb{R}^n$. The homomorphism given by the twisted adjoint representation yields an isomorphism of Lie algebras

$$\widetilde{\mathrm{Ad}}_*\colon \underline{\mathrm{spin}}(n) \to \underline{\mathrm{so}}(n).$$

Proposition 2.4.10. *The homomorphism* $\widetilde{\mathrm{Ad}}_*$ *is given by*

$$\widetilde{\mathrm{Ad}}_*(e_i \cdot e_j) = 2e_i \wedge e_j.$$

As a consequence, for $v, w \in \mathbb{R}^n$ *we have that*

$$\widetilde{\mathrm{Ad}}_*^{-1}(v \wedge w) = \frac{1}{4}[v, w] = \frac{1}{4}\left(v \cdot w - w \cdot v\right).$$

2.5 spin$^{\mathbb{C}}$groups

The group $\mathrm{Spin}(n)$ is contained in the complex Clifford algebra together with the group $S^1 \subset \mathbb{C}\ell_n$.

Definition 2.5.1. The group $\mathrm{Spin}^{\mathbb{C}}(n)$ is the subgroup of $\mathbb{C}\ell_n^\times$ generated by $\mathrm{Spin}(n)$ and S^1.

It is easy to check, that $\mathrm{Spin}(n) \cap S^1 = \{\pm 1\} \subset \mathbb{C}\ell_n$, thus we obtain an isomorphism

$$\mathrm{Spin}^{\mathbb{C}}(n) \cong \mathrm{Spin}(n) \times_{\{\pm 1\}} S^1.$$

The $\mathrm{Spin}^{\mathbb{C}}(n)$ group admits a surjective homomorphism

$$\mathrm{Spin}^{\mathbb{C}}(n) \xrightarrow{\widetilde{\mathrm{Ad}}_{\mathbb{C}}} SO(n) \times S^1$$

$$[g, z] \longmapsto (\widetilde{\mathrm{Ad}} g, z^2)$$

Proposition 2.5.2. *For* $n \geq 3$ *the fundamental group* $\pi_1(\mathrm{Spin}^{\mathbb{C}}(n))$ *is isomorphic to* $\mathbb{Z}$*. Let* $\alpha \in \pi_1(SO(n))$ *and* $\beta \in \pi_1(S^1)$ *be generators. If* $\gamma \in \pi_1(\mathrm{Spin}^{\mathbb{C}}(n))$ *generates the fundamental group, then*

$$(\widetilde{\mathrm{Ad}}_{\mathbb{C}})_*(\gamma) = \alpha + \beta.$$

Proof. To prove the first part consider the following exact sequence of Lie groups

$$1 \to \mathrm{Spin}(n) \to \mathrm{Spin}^{\mathbb{C}}(n) \to S^1 \to 1.$$

The above sequence is in fact a fibration sequence, thus we can apply long exact sequence of homotopy groups of a fibration which yields the desired result. □

Consider the following homomorphism of Lie groups

$$f: U(n) \to SO(2n) \times S^1, \qquad f(A) = (A, \det A).$$

Using the proposition 2.5.2 and covering space theory it is easy to prove the following proposition.

Proposition 2.5.3. *The homomorphism $f: U(n) \to SO(2n) \times S^1$ has a lifting to the homomorphism $g: U(n) \to \mathrm{Spin}^{\mathbb{C}}(2n)$.*

Using Clifford multiplication we obtain spinor representations of the group $\mathrm{Spin}^{\mathbb{C}}(n)$

$$\Delta_n^{\mathbb{C}}: \mathrm{Spin}^{\mathbb{C}}(n) \to GL(S_n^{\mathbb{C}}).$$

In fact complex spinor representations of $\mathrm{Spin}(n)$ extends to the complex spinor representations of $\mathrm{Spin}^{\mathbb{C}}(n)$. For n odd this representation is well defined, because the two irreducible $\mathbb{C}\ell_n$ representations are isomorphic when restricted to $\mathrm{Spin}^{\mathbb{C}}(n)$.

3 Spin and spin$^{\mathbb{C}}$structures

3.1 Spin structures

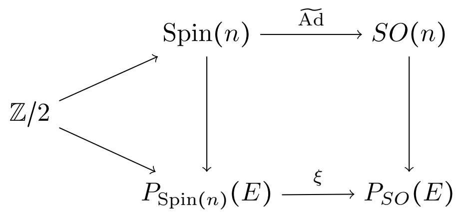
Our goal is to construct the Dirac operator on a compact manifold. In order to do that we need to construct a bundle of Clifford algebras and then construct a bundles of irreducible Clifford modules. It turns out that the first construction is quite easy to perform, while there are certain geometric obstructions which make the second one impossible in general. These geometric obstruction are related to the structure group of the tangent bundle of manifold. The aim of this section is to investigate these obstructions and describe the constructions of bundles of Clifford algebras and Clifford modules.

Definition 3.1.1. Let $n \geq 3$ and $E \to X$ be an n-dimensional oriented vector bundle equipped with a riemannian metric. Let $P_{SO}(E)$ be an associated principal $SO(n)$-bundle. A spin structure on E is a $\mathrm{Spin}(n)$-principal bundle $P_{\mathrm{Spin}(n)}(E)$ with a 2-sheeted covering

$$\xi: P_{\mathrm{Spin}(n)}(E) \to P_{SO}(E)$$

such that for all $p \in P_{\mathrm{Spin}(n)}$ and $g \in \mathrm{Spin}(n)$ we have $\xi(p \cdot g) = \xi(p)\widetilde{\mathrm{Ad}}(g)$.

It is easy to see, that if one restrict covering map ξ to any fiber of $P_{\mathrm{Spin}(n)}$, then it is exactly the universal covering of $SO(n)$. Furthermore if there exists a 2-sheeted covering $P_{\mathrm{Spin}(n)} \to P_{SO}(E)$, which is non-trivial when restricted to the fibres, such that the following diagram commutes.



Then using the covering space theory one can lift the action of $SO(n)$ on $P_{SO}(E)$ to the action of $\mathrm{Spin}(n)$ on $P_{\mathrm{Spin}(n)}$, thus making $P_{\mathrm{Spin}(n)}$ into a principal $\mathrm{Spin}(n)$-bundle over X. We conclude the following.

Theorem 3.1.2. *Let E be an n-dimensional, oriented and riemannian vector bundle. The spin structures on E are in natural bijection with 2-sheeted coverings of $P_{SO}(E)$ which restrict to non-trivial coverings on fibres $P_{SO}(E)_x$.*

In other words spin structures on E are in natural bijection with elements of $H^1(P_{SO}(E);\mathbb{Z}/2)$ whose restriction to fibre is non-trivial.

Serre spectral sequence associates to a fibration $SO(n) \to P_{SO}(E) \to X$ an exact sequence

$$0 \longrightarrow H^1(X,\mathbb{Z}/2) \longrightarrow H^1(P_{SO}(E),\mathbb{Z}/2) \xrightarrow{i^*} H^1(SO(n),\mathbb{Z}/2) \xrightarrow{w_E} H^2(X,\mathbb{Z}/2)$$

Using the naturality of the above sequence one can deduce that $w_E(g) = w_2(E)$, where g is a generator of $H^1(SO(n);\mathbb{Z}/2)$ and $w_2(E)$ is the second Stiefel-Whitney class of E.

Theorem 3.1.3. *Let E be an oriented, riemannian vector bundle of rank ≥ 3. Then E posses a spin structure if and only if $w_2(E) = 0$. Furthermore there exists a free and transitive action of $H^1(X;\mathbb{Z}/2)$ on the set of spin structures of E.*

Proof. There exists a spin structure on E if and only if there exists an element $x \in H^1(P_{SO}(E);\mathbb{Z}/2)$ such that $i^*(x) = g$. Exactness of the above sequence implies that $w_E(g) = 0$ is equivalent to the existence of the spin structure on E. If $x \in H^1(P_{SO}(E);\mathbb{Z}/2)$ is an element corresponding to a spin structure on E, then exactness of the above sequence shows that every element $y \in H^1(P_{SO}(E);\mathbb{Z}/2)$ such that $x - y \in H^1(X,\mathbb{Z}/2) \subset H^1(P_{SO}(E),\mathbb{Z}/2)$ also corresponds to a spin structure on E. □

Definition 3.1.4. Let X be a smooth, compact manifold of dimension ≥ 3. We say that X is a **spin manifold** if there exists a spin structure on the tangent bundle TX. In the remaining part of this paper, to make things shorter, a spin structure on manifold X will mean the spin structure on TX.

