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On Laver extension

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Abstract

We prove that uncountable subsets of 2^ω from the ground model are not strong null in the Laver extension. Then we show that countable support iteration of Laver forcing forces the Borel Conjecture. We find a subset of a product of Polish spaces with all vertical sections open dense, but with only countably many horizontal sections full in the sense of the Laver ideal.

0 Introduction

The main issue of this paper is the σ-ideal of strong measure zero sets in the Cantor space and the Laver forcing.

In section 2 we check if sets from the ground model have strong measure zero in the generic extension via Laver forcing and some other notions. From this point of view Laver and Cohen forcings act in opposite way.

In section 3 we prove the consistency of the Borel Conjecure, which says that all strong measure zero sets are countable. The proof is done by the countable support iteration of the Laver forcing.

In the last section we find a subset of a product of Polish spaces, with some "anti-Fubini" property. More precisely, all its vertical sections are open dense (so they are big in the Baire category sense), but only countably many of its horizontal sections are full in the sense of the Laver ideal, i.e. the σ-ideal connected to the Laver forcing.

We also state the fact that such a set exists for some other ideals, which are defined as the Fubini product of the Laver ideal and some $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ ideal.

1 Preliminaries

1.1 Basic definitions and notation

Definition 1.1. Let A (alphabet) be either 2 or ω.

- for $s \in A^{<\omega}$ let $[s] = \{t \in \omega^{<\omega}\colon\ t \subseteq s \text{ or } s \subseteq t\}$ and let $|s|$ denote the length of s.
- T is a tree if $T \subseteq A^{<\omega}$ and $(\forall s \in T)(\forall t \in A^{<\omega})(t \subseteq s \Rightarrow t \in T)$. We will consider only trees, in which each node has a proper extension (pruned trees).

- For a tree T we define $[T] = \{x \in A^\omega \colon (\forall n \in \omega)(x{\restriction}n \in T)\}$. An element of $[T]$ is called a branch of T. Pruned trees in $A^{<\omega}$ are in one-to-one corespondence with closed sets in A^ω (see [5]).
- If T is a tree then $Lev_n(T) = \{t \in T : |t| = n\}$ is the n-th level of T.
- For $s \in T$ we define $Succ_T(s) = \{t \in T\colon t \text{ is an immediate successor of } s\}$.
- $s \in T$ is a splitting node if $|Succ_T(s)| \geq 2$.
- $s \in T$ is an ω-splitting node if $|Succ_T(s)| = \aleph_0$.

We consider Cantor set 2^ω, or Baire space ω^ω with metric d, given by $d(x,y) = 2^{-min\{n\colon x(n)\neq y(n)\}}$ for $x \neq y$. For X a subset of a metric space, $diam(X)$ denotes its diameter. λ is Lebesque measure on 2^ω, i.e. the only measure given by $\lambda([s]) = diam([s]) = 2^{|s|}$ on basic clopen sets $[s]$, $s \in 2^{<\omega}$. If $\mathcal{I}$ is an ideal on a space X, $\mathcal{I}^c$ denotes its dual filter, i.e. $\mathcal{I}^c = \{Y \subset X\colon X \backslash Y \in \mathcal{I}\}$. For sets X, Y, $G \subseteq X \times Y$ and points $x \in X$, $y \in Y$ we define the vertical section of G at x by $G_x = \{y \in Y\colon (x,y) \in G\}$ and the horizontal section at y as $G^y = \{x \in X\colon (x,y) \in G\}$. For $X \subseteq 2^\omega$ and $t \in 2^\omega$ we have the translation $X + t = \{x + t,\ x \in X\}$, where the latter $+$ stays for addition in the Cantor set, i.e. coordinatewise addition modulo 2.

1.2 Strong measure zero sets

Definition 1.2. Let $X \subseteq \mathbb{R}$. X has strong measure zero (X is strong null, $X \in \mathcal{SN}$) iff $\forall (\varepsilon_n)_{n\in\omega}\ \exists (r_n)_{n\in\omega} \subseteq \mathbb{R}\ \ X \subseteq \bigcup_{n\in\omega} (r_n - \varepsilon_n, r_n + \varepsilon_n)$.

So X is strong null iff X can be covered by countably many arbitrarily small intervals. It is easy to see that each countable set is strong null and the family $\mathcal{SN}$ forms a σ-ideal (see [1]).

Now we will translate the notion of strong measure zero to subsets of the Cantor set.

Definition 1.3. Let $X \subseteq 2^\omega$. $X \in \mathcal{SN}$ iff
$\forall f \in \omega^\omega\ \ \exists g \colon \omega \to 2^{<\omega}\ (\forall n \in \omega\ \ g(n) \in 2^{f(n)} \wedge\ \forall x \in X\ \ \exists n \in \omega\ \ g(n) \subseteq x)$

The function $f \in \omega^\omega$ plays role of the sequence of ε_n's of the definition for the real line $\mathbb{R}$, $g(n)$ stays for n-th interval of "size" given by $f(n)$. Last part of the formula says that X is covered by $\bigcup_{n\in\omega} [g(n)]$. We can demand each x from X to be a member of infinitely many $[g(n)]$'s:

Fact 1.4. $X \in \mathcal{SN}$ iff
$\forall f \in \omega^\omega\ \ \exists g \colon \omega \to 2^{<\omega}\ (\forall n \in \omega\ \ g(n) \in 2^{f(n)} \wedge\ \forall x \in X\ \ \exists^\infty n \in \omega\ \ g(n) \subseteq x)$

