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# Stable classification of 4-manifolds obtained by the surgery on loops

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# Stable classification of 4-manifolds obtained by the surgery on loops

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**Abstract**

This paper is concerned with the problem of stable diffeomorphism classification of 4-manifolds obtained using the surgery on loops. The main theorem states that under the assumption that the normal 1-type of two 4-manifolds in question is the same, the only classifying invariant is the signature. This paper is an outgrowth of [7].

## 1 Introduction

A well known theorem states that for any finitely presented group $\pi$ and $n \geq 4$ there exists a smooth compact connected $n$-manifold whose fundamental group is isomorphic to $\pi$. The construction starts with the presentation $\mathcal{P}$ of $\pi$. Then the surgery is performed on

$$(S^1 \times S^3)\# \dots \#(S^1 \times S^3)$$

along loops representing relations of $\mathcal{P}$. The Seifert-Van Kampen theorem implies that the obtained manifold has fundamental group isomorphic to $\pi$.

The manifold obtained by the construction will depend on the presentation of $\pi$ and framings chosen during the surgery. In order to classify these manifolds one has to take care of all the data, improve the construction and make it more precise. This is done in section 4 where the construction of surgery on loops is defined.

In 1960's C.T.C. Wall proved in [12] that two closed smooth simply connected 4-manifolds which are h-cobordant are, in fact, stably diffeomorphic, i.e. diffeomorphic after taking the connected sum of the manifold with some copies of the product $S^2 \times S^2$. This result was generalized by Quinn [8] to the effect that two closed smooth 4-manifolds which are s-cobordant are stably diffeomorphic. Moreover, Gompf in [4] proved that two closed smooth 4-manifilds which are homeomorphic are always stably diffeomorphic. These results shows the importance of the stable diffeomorphism relation in 4-dimensional topology.

The main result of this paper is the following theorem.

**Theorem 1.1.** *Suppose that $M$ and $N$ are two closed connected smooth orientable 4-manifolds obtained by the surgery on loops starting with two manifolds $M'$ and $N'$ such that $\pi_1(M')$ and $\pi_1(N')$ are free. Suppose that the normal 1-type of $M$ and $N$ is the same. Then $M$ and $N$ are stably diffeomorphic if and only if*

$$\sigma(M) = \sigma(N).$$

**Corollary 1.2.** If we start with two different presentations of $G$ and perform surgery on loops on the connected sum of copies of $(S^1 \times S^3)$, then the resulting manifolds are stably diffeomorphic if their normal 1-types agree.

**Corollary 1.3.** If $M'$ and $N'$ are non-spin 4-manifolds with free fundamental groups and we perform surgery on loops on $M'$ and $N'$ using two different presentations then we always obtain stably diffeomorphic 4-manifolds.

The paper is organized as follows. In section 2 the normal 1-type is defined. This concept is essential for controlling changes made during the construction. It is also a starting point of the modified surgery of M. Kreck, see [5]. The advantage of using the normal 1-type comes from the stable diffeomorphism theorem of M. Kreck (thm. 5.2). One reduces the problem of stable diffeomorphism classification to the algebraic problem of computing the bordism groups of the fibration representing the normal 1-type.

In section 3 the construction of the James spectral sequence is described. This spectral sequence was originally constructed in [11]. The James spectral sequence plays a crucial role in later considerations.

Section 4 describes the surgery on loops under circumstances needed in this paper.

Section 5 contains the proof of the main theorem 1.1.

# 2 Normal 1-types

**Definition 2.1.** Let $k \in \mathbb{N}$ and $p\colon B_k \to BSO$ be a fibration. We say that $p$ is $k$-universal if the homotopy fibre of $p$ is connected and the relative homotopy groups $\pi_i(p)$ vanish for $i \geq k+1$.
Let $M$ be a manifold with a $B_k$-structure, i.e. the normal Gauss map

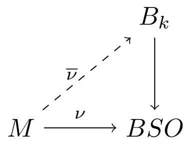
$$\nu\colon M \to BSO$$

admits a lifting $\overline{\nu}$ to $B_k$. The map $\overline{\nu}$ is called $k$-smoothing if it is a $(k+1)$-equivalence.