If X is a manifold with boundary ∂X, then every spin structure on X yields a spin structure on ∂X. Using the outward normal vector field on the boundary we get an embedding $P_{SO}(\partial X) \hookrightarrow P_{SO}(X)$. Now we can restrict the associated covering to $P_{SO}(\partial X)$.

Definition 3.1.5. Let X be a closed, smooth and spin manifold with fixed spin structure. We say that X is a spin boundary if there exists a spin manifold W with spin structure such that $X = \partial W$ and the spin structure on X is exactly the spin structure induced from W.

Analogously as in case of oriented manifolds one can construct a spin bordism ring Ω_*^{spin}. The oriented bordism class of X is determined by the Stiefel-Whitney and Pontriagin numbers. The spin bordism class of X is determined by the Stiefel-Whitney and KO-characteristic numbers (see [7]).

3.2 spin$^{\mathbb{C}}$structures

Definition 3.2.1. Let $P_{SO}(E)$ be the principal $SO(n)$ bundle associated to E. A spin$^{\mathbb{C}}$structure on E consists of a principal S^1 bundle P over X and a principal $\mathrm{Spin}^{\mathbb{C}}(n)$ bundle $P_{\mathrm{Spin}^{\mathbb{C}}(n)}$ over X together with a two folded equivariant covering map

$$\xi: P_{\mathrm{Spin}^{\mathbb{C}}(n)} \to P_{SO}(E) \tilde{\times} P,$$

where $P_{SO}(E) \tilde{\times} P$ denotes the fibre product of two principal bundles. The class $c \in H^2(X, \mathbb{Z})$ corresponding to the bundle P is called the canonical class of the spin$^{\mathbb{C}}$structure. The complex line bundle $P \times_{S^1} \mathbb{C}$ is called the determinant line bundle.

Consider the exact sequence

$$0 \to \mathbb{Z}/2 \to \mathrm{Spin}^{\mathbb{C}}(n) \to SO(n) \times S^1 \to 1,$$

which yields the following exact sequence of cohomology sets

$$H^1(X, \mathrm{Spin}^{\mathbb{C}}(n)) \to H^1(X, SO(n)) \times H^1(X, S^1) \xrightarrow{\partial} H^2(X, \mathbb{Z}/2),$$

where $\partial(P_{SO}, P) = w_2(P_{SO}) + \overline{c}(P)$. Here $\overline{c}(P)$ is the $\pmod 2$ reduction of the first Chern class $c_1(P)$. Recall that $H^1(X, S^1) \cong H^2(X, \mathbb{Z})$ and the isomorphism is given by the first Chern class. Consequently the boundary homomorphism can be expressed by the formula

$$\partial: H^1(X, SO(n)) \times H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}/2), \qquad \partial = w_2 + \rho_2,$$

where $\rho_2: H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}/2)$ is the $\pmod 2$ reduction.

Proposition 3.2.2. *Let E be oriented and riemannian vector bundle over the compact riemannian manifold X. The bundle E posses a* spin$^{\mathbb{C}}$*structure if and only if $w_2(E)$ is the* $\pmod 2$ *reduction of the integral cohomology class.*

Remark 3.2.3. There exists a commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}^{\mathbb{C}}(n) & \longrightarrow & \mathrm{Spin}(n+2) \\ \downarrow & & \downarrow \\ SO(n)\times S^1 & \longrightarrow & SO(n+2) \end{array}$$

where vertical arrows are canonical double coverings. Thus $E \oplus L$ possess a spin structure if and only if it is associated to the principal $\mathrm{Spin}^{\mathbb{C}}(n)$ bundle.

Definition 3.2.4. An oriented riemannian manifold is a spin$^{\mathbb{C}}$manifold if its tangent bundle possess a spin$^{\mathbb{C}}$structure.

Corollary 3.2.5. An oriented riemannian manifold X is a spin$^{\mathbb{C}}$manifold iff $w_2(X)$ is the (mod 2) reduction of the integral cohomology class.

Since every closed orientable 3-manifold is paralelizable it follows that every manifold of dimension 3 possess a spin structure and in particular spin$^{\mathbb{C}}$structure. In dimension 4 the situation is slightly different because not every oriented 4-manifold has a spin structure. Complex projective plane $\mathbb{C}P^2$ is an example of such manifold. Next theorem deals with the problem of existence of a spin$^{\mathbb{C}}$structure on arbitrary 4-manifold.

Theorem 3.2.6 (Hirzebruch, Hopf)**.** *Every closed and oriented* 4*-manifold possess a* spin$^{\mathbb{C}}$*structure.*

Proof. Poincaré duality implies that the torsion subgroups of $H^2(M,\mathbb{Z})$, $H_2(M,\mathbb{Z})$ and $H^3(M,\mathbb{Z})$ are isomorphic. Thus let T denote this torsion subgroup. Let us first prove the following equality

$$\mathrm{im}(r) = \{x \in H^2(M,\mathbb{Z}/2) \colon \forall_{y \in r(T)} x \cup y = 0\},$$

where r denotes the mod 2 reduction homomorphism. The inclusion

$$\mathrm{im}(r) \subset \{x \in H^2(M,\mathbb{Z}/2) \colon \forall_{y \in r(T)} x \cup y = 0\}$$

is obvious, the proof is completed by showing the equality

$$\dim_{\mathbb{Z}/2} \mathrm{im}(r) = \dim_{\mathbb{Z}/2}\{x \in H^2(M,\mathbb{Z}/2) \colon \forall_{y \in r(T)} x \cup y = 0\}. \tag{2}$$

Poincaré duality implies that

$$\dim_{\mathbb{Z}/2}\{x \in H^2(M,\mathbb{Z}/2) \colon \forall_{y \in r(T)} x \cup y = 0\} = \dim_{\mathbb{Z}/2} H^2(M,\mathbb{Z}/2) - \dim_{\mathbb{Z}/2} r(T).$$

Consider now the exact sequence

$$\cdots \longrightarrow H^2(M,\mathbb{Z}) \xrightarrow{\times 2} H^2(M,\mathbb{Z}) \xrightarrow{r} H^2(M,\mathbb{Z}/2) \xrightarrow{\beta} H^3(M,\mathbb{Z}) \xrightarrow{\times 2} \cdots$$

From this sequence we conclude that

$$\dim_{\mathbb{Z}/2} H^2(M, \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} \operatorname{im}(r) + \dim_{\mathbb{Z}/2} \operatorname{im}(\beta).$$

Furthermore $\operatorname{im}(\beta) = \{x \in T : 2x = 0\}$ and $r(T) = T/2T$. Exactness of the sequence

$$0 \longrightarrow \{x \in T : 2x = 0\} \longrightarrow T \xrightarrow{\times 2} T \longrightarrow T/2T \longrightarrow 0$$

shows that

$$\dim_{\mathbb{Z}/2} \operatorname{im}(\beta) = \dim_{\mathbb{Z}/2}\{x \in T : 2x = 0\} = \dim_{\mathbb{Z}/2} T/2T = \dim_{\mathbb{Z}/2} r(T).$$

This establishes the equality (2). To proceed further we will use the Wu formula.

Theorem 3.2.7 (Wu)**.** *Let X be a smooth closed connected and orientable 4-manifold. Then for every $x \in H^2(X, \mathbb{Z}/2)$ the following equality holds*

$$x \cup x = w_2(X) \cup x.$$

Choose $x \in r(T)$, then $w_2(X) \cup x = x \cup x = 0$. Consequently $w_2(X) \in \operatorname{im}(r)$ which completes the proof. □

3.3 Spinor bundles

Let $\rho_n : SO(n) \to \operatorname{Aut}(\mathbb{R}^n)$ be the standard representation of $SO(n)$. By the universal property 2.1.2 we obtain another representation

$$\mathrm{C}\ell(\rho_n) : SO(n) \to \operatorname{Aut}(\mathrm{C}\ell_n).$$

Definition 3.3.1. Let $E \to X$ be an oriented riemannian vector bundle. Then the Clifford bundle of E is the bundle of algebras

$$\mathrm{C}\ell(E) = P_{SO}(E) \times_{\mathrm{C}\ell(\rho_n)} \mathrm{C}\ell_n .$$

Using the naturality of the construction we easily obtain an automorphism $\alpha : \mathrm{C}\ell(E) \to \mathrm{C}\ell(E)$ which yields a $\mathbb{Z}/2$-grading on $\mathrm{C}\ell(E)$. Analogously we can construct an isomorphism of vector bundles

$$\Lambda^* E \to \mathrm{C}\ell(E).$$

At this point it is natural to ask for the bundles which are irreducible $\mathrm{C}\ell(E)$-modules. Although in general such bundles do not exist if E possess a spin structure it is possible to construct such objects.

