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Properties of some generic reals

Praca semestralna nr 1

(semestr zimowy 2010/11)

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UNIA EUROPEJSKA
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FUNDUSZ SPOŁECZNY

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Abstract

We check the minimality of the Cohen, Sacks, Miller and random forcing notions. We also characterize the generic reals and verify if they weakly dominate the ground model.

1 Introduction

In this paper we want to compare some properties of Cohen, Sacks, Miller and random forcings. We assume that Reader is familiar with the theory of forcing. We use some facts from topology, descriptive set theory and measure theory, which can be found in references.

The forcing notions are introduced combinatorially, but often we will look at them from the descriptive set theory point of view. We will show some facts, which allow us to switch the point of view.

Each of these notions can be seen as a quotient algebra of Borel sets on uncountable Polish space, divided by some σ-ideal. First we deal with Cohen, Sacks and Miller forcings, as the σ-ideals corresponding to them are σ-generated by closed sets. It fails for random forcing, which is considered in the last section.

We will present combinatorial proofs, that Sacks and Miller forcings are minimal. Then, using the property of continuous reading of names, we shall prove the same theorems in the descriptive set theory framework. We will figure out why Cohen and random reals can not be minimal. Then the generic reals will be characterized and verified if they weakly dominate all reals from the gound model.

2 Preliminaries

2.1 Trees and arboreal forcing notions

Let us start with some basic definitions:

Definition 2.1. Let A (alphabet) be either 2 or ω.

- for $s \in A^{<\omega}$ let $[s] = \{ t \in \omega^{<\omega} : t \subseteq s \text{ or } s \subseteq t\}$ and let $|s|$ denote the length of s.
- T is a tree if $T \subseteq A^{<\omega}$ and $(\forall s \in T)(\forall t \in A^{<\omega})(t \subseteq s \Rightarrow t \in T)$. We will consider only trees such that each node has a proper extension (pruned trees).
- For a tree T we define $[T] = \{x \in A^{\omega}: (\forall n \in \omega)(x \restriction n \in T)\}$. An element of $[T]$ is called a branch of T. Pruned trees in $A^{<\omega}$ are in one-to-one corespondence with closed sets in A^{ω} (see [4]).
- If T is a tree then $Lev_n(T) = \{t \in T : |t| = n\}$ is the n-th level of T.
- For $s \in T$ we define $Succ_T(s) = \{t \in T : t \text{ is an immediate successor of } s\}$.
- $s \in T$ is a splitting node if $|Succ_T(s)| \geq 2$.
- $s \in T$ is an ω-splitting node if $|Succ_T(s)| = \aleph_0$.
- A tree T is compact perfect if for each $s \in T$ the set $Succ_T(s)$ is finite and there is a splitting extension of s.
- T is a Miller tree if each splitting node is ω-splitting and each node has an ω-splitting extension.

We will consider arboreal forcing notions $\mathbb{P}$, i.e. partially ordered sets, isomorphic to some collection $\mathcal{T}$ of trees in $A^{<\omega}$ ordered by inclusion, which satisfies the condition: for each $S \in \mathcal{T}$ there is $T \in \mathcal{T}$ such that $T \subseteq S$ and $s \subsetneq t$ where s and t are the first splitting nodes in S and T respectively (definition taken from [5]). In these cases the generic filter G correspondes to one real $g \in A^{\omega}$ called the generic real. Indeed, $g = \bigcap\{[T] : T \in G\}$ and $G = \{T \in \mathbb{P} : g \in [T]\}$.

Let us introduce three arboreal forcing notions:

Definition 2.2.

1. The Cohen forcing $\mathfrak{C} = \{[s] : s \in \omega^{<\omega}\}$
2. The Sacks forcing $\mathfrak{S} = \{T \subseteq \omega^{<\omega} : T \text{ is a compact perfect tree}\}$
3. The Miller forcing $\mathfrak{M} = \{T \subseteq \omega^{<\omega} : T \text{ is a Miller tree}\}$

The Cohen and Sacks forcings can also be equivalently considered as consisting of subtrees of the Cantor tree $2^{<\omega}$. This is often usefull beacuse of the compactness of 2^{ω}, especially for $\mathfrak{S}$. Equivalence here means that the algebras of regulary open sets are isomorphic. For Borel sets and functions we often simplify the fraze "coded in V" to "from V". For $T, S \in \mathbb{P}$ we will write $T \parallel S$ if T and S are compatible and $T \perp S$ if not.

2.2 A descriptive set theory glance at the Cohen, Sacks and Miller forcings

Sometimes it is easier to look at all these forcing notions in a different way. They are forcingwise equivalent to the algebra of Borel sets divided by some σ-ideals. We will formulate this in the following fact:

Fact 2.3. The orderings $\mathfrak{C}$, $\mathfrak{S}$ and $\mathfrak{M}$ are forcingwise equivalent to algebras $Bor(\omega^\omega)/\mathcal{I}$ where $\mathcal{I}$ is the σ-ideal of meager, countable, and the σ-ideal generated by compact sets, respectively.

Remark 2.4. For Sacks forcing ω^ω can be replaced by any uncountable Polish space and for Cohen forcing by any uncountable Polish space without isolated points. We will often use 2^ω while considering $\mathfrak{S}$.

Before proving the fact let us state two theorems, which will be helpfull when dealing with the Miller forcing $\mathfrak{M}$:

Theorem 2.5 (Hurewicz)**.** Let X be a Polish space. Then X contains a closed subspace homeomorphic to ω^ω iff X is not σ-compact.□

Theorem 2.6 (Kechris-Louveau-Woodin)**.** Let X be a Polish space, let $A \subseteq X$ be an analytic and let $B \subseteq X$ be arbitrary with $A \cap B = \emptyset$. If there is no F_σ set separating A from B, then there is a copy of the Cantor set $C \subseteq X$ such that $C \subseteq A \cup B$ and $C \cap B$ is countable dense in C. In particular, $C \cap A$ is homeomorphic to ω^ω. □

Both theorems with proofs can be found in [4].

Proof of Fact 2.3: In each case it is enough to show that $\{[T]_{\equiv_\mathcal{I}} : T \in \mathbb{P}\}$ is a dense subset of $Bor(\omega^\omega)/\mathcal{I}$, where $\mathbb{P}$ is $\mathfrak{C}$, $\mathfrak{S}$ or $\mathfrak{M}$, $\mathcal{I}$ is the corresponding σ-ideal, and $[T]_{\equiv_\mathcal{I}}$ is the equivalence class of the Borel set $[T]$ in the quotient algebra.