Proof: The only implication to show is from left to right. Let $f \colon \omega \to \omega$. Take infinite sets A_i, $i \in \omega$ which form a partition of ω. Since X is strong null and $|A_i| = \omega$ we can find for each $i \in \omega$ a function $g_i \colon A_i \to 2^{<\omega}$ such that $g_i(n) \in 2^{f(n)}$ for $n \in A_i$ and $X \subseteq \bigcup_{n\in A_i} [g_i(n)]$. Now take $g = \bigcup_{i\in\omega} g_i \colon \omega \to \omega$. Then each $x \in X$ belongs to $[g(n)]$ for infinitely many n's. $\square$

A natural question is whether $\mathcal{SN}$ is just the same as the σ-ideal of measure zero sets. The answer is no, there holds the proper inclusion $\mathcal{SN} \subsetneq \mathcal{N}$. This is a consequence of the following lemma:

Lemma 1.5. Let $h: 2^\omega \to 2^\omega$ be a continous function. Let $X \subseteq 2^\omega$ be strong null. Then $h[X]$ is strong null.

Proof: As a continous function with compact domain, h is uniformly continous. Hence, for each $n \in \omega$ there exists $n^* \in \omega$ such that for each $Y \subseteq 2^\omega$ satisfying $diam(Y) \leq 2^{-n^*}$ we have $diam(h[Y]) \leq 2^{-n}$, so $h[Y]$ is contained in one basic clopen $[s]$, for some $s \in 2^n$.

Fix $f \in \omega^\omega$. We are looking for $g: \omega \to 2^{<\omega}$ as in definition 1.3. Since $X \in \mathcal{SN}$ we have $g^*: \omega \to 2^{<\omega}$, such that $g^*(n) \in 2^{f(n)^*}$ and $X \in \bigcup_{n\in\omega} [g^*(n)]$. By the uniform continuity, for each n we can find $g(n) \in 2^{f(n)}$ such that $h[[g^*(n)]] \subseteq [g(n)]$. Then $h[X] \subseteq \bigcup_{n\in\omega} [g(n)]$. □

The above proof in slightly different context (closed interval instead of the Cantor space) can be found in [1]. As a corollary we get a null set, which is not strong null.

Corollary 1.6. Define $X = \{x \in 2^\omega \colon \text{for each odd } n\ x(n) = 0\}$. Then $\lambda(X) = 0$ and $X \notin \mathcal{SN}$.

Proof: Take continous $h: 2^\omega \to 2^\omega$, $h(x)(n) = x(2n)$. If $X \in \mathcal{SN}$ then $h[X] = 2^\omega \in \mathcal{SN}$, which is a contradiction. □

2 Laver extension

2.1 Basic properties of the Laver forcing

Definition 2.1.

(i) We say that $T \subseteq \omega^{<\omega}$ is a Laver tree iff T is a tree and $\exists t \in T\ \forall s \in T\ (s \subseteq t \vee (t \subseteq s \wedge |Succ_T(s)| = \aleph_0))$.

(ii) For a Laver tree T, $stem(T)$ denotes the minimal splitting node of T, i.e. the only witness of the "$\exists t \in T$..." clause in the point (i) of this definition.

(iii) Define Laver forcing $\mathfrak{L} = \{T \subseteq \omega^{<\omega} : T \text{ is a Laver tree}\}$, with inclusion as ordering.

(iv) For $g: \omega^{<\omega} \to \omega$ we put $L_g = \{f \in \omega^\omega \colon \exists^\infty n \in \omega\ f(n) < g(f \restriction n)\}$. The Laver ideal is the σ-ideal generated by sets L_g over all possible g's.

We will refere to some properties of the poset $\mathfrak{L}$, which are described in [6]. Since trees with arbitrarily high *stem* are dense in $\mathfrak{L}$, the generic filter G correspondes to one real $g \in \omega^\omega$, called Laver real. Namely, $g = \bigcup \bigcap_{T\in G} T = \bigcup_{T\in G} stem(T)$ and $G = \{T \in \mathfrak{L} \colon g \in [T]\}$. Laver forcing, introduced as above, is forcingwise equivalent to $\mathcal{L}$-positive Borel sets in ω^ω ordered by inclusion $(Bor(\omega^\omega) \setminus \mathcal{L})$, and to the algebra of Borel sets modulo $\mathcal{L}$ $(Bor(\omega^\omega)/\mathcal{L})$(see [11]).

A common technique to deal with forcing notions consisting of trees is fusion. In the case of $\mathfrak{L}$ it looks as follows:

Definition 2.2.

(i) For $t \in T$ we put $T{\restriction}t = \{s \in T \colon s \subseteq t \text{ or } t \subseteq s\}$.

(ii) For $T \in \mathfrak{L}$ and $n \in \omega$ let $Split_n(T)$ be the n-th level of splitters in T, i.e. $Split_n(T) = Lev_{|stem(T)|+n}(T)$. So 0-th splitting level is $\{stem(T)\}$. 1-st splitting level consists of all immediate successors of $stem(T)$ and so on.

(iii) For each $n \in \omega$ we define an ordering $\leq_n$ by:
$T \leq_n S \Leftrightarrow stem(T) = stem(S)$ and T keeps first n levels of splitters. More precilesly:
$T \leq_n S \Leftrightarrow stem(T) = stem(S)$ and $Split_n(T) = Split_n(S)$.