Definition 3.3.2. Let $E \to X$ be an oriented and riemannian vector bundle with spin structure

$$\xi : P_{Spin}(E) \to P_{SO}(E).$$

A real spinor bundle is the bundle

$$P_{Spin}(E) \times_\mu M,$$

where M is a left $C\ell_n$-module and $\mu: \mathrm{Spin}(n) \to SO(M)$ is the representation given by the multiplication from the left by elements from $\mathrm{Spin}(n)$.

Analogously one can define a complex spinor bundle

$$S_{\mathbb{C}}(E) = P_{Spin}(E) \times_\mu M_{\mathbb{C}},$$

where $M_{\mathbb{C}}$ is a left $C\ell_n \otimes \mathbb{C}$-module.

Remark 3.3.3. When a manifold X possess a spin$^{\mathbb{C}}$structure then using the associated principal $\mathrm{Spin}^{\mathbb{C}}(n)$-bundle one can construct bundles of complex Clifford algebras $\mathbb{C}\ell_E$ and complex Clifford modules.

Proposition 3.3.4. *Let $S(E)$ be a real or complex spinor bundle. Then $S(E)$ is a bundle of modules over the bundle of algebras $C\ell(E)$. Consequently the set of sections of $S(E)$ are in natural way a module over the algebra of sections of $C\ell(E)$.*

Proof. The only thing to check is the equivariance of the multiplication by elements from $C\ell(E)$. Observe that the proposition follows from the commutativity of the following diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}(E) \times C\ell_n \times M & \longrightarrow & P_{\mathrm{Spin}(n)}(E) \times M \\ \downarrow & & \downarrow \\ P_{\mathrm{Spin}(n)}(E) \times C\ell_n \times M & \longrightarrow & P_{\mathrm{Spin}(n)}(E) \times M \end{array}$$

$$\begin{array}{ccc} (p, \phi, m) & \longrightarrow & (p, \phi \cdot m) \\ \downarrow & & \downarrow \\ (pg^{-1}, gpg^{-1}, gm) & \longrightarrow & (pg^{-1}, g\phi m) \end{array}$$

□

Definition 3.3.5. If $S(E)_1$ and $S(E)_2$ are two real or complex spinor bundles of E then they are equivalent if they are equivalent as bundles of $C\ell(E)$-modules. A spinor bundle $S(E)$ is irreducible if every fiber $S(E)_x$ is irreducible $C\ell(E)_x$-module.

Next proposition follows easily from remark 2.2.4.

Proposition 3.3.6. *Every spinor bundle (real, complex, $\mathbb{Z}/2$-graded) splits as a direct sum of irreducible Clifford bundles. Furthermore when X is connected the classification of irreducible Clifford bundles follows from theorems 2.3.3 and 2.3.7.*

Naturality of the construction of the spinor bundle implies that any complex or quaternionic structure on the Clifford module passes to a parallel (with respect to the spin connection to be defined later) complex or quaternionic structure on the spinor bundle.

Proposition 3.3.7. *Let E be an n-dimensional oriented and riemannian vector bundle. If $n \equiv 1, 5 \pmod 8$, then any real spinor bundle carries a complex struture such that Clifford multiplication is complex linear. If $n \equiv 2, 3, 4 \pmod 8$ then every real spinor bundle carries natural quaternionic structure such that Clifford multiplication is quaternionic linear.*

Volume elements defined in the paragraph 2.3 enables us to construct a section of every real or complex Clifford bundle. This section has the property that its restriction to every fibre is the volume element. We will denote this section by ω_E or $\omega_E^{\mathbb{C}}$ and call it the volume element. Using ω_E and $\omega_E^{\mathbb{C}}$ it is easy to obtain the splitting of spinor bundles analogously as in the case of representations.

Existence of the spin structure on the vector bundle implies that it is orientable with respect to the real K-theory. We shall sketch the construction of the universal KO-Thom class. Suppose that E is an oriented riemannian vector bundle with a spin structure. Define $D(E)$ to be the set of vectors of length less or equal to one. Analogously let $\partial D(E)$ be the set of vectors of length one. Consider the universal spin vector bundle $E_{8k} \to BSpin(8k)$. The volume element ω_E yields the splitting of the associated spinor bundle

$$S(E_{8k}) = S^+(E_{8k}) \oplus S^-(E_{8k}).$$

Using the bundle projection $\pi: D(E_{8k}) \to BSpin(8k)$ we can pull back both bundles $S^+(E_{8k})$ and $S^-(E_{8k})$. If $e \in \partial D(E_{8k})$ then the Clifford multiplication by e gives an isomorphism of the fibers

$$\mu(e): \pi^* S^+(E_{8k})_e \to \pi^* S^-(E_{8k})_e.$$

Thus

$$\eta(E_{8k}) = [\pi^* S^+(E_{8k}), \pi^* S^-(E_{8k}), \mu] \in KO(MSpin(8k))$$

where $MSpin(8k)$ is the Thom space of E_{8k}. The class $\eta(E_{8k})$ is the universal Thom class. Using this class we can define the characteristic number of every spin manifold X. By the Pontriagin-Thom construction a spin bordism class $[X]$ of X determines a map $f_X: S^{n+8k} \to MSpin(8k)$. Then we define

$$\mathcal{A}_n([X]) = f_X^* \eta(E_{8k}) \in \widetilde{KO}(S^{8k+n}) = \widetilde{KO}(S^n) = KO^{-n}(*).$$

This characteristic number yields a homomorphism of graded rings

$$\mathcal{A}_*: \Omega_*^{Spin} \to KO^{-*}(*).$$

The homomorphism $\mathcal{A}$ can be expressed analitically as the analytic index of the special differential operator defined on the spin manifold X. For details consult [3, ch. 2 par. 7]. This homomorphism turns out to be important in the theory of transformation groups.

Theorem 3.3.8 ([3])**.** *Let X be a compact, closed and spin manifold of dimension $4k$. Suppose that X admits an effective action of S^1, then $\mathcal{A}_{4k}([X]) = 0$.*

Theorem 3.3.9 ([3])**.** *Let X be a compact and spin manifold with effective action of S^3, then $\mathcal{A}_n([X]) = 0$.*

Thus $\mathcal{A}_*$ is the obstruction to the existence of a nontrivial group actions on a spin manifold. Notice that if a compact manifold admits an action of a compact Lie group of positive dimension then it also admits an action of the circle group S^1 (restrict the action to the maximal torus). Thus if X is a compact spin manifold of dimension $4k$ with $\mathcal{A}_{4k}([X]) \neq 0$, then X is very asymmetric because the only compact Lie groups that can act on X are finite groups. If the dimension is not divisible by 4 and $\mathcal{A}_*([X]) \neq 0$ then the only compact Lie groups that can act on X are tori.

Apart from being the obstruction to the existence of effective actions of compact Lie groups the map $\mathcal{A}$ plays a fundamental role in deciding whether a compact spin manifold admits a riemannian metric of positive scalar curvature.

Theorem 3.3.10 ([3])**.** *Let X be a compact spin manifold equipped with a metric of positive scalar curvature, then $\mathcal{A}_*([X]) = 0$.*

Let Θ_n be the group of n-dimensional homotopy spheres. Deep results of Milnor and Adams implies that for $n \equiv 1, 2, \pmod 0$ the homomorphism

$$\mathcal{A}_n\colon \Theta_n \to \mathbb{Z}/2$$

is surjective. According the the above theorem in these dimensions there are homotopy spheres which do not admit positive scalar curvature metrics. We can now state consequence of this fact.

Theorem 3.3.11. *Let X be a compact spin manifold of dimension $n \equiv 1, 2 \pmod 8$, $n > 8$. Then X is homeomorphic to a manifold which does not admit any metric of positive scalar curvature.*

3.4 Spin connections

Let $E \to X$ be a riemannian vector bundle over a compact manifold X. Suppose that E is equipped with a riemannian connection. Let ω be the $\underline{\mathrm{so}}(n)$- valued connection 1-form on $P_{SO}(E)$. Using the derivative

$$\mathrm{C}\ell(\rho_n)_*\colon \underline{\mathrm{so}}(n) \to \mathrm{Der}(\mathrm{C}\ell_n)$$

of the representation $\mathrm{C}\ell(\rho_n)$ we can construct a connection form $\omega_{\mathrm{C}\ell(E)}$ on $\mathrm{C}\ell(E)$. Here $\mathrm{Der}(\mathrm{C}\ell_n)$ denotes the space of derivations of $\mathrm{C}\ell_n$.