For the Cohen forcing $\mathfrak{C}$, $\mathcal{I}$ is the σ-ideal of meager sets:
We know that each Borel set B has the Baire property (see [9]), i.e. is open modulo a meager set. That means that $B \equiv_\mathcal{I} O$ for some open O. The family $\{[T] : T \in \mathfrak{C}\}$ forms a topological base of ω^ω, so there is a tree $T \in \mathfrak{C}$ such that $[T] \subseteq O$. But then $[T]_{\equiv_\mathcal{I}} \leq B_{\equiv_\mathcal{I}}$ in $Bor(\omega^\omega)/\mathcal{I}$.

For the Sacks forcing $\mathfrak{S}$, $\mathcal{I}$ is the σ-ideal of countable sets:
Take any uncountable $B \in Bor(\omega^\omega)$. Then B contains a copy of the Cantor set C (see [9]). Let T be a compact perfect tree such that $C = [T]$. Then $[T]_{\equiv_\mathcal{I}} \leq B_{\equiv_\mathcal{I}}$.

For the Miller forcing $\mathfrak{M}$, $\mathcal{I}$ is the σ-ideal σ-generated by compact sets. We see ω^ω as $X = 2^\omega \setminus \mathbb{Q}$, where $\mathbb{Q} = \{t \in 2^\omega : (\exists n < \omega)(\forall m \geq n)(t(m) = 0)\}$. In other words X is the set of all points from the Cantor set, which are not eventually zero (see [3]). Take $B \in Bor(X)$ which is outside $\mathcal{I}$. We want to apply Kechris-Louveau-Woodin theorem for space 2^ω. B is Borel, hence analytic. Since B is not in the σ-ideal σ-generated by compact sets from X, it is not F_σ separable from $\mathbb{Q}$. By theorem 2.6 there exists closed $C \subseteq 2^\omega$ such that $C \subseteq B \cup \mathbb{Q}$ and $C \cap B$ is homeomorphic to ω^ω. But $C \cap B = C \cap X$ so it is closed in X.

We showed that each $\mathcal{I}$-positive Borel set $B \subseteq \omega^\omega$ contains D - a closed copy of ω^ω. Hence there is a Miller tree T such that $[T] \subseteq D$. Then $[T]_{\equiv_\mathcal{I}} \leq B_{\equiv_\mathcal{I}}$. $\square$

Now we can see $\mathfrak{C}$, $\mathfrak{S}$ and $\mathfrak{M}$ as algebras $Bor(\omega^\omega)/\mathcal{I}$ where $\mathcal{I}$ is the respective σ-ideal. But we can also see these forcing notions as collections of all Borel $\mathcal{I}$-positive sets. This is formulated in the following fact (see [10]):

Fact 2.7. Let $\mathcal{I}$ be a σ-ideal on a Polish space X. Then algebra $Bor(X)/\mathcal{I}$ is forcingwise equialent to $Bor(X) \setminus \mathcal{I}$, i.e. the family of all Borel sets which are not in $\mathcal{I}$. $\square$

3 Minimality

In this section we will consider the minimality of generic extensions. This feature distinguishes $\mathfrak{C}$ from $\mathfrak{S}$ and $\mathfrak{M}$. We will show that only the last two notions have this property. That will be proved in two ways. The first one is combinatorial by some constructions on trees. The second one uses some advanced theorems of descriptive set theory.

Now we will define the minimality property (see [3]).

Definition 3.1. We say that the generic real g is minimal over V iff for every real $r \in \omega^\omega$ in $V[g]$, $r \in V$ or $g \in V[r]$.

The counterexample beneth shows that this property does not hold for the Cohen forcing:

Counterexample 3.2. $\mathfrak{C}$ is not minimal. Indeed, $\mathfrak{C} \cong \mathfrak{C} \star \dot{\mathfrak{C}}$, where $\cong$ means forcingwise equivalence and $\star$ stays for iteration (see [1]). Hence $V[g] = V[g_1][g_2]$, g and g_1 are $\mathfrak{C}$-generic reals over V and g_2 is $\mathfrak{C}$-generic real over $V[g_1]$.

In the following subsections we will present two independent proofs that $\mathfrak{S}$ and $\mathfrak{M}$ are minimal.

3.1 Minimality from the combinatorial point of view

Now we will prove the minimality of the Sacks forcing. The proof uses the fusion technique and can be found in [3].

Theorem 3.3 (Sacks).
The generic real for the Sacks forcing $\mathfrak{S} = \{T \subseteq 2^{<\omega} : T \text{ is a perfect tree}\}$ is minimal over the ground model.

It is easier to see $\mathfrak{S}$ as trees in $2^{<\omega}$ instead of $\omega^{<\omega}$. This is possible by Remark 2.4. Before we prove the theorem let us fix some notation and get familiar with fusion sequences.

Definition 3.4.

- For $t \in T$ let $T \restriction t = \{s \in T : s \subseteq t \text{ or } t \subseteq s\}$.
- A node t is called n-th splitting, if it is splitting and there are exactly n nodes $s \subseteq t$ such that s is splitting.
- $Split_n(T)$ is the set of all n-th splitting nodes in T.
- $Split(T)$ is the set all splitting nodes in T, i.e. $Split(T) = \bigcup_{n=1}^{\infty} Split_n(T)$.
- Having an arboreal forcing notion $\mathbb{P}$ and P, $Q \in \mathbb{P}$ we say $P \leq_n Q$ if $P \leq Q$ and $Split_n(P) = Split_n(Q)$.
- A fusion sequence is a sequence of conditions $\{P_n\}_{n<\omega}$ such that for all $n > 0$ we have $P_n \leq_n P_{n-1}$.

It is easy to see the following claim:

Claim 3.5. If $\{T_n\}_{n<\omega} \subseteq \mathfrak{S}$ (or $\mathfrak{M}$) is a fusion sequence then $\bigcap_{n<\omega} T_n \in \mathfrak{S}$ ($\mathfrak{M}$).

Now we can prove Theorem 3.3.