(iv) $(T_n)_{n\in\omega}$ is a fusion sequence iff $\forall n \in \omega \;\; T_{n+1} \leq_n T_n$.

Next fact is a straightfoward consequence of this definition.

Fact 2.3. For a fusion sequence $(T_n)_{n\in\omega}$, its intersection $T_\infty = \bigcap_{n\in\omega} T_n$ is a condition belonging to $\mathfrak{L}$ and $T_\infty \leq_n T_n$ for each $n \in \omega$. $\square$

Using fusion argument we shall prove a very usefull lemma about $\mathfrak{L}$, called the pure extension property:

Lemma 2.4. Let $T \in \mathfrak{L}$ and φ be a formula of the forcing language. There is $T' \leq_0 T$ deciding φ ($T' \Vdash \varphi$ or $T' \Vdash \neg\varphi$).

Notice that $T' \leq_0 T$ means just $stem(T') = stem(T)$.
Proof: Suppose the lemma is false. Fix T and φ which are a counterexample. Define

$$T^+_\varphi = \{t \in T \colon \exists S \leq_0 T{\restriction}t \;\; S \Vdash \varphi\} \text{ and } T^-_\varphi = \{t \in T \colon \exists S \leq_0 T{\restriction}t \;\; S \Vdash \neg\varphi\}$$

Finally let $T_\varphi = T^+_\varphi \cup T^-_\varphi$. Note that our assumption about T and φ means exactly that $stem(T) \notin T_\varphi$.

We will contruct inductively a fusion sequence T_n such that no $S \leq \bigcap_{n\in\omega} T_n$ decides φ, contradicting basic facts of forcing theory. Take $T_0 = T$. Notice that $Split_1(T_0) \cap T_\varphi$ is finite. Otherwise without loss of generality we may assume $Split_1(T_0) \cap T^+_\varphi$ is infinite. For $t \in Split_1(T_0) \cap T^+_\varphi$ we pick $S_t \leq_0 T{\restriction}t$ which forces φ. Then $\bigcup_{t\in Split_1(T_0)\cap T^+_\varphi} S_t \leq_0 T_0$ and it forces φ. A contradiction.

Hence we can put $T_1 = \bigcup_{t\in Split_1(T_0)\setminus T_\varphi} T_0{\restriction}t \leq_0 T_0$.

Suppose we have T_n. By previous argument repeated at $T_n{\restriction}s$ for each s from $Split_n(T_n)$ we can define

$$T_{n+1} = \bigcup_{t\in Split_{n+1}(T_n)\setminus T_\varphi} T_n{\restriction}t$$

Then T_{n+1} is a condition in $\mathfrak{L}$ and $T_{n+1} \leq_n T_n$. The inductive step is done.

By construction, no condition strenghtening $\bigcap_{n\in\omega} T_n$ can decide φ. $\square$

2.2 Laver forcing and strong measure zero sets

In this part we consider uncountable sets of reals from the ground model and the $\mathcal{SN}$ ideal in the $\mathfrak{L}$-generic extension. We prove the following theorem:

Theorem 2.5. Suppose $X \subseteq 2^\omega \cap V$ is uncountable. Let G be $\mathfrak{L}$-generic over V. Then $V[G] \models X \notin \mathcal{SN}$.

Proof: We will show that Laver forcing adds a function $f\colon \omega \to \omega$ such that no sequence $(s(n))_{n\in\omega}$, $s(n) \in 2^{f(n)}$ satisfy the requirement presented in Fact 1.4. It comes out that Laver real is such a function.

Recall that Laver real $f = \bigcup_{T\in G} stem(T)$, where G is $\mathfrak{L}$-generic over V. Let $\dot{f}$ be the canonical name for Laver real. Notice that each condition $T \in \mathfrak{L}$ "knows" values of $\dot{f}(n)$ for $n < |stem(T)|$, i.e. $T \Vdash stem(T) \subseteq \dot{f}$.

Fix a tree $T \in \mathfrak{L}$. Let $(\dot{s}(n))_{n\in\omega}$ be a name for a sequence such that $\mathfrak{L} \Vdash \forall n \in \omega \ \ \dot{s}(n) \in 2^{\dot{f}(n)}$. We will find conditions $T''' \leq_0 T'' \leq_0 T' \leq_0 T$, applying at each step the fusion technique. The final T''' will satisfy

$$T''' \Vdash \exists x \in \check{X} \ \ \forall^\infty n \in \omega \ \ x \notin [\dot{s}(n)]$$

Step 1: We shall obtain T' with the property: for each node $t \supseteq stem(T')$, $T' \restriction t \Vdash \dot{s}_{|t|-1} = s_t$ for some $s_t \in 2^{t(|t|-1)}$ ($T \restriction t$ forces $t \subseteq \dot{f}$, in particular $T \restriction t \Vdash \dot{f}(|t|-1) = t(|t|-1)$). So we want $T' \restriction t$ to decide not only the value of $\dot{f}(|t|-1)$ but also of $\dot{s}_{|t|-1}$.