Proposition 3.4.1. *Let $\nabla^{\mathrm{C}\ell(E)}$ be a covariant derivative on $\mathrm{C}\ell(E)$ induced by the connection form $\omega_{\mathrm{C}\ell(E)}$. Then $\nabla^{\mathrm{C}\ell(E)}$ acts as a derivation on the sections of $\mathrm{C}\ell(E)$. Furthermore after the identification $\mathrm{C}\ell(E) \cong \Lambda^*(E)$ connection $\nabla^{\mathrm{C}\ell(E)}$ becomes the standard connection induced by the exterior power $\Lambda^p \rho_n$ of the representation ρ_n.*

Proof. This is a simple consequence of the properties of the representation $\mathrm{C}\ell(\rho_n)$. □

Corollary 3.4.2. The subbundles $\mathrm{C}\ell(E)^0$ and $\mathrm{C}\ell(E)^1$ are preserved by $\nabla^{\mathrm{C}\ell(E)}$. Furthermore the volume element ω_E is globally parallel, that is

$$\nabla\omega_E = 0$$

and subbundles $\mathrm{C}\ell(E)^{\pm}$ are preserved by $\nabla^{\mathrm{C}\ell(E)}$ if they are defined.

Let $P_{Spin}(E) \to P_{SO}(E)$ be a spin structure on E. Then there is an associated spinor bundle $S(E) = P_{Spin}(E) \times_\mu M$. The connection ω on $P_{SO}(E)$ lifts to a connection ω_{spin} on $P_{Spin}(E)$. Then the form ω_{spin} induces a connection ω_S on $S(E)$ by the representation $\mu_*: \underline{\mathrm{spin}}(n) \to \underline{\mathrm{so}}(M)$.

Proposition 3.4.3. *Let ∇^S be a covariant derivative induced by the form ω_S. Then ∇^S acts as a derivative with respect to the module structure. In other words*

$$\nabla^S(\phi\cdot\sigma) = \nabla^{\mathrm{C}\ell(E)}(\phi)\cdot\sigma + \phi\cdot(\nabla^S\sigma).$$

Proof. The representation μ preserves the $\mathrm{C}\ell_n$-module structure on M in the sense that

$$\mu(g)(\phi\cdot\sigma) = (\mathrm{C}\ell(\rho_n)(g)\phi)\cdot(\mu(g)\sigma)\,,$$

where $\phi \in \mathrm{C}\ell_n$ and $\sigma \in M$. Taking derivative of this identity at $g = Id$ we obtain that for every $A \in \underline{\mathrm{spin}}(n)$ $\mu_*(A)$ acts as a derivative with respect to the $\mathrm{C}\ell_n$-module structure. Thus the same holds for the covariant derivative ∇^S. □

Corollary 3.4.4. For $n \equiv 3, 4 \pmod 4$ the eigenbundles

$$S^{\pm}(E) = \{\phi \in S(E): \omega_E\cdot\phi = \pm\phi\}$$

are preserved by the covariant derivative ∇^S.

Proof. This is the consequence of the corollary 3.4.2 and proposition 3.4.3. □

Sometimes it is inevitable to perform computations in local coordinates. Thus it is desirable to have explicit formulas for the connection ∇^S. Let $p \in X$ and let U be an open neighbourhood of p such that there exists an orthonormal frame $e_1, \dots, e_n$ over U. The covariant derivative ∇ on $P_{SO}(E)$ is given by the following formula with respect to the chosen orthonormal frame.

$$\nabla = d + \sum_{1\le i<j\le n} \omega_{ij} e_i \wedge e_j,$$

where $e_i \wedge e_j$ is an element of $\underline{\mathrm{so}}(n)$ defined by the formula

$$(e_i \wedge e_j)(v) = \langle v, e_i\rangle e_j - \langle v, e_j\rangle e_i.$$

Proposition 3.4.5. *Let ∇^S be a covariant derivative on the spinor bundle $S(E)$. Then locally ∇^S is given by the following formula.*

$$\nabla^S = d + \frac{1}{2} \sum_{1 \le i < j \le n} \omega_{ij} e_i \cdot e_j .$$

Proof. This is the consequence of the prop. 2.4.10. □

Suppose that X is a spin manifold. Then by the fundamental theorem of riemannian geometry there exists a canonical connection on TX, the Levi-Civita connection. Thus using the construction described above it is easy to see that the Levi-Civita connection yields a canonical connections on the Clifford bundle $C\ell(X)$ and the spinor bundle $S(X)$ associated to TX. In the remaining part of this paper we will always use these canonical connections on $C\ell(X)$ and $S(X)$ without mentioning it.

Now suppose that X is a spin$^{\mathbb{C}}$manifold. In order to introduce a connection on the complex spinor bundle on X we need more than just a Levi-Civita connection. Let L be the canonical complex line bundle of the spin$^{\mathbb{C}}$structure. Let A be a connection on the principal S^1-bundle P of L. Now consider bundle $Q = P_{SO}(E) \widetilde{\times} P$, which is the fiber product of P and $P_{SO}(E)$. Using the Levi-Civita connection and fixed connection A we can introduce connection on Q. Then we can introduce a connection on the principal $\mathrm{Spin}^{\mathbb{C}}(n)$-bundle P_{Spinc} by lifting connection from Q to P_{Spinc} by the double covering.

Let ∇^S be the covariant derivative on the complex spinor bundle $S(E)$ associated to the connection on the principal $\mathrm{Spin}^{\mathbb{C}}(n)$-bundle Q. Using similar arguments as in the spin case one can show, that ∇^S acts as a module derivation. Analogously one can obtain the following local formula for ∇^S.

$$\nabla^S = d + \frac{1}{2} \sum_{1 \le i < j \le n} \omega_{ij} e_i \cdot e_j + \frac{1}{2} A,$$

where ω is the Levi Civita connection on X and A denotes the Clifford multiplication by the 1-form associated to the connection A.

4 Dirac operator

4.1 Definition and elementary properties of Dirac operator

Definition 4.1.1. Let X be a closed, riemannian manifold and let S be a bundle of left modules over real or complex Clifford bundle. Suppose that S is equipped with riemannian (hermitian) metric h and metric connection ∇^S. Furtheromore suppose that Clifford multiplication is skew-adjoint with respect to h and ∇^S is a module derivation. Then the triple (S, h, ∇^S) is called the real or complex Dirac bundle.

Example 4.1.2. Every spinor bundle $S(E)$ posses a riemannian or hermitian metric h such that the triple $(S(E), h, \nabla^{S(E)})$ is a Dirac bundle. The existence of h is the consequence of the fact that $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ act by isometries on Clifford modules.

Remark 4.1.3. In the remaining part of this paper we will denote the riemannian or hermitian metric on the spinor bundle by $\langle \cdot, \cdot \rangle$.

Definition 4.1.4. Let S be a Dirac bundle over X and let $\Gamma(S)$ denote the set of smooth sections of S. The Dirac operator $\mathcal{D}: \Gamma(S) \to \Gamma(S)$ is a first order partial differential operator given by the formula

$$\mathcal{D} = c \circ \nabla^S$$

where c is the Clifford multplication

$$c: \Gamma(T^*X \otimes S) = \Gamma(TX \otimes S) \to S.$$

The second order operator $\mathcal{D}^2$ is called the Dirac laplacian.

Remark 4.1.5. The basic theory of partial differential operators on manifolds is sketched in appendix A.

Let $e_1, \dots, e_n$ be a local orthonormal frame. Then

$$\nabla^S = \sum_{k=1}^{n} e^k \otimes \nabla^S_{e_k},$$

where $e^1, \dots, e^n$ denotes the dual coframe. Thus locally the Dirac operator is given by the following formula

$$\mathcal{D}\psi = \sum_{k=1}^{n} e_k \cdot \nabla^S_{e_k} \psi.$$

A straightforward consequence if this formula is the following lemma.

Lemma 4.1.6. For any $f \in C^\infty(X)$ and every $\psi \in \Gamma(S)$ we have

$$\mathcal{D}(f\psi) = \mathrm{grad}(f) \cdot \psi + f\mathcal{D}\psi,$$

where $\mathrm{grad}(f)$ is a gradient vector field of f.

Proof.

$$\begin{aligned} \mathcal{D}(f\psi) &= \sum_k e_k \cdot \nabla^S_{e_k}(f\psi) = \sum_k e_k \cdot ((e_k f)\psi + f\nabla^S_{e_k}\psi) = \\ &= \sum_k (e_k f) e_k \cdot \psi + f \sum_k e_k \cdot \nabla^S_{e_k}\psi = \mathrm{grad}(f) \cdot \psi + f\mathcal{D}\psi. \end{aligned}$$

□

Proposition 4.1.7. *For any covector ξ the symbol of $\mathcal{D}$ and $\mathcal{D}^2$ is given by the following formula*

$$\sigma(\mathcal{D})(\xi) = \xi\cdot,$$
$$\sigma(\mathcal{D}^2)(\xi) = -|\xi|^2$$

where $\xi\cdot$ denotes the Clifford multiplication by the covector ξ. In consequence D and D^2 are elliptic operators.