Proof of Theorem 3.3: Take $r \in 2^\omega \cap (V[g] \setminus V)$ and $\dot{r}$ a name for r. Let $[T] \ni g$ and $T \Vdash \dot{r} \notin \check{V}$, i.e. $(\forall S \leq T)(\forall A \in V)(S \nVdash \dot{r} = \check{A})$. We want to reconstruct the generic real g from r. Take any condition $T' \subseteq T$. Inductively we will construct a fusion sequence $\{T_n\}_{n<\omega}$ and finite sets $A_n \subseteq \omega$ such that:
(i) $A = \bigcup_{n<\omega} A_n = \{n_s : s \in Split(S)\}$ where $S = \bigcap_{n<\omega} T_n \subseteq T'$
(ii) $S \restriction s^\frown\langle 0\rangle$ and $S \restriction s^\frown\langle 1\rangle$ decide whether $\dot{r}(n_s) = 1$ or $\dot{r}(n_s) = 0$ in opposite ways.

Take T' as T_0 and $A_0 = \emptyset$. Suppose we have T_{n-1} and A_{n-1}. For each $s \in Split_n(T_{n-1})$ pick some $n_s < \omega$ such that $n_s > max(A_{n-1})$ (we take 0 as $max(A_0)$), $n_s \neq n_t$ for $s \neq t$ in $Split_n(T_{n-1})$ and $T_{n-1} \restriction s$ does not decide the value of $\dot{r}(n_s)$ (We can find such numbers because condition $T_{n-1} \restriction s$ forces $\dot{r} \notin V$ so it does not decide infinitely many values of $\dot{r}$. Indeed, if $T_{n-1} \restriction s$ was indecisive only on finitely many values of $\dot{r}$, there would exist $T'' \leq T_{n-1} \restriction s$, which would decide them, hence $T'' \Vdash \dot{r} \in V$). Put $A_n = A_{n-1} \cup \{n_s : s \in Split_n(T_{n-1})\}$. For each $s \in Split_n(T_{n-1})$ find conditions $S_{s\hat{}\langle 0\rangle} \leq T_{n-1} \restriction s\hat{}\langle 0\rangle$ and $S_{s\hat{}\langle 1\rangle} \leq T_{n-1} \restriction s\hat{}\langle 1\rangle$ which decide the value of $\dot{r}(n_s)$ in opposite ways. Then we say

$$T_n = \bigcup_{s\in Split_n(T_{n-1})} S_{s\hat{}\langle 0\rangle} \cup S_{s\hat{}\langle 1\rangle} \quad \text{and} \quad S = \bigcap_{n<\omega} T_n$$

We can construct such conditions as S dense below T, so without lost of generality $g \in S$. Now we can recover the generic real g using r and S. Indeed, g can be characterized as the unique real $h \in [S]$ satisfying: for all $s \in Split(S)$ and $i \in \{0,1\}$ such that $s\hat{}\langle i\rangle \subseteq h$, the condition $S \restriction s\hat{}\langle i\rangle$ correctly decides the value $\dot{r}(n_s)$. That means if $S \restriction s\hat{}\langle i\rangle \Vdash \dot{r}(n_s) = 0$ then $\dot{r}^G(n_s) = 0$ and if $S \restriction s\hat{}\langle i\rangle \Vdash \dot{r}(n_s) = 1$ then $\dot{r}^G(n_s) = 1$. □

Now we will state and prove a similar theorem for the Miller forcing.

Theorem 3.6. The generic real for the Miller forcing $\mathfrak{M}$ is minimal over the ground model.

Proof: It is enough to consider only reals from the Cantor set, as in the ground model we have a Borel isomorphism between 2^ω and ω^ω. The beginning is similar to the Sacks' case. We take $r \in 2^\omega \cap (V[g] \setminus V)$ and $\dot{r}$, a name for r. Let $g \in [T]$, where T is forcing $\dot{r} \notin V$. Let $T' \subseteq T$ be a stronger condition. We will find $S \subseteq T'$ and $\{n_j^s < \omega :\ j < \omega \text{ and } s \in Split(S)\}$, which will be our tools to recover g from r. Again we construct inductively a fusion sequence, whose intersection is S. Let $T_0 = T'$. Suppose we have T_{i-1}. Take $s \in Split_i(T_{i-1})$.

By induction on j we will pick a sequence of natural numbers n_j^s, a descending sequence of infinite sets $A_j^s \subseteq \omega$, and conditions $T_{j,n}^s$ for $n \in A_j^s$. Take $A_0^s = \{n < \omega\colon\ s\hat{}\langle n\rangle \in T_{i-1}\}$ and let $a_0^s = min(A_0^s)$. Pick $n_0^s < \omega$ such that $T_{i-1} \restriction s\hat{}\langle a_0^s\rangle$ does not decide the value $\dot{r}(n_0^s)$. For each $n \in A_0^s \setminus \{a_0^s\}$ take a condition $T_{0,n}^s \leq T_{i-1} \restriction s\hat{}\langle n\rangle$, which decides the value $\dot{r}(n_0^s)$. At least one of the sets

$$B_0^s = \{n \in A_0^s\colon\ T_{0,n}^s \Vdash \dot{r}(n_0^s) = 1\}, \qquad C_0^s = \{n \in A_0^s\colon\ T_{0,n}^s \Vdash \dot{r}(n_0^s) = 0\}$$

is infinite. Let D_0^s be an infinite one among B_0^s and C_0^s. Now take $T_{0,a_0^s}^s \leq T_{i-1} \restriction s\hat{}\langle a_0^s\rangle$, which forces $\dot{r}(n_0^s) = 1$ iff $D_0^s = C_0^s$ and $\dot{r}(n_0^s) = 0$ iff $D_0^s = B_0^s$.

Suppose the $j-1$-th step is done. Take A_j^s as D_{j-1}^s and $a_j^s = min(A_j^s)$. Take $n_j^s < \omega$ such that $T_{j-1,a_j^s}^s$ does not decide the value $\dot{r}(n_j^s)$. For each $n \in A_j^s \setminus \{a_j^s\}$ choose $T_{j,n}^s \leq T_{j-1,n}^s$ which decides the value $\dot{r}(n_j^s)$. Again one of the sets

$$B_j^s = \{n \in A_j^s\colon\ T_{j,n}^s \Vdash \dot{r}(n_j^s) = 1\}, \qquad C_j^s = \{n \in A_j^s\colon\ T_{j,n}^s \Vdash \dot{r}(n_j^s) = 0\}$$

is infinite and D_j^s is an infinite one of those two. Now pick $T_{j,a_j^s}^s \leq T_{j-1,a_j^s}^s$, which forces $\dot{r}(n_j^s) = 1$ iff $D_j^s = C_j^s$ and $\dot{r}(n_j^s) = 0$ iff $D_j^s = B_j^s$.