Inductively we will build a fusion sequence T_n and T' will be its intersection. We start with $T_0 = T$. Suppose we have T_n. Take $t \in Split_n(T_n)$. Applying Fact 2.4 (pure extention property) $2^{t(|t|-1)}$-many times, we find $T_n^t \leq_0 T_n \restriction t$ such that $T_n^t \Vdash \dot{s}_{|t|-1} = s_t$ for some $s_t \in 2^{t(|t|-1)}$. We put $T_{n+1} = \bigcup_{t\in Split_n(T_n)} T_n^t$. Then $T_{n+1} \leq_n T_n$. Finally, $T' = \bigcap_{n\in\omega} T_n$.

Step 2: Now we want to find $T'' \leq_0 T'$ such that for each splitting $t \in T''$ the sequence $(s_{t^\frown\langle m\rangle}\colon\ t^\frown\langle m\rangle \in T'')$ approximates some $x_t \in 2^\omega$, strictly speaking: $\forall n \in \omega \ \ \exists k \in \omega \ \ \forall m \in \{m \in \omega\colon\ t^\frown\langle m\rangle \in T'' \ \&\ m > k\} \ \ x_t \restriction n \subseteq s_{t^\frown\langle m\rangle}$.

Again we start the inductive process with $T'_0 = T'$. Suppose we have constructed T'_n. Fix $t \in Split_n(T'_n)$. Look at the subtree of $2^{<\omega}$ defined as $A_t = \{s \in 2^{<\omega}\colon\ \exists m \in \omega \ \ t^\frown\langle m\rangle \in T'_n \ \ \& \ \ s \subseteq s_{t^\frown\langle m\rangle}\}$. So we take $s_{t^\frown\langle m\rangle}$'s for all m extending t in T'_n and A_t consists of all their initial segments. Since A_t is infinite ($s_{t^\frown\langle m\rangle}$ has length m and t is ω-splitting) it has a branch. Hence, we can pick some $x_t \in [A_t]$ and a sequence $(m_i^t)_{i\in\omega}$ such that $t^\frown\langle m_i^t\rangle \in T'_n$ and $(s_{t^\frown\langle m_i^t\rangle})_{i\in\omega}$ approximate x_t. Now we define

$$T'_{n+1} = \bigcup_{t\in Split_n(T'_n)} \bigcup_{i\in\omega} T'_n \restriction t^\frown\langle m_i^t\rangle$$

Obviously $T'_{n+1} \leq_n T'_n$, so we can define $T'' = \bigcap_{n\in\omega} T'_n$.

Step 3: Now we look for $x \in X$ and $T''' \leq_0 T''$, which forces x to be only in finitely many clopens $[\dot{s}(n)]$. X is uncountable, so fix $x \in X$ different from each x_t for all splitting $t \in T''$. Now we shall construct T'''.

Put $T''_0 = T''$. Suppose we have T''_n. Take $t \in Split_n(T''_n)$. Since $x \neq x_t$ and the latter is approximated by $(s_{t^\frown\langle m^t_i\rangle})_{i\in\omega}$, there is $j_t \in \omega$ such that $s_{t^\frown\langle m^t_i\rangle} \nsubseteq x$ for $i > j_t$. We define

$$T''_{n+1} = \bigcup_{t\in Split_n(T''_n)} \bigcup_{i>j_t} T''_n \restriction t^\frown\langle m^t_i\rangle$$

Again $T''_{n+1} \leq_n T''_n$ and we take $T''' = \bigcap_{n\in\omega} T''_n$. It follows from the construction that $T''' \Vdash \forall n \geq |stem(T''')| \quad \check{x} \notin [\dot{s}_n]$. □

Remark 2.6. (i) In the above proof we do not care about $\dot{s}_n$ for $n < |stem(T)|$. Once T is fixed, we have no control on them. That is why we need the characterisation of strong null sets from Fact 1.4.
(ii) It is clear from the proof that Laver extension provides one witness (namely Laver real) which guarantee that all uncountable sets from the ground model are not strong null.

Now we will introduce Mathias forcing, which often behaves similarily to Laver forcing.

Definition 2.7. Define Mathas forcing as

$$\mathfrak{M} = \{\langle s, A\rangle\colon\ s \in [\omega]^{<\omega},\ A \in [\omega]^\omega \quad \&\quad max(s) < min(A)\}$$

with ordering

$$\langle t, B\rangle \leq \langle s, A\rangle \ \Leftrightarrow\ t \supseteq s \ \ \&\ \ B \subseteq A \ \ \&\ \ t \setminus s \subseteq A$$

Fact 2.8.
(i) $\mathfrak{M}$ has pure extension property. In this case it is formulated as follows: For any condition $\langle s, A\rangle \in \mathfrak{M}$ and formula φ there exists an infinite set $B \subseteq A$ such that $\langle s, B\rangle \Vdash \varphi$ or $\langle s, B\rangle \Vdash \neg\varphi$.
(ii) Working with $\mathfrak{M}$ we can use fusion. Define orderings

$$\langle t, B\rangle \leq_n \langle s, A\rangle \ \Leftrightarrow\ t = s \ \ \&\ \ \text{the first } n \text{ elements of } A \text{ are in } B$$

For a fusion sequence ... $\leq_{n+1} \langle s, A_{n+1}\rangle \leq_n \langle s, A_n\rangle \leq_{n-1} ... \leq_0 \langle s, A_0\rangle$ we have $\langle s, \bigcap_{n\in\omega} A_n\rangle \in \mathfrak{M}$.
(iii) The fact similar to Theorem 2.5 holds for Matias forcing $\mathfrak{M}$. □

Above statements are described and proved in [1].