Proof. This is the consequence of the lemma 4.1.6 and metric duality between TX and T^*X. □

Remark 4.1.8. In fact one can define a Dirac operator to be any first order partial differential operator such that its square is a generalized laplasian (for the definition see A). It is straightforward to check that the symbol of this operator defines the Clifford multiplication on the bundle.

Proposition 4.1.9. *The Dirac operator $\mathcal{D}$ is formally self adjoint.*

Proof.

$$\begin{aligned}\langle \mathcal{D}\psi_1, \psi_2\rangle &= \sum_k \langle e_k \cdot \nabla^S_{e_k}\psi_1, \psi_2\rangle = -\sum_k \langle \nabla^S_{e_k}\psi_1, e_k \cdot \psi_2\rangle = \\ &= -\sum_k \left(e_k\langle \psi_1, e_k\cdot\psi_2\rangle - \langle \psi_1, (\nabla_{e_k} e_k)\psi_2\rangle - \langle \psi_1, e_k \cdot \nabla^S_{e_k}\psi_2\rangle \right)\end{aligned}$$

Now let X be a vector field defined by the condition $\langle X, Y\rangle = -\langle \psi_1, Y\cdot\psi_2\rangle$ for all $Y \in \Gamma(TM)$.

$$\begin{aligned} divX &= \sum_k \langle e_k, \nabla_{e_k} X\rangle = \\ &= \sum_k \left(e_k\langle e_k, X\rangle - \langle \nabla_{e_k} e_k, X\rangle\right) = \\ &= -\sum_k e_k\langle \psi_1, e_k\cdot\psi_2\rangle + \sum_k \langle \psi_1, \nabla_{e_k} e_k \cdot \psi_2\rangle. \end{aligned}$$

This yields the equality

$$\langle \mathcal{D}\psi_1, \psi_2\rangle = divX + \langle \psi_1, \mathcal{D}\psi_2\rangle.$$

The proof will be finished once we use the following theorem.

Theorem 4.1.10 (Divergence Theorem)**.** *thm Let (M, g) be a connected and oriented riemannian manifold. Let X be a smooth vector field on M. Then the following equality holds*

$$\int_M divX \,\mathrm{dvol}_g = \int_{\partial M} g(X, \nu)$$

where ν is the outer normal vector field on ∂X.

□

Proposition 4.1.7 implies that the square of the Dirac operator is a generalized laplacian. There is another natural generalized laplasian $\left(\nabla^S\right)^* \nabla^S$. It is reasonable to hope that there exists Weitzenböck formula relating these two operators. Following theorem confirms this prediction.

Theorem 4.1.11 (Lichnerowicz formula)**.** *Let R denote the scalar curvature of the* spin$^{\mathbb{C}}$*manifold X and let dA be the imaginary-valued curvature* 2*-form of the unitary connection A on the determinant bundle. Then*

$$\mathcal{D}^2\psi = (\nabla^S)^*\nabla^S\psi + \frac{R}{4}\psi + \frac{1}{2}dA \cdot \psi$$

where $(\nabla^S)^*$ denotes the formal adjoint of the covariant derivative.

The above formula plays crucial role in establishing analytic properties of $\mathcal{D}$. Of course one can use the general theory of elliptic operators to obtain these results, however we would like to omit going into this complicated theory.

The Lichnerewicz formula is extremely important in many applications of Dirac operators. One example is the so called Bochner technique. We will elaborate on this a bit more in section 4.3.

4.2 Analytic properties of Dirac operator

The Dirac operator was defined as an operator acting on smooth sections of the Dirac bundle S. However we can complete $\Gamma(S)$ with respect to the L^2-norm

$$||\psi||_2^2 = \langle\psi,\psi\rangle_2 = \int_X \langle\psi,\psi\rangle \,\mathrm{dvol}_g$$

and obtain a Hilbert space $L^2(S)$. Dirac operator D has a natural extension to the symmetric unbounded operator

$$\mathcal{D}: L^2(S) \to L^2(S)$$

with domain $\Gamma(S) \subset L^2(S)$.

Lemma 4.2.1. The Dirac operator $\mathcal{D}$ possess a closure

$$\overline{\mathcal{D}}: \mathrm{dom}(\overline{\mathcal{D}}) \subset L^2(S) \to L^2(S).$$

Proof. It is sufficient to prove that if $(\psi_n)_n$ is a sequence of smooth spinors such that $\psi_n \to 0$ and $\mathcal{D}\psi_n \to \psi_0$ as $n \to \infty$, then $\psi_0 = 0$. Choose $\phi \in \Gamma(S)$.

$$\int_X \langle\psi_0,\phi\rangle \,\mathrm{dvol}_g = \lim_{n\to\infty}\int_X \langle\mathcal{D}\psi_n,\phi\rangle \,\mathrm{dvol}_g = \lim_{n\to\infty}\int_X \langle\psi_n,\mathcal{D}\phi\rangle \,\mathrm{dvol}_g = 0.$$

Thus $\psi_0 = 0$, because $\Gamma(S)$ is dense in $L^2(S)$. □

Theorem 4.2.2. *The closure $\overline{\mathcal{D}}$ of a Dirac operator $\mathcal{D}$ on a closed riemannian manifold X is self-adjoint.*

Remark 4.2.3. The above theorem is true for any complete manifold, see [3, thm. 5.7.].

Lemma 4.2.4. Suppose that $\mathcal{D}^*$ is the adjoint of the Dirac operator $\mathcal{D}$. Define the norm N on the domain of $\mathcal{D}^*$ $\mathrm{dom}(\mathcal{D}^*)$ by the formula

$$N(\psi) = \sqrt{\|\psi\|_2^2 + \|\mathcal{D}^*\psi\|_2^2}.$$

If $\Gamma(S) \subset \mathrm{dom}(\mathcal{D}^*)$ is dense with respect to N, then $\overline{\mathcal{D}}$ is self-adjoint.

Proof. The operator $\mathcal{D}$ is symmetric, thus it is sufficient to prove that $\mathrm{dom}(\mathcal{D}^*) \subset \mathrm{dom}(\overline{\mathcal{D}})$. Let $\psi \in \mathrm{dom}(\mathcal{D}^*)$ and let $(\psi_n)_n$ be a sequence of smooth sections of S such that

$$\lim_{n\to\infty} N(\psi - \psi_n) = 0.$$

Thus $\lim_n \mathcal{D}\psi_n = \mathcal{D}^*\psi$ in $L^2(S)$. $\mathcal{D}$ is essentially self-adjoint hence $\mathcal{D}^*\psi_n = \mathcal{D}\psi_n$. In consequence the sequence $\mathcal{D}\psi_n$ is convergent in $L^2(S)$ and $\psi \in \mathrm{dom}(\mathcal{D})$. □

Proof of thm. 4.2.2 (Sketch). From the lemma 4.2.4 it is sufficient to prove that $\Gamma(S)$ is dense in $\mathrm{dom}(\mathcal{D}^*)$ with respect to the norm N. Choose an open covering $V_1, \dots, V_r$ of X by open balls such that $S|_{V_k}$ is trivial for each k and let $\psi_1, \dots, \psi_r$ be a partition of unity subordinate to the chosen covering. Choose $\sigma \in \mathrm{dom}(\mathcal{D}^*)$ and denote $\sigma_k = \psi_k\sigma$. Then σ_k has support conatained in V_k, so it can be treated as a map

$$\sigma_k : V_k \to \mathbb{R}^{\dim S}.$$

Now using the convolution with a sequence of bump functions we can approximate σ_k in the norm N with a sequence of smooth sections $(\sigma_k^n)_n$. Then the sequence $\sigma_n = \sum_{k=1}^r \sigma_k^n$ approximates σ in the norm N. □

Self-adjointness has a remarkable consequence which is described by the following lemma.

Lemma 4.2.5. Let $\overline{\mathcal{D}}$ denote the closure of the Dirac operator. Then $\ker \overline{\mathcal{D}}^2 = \ker \overline{\mathcal{D}}$.

Proof. Let $\sigma \in \ker \overline{\mathcal{D}}^2$, then

$$0 = \langle \overline{\mathcal{D}}^2\sigma, \sigma\rangle_2 = \langle \overline{\mathcal{D}}\sigma, \overline{\mathcal{D}}\sigma\rangle_2 = \|\overline{\mathcal{D}}\sigma\|_2.$$

Thus $\sigma \in \ker \overline{\mathcal{D}}$. □

As was mentioned earlier the Lichnerowicz formula 4.1.11 can be used to establish basic analytic properties of Dirac operator. The remaining part of this section is devoted to stating and proving these theorems. However before doing that we have to introduce some function spaces.