The inner induction (on j) is over. Define

$$T_i = \bigcup_{s \in Split_i(T_{i-1})} \bigcup_{j<\omega} T^s_{j,a^s_j} \quad \text{and} \quad S = \bigcap_{i<\omega} T_i$$

As in the previous theorem by a density argument we may assume $g \in S$. Now we want to reconstruct g from r using S. Look at s - the first splitting node in S. Take minimal j such that $S \restriction s\hat{}\langle a^s_j\rangle$ correctly guessed the value $\dot{r}(n^s_j)$. Then $s\hat{}\langle a^s_j\rangle \subseteq g$. Go to the first splitting node in S extending $s\hat{}\langle a^s_j\rangle$ and continue in this way. □

3.2 Continuous reading of names

We want to prove theorems 3.3 and 3.6 in a different, more descriptive way. For this purpose we need a property of continuous reading of names, which is described in [10]. First of all we need a definition of properness. This concept was introduced by Shelah and can be found in [3] or [8].

Definition 3.7. A notion of forcing $\mathbb{P}$ is proper iff for every uncountable cardinal λ, every stationary subset of $[\lambda]^\omega$ remains stationary in the generic extension.

Definition 3.8 (see [10])**.** Suppose $\mathcal{I}$ is a σ-ideal on a Polish space X such that the forcing $Bor(X)\setminus\mathcal{I}$ is proper. $Bor(X)\setminus\mathcal{I}$ has continuous reading of names (CRN) iff for every $\mathcal{I}$-positive Borel $B \subseteq X$ and every Borel function $f\colon B \to 2^\omega$ there is $\mathcal{I}$-positive Borel set $C \subseteq B$ such that $f \restriction C$ is continuous.

For our purposes we do not need to learn more about properness. We just need to know that our three notions are proper and have CRN. Again by [10]:

Theorem 3.9. Suppose $\mathcal{I}$ is a σ-ideal on a Polish space X σ-generated by closed sets. Then the forcing $Bor(X) \setminus \mathcal{I}$ is proper and has continuous reading of names. □

So $\mathfrak{C}$, $\mathfrak{S}$, and $\mathfrak{M}$ are proper and have CNR as they forcingwise equivalent to $Bor(X)\setminus\mathcal{I}$ where $\mathcal{I}$ is the σ-ideal generated by closed sets (Fact 2.3). We need one more proposition also stated and proved in [10]:

Proposition 3.10. Suppose $\mathcal{I}$ is a σ-ideal on a Polish space X such that the forcing $Bor(X)\setminus\mathcal{I}$ is proper. Suppose Y is a Polish space, $B \in Bor(X)\setminus\mathcal{I}$ and $B \Vdash \dot{y} \in Y$. Then there is a condition $C \subseteq B$ in $Bor(X)\setminus\mathcal{I}$ and a Borel function $f\colon C \to Y$ such that $C \Vdash \dot{y} = f(\dot{g})$ where $\dot{g}$ is the $Bor(X)\setminus\mathcal{I}$-name for the generic real. □

The above results give us the following corrolary:

Corollary 3.11. Suppose $\mathbb{P}$ is $\mathfrak{C}$, $\mathfrak{S}$ or $\mathfrak{M}$ (seen as $Bor(\omega^\omega) \setminus \mathcal{I}$, where $\mathcal{I}$ is the respective ideal), and g is $\mathbb{P}$-generic real. Let $r \in \omega^\omega \cap V[g]$. Then there is $D \in \mathbb{P}$ and a continuous $f\colon D \to \omega^\omega$ such that D and f are in the ground model V, $g \in D$ and $r = f(g)$.

Proof: Take $\dot{r}$ - $\mathbb{P}$-name for r and $B \in G$ which forces that $\dot{r} \in \omega^\omega$. Take any condition $B' \subseteq B$. By Proposition 3.10 there exists a stronger condition $C \subseteq B'$ and a Borel function $f\colon C \to \omega^\omega$ such that $C \Vdash \dot{r} = f(\dot{g})$. By Theorem 3.8 $\mathbb{P}$ has CRN so there is a condition $D \subseteq C$ such that $f \restriction D$ is continuous. $\square$

3.3 Minimality from the descriptive point of view

We are close to give different proofs of minimality for the Sacks and Miller forcings. First we will handle $\mathfrak{S}$. We need the following lemma, for which we will give two independent proofs:

Lemma 3.12. Suppose we have a continuous function $f\colon 2^\omega \to 2^\omega$. Then there exist perfect set $D \subseteq 2^\omega$ such that $f \restriction D$ is either constant or is a homeomorphism onto $f[D]$.

First proof (descriptive): Let us write $X = Y = 2^\omega$ and $f\colon X \to Y$. Suppose we have $\sigma \in Y$ such that $f^{-1}[\{\sigma\}]$ is uncountable. Being closed and uncountable this preimage must contain a perfect set D. Clearly $f \restriction D$ is constant.

If there is no such σ then $f[X]$ is uncountable. It is compact, as continuous image of a compact space. Hence it is closed and contains a perfect subset A. Define $B \subseteq X \times Y$

$$B = \bigcup_{\sigma \in A} f^{-1}[\{\sigma\}] \times \{\sigma\}$$

B is Borel as a graph of a continuous function $f \restriction f^{-1}[A]$. Each section B_y for $y \in A$ is countable and $\pi_Y[B] = A$. By Lusin's theorem (see [9]) there exists a Borel uniformization $C \subseteq B$, i.e. $\pi_Y[C] = A$ and C is a graph of a Borel function from A into X. $\pi_X \restriction C$ is one-to-one because sections B_y are pairwise disjoint. So by Lusin-Souslin Theorem (see [4]) $\pi_X[C]$ is Borel. It is also uncountable so it contains a perfect subset D. Then $f \restriction D$ is one-to-one, D is compact, so $f \restriction D$ is a homeomorpism onto its image. $\square$

We can also prove Lemma 3.12 more directly, by some "tree combinatorics":

Second proof (combinatorial): Suppose f is constant on some basic $[s]$, $s \in 2^{<\omega}$. Then we take $D = [s]$. If f is not constant on any basic $[s]$ then we perform the following inductive construction.