From the "strong null point of view", notions $\mathfrak{L}$ and $\mathfrak{M}$ are antipodic to Cohen forcing $\mathfrak{C}$. The last one forces all sets from the ground model to have strong measure zero:

Proposition 2.9. $\mathfrak{C} \Vdash 2^\omega \cap \check{V} \in \mathcal{SN}$

Proof: We see Cohen forcing as finite partial functions from $\omega \times \omega$ into 2, i.e. $\mathfrak{C} = \{f\colon dom(f) \to 2\colon\ dom(f) \in [\omega \times \omega]^{<\omega}\}$. Take $\dot{f}$ forced to be a real in ω^ω. We must find names $\dot{s}_n$ such that $\mathfrak{C} \Vdash \dot{s}_n \in 2^{\dot{f}(n)}$ and each $x \in 2^\omega$ from the ground model is forced to be in some $[\dot{s}_n]$.

Let $\tilde{G}$ be the cannonical name for the generic filter. For $n \in \omega$ take $\dot{g}_n$, a name for an element of 2^ω, which satisfies $\mathfrak{C} \Vdash \forall i \in \omega \quad \dot{g}_n(i) = \bigcup \tilde{G}(n,i)$. So $\dot{g}_n$ is supposed to be n-th section of $\bigcup \tilde{G}$. For each $n \in \omega$ take a maximal antichain $\{p_i^n, i \in \omega\}$ (it is countable since $\mathfrak{C}$ has ccc) of conditions, which decide the value $\dot{f}(n)$, i.e. $p_i^n \Vdash \dot{f}(n) = m$ for some $m \in \omega$.

We want $\dot{s}_n$ to have the following property: For $i \in \omega$, $p_i^n \Vdash \dot{s}_n = \dot{g}_j \restriction m$, where $p_i^n \Vdash \dot{f}(n) = m$ and $j > n$ is such big that $dom(p_i^n) \subseteq j \times \omega$. Since $\{p_i^n, i \in \omega\}$ forms an antichain, we can find such a name.

Now take any $x \in 2^\omega \cap V$ and $p \in \mathfrak{C}$. Find n such that $dom(p) \subseteq n \times \omega$, and i that $p \parallel p_i^n$. Then $p' = p \cup p_i^n \leq p, p_i^n$. Let m be forced by p_i^n to be the value of $\dot{f}(n)$. Take $q \in \mathfrak{C}$, $q\colon \{j\} \times m \to 2$ given by $q(j,i) = x(i)$ for $i < m$. Then $p' \cup q \leq p$ and it forces $x \in [\dot{s}_n]$. □

3 Borel Conjecture

As we mentioned in section 1, every countable set of reals has strong measure zero. In we will work on the problem if this inclusion can be reversed.

Definition 3.1. "Borel Conjecture" (BC) is a sentence, which says that each strong null subset of reals is countable.

Borel Conjecture turns out to be independent of ZFC. It is not difficult to proof that BC fails in models of CH. Much harder is to find a model satisfying BC. It was done by R. Laver in [6].

Let us see that Borel Conjecture contradicts the Continuum Hypothesis.

Fact 3.2. ZFC + CH $\vdash \neg$BC

Proof: Construct a Luzin set $X \subseteq 2^\omega$, i.e. X is uncountable and $X \cap F$ is countable for every meager set F.

Take $\{F_\xi, \xi < \omega_1\}$ - ennumeration of all Borel meager sets. Pick by induction $x_\xi \in 2^\omega \setminus \bigcup_{\eta<\xi} F_\xi$ for $\xi < \omega_1$. Then $X = \{x_\xi, \xi < \omega_1\}$ is a Luzin set.

We claim $X \in \mathcal{SN}$: Fix $f \in \omega^\omega$ and $\{q_n, n \in \omega\}$ a countable dense subset of 2^ω. Set $X \setminus \bigcup_{n\in\omega} [q_n \restriction f(2n)]$ is countable (as the union is open dense). Let $\{x_n\}_{n\in\omega}$ ennumerate its elements. We have

$$X \subseteq \bigcup_{n\in\omega} [q_n \restriction f(2n)] \cup \bigcup_{n\in\omega} [x_n \restriction f(2n+1)]$$

which finishes the proof. □

Now we are going to proof the consistency of Borel Conjecture with ZFC. That will be done by countable support iteration of Laver forcing. We shall prove the following theorem:

Theorem 3.3. Suppose $V \models CH$. Let $\mathfrak{L}_{\omega_2}$ be the countable support iteration of poset $\mathfrak{L}$ of length ω_2. Let G_{ω_2} be $\mathfrak{L}_{\omega_2}$-generic over V. Then $V[G_{\omega_2}] \models$ BC.

Before we give the proof we must introduce some concepts and recall some facts. First of all we will define the Laver property:

Definition 3.4. A notion of forcing $\mathbb{P}$ has the Laver property iff it satisfies the condition: For each name $\dot{f}$, real $g \in \omega^\omega \cap V$ and $P \in \mathbb{P}$, if $P \Vdash \forall n \in \omega\ \dot{f}(n) \in g(n)$ then there is a condition $Q \leq P$ and a sequence $(S(n))_{n\in\omega} \in V$ such that $S(n) \subseteq g(n)$, $|S(n)| \leq n$ and $Q \Vdash \forall n \in \omega\ \ \dot{f}(n) \in S(n)$.