For $k \in \mathbb{Z}_+$ and $p \geq 1$ consider the norm

$$\|\psi\|_{k,p} = \left(\int_X \left(|\psi|^p + |\nabla^S \psi|^p + \cdots + |\left(\nabla^S\right)^k \psi|^p \right) \mathrm{dvol}_g \right)^{\frac{1}{p}}$$

on $\Gamma(S)$. The completion of $\Gamma(S)$ with respect to $\|\cdot\|_{k,p}$ is denoted by $L^{k,p}(S)$ and is called the Sobolev space. The Sobolev spaces are essential in global analysis, especially in the theory of elliptic partial differential operators. In this paper we will be mainly interested in the case $k = 1$ and $p = 2$.

From the Lichnerowicz formula we obtain

$$\|\mathcal{D}\psi\|_2^2 = \|\nabla^S \psi\|_2^2 + \frac{1}{4}\|R\psi\|_2^2 + \int_X \frac{1}{2}\langle dA \cdot \psi, \psi\rangle.$$

Clifford multiplication by 1-forms is skew-adjoint which implies that the Clifford multiplication by 2-forms is also skew-adjoint. Hence

$$\langle dA \cdot \psi, \psi\rangle$$

is always a real number. Consequently there exists a constant $C > 0$ such that

$$-C\|\psi\|_2^2 \leq \frac{1}{2}\langle dA \cdot \psi, \psi\rangle_2 \leq C\|\psi\|_2^2.$$

for all ψ.

Proposition 4.2.6. *Let R_{max} and R_{min} denote the maximal and minimal value of the scalar curvature function R on X. Then the following inequalities hold.*

$$\|\psi\|_{1,2}^2 + \left(\frac{R_{min}}{4} - C - 1\right)\|\psi\|_2^2 \leq \|D\psi\|_2^2 \leq \|\psi\|_{1,2}^2 + \left(\frac{R_{max}}{4} + C - 1\right)\|\psi\|_2^2.$$

Proposition 4.2.7. *The domain of the closure $\overline{\mathcal{D}}$ is precisely $L^{1,2}(S)$.*

Proof. The inclusion $L^{1,2} \subset \mathrm{dom}(\overline{\mathcal{D}})$ is obvious.

Let $\psi \in \mathrm{dom}\,\overline{\mathcal{D}}$. Then there exists a sequence $(\psi_n) \subset \Gamma(S)$, which converge to ψ is $L^2(S)$ and $\mathcal{D}\psi_n$ is also convergent in $L^2(S)$. Using the inequalities 4.2.6 we deduce that ψ_n is a Cauchy sequence in $L^{1,2}(S)$. Using the continuity of the inclusion $L^{1,2}(S) \hookrightarrow L^2(S)$, we obtain that $\psi \in L^{1,2}(S)$. □

Proposition 4.2.8. *The Dirac operator $\overline{\mathcal{D}}$ is bounded when restricted to $L^{1,2}(S)$.*

Proof. This is a straightforward consequence of the inequalities (4.2.6). □

We will now consider the spectral properties of $\mathcal{D}$. The Dirac operator is a self-adjoint operator so its residual spectrum is empty. Thus $\sigma(\overline{\mathcal{D}})$ is equal to the approximate point spectrum. In fact more is true.

Proposition 4.2.9. *The spectrum of the Dirac operator consists only of eigenvalues.*

Proof. Let λ be an approximate eigenvalue. Then there exists a sequence of smooth spinor fields $(\psi_n)_n$ such that $\|\psi_n\|_2 = 1$ and

$$\lim_{n\to\infty} \|\overline{\mathcal{D}}\psi_n - \lambda\psi_n\|_2^2 = 0.$$

From the inequality (4.2.6) we obtain the following estimate.

$$\begin{aligned}\|\psi_n\|_{1,2} &\leq \|\overline{\mathcal{D}}\psi_n\|_2^2 + \left|\frac{R_{\text{min}}}{4} - c - 1\right| \|\psi_n\|_2^2 \leq \\ &\leq \left(\|\overline{\mathcal{D}}\psi_n - \lambda\psi_n\|_2 + \lambda\|\psi_n\|_2\right)^2 + \left|\frac{R_{\text{min}}}{4} - c - 1\right| \leq \\ &\leq 2\|\overline{\mathcal{D}}\psi_n - \lambda\psi_n\|_2^2 + 2\lambda^2 + \left|\frac{R_{\text{min}}}{4} - c - 1\right|\end{aligned}$$

Now we can use the fact that λ is an approximate eigenvalue and conclude that the sequence $(\psi_n)_n$ is bounded in $L^{1,2}(S)$. By the Rellich lemma [REFERENCJE] the embedding $L^{1,2}(S) \hookrightarrow L^2(S)$ is compact, thus there exists a spinor field ψ such that (after passing to a subsequence) $\psi_n \to \psi$ as $n \to \infty$ in $L^2(S)$. Consequently the sequence $(\overline{\mathcal{D}}\psi_n)$ is convergent in $L^2(S)$. The operator $\overline{\mathcal{D}}$ is closed thus $\overline{\mathcal{D}}\psi = \lambda\psi$. Consequently we obtain that λ is the eigenvalue of $\overline{\mathcal{D}}$. □

Proposition 4.2.10. *The spectrum $\sigma(\overline{\mathcal{D}})$ is an infinite, discrete set containing only real eigenvalues of $\overline{\mathcal{D}}$ each with finite multiplicity. Moreover there exists an orthonormal basis of $L^2(S)$ consisting of eigenvectors of the Dirac operator.*

Proof. From the last proposition and self-adjointness of $\overline{\mathcal{D}}$ we know that $\sigma(\overline{\mathcal{D}})$ consists of real eigenvalues. Suppose now that $\lambda \notin \sigma(\overline{\mathcal{D}})$. We will show that $\left(\overline{\mathcal{D}} - \lambda\right)^{-1}$ is a compact operator. The spectral theorem for compact operators will imply the desired result.

From the Lichnerowicz formula we obtain

$$\|(\overline{\mathcal{D}} - \lambda)^{-1}(\overline{\mathcal{D}} - \lambda)\psi\|_{1,2}^2 = \|\psi\|_{1,2}^2 \leq 2\|\overline{\mathcal{D}}\psi - \lambda\psi\|_2^2 + \left(2\lambda^2 + \left|\frac{R_{\text{min}}}{4} - c - 1\right|\right)\|\psi\|_2^2$$

Let $\varphi = (\overline{\mathcal{D}} - \lambda)\psi$. Then we have the following inequality

$$\|(\overline{\mathcal{D}} - \lambda)^{-1}\varphi\|_{1,2}^2 \leq \|\varphi\|_2^2 + C'\|(\overline{\mathcal{D}} - \lambda)^{-1}\varphi\|_2^2$$

for some $C' > 0$. The operator $(\overline{\mathcal{D}} - \lambda)^{-1}$ is continous in $L^2(S)$, thus there exists $C'' > 0$ such that

$$\|(\overline{\mathcal{D}} - \lambda)^{-1}\varphi\|_{1,2}^2 \leq C''\|\varphi\|_2^2.$$

In conclusion the image of $(\overline{\mathcal{D}} - \lambda)^{-1}$ is contained in $L^{1,2}(S)$, thus by the compactness of the embedding $L^{1,2}(S) \hookrightarrow L^2(S)$ we obtain that $(\overline{\mathcal{D}} - \lambda)^{-1}$ is a compact operator. By the spectral theorem there exists a sequence $(\psi_n)_n$ of eigenvectors of $\overline{\mathcal{D}}$, which is an orthonormal basis of $L^2(S)$. Every ψ_n is also an eigenvector of $\overline{\mathcal{D}}$ thus the Dirac operator is diagonal with respect to the basis $(\psi_n)_n$. □

Corollary 4.2.11 (Poincaré inequality)**.** There exists a constant $C > 0$ such that if ψ is a spinor field orthogonal to the kernel $\ker(\overline{\mathcal{D}})$, then we have the following inequality

$$\|\overline{\mathcal{D}}\psi\|_2 \geq C\|\psi\|_2.$$

Let now $d = \dim X$. Then

$$\|\overline{\mathcal{D}}\psi\|_2^2 = \int_X \left\langle \sum_i e_i \cdot \nabla_{e_i}\psi, \sum_j e_j \cdot \nabla_{e_j}\psi \right\rangle = \sum_{i,j} \int_X \left\langle e_i \cdot \nabla_{e_i}\psi, e_j \cdot \nabla_{e_j}\psi \right\rangle \leq$$

$$\leq \sum_{i,j} \int_X |\nabla_{e_i}\psi||\nabla_{e_j}\psi| \leq \frac{1}{2} \sum_{i,j} \int_X \left(|\nabla_{e_i}\psi|^2 + |\nabla_{e_j}\psi|^2\right) = d\|\nabla\psi\|_2^2.$$

Thus we obtain the following inequality

$$\frac{1}{d}\left(\|\psi\|_2^2 + \|\overline{\mathcal{D}}\psi\|_2^2\right) \leq \|\psi\|_{1,2}^2. \tag{3}$$

Following proposition is a consequence of the proposition (4.2.6) and (3).