Pick x, $y \in f[2^\omega]$, $x \neq y$. Let U_x, U_y be disjoint open neighbourhoods of x and y. Take t_x, $t_y \in 2^{<\omega}$ such that $x \in f[[t_x]] \subseteq U_x$ and $y \in f[[t_y]] \subseteq U_y$. This is possible because f is continuous. $U_x \cap U_y = \emptyset$ so t_x and t_y are incomparable. We define $T_1 = \{t_x, t_y\}$.

Suppose we have T_i. Take $t \in T_i$ and x, $y \in f[[t]]$, $x \neq y$. Let U_x, U_y - disjoint open neighbourhoods of x and y contained in the neighbourhood chosen at the very previous step. Again we pick t_x, $t_y \in 2^{<\omega}$, t_x, $t_y \supseteq t$, $x \in f[[t_x]] \subseteq U_x$ and $y \in f[[t_y]] \subseteq U_y$. Then

$$T_{i+1} = \bigcup_{t \in T_i} \{t_x, t_y\} \qquad \text{and} \qquad T = \{s \in 2^{<\omega} \colon\, (\exists t \in \bigcup_{i=1}^{\infty} T_i)(s \subseteq t)\}.$$

T is a perfect tree and f is one-to-one on $D = [T]$. Now $f \restriction D$ is continuous one-to-one function with compact domain, so it is a homeomorphism. □

Lemma 3.12 together with CRN property will give us the promised another proof that the Sacks forcing is minimal:

Proof of Theorem 3.3: Suppose $r \in 2^\omega \cap V[g]$. By CRN (corollary 3.11) we have in the ground model a perfect set $D \subseteq 2^\omega$ and $f \colon D \to 2^\omega$ such that $f(g) = r$. D is homeomorphic to 2^ω so by lemma 3.12 there is perfect $P \subseteq D$ such that $f \restriction P$ is either constant or one-to-one. If it is constant then $r = f(g)$ is in the ground model V. If it is one-to-one then we have in V the inverse function f^{-1}. Then $g = f^{-1}(r)$, so $V[g] = V[r]$. □

Now we will do the same for the Miller forcing. We need some fact analogous to Lemma 3.12. Before we formulate and prove it let us state a very easy topological claim:

Claim 3.13. Let X be an infinite metrizable separable space. Then X contains an infinite discrete subspace Y, i.e. each subset $A \subseteq Y$ is open in Y.

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of pairwise different points from X. Let $\bar{X}$ be a metrizable compactification of X. (We can embed X into Hilbert's cube $\mathcal{H}$ and take its closure as $\bar{X}$.) (see [2]). There is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$, which is convergent in $\bar{X}$. But then $\{x_{n_k}\}_{k=1}^{\infty} \subseteq X$ is the infinite discrete subspace. □

Lemma 3.14. Suppose we have a continuous function $f \colon \omega^\omega \to \omega^\omega$. Then there exist a Miller tree $T \subseteq \omega^{<\omega}$ such that $f \restriction [T]$ is constant or is a homeomorphism onto its image.

Proof: Case 1: Suppose we have $s \in \omega^{<\omega}$ such that $f[[s]]$ is finite, say $f[[s]] = \{\sigma_1, \sigma_2, ..., \sigma_n\}$. Let $A_i = [s] \cap f^{-1}[\{\sigma_i\}]$. A_i's are closed. We have

$[s] = \bigcup_{i=1}^{n} A_i$. $[s]$ is not σ-compact so there exist $m \leq n$ that A_m is not σ-compact. By Theorem 2.5 A_m contains a closed subspace homeomorphic to ω^ω, so there is a Miller tree T, such that $[T] \subseteq A_m$. Then $f \restriction [T]$ is constant.
Case 2: For each $s \in \omega^{<\omega}$ $f[[s]]$ is infinite. Perform the following inductive construction. Take $\{\emptyset\}$ as T_1. Suppose we have T_i. Fix $s \in T_i$. For $i < \omega$ we choose $\sigma_i \in f[[s\hat{}\langle i\rangle]]$ such that σ_i's are pairwise different. By claim 3.13 we can take infinite $A_s \subseteq \omega$ such that the space $\{\sigma_i : i \in A_s\}$ is discrete. Take open, pairwise disjoint sets $U_i^s \ni \sigma_i$, $i \in A_s$. Let $s_i \supseteq s\hat{}\langle i\rangle$ be such that $f[[s_i]] \subseteq U_i^s$ (here we use the continuity of f). We define

$$T_{i+1} = \bigcup_{s \in T_i} \{s_i : i \in A_s\} \qquad \text{and} \qquad T = \{t \in \omega^{<\omega} :\ (\exists s \in \bigcup_{i=1}^{\infty} T_i)(t \subseteq s)\}.$$

Then $f \restriction [T]$ is obviously one-to-one. The set $\{[T] \cap [s] : s \in \bigcup_{i=1}^{\infty} T_i\}$ form an open base of $[T]$. But $f[[s] \cap [T]] = f[[T]] \cap (\bigcup_{i \in A_s} U_i^s)$ so its open in $f[[T]]$. Hence $f \restriction [T]$ is a homeomorphism onto its image. □

Now we will give a proof that $\mathfrak{M}$ is minimal, very similar to the one for $\mathfrak{S}$.

Proof of Theorem 3.6: Suppose $r \in \omega^\omega \cap V[g]$. By CRN (corollary 3.11) in the ground model there is a Miller tree $T \subseteq \omega^{<\omega}$ and $f : [T] \to \omega^\omega$ such that $f(g) = r$. By lemma 3.14 there is a Miller subtree $S \subseteq T$ such that $f \restriction [S]$ is either constant or one-to-one. If it is constant then $r = f(g)$ is in the ground model V. If it is one-to-one then we have in V the inverse function f^{-1}. Then $g = f^{-1}(r)$, so $V[g] = V[r]$. □

In both Sacks and Miller forcings the proof uses the same idea. By the continuous reading of names a new real r is the value at the generic real of some continuous function f coded in the ground model. By lemmata 3.12 and 3.14 we can assume f is constant or one-to-one. Hence either r is in the ground model, or the generic real can be recovered from r via f^{-1}. This technique fails with the Cohen forcing. $\mathfrak{C}$ has CRN but we do not have such a lemma about constant or one-to-one functions as for $\mathfrak{S}$ and $\mathfrak{M}$. Beneth we give an example of a function which is neither constant nor one-to-one on any condition from $\mathfrak{C}$.

