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On the hyperspace dimension

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ON THE HYPERSPACE DIMENSION

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Abstract. We present a detailed proof of the theorem of M. Levin and J. T. Rogers, Jr. [12] that the hyperspace $C(X)$, where X is a 2-dimensional metric continuum, is not a C-space. Moreover, if X is a 1-dimensional hereditarily indecomposable metric continuum, then $C(X)$ either is 2-dimensional, or is not a C-space.

In the case of a non-metrizable continuum X we show that if $\dim X \geq 2$, then $\dim C(X) = \infty$ and if $\dim X = 1$ and X is hereditarily indecomposable, then $\dim C(X) \in \{2, \infty\}$.

1. Introduction

Throughout the paper all spaces are normal. We assume the reader is familiar with basics of dimension theory. We keep the notation from [5]. A continuum is a compact, connected Hausdorff space. By dimension we always mean the covering dimension dim. A continuum X is hereditarily indecomposable if for each subcontinua $A, B \subseteq X$ if $A \cap B \neq \emptyset$ then $A \subseteq B$ or $B \subseteq A$. If X is compact, then $K(X)$ is the hyperspace of all non-empty subcompacta of X, equipped with the Vietoris topology.

If X is metric then the Vietoris topology coincides with the one generated by the Hausdorff metric ρ_H (see, e.g. [7]). By $C(X)$ we denote the hyperspace of all non-empty subcontinua of X, with the topology inherited from $K(X)$. Given a family $\mathcal{F}$ of subsets of X, its order $ord(\mathcal{F})$ is the least $n \in \mathbb{N}$, such that the intersection of any $n+2$ pairwise distinct elements of $\mathcal{F}$ is empty. Define $\operatorname{mesh} \mathcal{F} = \sup\{\operatorname{diam} F \colon F \in \mathcal{F}\}$, where diam is the diameter. A light mapping means a continuous function with 0-dimensional fibers. The unit interval $[0,1]$ is denoted by I.

Definition 1.1. A space X is a C-space (or has property C) if and only if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$, such that each $\mathcal{V}_i$ is a family of pairwise disjoint open subsets of X, $\mathcal{V}_i \prec \mathcal{U}_i$ ($\mathcal{V}_i$ refines $\mathcal{U}_i$, i.e. $\forall\ V \in \mathcal{V}_i\ \exists U \in \mathcal{U}_i\ V \subseteq U$) and $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a cover of X.

It is easy to show that an F_σ subset of a C-space is a C-space. We refer to [5] for basic properties of C-spaces.

The notion of a C-space is close to the notion of weakly infinite dimensional space. Indeed, the following fact characterizes weakly infinite dimensional spaces:

Fact 1.2. *A space X is weakly infinite dimensional if and only if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of two-element open covers of X, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$, such*

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that each $\mathcal{V}_i$ *is a family of pairwise disjoint open subsets of* X*,* $V_i \prec \mathcal{U}_i$ *and* $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ *is a cover of* X*.*

We still do not have any counterexample, which would distinguish these two notions in the class of metric compact spaces.

We also know that countable dimensional metric separable spaces (i.e. countable unions of finite dimensional spaces) have the property C. So we have (for metric separable spaces):

$$\text{countable dimension} \Rightarrow \text{property } C \Rightarrow \text{weakly infinite dimension}$$

2. Hyperspaces of 2-dimensional metric continua

The purpose of this section is to present a full, self-contained proof of the following theorem.

Theorem 2.1 ([12, Theorem 1.4])**.**

(i) *If* X *is a metric continuum and* $\dim X \geq 2$*, then* $C(X)$ *is not a* C*-space.*

(ii) *Moreover, there is* $Y \in C(X)$ *of dimension* 1*, such that* $C(Y)$ *is not a* C*-space.*

This theorem was stated in [12] without a detailed proof. An idea of the proof comes from [13] and it uses the technique of dominating sequences introduced in [12].

Definition 2.2.