Laver property means that if a function $f \in V[G]$ is placed "under" some function g from the ground model ($f(n) < g(n)$ for each n), then there is a sequence $(S(n))_{n\in\omega} \in V$ such that $f \in \prod_{n\in\omega} S(n)$, and the product is "not too big" (each $S(n)$ has at most n elements).

It is well known that both Laver and Mathias forcings have the Laver property:

Fact 3.5. The notions $\mathfrak{L}$ and $\mathfrak{M}$ have the Laver property. □

For the proof for Laver forcing we refer to [6], proof for Mathias is described in [1]. We need one more Lemma, proved in [1] or [3]:

Lemma 3.6. The Laver property is preserved under countable support iteration of proper forcing notions. □

The concept of proper forcing is described in [3]. For our purposes it is enough to know that $\mathfrak{L}$ and $\mathfrak{M}$ are proper (see [2]). In fact, all the notions considered in this paper are proper. Properness is preserved under countable support iteration, and proper forcing does not collapse ω_1 (see [3] and [4]).

The last component we need to build the proof of Theorem 3.3 is the following Lemma:

Lemma 3.7. Let $\mathbb{P}$ be a forcing notion which satisfies the Laver property and G is $\mathbb{P}$-generic over V. Suppose $X \subseteq 2^\omega$ is in V and $V \models X \notin \mathcal{SN}$. Then $V[G] \models X \notin \mathcal{SN}$.

Proof: Take $f \in \omega^\omega \cap V$ witnessing that $X \notin \mathcal{SN}$. That means for each sequence $(s_n)_{n\in\omega}$, $s_n \in 2^{f(n)}$ there is $x \in X$ such that $x{\restriction}f(n) \neq s_n$ for all but finitely many n. Without loss of generality we may assume that f is strictly increasing.

Suppose a contrario that $V[G] \models X \in \mathcal{SN}$. Pick $g \in \omega^\omega \cap V$, $g(n) = f(n^2)$. Take a sequence $(s_n)_{n\in\omega} \in V[G]$ such that

$$V[G] \models \forall n\ \ s_n \in 2^{g(n)}\ \ \&\ \ \forall x \in X\ \ \exists^\infty n\ \ x{\restriction}g(n) = s_n$$

By Laver property, there is a sequence $(A_n)_{n\in\omega} \in V$ such that $s_n \in A_n$ and $|A_n| \leq n$, for all n. Take $(t_i)_{i\in\omega} \in V$, a sequence ennumerating all elements of $\bigcup_{n\in\omega} A_n$, with respect to their length (longer elements have larger indexes).

If $|t_i| = g(l) = f(l^2)$ then $i \leq 1+2+...+l \leq l^2$, so $|t_i| \geq f(i)$ (f is increasing). On the other hand $X \subseteq \{y \in 2^\omega\colon \exists^\infty k\ \ y{\restriction}k = t_k\}$ contradicting the assumption that f is the witness of $V \models X \notin \mathcal{SN}$. □

The above proof is written in [1]. Now we can prove the consistency of Borel Conjecture.

Proof of Theorem 3.3: It is enough to show that in $V[G]$ no subset of 2^ω of size $\aleph_1$ has strong measure zero. Suppose X is such a set. We can think X is a function from ω_1 into 2^ω.

Claim: $X \in V[G_\alpha]$ for some $\alpha < \omega_2$.

$V[G_\alpha]$ is the intermediate step of the iteration, (see [2]).

Once claim is proved, the theorem results easily from previous lemmata. By Theorem 2.5, $V[G_{\alpha+1}] \models X \notin \mathcal{SN}$. As a consequence of Fact 3.5 and Lemma 3.6 the notion $\mathfrak{L}_{\omega_2}$ has the Laver property. Since $V[G_{\omega_2}] = V[G_{\alpha+1}][G^*_{\omega_2}]$, where $G^*_{\omega_2}$ is $\mathfrak{L}_{\omega_2}$-generic over $V[G_{\alpha+1}]$, we have $V[G_{\omega_2}] \models X \notin \mathcal{SN}$.

It remains to prove the Claim.

Since $V \models CH$, $\mathfrak{L}_{\omega_2}$ satisfies $\aleph_2$-chain condition, as a countable support iteration of proper forcings of size $\aleph_1$ (see [3] or [9]). Take $\dot{X}$ a name for X. So $\mathfrak{L}_{\omega_2} \Vdash \dot{X}\colon \omega_1 \to 2^\omega$ (we know $\mathfrak{L}_{\omega_2}$ it does not collapse ω_1). For $\alpha < \omega_1$ and $n \in \omega$ let $A_{\alpha,n}$ be a maximal antichain of conditions, which decide the value $\dot{X}(\alpha)(n)$. By $\aleph_2$-chain condition, $|A_{\alpha,n}| \leq \aleph_1$. Hence there is $\alpha < \omega_2$ such that all conditions, which appeared in $A_{\alpha,n}$'s, have supports contained in α. So $\dot{X} \in \mathfrak{L}_\alpha$ and $X \in V[G_\alpha]$. $\square$

Remark 3.8. The same proof works for Mathias forcing. The only difference is that instead of Theorem 2.5 we need to use its analogue for $\mathfrak{M}$ (Fact 2.8). The whole proof is in [1].