Proposition 4.2.12. *The following two norms are equivalent on* $L^{1,2}(S)$

$$\|\psi\|_{1,2}, \qquad \sqrt{\|\psi\|_2^2 + \|\overline{\mathcal{D}}\psi\|_2^2}.$$

Corollary 4.2.13 (Elliptic estimate)**.** There exists a constant $C > 0$ such that for every $\psi \in L^{1,2}(S)$ the following inequality holds

$$\|\psi\|_{1,2} \leq C\left(\|\overline{\mathcal{D}}\psi\|_2 + \|\psi\|_2\right).$$

Remark 4.2.14. In fact if E and F are arbitrary vector bundles over X and $L\colon\Gamma(E) \to \Gamma(F)$ is any elliptic partial differential operator of order m, then for every $k \in \mathbb{Z}_+$ and $p \geq 1$ there exists a constant $C_{k,p,L} > 0$ such that for every $\psi \in L^p(S)$ we have

$$\|\psi\|_{k+m,p} \leq C_{k,p,L}\left(\|L\psi\|_{k,p} + \|\psi\|_p\right).$$

These inequalities are used to obtain very strong regularity results for solutions of elliptic partial differential operators. For example if $\psi \in L^p(E)$ such that $L\psi \in \Gamma(F)$, then $\psi \in \Gamma(E)$. In particular every eigenspace of L consists only of smooth sections of E. For more on elliptic regularity theory see [REFERENcJE].

Definition 4.2.15. Let $A\colon H \to H$ be a linear operator defined on a dense subspace of a Hilbert space H. The operator A is a Fredholm operator if its image is closed and the kernel and cokernel are finite dimensional. The index of A is defined as

$$\operatorname{ind} A = \dim\ker A - \dim\operatorname{coker} A.$$

Proposition 4.2.16. *The Dirac operator over a compact manifold is a Fredholm operator. That is the kernel and cokernel are finite-dimensional. Furthermore the index of* $\overline{\mathcal{D}}$ *is zero.*

Proof. Consider two subsets in $L^{1,2}(S)$.

$$K^1 = \{\psi \in L^{1,2}(S) : \overline{\mathcal{D}}\psi = 0, \|\psi\|_2 + \|\overline{\mathcal{D}}\psi\|_2 \le 1\},$$
$$K^0 = \{\psi \in L^{1,2}(S) : \overline{\mathcal{D}}\psi = 0, \|\psi\|_2 \le 1\}.$$

These sets are equal and the compactness of the embedding $L^{1,2}(S) \hookrightarrow L^2(S)$ yields the compactness of $\ker \overline{\mathcal{D}}$.

Now we will show that $\overline{\mathcal{D}}(L^{1,2}(S))$ is a closed subspace in $L^2(S)$. Let $(\psi_n)_n$ be a sequence of spinor fields such that $\overline{\mathcal{D}}\psi_n$ converges to ψ in L^2 norm. Without loss of generality we can assume that all ψ_n are orthogonal to $\ker(\overline{\mathcal{D}})$. From the Poincare inequality we obtain

$$\|\overline{\mathcal{D}}\psi_n\|_2 \ge C\|\psi_n\|_2.$$

Thus ψ_n is a Cauchy sequence in $L^2(S)$. From the inequality 4.2.6 we conclude that ψ_n is a Cauchy sequence in $L^{1,2}(S)$. Consequently the sequence ψ_n has a limit ψ_0 in $L^{1,2}(S)$. From the continuity of $\overline{\mathcal{D}}$ we obtain

$$\psi = \lim_{n\to\infty} \overline{\mathcal{D}}\psi_n = \overline{\mathcal{D}}(\lim_{n\to\infty} \psi_n) = \overline{\mathcal{D}}\psi_0.$$

Thus the image of $\overline{\mathcal{D}}$ is a closed subspace in $L^2(S)$. Finally we will prove that $\left(\operatorname{im}\overline{\mathcal{D}}\right)^\perp = \ker \overline{\mathcal{D}}$. The inclusion $\ker \overline{\mathcal{D}} \subset \left(\operatorname{im}\overline{\mathcal{D}}\right)^\perp$ is obvious, thus we will focus on the second inclusion. Let $\psi \in V = \left(\operatorname{im}\overline{\mathcal{D}}\right)^\perp \cap \Gamma(S)$, then for every $\phi \in L^{1,2}(S)$ we have

$$0 = \langle \psi, \overline{\mathcal{D}}\phi\rangle = \langle \overline{\mathcal{D}}\psi, \phi\rangle.$$

This yields the equation $\overline{\mathcal{D}}\psi = 0$. Thus $V \subset \ker\overline{\mathcal{D}}$. However notice that V is dense subspace in $\left(\operatorname{im}\overline{\mathcal{D}}\right)^\perp$. This impliest that

$$\left(\operatorname{im}\overline{\mathcal{D}}\right)^\perp = \overline{V} \subset \ker\overline{\mathcal{D}}.$$

The equality

$$\operatorname{coker}\overline{\mathcal{D}} \cong \left(\operatorname{im}\overline{\mathcal{D}}\right)^\perp = \ker\overline{\mathcal{D}}$$

implies that $\operatorname{ind}\overline{\mathcal{D}} = 0$, which finishes the proof. □

Corollary 4.2.17. There exists a splitting

$$L^2(S) = \ker\overline{\mathcal{D}} \oplus \operatorname{im}\overline{\mathcal{D}}.$$

Suppose now that the manifold X is even dimensional. Then we can consider Dirac operators

$$\overline{\mathcal{D}}^{\pm} : L^{1,2}(S^{\pm}) \to L^2(S^{\mp}).$$

The operator $\overline{\mathcal{D}}$ is self-adjoint, thus $(\overline{\mathcal{D}}^+)^* = \overline{\mathcal{D}}^-$. Both of these operators are Fredholm. By the celebrated Atiyah-Singer Index theorem the index of $\overline{\mathcal{D}}^+$

can expressed in terms of characteristic classes of X. Let $p_1, \dots, p_k$ denote the Pontryagin classes of X. Then there are classes $x_1, \dots, x_k \in H^2(X)$ such that

$$(1 + x_1^2) \dots (1 + x_k^2) = 1 + p_1 + \dots + p_k.$$

Define

$$\widehat{A}(X) = \prod_{i=1}^{k} \frac{x_i}{\exp \frac{x_i}{2} - \exp \frac{x_i}{2}}.$$

This expression is a power series in variables $x_1^2, \dots x_k^2$, so it can be expressed as a polynomial in Pontriagin classes. Let c_1 denote the first Chern class of the line bundle determined by the spin$^{\mathbb{C}}$structure on TX.

Theorem 4.2.18. *Let X be even dimensional riemannian manifold. Then the index of the Dirac operator $\overline{\mathcal{D}}^{+}$ is given by the following formula*

$$\operatorname{ind}(\overline{\mathcal{D}}^{+}) = \int_X \exp\left(\frac{1}{2}c_1\right) \wedge \widehat{A}(X)$$

4.3 Bochner technique

The Bochner technique is a very general and efficient method to prove many kinds of vanishing theorems. The simplest example of the vanishing theorem in geometry is the theorem which says that is a surface admits a metric of positive gaussian curvature then its genus equals zero.

The idea behind the Bochner technique is very simple. Suppose we have a self-adjoint second order elliptic differential operator T acting on a riemannian vector bundle $E \to X$. Sometimes one can prove that T satisfies the so called Weitzenböck formula

$$T = (\nabla)^* \nabla + R,$$

where ∇ is a metric covariant derivative on E and R is a section of the bundle $\operatorname{End}(E)$. Often R is expressible in terms of curvature of X and curvature of ∇. Now if one makes an appropriate assumptions on the curvature then one can the above formula to prove vanishing of some geometric invariant of X connected somehow with the operator T.

The next theorem is an example of the use of the Bochner technique.

Theorem 4.3.1. *Let X be a closed riemannian spin manifold of dimension $4k$. Suppose furhter that X posses a metric of positive scalar curvature. Then*

$$\int_X \widehat{A}(X) = 0.$$

Equivalently the index of $\mathcal{D}^+$ vanishes.