**Definition 2.2.** Let $M$ be a closed connected and oriented $n$-manifold. The normal $k$-type of $M$ is the fibre homotopy type of the $k$-universal fibration

$$B_k \to BSO,$$

such that there exists a $k$-smoothing of $M$

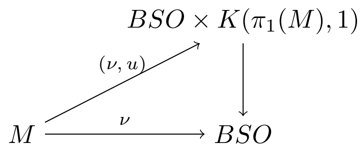



In this paper we will be mostly interested in normal 1-types of 4-manifolds. To construct a fibration, which represents the normal 1-type of a given 4-manifold $M$ we have to consider two separate cases.

First suppose that $w_2(\widetilde{M}) \neq 0$, where $\widetilde{M}$ is the universal covering space of $M$ and $w_2$ is the second Stiefel-Whitney class.

**Definition 2.3.** If the universal covering of a 4-manifold $M$ is non-spin, then we say that this manifold if of odd type.

In this case the normal 1-type of $M$ is represented by the following fibration.




Here $u\colon M \to K(\pi_1(M), 1)$ is the classifying map for the universal covering.

Let $w_2(\widetilde{M}) = 0$. Again let $u\colon M \to K(\pi_1(M), 1)$ classify the universal covering.

**Definition 2.4.** Such a manifold $M$ is said to have even type.

Consider the following lemma.

**Lemma 2.5.** Suppose $M$ is a finite CW-complex and let $A$ be a trivial $\mathbb{Z}\pi_1(M)$-module. Then the following sequence is exact

$$0 \to H^2(\pi_1(M), A) \xrightarrow{u^*} H^2(M, A) \xrightarrow{p^*} H^2(\widetilde{M}, A).$$

*Proof.* Start with the following exact $\mathbb{Z}\pi_1(M)$-chain complex $\mathcal{C}$:

$$C_2(\widetilde{M}) \to C_1(\widetilde{M}) \to C_0(\widetilde{M}) \to \mathbb{Z} \to 0.$$

It can be extended to an acyclic $\mathbb{Z}\pi_1(M)$ chain complex $\mathcal{C}'$ of projective modules. The short exact sequence of chain complexes

$$0 \to \mathcal{C} \to \mathcal{C}' \to \mathcal{C}'/\mathcal{C} \to 0$$

yields the following exact sequence

$$0 \to \hom_{\mathbb{Z}\pi_1(M)}(\mathcal{C}'/\mathcal{C}, A) \to \hom_{\mathbb{Z}\pi_1(M)}(\mathcal{C}', A) \to \hom_{\mathbb{Z}\pi_1(M)}(\mathcal{C}, A) \to 0. \quad (1)$$

The snake lemma gives the long exact sequence in cohomology. Consider the following part of this exact sequence

$$0 \to H^2(\pi_1(M), A) \to X \xrightarrow{\delta} Y \tag{2}$$

where

$$\begin{aligned} X &= \operatorname{coker}\Big(\hom_{\mathbb{Z}\pi_1(M)}(C_1(\widetilde{M}), A) \to \hom_{\mathbb{Z}\pi_1(M)}(C_2(\widetilde{M}), A)\Big) \\ Y &= \ker\big(\hom_{\mathbb{Z}\pi_1(M)}(C_3', A) \to \hom_{\mathbb{Z}\pi_1(M)}(C_4', A)\big) \end{aligned}$$

Chain complex $\mathcal{C}'$ is acyclic, thus there exists a diagram of exact sequences.

$$\begin{array}{ccccccc} C_4' & \longrightarrow & C_3' & \xrightarrow{\qquad\partial\qquad} & C_2(\widetilde{X}) & \longrightarrow & C_2(\widetilde{X}) \\ & & & \searrow \qquad \nearrow & & & \\ & & & Z_2(\widetilde{X}) & & & \\ & & \nearrow & & \searrow & & \\ & & 0 & & 0 & & \end{array}$$