Counterexample 3.15. Let $f : \omega^\omega \to \omega^\omega$, $f(n_0, n_1, n_2, ...) = (n'_0, n'_1, n'_2, ...)$, where $n'_i = 0$ iff n_i is even, and $n'_i = 1$ iff n_i is odd. For no $s \in \omega^{<\omega}$ $f \restriction [s]$ is constant or injective.

4 Characterization of generic reals

In this section we ask which new reals $r \in V[G] \setminus V$ are also generic over V, i.e. for which r the filter $\{P \in \mathbb{P} :\ r \in P\}$ intersects all open dense $D \subseteq \mathbb{P}$

from the ground model V. Each notion $\mathfrak{C}$, $\mathfrak{S}$ and $\mathfrak{M}$ gives different answer. First we will deal with $\mathfrak{S}$.

4.1 Sacks reals

Theorem 4.1. Let $r \in 2^\omega \cap (V[G] \setminus V)$, where G is $\mathfrak{S}$-generic over V. Then r is also $\mathfrak{S}$-generic.

Proof: By CRN (corollary 3.11) we have a continuous $f \in V$, $f(g) = r$. As usually g is the generic real connected to G. By lemma 3.12 the collection of conditions $T \in \mathfrak{S}$ such that $f \restriction [T]$ is constant or injective forms in V a set, which is dense below $dom(f)$. Since $r = f(g) \notin V$, there exists $S \in G$ such that $f \restriction [S]$ is one-to-one. By compactness it is a homeomorphism onto $[S'] = f[[S]]$, where $S' \in \mathfrak{S}$.

We want to show that $G_r = \{T \in \mathfrak{S} : \ r \in [T]\}$ intersects each open dense set $D \in V$. Note that $D^{-1} = \{T \in \mathfrak{S} : \ f[[T]] \in D\}$ is open dense below S. So we have a condition T in $D^{-1} \cap G$, i.e. $g \in [T]$. Then $r = f(g) \in f[T] = [T']$ where $T' \in D \cap G_r$. □

4.2 Miller reals

Characterizing generic reals for the Miller forcing is not as easy as for the Sacks case. It can be found in [6]. Anyway here we will give another proof, but first some definition (see [6]):

Definition 4.2. Let $r \in \omega^\omega \cap (V[G] \setminus V)$, where G is $\mathfrak{M}$-generic over V. We say that r weakly dominates the ground model V iff for any function $f \in \omega^\omega \cap V$ there are infinitely many $n < \omega$ such that $r(n) > f(n)$.

Theorem 4.3. Let $r \in \omega^\omega \cap (V[G] \setminus V)$, where G is $\mathfrak{M}$-generic over V. Then r is $\mathfrak{M}$-generic over V if and only if r weakly dominates V.

Proof: ($\Rightarrow$): Pick $h \in \omega^\omega \cap V$. We will show that set

$$D_h = \{T \in \mathfrak{M} : \ (\forall r \in [T])(\exists^\infty n < \omega)(r(n) > h(n))\}$$

is dense in $\mathfrak{M}$. Take any $S \in \mathfrak{M}$. We will build a fusion sequence $T_i \subseteq S$ such that $T = \bigcap_{i<\omega} T_i \in D_h$. Let $T_0 = S$. Suppose we have T_i. To obtain T_{i+1} for each $s \in Split_{i+1}(T_i)$ we cut off all nodes $t \supseteq s\hat{\ }\langle n\rangle$ for $n \leq h(|s|+1)$. More precisely

$$T_{i+1} = \bigcup_{s \in Split_{i+1}(T_i)} \ \bigcup_{n > h(|s|+1)} T_i \restriction s\hat{\ }\langle n\rangle$$

We claim $T = \bigcap_{i<\omega} T_i \in D_h$. Indeed, take $r \in [T]$. Then $r(n) > h(n)$ for each $n < \omega$ such that $n = |s|+1$ for some $s \subseteq r$, which is a splitting node in T.

($\Leftarrow$): As in the Sacks case by CRN we have in V a Miller tree T and $f\colon [T] \to \omega^\omega$, such that $f(g) = r$ and f is a homeomorphism onto $f[[T]]$. As a Polish space this image is G_δ in ω^ω (see [2]). However, we do not know if it is closed.
Claim: $f[[T]]$ is not σ-compact: Suppose $f[[T]] \subseteq \bigcup_{n<\omega} A_n$, where A_n's are compact. Compact sets in ω^ω are dominated by functions from ω^ω. So for each n we can take $h_n \in \omega^\omega \cap V$ such that $(\forall h' \in A_n)(\forall i < \omega)(h_n(i) > h'(i))$. Take $h \in \omega^\omega \cap V$ such that $h(n) > max\{h_0(n), h_1(n), h_2(n), ..., h_n(n)\}$ for each $n < \omega$. But $r \in f[[T]]$ so $r(n) > h(n)$ only for finitely many n. That gives a contradiction with the assumption that r weakly dominates V. The claim is proved.

Now, $f[[T]]$ is Polish and not σ-compact, so by Hurewicz Theorem (2.5) it contains a closed subset homeomorphic to ω^ω. So there is a Miller tree $S' \in \mathfrak{M}$ such that $[S'] \subseteq f[[T]]$. Take S such that $[S] = f^{-1}[[S']]$. Then $f \restriction [S]$ is a homeomorphism onto $[S']$. The rest of the proof is the same as in the case of $\mathfrak{S}$ (Theorem 4.1) even with similar notation. We must only change $\mathfrak{S}$ to $\mathfrak{M}$. $\square$

4.3 Cohen reals

Cohen reals are those, which avoid all meager Borel sets coded in the ground model V. It follows from a little bit more general theorem:

Theorem 4.4. Let $\mathcal{I}$ be a σ-ideal on a Polish space X, such that $\mathbb{P} = Bor(X) \setminus \mathcal{I}$ satisfy the countable chain condition (see [3]). Then for any real $r \in \omega^\omega$ the set $G_r = \{B \in \mathbb{P} : r \in B\}$ is $\mathbb{P}$-generic over V if and only if $r \notin A$ for any Borel $A \in \mathcal{I}$ coded in V.