Proof. Suppose that ψ is a spinor field such that $\mathcal{D}\psi = 0$. Then

$$0 = ||\mathcal{D}^2\psi||_2^2 = ||\nabla\psi||_2^2 + \int_X \frac{R}{4}|\psi|^2.$$

Thus

$$\|\nabla\psi\|_2^2 = -\int_X \frac{R}{4}|\psi|^2.$$

The above equality implies that $\|\nabla\psi\|_2 = 0$ and $|\psi| = 0$ at every point $x \in X$. Thus $\psi = 0$. Consequently $\ker(\mathcal{D}) = 0$. Now $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$ and $\ker(\mathcal{D}) = \ker(\mathcal{D}^+) \oplus \ker(\mathcal{D}^-)$. Thus $\mathcal{D}^+$ and $\mathcal{D}^-$ are injective. Since $(\mathcal{D}^+)^* = \mathcal{D}^-$ the cokernel of $\mathcal{D}^+$ is trivial and by the Atiyah Singer Index Theorem we obtain the desired result. □

The above theorem is only the tip of the iceberg. For more information about the Bochner technique see [3].

A Elements of global analysis

A.1 Partial differential operators

Let E, F be two K-vector bundles over a compact manifold X, where $K = \mathbb{R}, \mathbb{C}$. Let $\mathrm{OP(E,F)}$ denote the space of K-linear operators $\Gamma(E) \to \Gamma(F)$, where $\Gamma(E)$ and $\Gamma(F)$ denotes the $C^\infty(X)$-modules of sections of E and F, respectively. Let $f \in C^\infty(X)$ be a smooth function on X. Define $\mathrm{ad}(f)\colon \mathrm{OP(E,F)} \to \mathrm{OP(E,F)}$ by the formula

$$\mathrm{ad}(f)(T)\sigma = T(f\sigma) - fT\sigma.$$

Definition A.1.1. Let $T \in \mathrm{OP(E,F)}$. We say that T is a partial differential operator of order $\leq n$ if $\mathrm{ad}(f_0) \circ \ldots \circ \mathrm{ad}(f_n)T = 0$ for all $f_0, \ldots f_n \in C^\infty(X)$. Furthermore we define $\mathrm{PDO}^{\leq n}(E,F)$ to be the vector space of partial differential operators of order $\leq n$.

Now let $f_1, \ldots, f_n \in C^\infty(X)$ be a functions on X and let $T \in \mathrm{PDO}^{\leq n}(E,F)$. Then for every point $x_0 \in X$ $\mathrm{ad}(f_1) \circ \ldots \circ \mathrm{ad}(f_n)|_{x_0}$ is a linear map $E_{x_0} \to F_{x_0}$, which is symmetric in $f_1, \ldots, f_n$ and depend only on the values of $df_1|_{x_0}, \ldots, df_n|_{x_0}$. Using the polarization identities it is easy to see that the values of ad^n is determined by the values of $\mathrm{ad}^n(\xi, \ldots, \xi)$, where $\xi \in T^*_{x_0}X$.

Definition A.1.2. Let $T \in \mathrm{PDO}^{\leq n}(E,F)$. Then the symbol of T is the map

$$\sigma_n(T) \in \hom(\pi^*E, \pi^*F), \qquad \sigma_n(P)(\xi) = \mathrm{ad}^n\, T(\xi),$$

where $\pi\colon T^*X \to X$. We say that T is of order n if $\sigma_n(T) \neq 0$ and denote the space of PDO's of order n by $\mathrm{PDO}^n(E,F)$.

Definition A.1.3. Let $T \in \mathrm{PDO}^n(E,F)$. We say that T is elliptic of order n if $\sigma_n(T)(\xi)\colon E_x \to F_x$ is a linear isomorphism for every x and $\xi \in T^*_xX \setminus \{0\}$.

If $E = F$ $n = 2$ and $\sigma(T)(\xi) = -|\xi| \cdot$, then we say that T is a generalized laplasian. Here $|\cdot|$ denote the fiberwise norm defined by the riemannian or hermitian metric on the vector bundle.

Definition A.1.4. Suppose (X, g) is a compact riemannian manifold and let E be a vector bundle equipped with riemannian (hermitian) metric. Let $T \in \mathrm{PDO}^m(E, F)$. We say that that $T^* \in \mathrm{PDO}^m(F, E)$ is a formal adjoint of T if

$$\int_X \langle T\sigma_1, \sigma_2\rangle_E \,\mathrm{dvol}_g = \int_X \langle \sigma_1, T^*\sigma_2\rangle_E \,\mathrm{dvol}_g$$

for all $\sigma_1 \in \Gamma(E)$ and $\sigma_2 \in \Gamma(F)$, where dvol_g denotes the metric volume form on X. When $E = F$ and $T = T^*$ we say that T is formally self-adjoint.

Suppose that E is a riemannian (hermitian) vector bundle over the closed riemannian manifold X and T is a PDO acting on E. Using the metric on E we can introduce a pre Hilbert space structure on the space of secions of E. The inner product is defined by the formula

$$(\sigma_1, \sigma_2) = \int_X \langle \sigma_1, \sigma_2\rangle \,\mathrm{dvol}_g \,.$$

Let $L^2(E)$ be the completion of $\Gamma(E)$ in the norm given by the inner product $(\cdot, \cdot)$. It is easily to check that $L^2(E)$ is a Hilbert space.

The operator T can be considered as an unbounded operator $T\colon L^2(E) \to L^2(E)$ with the domain $\Gamma(E)$. We can use methods from the functional anlysis to investigate the properties of T. Firstly we can define the closure of T denoted by $\overline{T}$ and the adjoint of T denoted by T^*. The domain of $\overline{T}$ is the set

$$\mathrm{dom}(\overline{T}) = \{\sigma \in L^2(E)\colon \exists_{\psi \in L^2(E)} \exists_{(\sigma_n)_n \subset \Gamma(E)} \sigma_n \to \sigma, \quad T\sigma_n \to \psi \text{ in } L^2(E)\}$$

and $\overline{T}\sigma = \psi$. The domain of the adjoint T^* is the set

$$\mathrm{dom}(T^*) = \{\sigma \in L^2(E)\colon \exists_{\psi \in L^2(E)} \forall_{\phi \in \Gamma(E)} (T\phi, \sigma) = (\phi, \psi)\}.$$

Then we have $T^*\sigma = \psi$.

If T is formally self adjoint then $\mathrm{dom}(T) \subset \mathrm{dom}(T^*)$ and $T^*|_{\mathrm{dom}(T)} = T$, which we denote by $T \subset T^*$. Such operators are called symmetric.

Theorem A.1.5. *Let T be a symmetric densly defined operator acting on the Hilbert space. Then*

$$\overline{T} = T^{**} \subset T^*.$$

A.2 Functional spaces

Definition A.2.1. Let E be a riemannian or hermitian vector bundle over the compact manifold X with fixed metric covariant derivative ∇. Choose $k \in \mathbb{Z}_+$ and $p \geq 1$. Define $L^{k,p}(E)$ to be the completion of $\Gamma(S)$ with respect to the norm

$$\|\psi\|_{k,p} = \left(\int_X \left(|\psi|^p + |\nabla^S\psi|^p + \cdots + |\left(\nabla^S\right)^k \psi|^p\right) \mathrm{dvol}_g\right)^{\frac{1}{p}}$$

Theorem A.2.2. *Suppose that the base manifold X is compact. For $k \in \mathbb{Z}_+$ and $p \geq 1$ $L^{k,p}(S)$ does not depend neither on the chosen connection nor on the riemannian or hermitian matric. Furthermore $L^{k,p}(S)$ is a Banach space. For $k = 2$ it is a Hilbert space with the inner product*

$$\langle \psi, \phi \rangle_{k,p} = \int_X \left(\langle \psi, \phi \rangle + \langle \nabla^S \psi, \nabla^S \phi \rangle + \cdots + \langle \left(\nabla^S\right)^k \psi, \left(\nabla^S\right)^k \phi \rangle \right) \mathrm{dvol}_g$$

Theorem A.2.3 (Rellich Theorem)**.** *Let $k_0, k_1 \in \mathbb{Z}_+$ and $p_0, p_1 \geq 1$. Denote $N = \dim X$. If $k_0 \geq k_1$ and $k_0 - \frac{N}{p_0} \geq k_1 - \frac{N}{p_1}$ then the embedding*

$$L^{k_0,p_0}(S) \hookrightarrow L^{k_1,p_1}(S)$$

is continous. When both inequalities are strict the embedding is compact.

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