(i) A sequence of natural numbers $l = (l_1, l_2, \ldots)$ dominates a sequence of $\mathcal{V}_1, \mathcal{V}_2, \ldots$, where each $\mathcal{V}_i$ is a family of subsets of a topological space X iff for each $j \leq i$ and $V \in \mathcal{V}_i$ we have $\{V' \in \mathcal{V}_j : V' \cap V \neq \emptyset\}$ has at most l_j elements. In other words, each $V \in \mathcal{V}_i$ intersects at most l_j elements of $\mathcal{V}_j$ for $j \leq i$.

(ii) A sequence of natural numbers $l = (l_1, l_2, \ldots)$ dominates X iff for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X, there is a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ dominated by l, consisting of families of open sets such that $V_i \prec \mathcal{U}_i$ and $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a cover of X.

Proposition 2.3 ([12, Proposition 2.1])**.** *There exists a sequence* $l = (l_1, l_2, \ldots)$ *which dominates every compact metric* C*-space.*

Proof. For $m \in \mathbb{N}$, let $r(m)$ be a natural number for which there exists an arbitrarily small open cover of $\mathbb{R}^{2m+1}$ such that each element of the cover intersects at most $r(m)$ elements of the cover. Then we have arbitrarily small open covers with the same property for every m-dimensional compactum, since it can be embedded in $\mathbb{R}^{2m+1}$.

The sequence l is defined by induction: $l_1 = 1$ and $l_{i+1} = r(l_1 + l_2 + \ldots + l_i + 1)$.

Take a compactum X and a sequence of its open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$. By the property C there is a sequence $\mathcal{V}_1^*, \mathcal{V}_2^*, \ldots$ of disjoint open refinements of $\mathcal{U}_1, \mathcal{U}_2, \ldots$, such that $\bigcup_{i=1}^{\infty} \mathcal{V}_i^*$ is a cover of X. By compactness of X, we can assume that for some m all $\mathcal{V}_{m+1}^*, \mathcal{V}_{m+2}^*, \ldots$ are empty and $\mathcal{V}_1^*, \ldots, \mathcal{V}_m^*$ are finite. Finally, for $i \leq m$, we can obtain a family $\mathcal{V}_i$ by shrinking elements of $\mathcal{V}_i^*$ to closed subsets, so that $\bigcup_{i=1}^{m} \mathcal{V}_i$ still forms a cover of X.

For every $V \in \mathcal{V}_i$ we will find a closed cover $\mathcal{F}_V$ of V such that the sequence $\mathcal{V}_1', \mathcal{V}_2', \ldots, \mathcal{V}_n'$, where $\mathcal{V}_i' = \bigcup\{\mathcal{F}_V : V \in \mathcal{V}_i\}$, is dominated by l. This will finish the proof, since we can slightly enlarge closed sets from $\mathcal{V}_i'$ to open sets so that they refine $\mathcal{U}_i$ and add no new non-empty intersections (thus the enlarged families are still dominated by l).

We begin the induction putting $\mathcal{V}_1' = \mathcal{V}_1$. Now suppose we have already constructed families $\mathcal{V}_1', \mathcal{V}_2', \ldots, \mathcal{V}_i'$ which are dominated by $(l_1, l_2, \ldots, l_i)$.

Let $\mathcal{W} = \mathcal{V}_1' \cup \mathcal{V}_2' \cup \ldots \cup \mathcal{V}_i' \cup \mathcal{V}_{i+1}$. For $W \in \mathcal{W}$ take open $W' \supseteq W$ such that for every $j \leq i$ we have $ord(\{W' \colon W \in \mathcal{V}_j'\}) = ord(\mathcal{V}_j')$ (in particular this order is $\leq l_j$) and $ord(\{W' \colon W \in \mathcal{W}\}) = ord(\mathcal{W}) \leq l_1 + l_2 + \ldots + l_i + 1 =: m$. Again, we can get sets $\mathcal{W}'$ by gently enlarging elements of $\mathcal{W}$ so that no new non-empty intersections appear.

For $W \in \mathcal{W}$ take a continuous function $f_W \colon X \to I$ such that $f_W^{-1}(1) = W$ and $f_W^{-1}(0) = X \setminus W'$. Let $f = \{f_W\}_{W \in \mathcal{W}} \colon X \to I^{|\mathcal{W}|}$ be the product function. We have $ord(\{f[W] \colon W \in \mathcal{W}\}) \leq m$ and $ord(\{f[W] \colon W \in \mathcal{V}_j'\}) \leq l_j$ for $j \leq i$. Define M as the image of X under f. Note that each point of M has at most m non-zero coordinates, so $\dim M \leq m$.