For the Cohen case $\mathcal{I}$ is the σ-ideal of meager sets and $\mathfrak{C} = Bor(\omega^\omega) \setminus \mathcal{I}$ satisfy ccc. Above theorem will be used again in the next section, where we deal with the random forcing.

Proof of 4.4: ($\Rightarrow$): Pick a Borel $A \in \mathcal{I}$ coded in the ground model V. The set $\mathcal{D} = \{B \in \mathbb{P} :\ B \cap A = \emptyset\}$ is dense in $\mathbb{P}$. As G_r is $\mathbb{P}$-generic, there is $B \in G_r \cap \mathcal{D}$. Then $r \in B \in \mathcal{D}$ so $r \notin A$.

($\Leftarrow$): G_r is a filter. If not we would have B_1, $B_2 \in G_r$ such that $B_1 \cap B_2 \notin G_r$. That means $B_1 \cap B_2 \in \mathcal{I}$ is a Borel set from V and $r \in B_1 \cap B_2$.

Suppose G_r is not $\mathbb{P}$-generic. We will find a Borel $A \in \mathcal{I}$ from V containing r. Let $\mathcal{D} \in V$ be an open dense subset of $\mathbb{P}$ such that $\mathcal{D} \cap G_r = \emptyset$. Take $\mathcal{A} \subseteq \mathcal{D}$ a maximal antichain. $\mathbb{P}$ has ccc, hence $\mathcal{A}$ is countable. This implies $A = X \setminus \bigcup \mathcal{A}$ is a Borel set from $\mathcal{I}$ coded in V. For each $A' \in \mathcal{A}$ we have $A' \notin G_r$, so $r \notin A'$. That means $r \in A$. $\square$

5 Random forcing

Let us recall a well known fact from measure theory, the proof can be found in [9]:

Theorem 5.1. Let X be a Polish space, and μ a probabilistic continuous Borel measure on X. (Continuity means $\mu(\{x\}) = 0$ for each $x \in X$). Then there exists a Borel isomorphism $h\colon X \to I$ such that $\forall B \in Bor(I)$ $\lambda(B) = \mu(h^{-1}(B))$, where I is the unit interval $[0,1]$ and λ denotes the Lebesgue measure. $\square$

We introduce the random forcing notion as $\mathfrak{R} = Bor(I) \setminus \mathcal{N}$ ordered by $\subseteq$, $\mathcal{N}$ is the σ-ideal of sets of Lebesgue measure zero. By theorem 5.1 for any Polish space X with probabilistic continuous measure λ, the collection of λ-positive Borel subsets of X is forcingwise equivalent to $\mathfrak{R}$. Sometimes we will consider I^2 with the product of Lebesgue measure, or ω^ω with the unique measure extending λ', where λ' is given on basic clopen sets in the following way: $\lambda'([\emptyset]) = 1$, $\lambda'([s\hat{\ }\langle n\rangle]) = \lambda'([s]) \cdot 1/2^{n+1}$. Each measure we use is called λ and $\mathcal{N}$ is the σ-ideal of λ-zero sets. There is no confusion, since we never take two distinct measures on the same space.

Remark 5.2. Each finite measure λ on a metric space X is inner and outer regular, i.e. for each $B \in Bor(X)$:
$\lambda(B) = sup\{\lambda(F) : F \subseteq B,\ F \text{ - closed}\} = inf\{\lambda(O) : B \subseteq O \subseteq X,\ O \text{ - open}\}$
Morover, if X is Polish then λ is tight, i.e. for each $B \in Bor(X)$:
$\lambda(B) = sup\{\lambda(F) : F \subseteq B,\ F \text{ - compact}\}$. $\square$

The above remark is a well known fact of measure theory. It implies that $\mathfrak{R}$ is an arboreal forcing notion. Indeed:

$$\mathfrak{R} \cong \{T \subseteq \omega^{<\omega} : T \text{ is a tree and } \lambda([T]) > 0\}$$

since λ-positive closed sets form a dense subcollection of λ-positive Borel sets in the Baire space ω^ω. The real which correspondes to $\mathfrak{R}$-generic filter is called the random real.

The next fact is an immediate consequence of Theorem 4.4, as $\mathfrak{R}$ satisfy ccc:

Fact 5.3. Let $r \in X$, where X is a Polish space. r is random over V if and only if $r \notin N$ for any Borel $N \in \mathcal{N}$ coded in V. $\square$

Now we will see, that random real is not minimal. We will figure out properties of the σ-ideal $\mathcal{N}$, which imply $Bor(\omega^\omega) \setminus \mathcal{N}$ is not minimal.

Theorem 5.4. $\mathfrak{R} \cong \mathfrak{R} \star \dot{\mathfrak{R}}$. Hence $\mathfrak{R}$ is not minimal.

Proof: Denote $\mathfrak{R}' = Bor(I^2) \setminus \mathcal{N}$ and $\mathfrak{R} = Bor(I) \setminus \mathcal{N}$. For $B \in \mathfrak{R}'$ let $\bar{B} = \{x \in I : \lambda(B_x) > 0\}$, where $B_x = \{y \in I : (x, y) \in B\}$ is the vertical section of B at x. Let $\dot{r}$ be the canonical $\mathfrak{R}$-name for the generic real, i.e. for each $B \in \mathfrak{R}$, $B \Vdash \dot{r} \in \check{B}$.

We already know $\mathfrak{R} \cong \mathfrak{R}'$. We will prove that the mapping $E: \mathfrak{R}' \to \mathfrak{R} \star \dot{\mathfrak{R}}$ given by $E(B) = (\bar{B}, B_{\dot{r}})$ is a dense embedding. First of all, let us show that E is well defined: Take any $B \in \mathfrak{R}'$. By Fubini theorem (see [7]) $\lambda(B) > 0$ if and only if $\lambda(\{x \in I : \lambda(B_x) > 0\}) > 0$, so $\bar{B} \in \mathfrak{R}$. Since $\bar{B} \Vdash (\forall x \in \bar{B})(\lambda(B_x) > 0)$ we have $\bar{B} \Vdash B_{\dot{r}} \in \dot{\mathfrak{R}}$. So $E(B) = (\bar{B}, B_{\dot{r}}) \in \mathfrak{R} \star \dot{\mathfrak{R}}$.