The diagram above implies the following equality

$$\ker\big(\hom_{\mathbb{Z}\pi_1(M)}(C_3', A) \to \hom_{\mathbb{Z}\pi_1(M)}(C_4', A)\big) = \hom_{\mathbb{Z}\pi_1(M)}(Z_2(\widetilde{X}), A).$$

We have

$$\begin{aligned} Z^2(\widetilde{X}, A) = \{\phi \in \hom_{\mathbb{Z}\pi_1(M)}(C_2(\widetilde{M}), A) \colon B_3(\widetilde{X}) \subset \ker\phi\} \subset \\ \subset \hom_{\mathbb{Z}\pi_1(M)}(C_2(\widetilde{M}), A). \end{aligned}$$

Examination of the exact sequence (1) shows that the connecting homomorphism $\delta$ is equal to $\hom(\partial, A)$. Thus

$$\operatorname{Im}(\delta) = \hom_{\mathbb{Z}\pi_1(M)}(Z_2(\widetilde{X}), A) \subset \hom_{\mathbb{Z}\pi_1(M)}(C_3', A)$$

and $\delta$ is equal to the restriction to

$$\phi \mapsto \phi|_{Z_2(\widetilde{X})}.$$

Consequently (2) becomes

$$0 \to H^2(\pi_1(M), A) \to H^2(M, A) \xrightarrow{f} \hom_{\mathbb{Z}\pi_1(M)}(Z_2(\widetilde{X}), A).$$

Now the image of $f$ consists of those homomorphisms $\phi \colon Z_2(\widetilde{X}) \to A$ which vanish on $B_3(\widetilde{X})$. Thus

$$\begin{aligned} \operatorname{Im}(f) \subset \hom_{\mathbb{Z}\pi_1(M)}(Z_2(\widetilde{X})/B_3(\widetilde{X}), A) \subset \\ \subset \hom_{\mathbb{Z}}(Z_2(\widetilde{X})/B_3(\widetilde{X}), A) \subset H^2(\widetilde{X}, A). \end{aligned}$$

Consequently we obtain the following exact sequence

$$0 \to H^2(\pi_1(M), A) \to H^2(M, A) \to H^2(\widetilde{M}, A).$$

$\square$

**Remark 2.6.** It is obvious that spin manifolds have even type, however the reverse implication is not true in general. Nevertheless sometimes it can be reversed. The next corollary follows easily from lemma 2.5.

**Corollary 2.7.** Every manifold with even type and fundamental group $\pi$ is spin manifold if and only if $H^2(\pi, \mathbb{Z}/2) = 0$.

From the lemma 2.5 we can conclude that there exists a unique class $w \in H^2(\pi, \mathbb{Z}/2)$ such that $u^*w = w_2(M)$. Consider now the following pullback square.

$$\begin{array}{ccc} B(\pi, w) & \longrightarrow & K(\pi, 1) \\ \downarrow & & \downarrow{\scriptstyle w} \\ BSO & \xrightarrow{w_2} & K(\mathbb{Z}/2, 2) \end{array}$$

**Proposition 2.8.** *The fibration $B \to BSO$ from the diagram above represents the normal 1-type of $M$.*

*Proof.* Two maps $u$ and $\nu$ determine the lifting $\overline{\nu}\colon M \to B(\pi, w)$ of the normal map $\nu$ by the universal property of the fibration. Observe now that $BSpin$ is the homotopy fibre of the map $B(\pi, w) \to K(\pi, 1)$, which implies that this map induces an isomorphism on $\pi_1$ and $\pi_2$. The map $u$ is a 2-equivalence, thus $\overline{\nu}$ must be a 2-equivalence. Additionally the homotopy groups of the homotopy fibre of the map $B(\pi, w) \to BSO$ vanish in dimension greater than 1. This proves that $B(\pi, w) \to BSO$ is a 1-universal fibration and $\overline{\nu}$ is a 2-smoothing. $\square$

# 3 The James spectral sequence (JSS)

The proof of the main theorem requires some knowledge of the bordism group $\Omega_4(B, \pi, w)$ of the 1-universal fibration $B(\pi, w) \to BSO$. When $w = 0$ this group is isomorphic to the spin bordism group of the classifying space $K(\pi, 1)$ and we can use the classical Atiyah-Hirzebruch spectral sequence to extract the necessary information. To deal with the case when $w \neq 0$ we need the James spectral sequence, which was constructed in [11]. We will briefly describe the construction.