Let $\mathcal{M}$ be a finite closed cover of M such that each element of $\mathcal{M}$ intersects at most $r(m)$ elements of $\mathcal{M}$ and at most l_j elements of $\{f[W] \colon W \in \mathcal{V}_j'\}$ for $j \leq i$. Such a cover exists: we can start with an open cover, whose elements intersect at most l_1 elements of $\{f[W] \colon W \in \mathcal{V}_1'\}$, then take its refinement, whose elements intersect at most l_2 elements of $\{f[W] \colon W \in \mathcal{V}_2'\}$, and so on. At the final step take a refinement $\mathcal{M}^*$ whose elements intersect at most $l_{i+1} = r(m)$ elements of $\mathcal{M}^*$ (since $\dim \mathcal{M} \leq m$) and shrink it to a closed cover $\mathcal{M}$.

Now for $V \in \mathcal{V}_{i+1}$ we define $\mathcal{F}_V = \{V \cap f^{-1}[F] \colon F \in \mathcal{M},\ F \cap f[V] \neq \emptyset\}$. It follows that $\mathcal{V}_1', \mathcal{V}_2', \ldots, \mathcal{V}_{i+1}' = \bigcup\{\mathcal{F}_V \colon V \in \mathcal{V}_{i+1}\}$ is dominated by $(l_1, l_2, \ldots, l_{i+1})$. □

Now we begin to follow [13].

Proposition 2.4 ([13, Theorem 1.1])**.** *For each n-dimensional compact metric space X there exists an n-dimensional hereditarily indecomposable metric continuum Y and a light mapping $f \colon Y \to X$.*

Proof. There exists an n-dimensional hereditarily indecomposable continuum $Y \subseteq X \times I$ (since $\dim X \times Y = n+1$, see [2]). Let $f = \pi_X \restriction Y \colon Y \to X$ be the restriction of the projection $\pi_X \colon X \times I \to X$ to the subspace Y. We claim f is light: A component of a fiber of f is a subcontinuum of Y, hence it is indecomposable. On the other hand, it is a subcontinuum of I, so it must be a singleton. By compactness, it means each fiber of f is 0-dimensional. □

We recall the definition of a Whitney mapping for a continuum Y. It is a continuous function $W \colon C(Y) \to \mathbb{R}$ such that $W(\{y\}) = 0$ for all $y \in Y$ and $W(A) < W(B)$ if $A \subsetneq B$. We can construct such a mapping for any metric continuum Y (see [7]).

We collect some well-known facts and easy observations in the following remark.

Remark 2.5. Assume that Y is a hereditarily indecomposable metric continuum and $W \colon C(Y) \to \mathbb{R}$ is a Whitney mapping. Then:

(*i*) For $t \in [0, W(t)]$, $\mathcal{K}_t = W^{-1}(t)$ is a decomposition of Y into subcontinua.

(*ii*) $\mathcal{K}_t$ is closed in $C(Y)$. Hence the quotient topology τ_q on $\mathcal{K}_t$ (i.e. τ_q is the maximal topology such that the quotient mapping $q \colon Y \to \mathcal{K}_t$ is continuous) equals the topology τ_c inherited from $C(Y)$.

(*iii*) If $t \in (0, W(t)]$ then, by compactness, we have $\inf\{\operatorname{diam} K \colon K \in \mathcal{K}_t\} > 0$.

Definition 2.6. For a metric space Z we say $d_1(Z) < \epsilon$ (the 1-dimensional degree is less than ϵ) if for every component C of Z we have $\operatorname{diam} C < \epsilon$.

Now we are ready to prove a crucial lemma.