Next we show that E is an embedding. Take $B, C \in \mathfrak{R}'$, $B \perp C$. Suppose a contrario that $(\bar{B}, B_{\dot{r}}) \parallel (\bar{B}, B_{\dot{r}})$. Then there exists a Borel $\mathcal{N}$-positive $A \subseteq \bar{B}, \bar{C}$ and $\dot{A} \in \dot{\mathfrak{R}}$, such that $A \Vdash \dot{A} \subseteq B_{\dot{r}} \cap C_{\dot{r}}$. Define $N = \{x \in X : \lambda(B_x \cap C_x) > 0\}$. Then $\lambda(N) = 0$ by incompatibleness of B and C and the Fubini theorem. We have $A \Vdash \dot{r} \in \check{N}$ which leads to a contradiction with fact 5.3.

We still need to show that the image of E is dense in the iteration. Take an arbitrary maximal antichain $\{B^n, n \in \omega\}$ (it is countable since $\mathfrak{R}'$ is ccc). $\{E(B^n), n \in \omega\}$ is an antichain because E is an embedding. We claim that it is maximal. Suppose not. Then find $(C, \dot{D}) \in \mathbb{P} \star \dot{\mathbb{P}}$ incompatible with each $(\bar{B^n}, B^n_{\dot{r}})$. If $\lambda(\bar{B^n} \cap C) > 0$ then $\bar{B^n} \cap C \Vdash \lambda(B^n_{\dot{r}} \cap \dot{D}) = 0$. Take $G \ni C$, $\mathfrak{R}$-generic over V, and denote $r = \dot{r}^G$ (so r is a random over V and $r \in B \Leftrightarrow B \in G$ for each $B \in \mathfrak{R}$). In $V[G]$, for each n holds the following:
If $r \in \bar{B^n}$ then $\lambda(B^n_r \cap \dot{D}^G) = 0$.
If $r \notin \bar{B^n}$ then $\lambda(B^n_r) = 0$ as $(\bar{B^n})^c = \{x \in X : \lambda(B^n_x) = 0\}$.
It follows that

$$\lambda(\bigcup_{n \in \omega} B^n_r \cap \dot{D}^G) = 0$$

But $\lambda(\dot{D}^G) > 0$, so

$$r \in \{x \in X : \lambda(\bigcup_{n \in \omega} B^n_x) < 1\}$$

The last set is in $\mathcal{N}$ because of the Fubini theorem and maximality of the antichain $\{B^n, n \in \omega\}$. Again this is a contradiction with fact 5.3. □

A similar theorem is true for Cohen forcing, as we mentioned in section 3. To convince oneself that $\mathfrak{C} \cong \mathfrak{C} \star \dot{\mathfrak{C}}$, one can follow the above proof considering category instead of measure, the σ-ideal of meager sets instead of $\mathcal{N}$, and the Kuratowski-Ulam theorem (see [7] or [9]) instead of the Fubini theorem.

Since $\mathfrak{S}$ and $\mathfrak{M}$ are minimal, they do not admit anything like Theorem 5.4. In these cases the generic real also avoids sets from the corresponding σ-ideal $\mathcal{I}$ which are in the ground model, but neither these notions satisfy ccc, nor we have facts analogous to Fubini and Kuratowski-Ulam theorems. Indeed, we can find a perfect compact or non-σ-compact subset of $\omega^\omega \times \omega^\omega$, contained in just one vertical section of the product.

As the other notions we considered, the random forcing admits the continuous reading of names. The σ-ideal of measure zero sets is not σ-generated by closed sets (counterexample below), so we can not use Theorem 3.9. The CRN property for $\mathfrak{R}$ is stated in the following theorem of Luzin. A proof can be found in [7] or any handbook on measure theory:

Theorem 5.5. Suppose X is metric separable, λ is a finite measure on X and $f\colon X \to \mathbb{R}$ is λ-measurable. Then for each $\varepsilon > 0$ there is a closed $F \subseteq X$ such that $\lambda(X \setminus F) < \varepsilon$ and $f \restriction F$ is continuous. Moreover, if X is Polish we can demand F to be compact. □

Counterexample 5.6. The σ-ideal $\mathcal{N}$ of λ-zero subsets of I is not σ-generated by closed sets. Otherwise, each for each $N \in \mathcal{N}$ we can find F_σ set N^* satisfying $N \subseteq N^* \in \mathcal{N}$. We can find A and B such that $I = A \dot{\cup} B$, A is comeager of measure 0 and B is meager of measure 1 (the Marczewski decomposition, see [7]). Take any F_σ set $A^* \supseteq A$ of measure 0. Then $I \setminus A^*$ is a meager dense (because it is of full measure) G_δ set. A contradiction with the Baire category theorem.

Having CRN and not being minimal, the random forcing must fail the property of shrinking continuous functions to constant or homeomorphic. Otherwise, descriptive arguments used in proving the minimality of $\mathfrak{S}$ and $\mathfrak{M}$ would work also for $\mathfrak{R}$.

Counterexample 5.7. Let $f\colon I^2 \to I^2, f(x,y) = (x,0)$. Then f is neither constant nor 1-1 on any Borel B of positive measure. Indeed, if f is constant on some B then B is contained in one vertical section, which has measure zero. If f is 1-1 on B then each section B_x is one-point, hence by Fubini theorem B has measure zero.

All these properties of the random forcing make it appear to be very close to the Cohen forcing. We look for something discerning the two notions:

Fact 5.8. The Cohen real c weakly dominates the ground model V, while the random real r does not. (see [3])

Proof: We see the forcing notions as collections of trees on ω. Let us first deal with $\mathfrak{C}$. Take any $f\colon \omega \to \omega$ from V and fix arbitrary $n_0 \in \omega$. For each $[s] \in \mathfrak{C}$ we can find a stronger condition $[t]$ satisfying $t(n) > f(n)$ for some $n > n_0$. Since n_0 was arbitrary we have $c(n) > f(n)$ for infinitely many n.

Now for $\mathfrak{R}$ we will find in V a function dominating r. By remark 5.2, λ is tight. That means $\{K \in \mathfrak{R} : [K] \text{ is compact}\}$ is a dense subset of $\mathfrak{R}$. Each such K has finite levels so we can take $f_K(n) > t(n)$ for all $t \in Lev_{n+1}(K)$. We obtain $f_K\colon \omega \to \omega$ from V, dominating all branches of K. Hence there exists such a function which is not weakly dominated by r. □

Remark 5.9. By theorem 4.3 the Miller real also weakly dominates V. The Sacks real does not, by the same argument as for random, as the Sacks forcing consists of compact sets.

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