Suppose that $h$ is a generalized homology theory s.t. $h_i(*) = 0$ for $i < 0$. Let $F \to B \to K$ be an $h$-orientable fibration. Consider a stable vector bundle $\xi\colon B \to BSO$. Using the pullback construction one can construct a sequence of spaces $B_n$ and vector bundles $\xi_n\colon B_n \to BSO(n)$ such that the following diagram is commutative for each $n = 1, 2, \ldots$

$$\begin{array}{ccccc} B_n & \longrightarrow & B_{n+1} & \longrightarrow & B \\ \downarrow{\scriptstyle \xi_n} & & \downarrow{\scriptstyle \xi_{n+1}} & & \downarrow{\scriptstyle \xi} \\ BSO(n) & \longrightarrow & BSO(n+1) & \longrightarrow & BSO \end{array}$$

Choose arbitrary $n$, then there exists a relative fibration

$$(D^n, S^{n-1}) \to (D(\xi_n), S(\xi_n)) \xrightarrow{p_n} B_n,$$

where $D(\xi_n)$ and $S(\xi_n)$ denote the disk and the sphere bundle of $\xi_n$. Composition of $p_n$ and the map $B_n \to B \to K$ yields another relative fibration.

$$(D(\xi_n)|_{F_n}, S(\xi_n)|_{F_n}) \to (D(\xi_n), S(\xi_n)) \to K. \tag{3}$$

Here $F_n$ is a homotopy fibre of the fibration

$$F_n \to B_n \to K.$$

Consider the relative Serre spectral sequence for the fibration (3) and homology theory $h$

$${}_nE^2_{p,q} = H_p(K, h_q(D(\xi_n)|_{F_n}, S(\xi_n)|_{F_n})) \Longrightarrow h_{p+q}((D(\xi_n), S(\xi_n))).$$

Twofold application of Thom isomorphism

$$h_{q+n}(D(\xi_n)|_{F_n}, S(\xi_n)|_{F_n}) \xrightarrow{\cong} h_q(F_n) \xrightarrow{\cong} h_{q+n+1}((D(\xi_n \oplus \underline{\mathbb{R}}^1)|_{F_n}, S(\xi_n \oplus \underline{\mathbb{R}}^1)|_{F_n}))$$

gives the identification (modulo the vertical shift) of the Serre spectral sequence for $\xi_n$ and $\xi_n \oplus \underline{\mathbb{R}}^1$. Consider now the following commutative diagram of vector bundles.

$$\begin{array}{ccc} \xi_n \oplus \underline{\mathbb{R}} & \longrightarrow & \xi_{n+1} \\ \downarrow & & \downarrow \\ B_n & \longrightarrow & B_{n+1} \end{array}$$

It yields a morphism of spectral sequences ${}_nE \to {}_{n+1}E$. To be more precise for each $n$ and $r$ there exists the following commutative diagram.

$$\begin{array}{ccc} {}_nE^r_{p,q} & \longrightarrow & {}_{n+1}E^r_{p,q} \\ \downarrow{\scriptstyle d^n_r} & & \downarrow{\scriptstyle d^{n+1}_r} \\ {}_nE^r_{p-r,q+r-1} & \longrightarrow & {}_{n+1}E^r_{p-r,q+r-1} \end{array}$$