Lemma 2.7 (cf. [13, Lemma 1.3])**.** *Let Y and X be continua and let $f: Y \to X$ be a light mapping. Let $\mathcal{K}_t = W^{-1}(t)$ for some $t \in (0, W(Y)]$, where $W: C(Y) \to \mathbb{R}$ is a Whitney mapping. Define $g: Y \to C(X)$ by $g(y) = f[q(y)]$ where $q: Y \to \mathcal{K}_t$ is the quotient mapping. If $\mathcal{H} = g[Y] \subseteq C(X)$ is a C-space, then for every $\epsilon > 0$ there exists a closed $Z \subseteq Y$ such that $d_1(Z) < \epsilon$ and $Z \cap K \neq \emptyset$ for every $K \in \mathcal{K}_t$.*

Proof. By Proposition 2.3, there exists $l = (l_1, l_2, \ldots)$ which dominates $\mathcal{H}$. In particular, for every sequence of positive reals $\delta_1, \delta_2, \ldots > 0$ there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$, where $\mathcal{V}_i$ is a familiy of open subsets of $\mathcal{H}$, $\operatorname{mesh} \mathcal{V}_i < \delta_i$ and each $V \in \mathcal{V}_i$ intersects at most l_j elements of $\mathcal{V}_j$ for $j \leq i$.

Fix $\epsilon > 0$. Define $\alpha = \inf\{\operatorname{diam} A : A \in \mathcal{H}\} = \inf\{\operatorname{diam} f[K] : K \in \mathcal{K}_t\}$. By Remark 2.5 (iii) and the lightness of f, we have $\alpha > 0$. It is easy to find by induction a sequence of reals $\delta_1 > \delta_2 > \ldots > 0$ satisfying:

(i) if $T \subseteq X$ and $\operatorname{diam} T < \delta_1$ then $d_1(f^{-1}[T]) < \epsilon$. Such δ_1 exists by the lightness of f.

(ii) $\sum_{j=1}^{i} 3\delta_j l_j < \alpha(1 - \frac{1}{2^i})$. This is needed because of the following reason: We want

$(*)$ for any $A \in \mathcal{H}$ and $B_1^1, B_2^1, \ldots, B_{l_1}^1, B_1^2, B_2^2, \ldots, B_{l_2}^2, \ldots, B_1^i, B_2^i, \ldots, B_{l_i}^i$ such that $\operatorname{diam} B_k^j < 3\delta_j$ we have $A \setminus \bigcup_{j=1}^{i} \bigcup_{k=1}^{l_j} B_k^j \neq \emptyset$.

Since each $A \in \mathcal{H}$ is a continuum of diameter $\geq \alpha$, inequality (ii) implies $(*)$.

Now take a sequence $\{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m\}$ dominated by l such that each $\mathcal{V}_i$ is finite, $\operatorname{mesh} \mathcal{V}_i < \delta_i$ and $\bigcup_{i=1}^{m} \mathcal{V}_i$ covers $\mathcal{H}$ (everything is finite by compactness). In particular each $V \in \mathcal{V}_i$ intersects at most $N_i = l_1 + l_2 + \ldots + l_i$ sets from $\bigcup_{j=1}^{i} \mathcal{V}_i$. Observe that:

(iii) for each i, $A \in \mathcal{H}$ and $x \in X$ such that $x \in A \in V \in \mathcal{V}_i$, we have $\bar{B}(x, \delta_i)$ (the closed ball in X with center at x and radius δ_i) intersects every $B \in V$, since $\operatorname{diam} V < \delta_i$ (diameter with respect to the Hausdorff metric in $\mathcal{H} \subseteq C(X)$.)

Enumerate $\bigcup_{i=1}^{m} \mathcal{V}_i$ as $\{V_1, V_2, \ldots, V_M\}$ listing at first elements of $\mathcal{V}_1$, then elements of $\mathcal{V}_2$ and so on. For $V \in \bigcup_{i=1}^{m} \mathcal{V}_i$ let $i(V)$ denote the first i such that $V \in \mathcal{V}_i$. So this enumeration satisfies: if $k \leq l$ then $i(V_k) \leq i(V_l)$.

We will construct Z inductively as a union $Z = \bigcup_{j=1}^{M} W_j$, where W_j's are closed, pairwise disjont and $d_1(W_j) < \epsilon$.