4 Some properties of ideals

4.1 Laver ideal

In this subsection we will derive some "anti-Fubini" property for the Laver ideal $\mathcal{L}$. Strictly speaking, we will find a set in the product of Polish spaces, which has each vertical section open dense, and only countably many horizontal sections are full in the sense of the Laver ideal. The latter remains true even if we move each vertical section, in a manner that the whole movement is given by a Borel function. So by the Kuratowski-Ulam theorem (see [10]) the set is comeager.

The fact we are going to state and prove is somehow related to the Galvin--Mycielski-Solovay theorem. Let us formulate this theorem.

Theorem 4.1. (Galvin-Mycielski-Solovay)
Suppose $X \subseteq 2^\omega$. Then X has strong measure zero if and only if for all comeager $G \subseteq 2^\omega$, there exists $t \in 2^\omega$ such that $X + t \subseteq G$. $\square$

A short proof of the above is sketched in [8]. The theorem means that we can translate each strong null set to subset of any comeager set.

Now we will proof the existance of the announced set. Let $\omega^{\omega}\!\!\nearrow$ denote the subset of the Baire space, consisting of all strictly increasing functions. It is closed in ω^ω, hence it is Polish. We consider the Polish space $\omega^{\omega}\!\!\nearrow \times 2^\omega$.

Theorem 4.2. There exists a Borel $\tilde{G} \subseteq \omega^{\omega}\!\!\nearrow \times 2^\omega$ which satisfies:

(*i*) For each $x \in \omega^{\omega}\!\!\nearrow$ the section $\tilde{G}_x$ is open dense

(*ii*) For any Borel function $\tilde{t}\colon \omega^{\omega}\!\!\nearrow \to 2^\omega$ the set $\{y \in 2^\omega\colon\ \{x : y + \tilde{t}(x) \in \tilde{G}_x\} \in \mathcal{L}^c\}$ is countable.

Proof: Define $\tilde{G}$ by setting its vertical sections:

$$\tilde{G}_x = \{y \in 2^\omega\colon\ \exists n \in \omega\ \ y{\restriction}[x(n), x(n+1)) \equiv 0\},\ \text{ for each } x \in \omega^{\omega}\!\!\nearrow .$$

$[x(n), x(n+1))$ denotes the "interval" consisting of natural numbers greater or equall to $x(n)$ and smaller than $x(n+1)$ (recall that x is strictly increasing). $\tilde{G}$ is given by a Borel formula and obviously $\tilde{G}_x$ is open dense for each $x \in \omega^{\omega}\!\nearrow$, so we have (i).

Suppose (ii) is false. Pick a Borel function $\tilde{t}\colon \omega^{\omega}\!\nearrow \to 2^{\omega}$ such that the set $Y = \{y \in 2^{\omega} \colon \{x : y + \tilde{t}(x) \in \tilde{G}_x\} \in \mathcal{L}^c\}$ is uncountable. We think Y is a ground model set in the Laver extension. Let g be the Laver real over V. By Theorem 2.5, $V[g] \models Y \notin \mathcal{SN}$.

We need one more characterization of strong measure zero sets in 2^{ω}:

Claim: $X \in \mathcal{SN}$ iff for each $f \in \omega^{\omega}\!\nearrow$ there is a sequence $(s_n)_{n\in\omega}$ such that $s_n \in 2^{[f(n), f(n+1))}$ and $\forall x \in X \;\; \exists n \in \omega \;\; x\restriction[f(n), f(n+1)) = s_n$.

The proof is an easy exercise, which is left to the Reader.

Define $G \in V[g]$ as $G = \tilde{G}_g = \{y \in 2^{\omega} \colon \exists n \in \omega \;\; y\restriction[g(n), g(n+1)) \equiv 0\}$. Since g witnesses $Y \notin \mathcal{SN}$ (Remark 2.6) we have $Y + \tilde{t}(g) \nsubseteq G$. Otherwise the sequence $\langle \tilde{t}(g)\restriction[g(n), g(n+1))\rangle_{n\in\omega}$ would deny that g witnesses $Y \notin \mathcal{SN}$ in the Laver extension $V[g]$.

But for each $y \in Y$ the set $A_y = \{x : y + \tilde{t}(x) \in \tilde{G}_x\}$ is a Borel set coded in V, and it is full in sense of the Laver ideal $\mathcal{L}$. Hence, $g \in A_y$, since g avoids all Borel sets from $\mathcal{L}$, which are coded in V. That means $\forall y \in Y \;\; y + \tilde{t}(g) \in G$, so $Y + \tilde{t}(g) \subseteq G$. A contradiction. $\square$

Remark 4.3.

(i) The comeager set $G = \tilde{G}_g$ from the proof witnesses that no uncountable $Y \subseteq 2^{\omega}$ from the ground model V has strong measure zero in the Laver extension $V[g]$, according to the characterization of strong null sets from the Galvin-Mycielski-Solovay theorem. In other words, for any uncountable $Y \subseteq 2^{\omega}$, $Y \in V$ and for any $t \in 2^{\omega} \cap V[g]$ we have $Y + t \nsubseteq G$. Indeed, each real $t \in 2^{\omega} \cap V[g]$ is given as a value of some gound model Borel function $\tilde{t} \in V$ on the generic real g, i.e. $t = \tilde{t}(g)$ (see [12]). According to the previous proof we conclude that $Y + \tilde{t}(g) \nsubseteq G$.