Passing to the limit over all possible diagrams simultaneosuly we obtain a new spectral sequence

$$E = \lim_{\vec{n}}({}_nE).$$

Exactness of the colimit functor implies that $E$ is indeed a spectral sequence. To identify the $E^2$ page consider the following limit

$$\begin{aligned}\lim_{\vec{n}}({}_nE^2_{p,q}) &= \lim_{\vec{n}} H_p(K, h_q(D(\xi_n)|_{F_n}, S(\xi_n)|_{F_n})) = \\ &= H_p(K, \lim_{\vec{n}} h_q(D(\xi_n)|_{F_n}, S(\xi_n)|_{F_n})) = \\ &= H_p(K, \lim_{\vec{n}} h_q(Th(\xi_n|_{F_n}))) = \\ &= H_p(K, h_q(M\xi|_F)),\end{aligned}$$

where $Th(\xi_n)$ is the Thom space of $\xi_n$ and $M\xi$ is the Thom spectrum of the stable vector bundle $\xi$. Exactness of the colimit functor and the fact that ${}_nE$ are first quadrant spectral sequences implies that

$$E^\infty_{p,q} = \lim_{\vec{n}}({}_nE^\infty_{p,q}).$$

Additionally the graded object of the spectral sequence $E$ is isomorphic to

$$\lim_{\vec{n}} h_{p+q}(D(\xi_n), S(\xi_n)) = \lim_{\vec{n}} h_{p+q}(Th(\xi_n)) = h_{p+q}(M\xi).$$

Thus we obtain the following proposition.

**Proposition 3.1.** *Let $F \to B \to K$ be a fibration and $B \to BSO$ be a stable vector bundle. Suppose that $h$ is a generalized homology theory whose coefficients vanish in negative dimensions. Then there exists a spectral sequence*

$$E^2_{p,q} = H_p(K, h_q(M\xi|_F)) \Longrightarrow h_{p+q}(M\xi).$$

*Furthermore this spectral sequence is natural with respect to the maps of fibrations.*

Let now $\xi\colon B(\pi, w) \to BSO$ be the 1-universal fibration. By the construction there exists a fibration

$$BSpin \xrightarrow{i} B(\pi, w) \xrightarrow{p} K(\pi, 1).$$

Additionally the composition $\xi \circ p$ is the classifying map for the universal bundle over $BSpin$. Thus applying the James spectral sequence to the above fibrations with $h = \pi^s$ we obtain the following spectral sequence

$$E^2_{p,q} = H_p(K(\pi, 1), \Omega^{Spin}_q) \Longrightarrow \Omega_{p+q}(B(\pi, w)).$$

# 4 Surgery on loops

First suppose that we are given a pair $(\pi, w)$, where $\pi$ is a finitely presented group and $w \in H^2(\pi, \mathbb{Z}/2)$. Using this data one can construct a 1-universal fibration over $BSO$ as a pullback of the following diagram.

$$\begin{array}{ccc} B(\pi, w) & \longrightarrow & K(\pi, 1) \\ \big\downarrow & & \big\downarrow{\scriptstyle w} \\ BSO & \xrightarrow{w_2(\gamma)} & K(\mathbb{Z}/2, 2) \end{array}$$

Let $(G, w')$ be another such pair and suppose that there exists an epimorphism $\psi\colon \pi \to G$ such that $w = \psi^* w'$. Then by the universal property of the pullback there exists a map of fibrations

$$B(\pi, w) \stackrel{\psi_\#}{\to} B(G, w')$$

Suppose now that $M$ is a smooth, closed, oriented, even 4-manifold whose normal 1-type is fibre-homotopy equivalent to $B(\pi, w) \to BSO$. Let

$$\overline{\nu}\colon M \to B(\pi, w)$$

be the lift of the normal Gauss map. Then the composition

$$\psi_\# \circ \overline{\nu}\colon M \to B(G, w')$$

is a normal map. We can perform surgery below the middle dimension on this map to obtain 1-smoothing $Y(M, \psi, w') \to B(G, w)$.

**Definition 4.1.** We say that manifold $Y(M, \psi, w')$ was obtained from $M$ by surgery on loops of type I.

**Proposition 4.2.** *Surgery on loops of type I yields a homomorphism of bordism groups* $\psi_*\colon \Omega_4(B(\pi, w)) \to \Omega_4(B(G, w'))$

**Remark 4.3.** To be precise one has to take care of the framings to be sure that the obtain manifold has the desired normal 1-type. If $\gamma\colon S^1 \to M$ is an embedding whose homotopy class represents one of the relations, then the framing of the stable normal bundle of the loop gamma comes from the nullhomotopy $H$ of the composition $\psi_\# \circ \overline{\nu} \circ \gamma$. If we take different nullhomotopy $H'$, then these two maps determine a map $S^2 \to B(\pi, w)$. However it is easy to check that $\pi_2(B(\pi, w)) = 0$, which implies that any two nullhomotopies are homotopic relative to the boundary, thus the homotopy class of the stable framing is uniquely defined.