For $j = 1$ pick some $A_1 \in V_1$ and $x_1 \in A_1$. Put $W_1 = f^{-1}[\bar{B}(x_1, \delta_{i(V_1)})] \cap g^{-1}[V_1]$. Assume we have $W_1, \ldots, W_{j-1}$ constructed as $W_k = f^{-1}[\bar{B}(x_k, \delta_{i(V_k)})] \cap g^{-1}[V_k]$. Take $A_j \in V_j$. Let $I_j = \{k < j : V_k \cap V_j \neq \emptyset\}$. So I_j has at most $N_{i(V_j)} = l_1 + l_2 + \ldots + l_{i(V_j)}$ elements. By (ii) we can find $x_j \in A_j \setminus \bigcup_{k \in I_j} \bar{B}(x_k, 3\delta_{i(V_k)})$. Define $W_j = f^{-1}[\bar{B}(x_j, \delta_{i(V_j)})] \cap g^{-1}[V_j]$. We claim $W_j \cap W_k = \emptyset$ for $k < j$. Indeed, if $k \in I_j$ then $\bar{B}(x_j, \delta_{i(V_j)}) \cap \bar{B}(x_k, \delta_{i(V_k)}) = \emptyset$ and if $k \notin I_j$ then $V_j \cap V_k = \emptyset$.

By (i) we have $d_1(W_j) < \epsilon$, hence $d_1(Z) < \epsilon$.

By *(iii)* the set W_j intersects every $K \in \mathcal{K}$ such that $f[K] \in V_j$. Since $V_1, V_2, \dots, V_M$ cover $\mathcal{H}$, the set Z intersects every $K \in \mathcal{K}_t$. $\square$

Now we are ready to prove the first part of Theorem 2.1 repeating the proof in [13]:

Proof of Theorem 2.1(i). It is enough to show that for any 2-dimensional metric continuum X, the hyperspace $C(X)$ is not a C-space. Suppose $C(X)$ has property C. By Proposition 2.4, there exist 2-dimensional hereditarily indecomposable continuum Y and a light mapping $f\colon Y \to X$. Fix also a Whitney mapping $W\colon C(Y) \to \mathbb{R}$.

There are closed disjoint sets $F_1, F_2 \subseteq Y$ and $r > 0$ such that every partition $L \subseteq Y$ between F_1 and F_2 satisfies $d_1(L) > r$. Otherwise, by compactness of 2^Y, for each pair of closed disjoint subsets of Y, we could find a 0-dimensional partition between them. But that means $\dim Y \leq 1$ (see [5]).

Take $t \in (0, W(Y)]$ so small that $\operatorname{diam} A < r$ for any $A \in \mathcal{K}_t = W^{-1}(t)$. Such t exists. Otherwise, by compactness of $C(Y)$ and continuity of W, there is $A \in C(Y)$ that $\operatorname{diam} A \geq r$ and $W(A) = 0$. But $W^{-1}(0)$ consists of singletons.

Set $\epsilon = \inf\{\rho(y_1, y_2)\colon y_1 \in F_1, y_2 \in F_2\}$, where ρ is a metric in Y. Define $g\colon Y \to C(X)$ as in Lemma 2.7. As $\mathcal{H} = g[Y]$ is closed in $C(X)$, it is a C-space. By Lemma 2.7, we get closed $Z \subseteq Y$ which intersects each $K \in \mathcal{K}_t$ and $d_1(Z) < \epsilon$.

Write Z as a union of closed, pairwise disjoint sets $Z_1, Z_2, \dots, Z_m$ of diameters less than ϵ.

Put $F_1' = F_1 \cup \bigcup\{Z_i\colon Z_i \cap F_1 \neq \emptyset\}$ and $F_2' = F_2 \cup \bigcup\{Z_i\colon Z_i \cap F_1 = \emptyset\}$. Then F_1', F_2' are closed disjoint. Take a partition L between F_1' and F_2'. So $Z \cap L = \emptyset$. But L is also a partition between F_1 and F_2, hence $d_1(L) \geq r$. Let L_1 be a component of L of diameter $\geq r$. Pick $A \in \mathcal{K}_t$ intersecting L_1 (recall $\mathcal{K}_t$ is a decomposition of Y). Summarizing, A and L_1 are subcontinua of the hereditarily indecomposable continuum Y and $A \cap L_1 \neq \emptyset$. Hence $A \subseteq L_1$ or $L_1 \subseteq A$. Since $\operatorname{diam} A < r$ and $\operatorname{diam} L_1 \geq r$, it must be the case $A \subseteq L_1$. Moreover, $Z \cap A \neq \emptyset$ since Z intersects each element in $\mathcal{K}_t$. Concluding,

$$\emptyset \neq Z \cap A \subseteq Z \cap L_1 \subseteq Z \cap L = \emptyset,$$

a contradiction. $\square$

Remark 2.8. In fact, we showed in the above proof that for sufficently small $t > 0$, the space $W^{-1}(t) \subseteq C(X)$ is not a C-space. We will use this fact later, while dealing with 1-dimensional hereditarily indecomposable continua.