(ii) The only reason why we take only Borel functions $\tilde{t}$, is that we need each sets A_y from the proof to be Borel. In such case it has a code which is a real from the ground model. Saying $g \in A_y$ we mean g belongs to the interpretation of the code of A_y in the generic extension.

4.2 Other ideals

In this subsection we will conclude facts analogous to Theorem 4.2 for some other σ-ideals. We will deal mostly with the Fubini products of ideals. Let us start with some definitions:

Definition 4.4. We say an ideal $\mathcal{I}$ on a Polish space X is $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$ if for every analytic $A \subseteq X \times X$ the set $\{x \in X \colon A_x \in \mathcal{I}\}$ is coanalytic.

Definition 4.5. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on a Polish spaces X and Y respectively. We define the ideal $\mathcal{I} * \mathcal{J}$ on the space $X \times Y$ by

$$\mathcal{I} * \mathcal{J} = \{A \subseteq X \times Y \colon \{x \in X \colon A_x \in \mathcal{J}\} \in \mathcal{I}^c\}$$

Fact 4.6. If the ideal $\mathcal{J}$ in the definition is $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$ then $Bor(X \times Y)/\mathcal{I} * \mathcal{J}$ is forcingwise equivalent to the iteration $Bor(X)/\mathcal{I} * Bor(Y)/\mathcal{J}$ (see [11]). $\square$

Let $\mathcal{M}$ stand for the σ-ideal on the Baire space ω^ω, which is σ-generated by compact sets. The notion $Bor(\omega^\omega)/\mathcal{M}$ is equivallent to the Miller forcing (see [7]). It is well known that Miller forcing is $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$ and satisfies the Laver property. Now we can prove a fact similar to the Theorem 4.2 for the ideal $\mathcal{L} * \mathcal{M}$:

Fact 4.7. There exists a Borel $\tilde{G} \subseteq \omega^{\omega\nearrow} \times 2^\omega$ with open dense vertical sections and for any Borel function $\tilde{t}\colon \omega^{\omega\nearrow} \to 2^\omega$ the set $\{y \in 2^\omega\colon\ \{x : y + \tilde{t}(x) \in \tilde{G}_x\} \in \mathcal{L}^c\}$ is countable.

Proof: The proof is similar to the one of the Theorem 4.2. The only difference is why the set Y is not strong null in the generic extension, given by the algebra of Borel sets modulo the ideal $\mathcal{L} * \mathcal{M}$. By Fact 4.6 the algebra is forcingwise equivalent to the Laver forcing iterated with the Miller forcing. So, the extension equals to $V[H_1][H_2]$, where H_1 is $\mathfrak{L}$-generic over V and H_2 is $Bor(\omega^\omega)/\mathcal{M}$-generic over $V[H_1]$.

The set Y from the proof is in V. By Theorem 2.5 $V[H_1] \models X \notin \mathcal{SN}$ Since the Miller forcing satisfies the Laver property, by Lemma 3.7 we obtain $V[H_1][H_2] \models X \notin \mathcal{SN}$. □

It is easy to observe, that the same fact holds for any ideal of the form $\mathcal{L}*\mathcal{I}$, where $\mathcal{I}$ is $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$ and $Bor(X)/\mathcal{I}$ satisfies the Laver property.

In case of the ideal of measure zero sets $\mathcal{N}$ we can not use Lemma 3.7, because random forcing $Bor(2^\omega)/\mathcal{N}$ does not have the Laver property. But there is a folklore fact, which say that for a subset of the Cantor space $Y \in V$ we have $V[G] \models Y \in \mathcal{SN}$ if and only if $V \models Y \in \mathcal{SN}$, where G is $Bor(2^\omega)/\mathcal{N}$-generic over V. Hence, the proof analogic to the one of Fact 4.7 works for the ideal $\mathcal{L}*\mathcal{N}$.

The fact holds also for the σ-ideal $\mathcal{V}$ of nowhere dense sets in ω^ω, equipped with the Vietoris topology. Indeed, $Bor(\omega^\omega)/\mathcal{V}$ is forcingwise equivalent to the Mathias forcing $\mathfrak{M}$. The proof of Theorem 4.2 works if we change the Laver real to the Mathias real. The same holds for $\mathcal{V} * \mathcal{M}$ and $\mathcal{V} * \mathcal{N}$.

References

[1] T. Bartoszyński, H. Judah, *Set theory: on the structure of the real line.*

[2] J. E. Baumgartner, *Iterated forcing.*

[3] M. Foreman, A. Kanamori, M. Magidor, *Handbook of Set Theory*, 2001.

[4] T. Jech, *Set Theory*, The Third Millenium Edition, Springer.

[5] A. S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag New York, 1995.

[6] R. Laver, *On the consistency of Borel's conjecture.*

[7] A. W. Miller, *Rational Perfect Set Forcing*, Contemporary Mathematics, 1984.

[8] J. Pawlikowski, *A characterization of strong measure zero sets*, Israel Journal of Mathematics, 1996.

[9] S. Shelah, *Proper and Improper Forcing*, Springer, 1998.

[10] S.M. Srivastava, *A Course on Borel Sets*, Springer-Verlag New York, 1998.

[11] J. Zapletal, *Descriptive Set Theory and Definable Forcing.*

[12] J. Zapletal, *Forcing idealized.*