In the second case the surgery construction is analogous. Let $Y(M', \psi)$ be the result of the construction performed on $M'$ using the homomorphism $\psi$.

**Definition 4.4.** We say that $Y(N, \psi)$ was obtained from $N$ by surgery on loops of type II.

**Proposition 4.5.** *If $\psi\colon \pi \to G$ is an epimorphism, then the surgery on loops yields a homomorphism of bordism groups*

$$\psi_*\colon \Omega_4^{SO}(K(\pi,1)) \to \Omega_4^{SO}(K(G,1)).$$

# 5 Stable classification of 4-manifolds

**Definition 5.1.** Let $M$ and $N$ be two smooth manifolds of dimension $2n$. We say $M$ and $N$ are stably diffeomorphic if there are two positive integers $k, l$ and a diffeomorphism

$$f\colon M\# k(S^n \times S^n) \to N\# l(S^n \times S^n).$$

Using the modified surgery [5] of M. Kreck one can give a sufficient condition for two manifolds $M$ and $N$ to be stably diffeomorphic.

**Theorem 5.2.** *Suppose $M$ and $N$ are two smooth, connected and closed $2q$-manifolds with the same normal $(q-1)$-type $B^{q-1} \to BO$, $q \geq 2$. If $(M, \overline{\nu}_M) = (N, \overline{\nu}_N)$ in $\Omega_{2q}^{B^{q-1}}$, then $M$ and $N$ are stably diffeomorphic.*

Theorem 5.2 yields the following corollary.

**Corollary 5.3.** Let $M$ and $N$ be two 4-manifolds whose normal 1-type is fibre-homotopy equivalent to $B(\pi, w) \to BSO$. Let $\psi\colon \pi \to G$ be an epimorphism of finitely presented groups and $w' \in H^2(G, \mathbb{Z}/2)$ such that $w = \psi^* w'$. Then $Y(M, \psi, w')$ and $Y(N, \psi, w')$ are stably diffeomorphic if and only if

$$[M, \overline{\nu_M}] - [N, \overline{\nu_N}] \in \ker(\psi_*\colon \Omega_4(B(\pi, w)) \to \Omega_4(B(G, w'))).$$

In order to prove the main theorem one has to investigate the properties of the homomorphisms from propositions 4.2 and 4.5.

**Proposition 5.4.** *Let $\pi$ be a finitely presented group. Let $\psi\colon F_\alpha \to \pi$ be an epimorphism. Then the homomorphism $\psi_*\colon \Omega_4^{Spin}(K(\pi,1)) \to \Omega_4(B(\pi, w))$ is injective. For any two epimorphisms $\psi_i\colon F_{\alpha_i} \to \pi$, $i = 1, 2$, we have*

$$\operatorname{im}((\psi_1)_*) = \operatorname{im}((\psi_2)_*)\,.$$

*Proof.* The proof uses the naturality of the JSS. First observe that $H_p(F_\alpha) = 0$ for $p \geq 1$. Thus $E^2_{1,3} = E^2_{2,2} = E^2_{3,1} = E^2_{4,0} = 0$. Naturality of the JSS implies that $\operatorname{im}(\psi_*) \subset E^\infty_{0,4} = E^2_{0,4} = \Omega_4^{Spin}$. The edge homomorphism corresponding to this entry is given by the signature. □

As a corollary we obtain the following theorem.