The second part of Theorem 2.1 says that 1-dimensional subcontinua of higher-dimensional metric continua are responsible for the lack of property C in the hyperspace of subcontinua. It is obtained in the same way as a similar result in [11], which says that each 2-dimensional metric continuum X contains a 1-dimensional continuum Y with $\dim C(Y) = \infty$.

Let us quote the main result of [11] without a proof.

Proposition 2.9 ([11, Theorem 3.1])**.** *Every 2-dimensional compact metric space X contains a 1-dimensional subcontinuum Y which admits an open monotone mapping $q : Y \to T$ onto some continuum T of dimension ≥ 2.*

Now, the second part of Theorem 2.1 follows as a corollary to the first part and the above proposition:

Proof of Theorem 2.1(ii). Take $Y \subseteq X$, a continuum T and $q : Y \to T$ as in Proposition 2.9. Then $\dim T \leq 2$, hence by Theorem 2.1(i), we have $C(T)$ is not a C-space. Define $q^* \colon C(T) \to C(Y)$ by $q^*(D) = q^{-1}[D]$. Since q is monotone and onto, q^* is well-defined and injective. Moreover, q is open, hence q^* is continuous. So q^* is an embedding. That implies $C(Y)$ is not a C-space. □

3. Hyperspaces of 1-dimensional metric continua

Eberhart and Nadler showed in [3] that the hyperspace $C(X)$ of a 1-dimensional hereditarily indecomposable metric continuum X is either two or infinite dimensional. The assumption of hereditary indecomposability is essential. For example, it is easy to see that the hyperspace of a 3-od is 3-dimensional (by an n-od we mean the union of n arcs with one common end-point).

In this section we strengthen the result of [3].

Theorem 3.1. *Let X be a 1-dimensional hereditarily indecomposable metric continuum. Then $C(X)$ is either 2-dimensional or it is not a C-space.*

Outside the class of hereditarily indecomposable continua we can find a 1-dimensional metric continuum X such that $C(X)$ is infinite dimensional, but it is a C-space, since it is countable dimensional.

Example 3.2. We start with the interval $[0, 1]$. For $n > 1$ we replace each point $\frac{1}{n} \in [0, 1]$ with an n-od T_n of diam $< \frac{1}{n}$ and each interval $(\frac{1}{n+1}, \frac{1}{n})$ with a space X_n which is homeomorphic to $(\frac{1}{n+1}, \frac{1}{n})$ such that $cl(X_n) = T_{n+1} \cup X_n \cup T_n$. One can easily check that for the resulting continuum X the hyperspace $C(X)$ is infinite dimensional, countable dimensional.

As it was done in the previous section, we obtain the theorem modifying the proof of its weaker version [3, Theorem 2].

Proof of Theorem 3.1. Let X be a 1-dimensional hereditarily indecomposable metric continuum. By [3, Theorem 1], we have $\dim C(X) > 1$. We will show that if $\dim C(X) > 2$ then $C(X)$ is not a C-space.

So assume that the dimension of $C(X)$ is greater than 2. Let $W_X \colon C(X) \to \mathbb{R}$ be a Whitney mapping for X. By the theorem on dimension-lowering mappings [4, Theorem 1.12.4, p.136], there is $t_0 > 0$ with $\dim W_X^{-1}(t_0) \geq 2$ (observe that $\dim W_X^{-1}(0) = 1$ because $W_X^{-1}(0)$ is homeomorphic to X).