**Theorem 5.5.** *Let $G$ be a finitely presented group and $w \in H^2(G, \mathbb{Z}/2)$. Choose two presentations of $G$ and let $\psi_M \colon F_{k_M} \to G$ and $\psi_N \colon F_{k_N} \to G$ be homomorphisms induced by these presentations. Choose two spin 4-manifolds $M$ and $N$ such that $\pi_1(M) = F_{k_M}$ and $\pi_1(N) = F_{k_N}$. Then $Y(M, \psi_M, w)$ and $Y(N, \psi_N, w)$ are stably diffeomorphic iff $\sigma(M) = \sigma(N)$.*

**Remark 5.6.** It is natural to ask whether every 4-manifold of even type and with fundamental group $\pi$, is stably diffeomorphic to a one obtained from a spin manifold with free fundamental group. The answer to this question is no. A counter example is the Enriques surface (see [1]). Signature of the Enriques surface is equal to $-8$. On the other hand, signature of manifold obtained by surgery on loops is alway divisible by 16. However when $H_2(\pi; \mathbb{Z}/2) = H_3(\pi; \mathbb{Z}/2) = 0$ and $H_4(\pi; \mathbb{Z}) = 0$ then the map

$$\psi_* \colon \Omega_4^{Spin}(K(F_\alpha, 1)) \to \Omega_4^{Spin}(K(\pi, 1))$$

is surjective.

In the odd case we have the following lemma.

**Lemma 5.7.** There exists an isomorphism

$$\begin{gathered}\Omega_4^{SO}(K(\pi, 1)) \to \mathbb{Z} \oplus H_4(K(G, 1), \mathbb{Z}) \\ (M, f) \mapsto (\sigma(M), f_*[M]).\end{gathered}$$

*Proof.* The lemma follows easily from the investigation of the Atiyah-Hirzebruch spectral sequence. Indeed notice that $\Omega_1^{SO} = \Omega_2^{SO} = \Omega_3^{SO} = 0$ implies that $E^2_{1,3} = E^2_{2,2} = E^2_{3,1} = 0$. Thus there are no nonzero differentials. Therefore we obtain the following exact sequence.

$$0 \to \Omega_4^{SO} \to \Omega_4^{SO}(K(\pi, 1)) \to H_4(\pi; \mathbb{Z}) \to 0.$$

However the constant map $K(\pi, 1) \to *$ induces a map on the bordism groups which splits the inclusion $\Omega_4^{SO} \to \Omega_4^{SO}(K(\pi, 1))$. Also one can show that the epimorphism $\Omega_4^{SO}(K(\pi, 1)) \to H_4(\pi \colon \mathbb{Z})$ is the Hurewicz map. □

**Corollary 5.8.** Let $\psi \colon \pi \to G$ be an epimorphism. Then the following diagram is commutative

$$\begin{array}{ccc} \Omega_4^{SO}(K(\pi, 1)) & \xrightarrow{\psi_*} & \Omega_4^{SO}(K(G, 1)) \\ \downarrow & & \downarrow \\ \mathbb{Z} \oplus H_4(K(\pi, 1), \mathbb{Z}) & \xrightarrow{\mathrm{id} \oplus \psi_*} & \mathbb{Z} \oplus H_4(K(G, 1), \mathbb{Z}) \end{array}$$

In this diagram the $\psi_*$ homomorphism occurring in $\mathrm{id} \oplus \psi_*$ is the homomorphism induced by $\psi$ on homology of $K(\pi, 1)$ and $K(G, 1)$.

Corollary 5.8 yields the following theorem.

**Theorem 5.9.** *Let $M$ and $N$ be two 4-manifolds of odd type with fundamental group $\pi$. Then $Y(M, \psi_M)$ and $Y(N, \psi_N)$ are stably diffeomorphic if and only if $\sigma(M) = \sigma(N)$ and $\psi_*[M] = \psi_*[N]$.*

**Remark 5.10.** From the corollary 5.8 it is easy to deduce that the map

$$\psi_* \colon \Omega_4^{SO}(K(F_\alpha, 1)) \to \Omega_4^{SO}(K(G, 1))$$

is surjective if and only if $H_4(\pi; \mathbb{Z}) = 0$. Thus every 4-manifold with fundamental group $\pi$ and odd type is stably diffeomorphic to one obtained by the surgery on loops from a manifold with free fundamental group if and only if $H_4(\pi; \mathbb{Z}) = 0$.

Now the proof of the theorem 1.1 follows easily from theorems 5.5 and 5.9.

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