Continuum X is hereditarily indecomposable, so $W_X^{-1}(t)$ is a decomposition of X for any $t \in [0, W_X(X)]$. Let $Y = W_X^{-1}(t_0)$ and $q_0 \colon X \to Y$ be the quotient mapping. We know q_0 is open and monotone. Hence Y is also a hereditarily indecomposable continuum. Take a Whitney mapping $W_Y \colon C(Y) \to \mathbb{R}$. Since $\dim Y \geq 2$, by Remark 2.8 there exists $t_1 > 0$ such that $Z = W_Y^{-1}(t_1)$ is not a C-space. Again, Z is a decomposition of Y and the quotient mapping $q_1 \colon Y \to Z$ is open and monotone.

Summarizing, we have an open and monotone mapping $q = q_1 \circ q_0$ from X onto a non-C-space Z. Then $Z' = \{q^{-1}(z) \colon z \in Z\}$ is a subspace of $C(X)$ homeomorphic to Z (by Remark 2.5(ii) the decomposition topology on Z' coincides with the one inherited from $C(X)$, and the decomposition space is homeomorphic to Z (see [10, p. 184]). We know Z' is compact, hence closed in $C(X)$ and it is not a C-space, so X cannot be a C-space. □

4. Hyperspaces of non-metric continua

In this section we generalize the results of [13] and [3] to non-metric continua. Namely, we give a proof of the following theorem:

Theorem 4.1.
(i) Suppose X is a (non-metric) continuum of dimension ≥ 2. Then $\dim C(X) = \infty$.
(ii) Suppose X is a 1-dimensional hereditarily indecomposable (non-metric) continuum. Then either $\dim C(X) = 2$ or $\dim C(X) = \infty$.

We use the technique of lattices and Wallman representations as well as some approach of modern set theory, as it was done in [1]. There appear some facts, which are well known. However, we give their proofs for reader's convenience.

4.1. **Lattices and Wallman spaces.**

Definition 4.2. A structure $(L, \sqcup, \sqcap, 0_L, 1_L)$ is a distributive and separative lattice iff $\sqcup, \sqcap \colon L \times L \to L$, $0_L, 1_L \in L$ and for all $a, b, c \in L$ the following hold:

- (i) $a \sqcup b = b \sqcup a$ and $a \sqcap b = b \sqcap a$ *(commutativity)*,
- (ii) $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$ and $a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c$ *(associativity)*,
- (iii) $a \sqcup a = a$ and $a \sqcap a = a$ *(idempotence)*,
- (iv) $a \sqcup 0_L = a$ and $a \sqcup 1_L = 1_L$, $a \sqcap 0_L = 0_L$ and $a \sqcap 1_L = a$ *(zero and unit)*,
- (v) $a \sqcup (a \sqcap b) = a$ and $a \sqcap (a \sqcup b) = a$ *(absorption)*,
- (vi) $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$ and $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$ *(distributivity)*,
- (vii) $a \sqcap b = a \vee \exists d \in L\, (d \neq 0_L \wedge d \sqcap a = d \wedge d \sqcap b = 0_L)$ *(separativity)*.

We will always assume that a lattice is distributive and separative.

For a compact T_2 space X we consider the lattice 2^X of closed subsets of X with $\cup$ and $\cap$ as lattice operations, $0_L = \emptyset$ and $1_L = X$. On the other hand each lattice L corresponds to the Wallman space wL consisting of all ultrafilters on L. For $a \in L$ let $\bar{a} = \{u \in wL \colon a \in u\}$. We define the topology in wL taking the family $\{\bar{a} \colon a \in L\}$ as a base for closed sets.

It is easy to show that $w2^X$ is homeomorphic to X. Many lattices may correspond to one space:

Fact 4.3. *If $\mathcal{F}$ is a base for closed sets in X which is closed under finite unions and intersections (so $\mathcal{F}$ is a lattice), then $w\mathcal{F}$ is homeomorphic to X.*

Proof. We define the homeomorphism $h \colon X \to w\mathcal{F}$ in the only possible way: $h(x) = \{F \in \mathcal{F} \colon x \in F\}$. It is not difficult but tedious to verify that h is a well-defined homeomorphism indeed. We leave it as an exercise. □

Definition 4.4. A lattice L is normal iff

$$L \models \forall a, b (a \sqcap b = 0_L \to \exists c, d (c \sqcup d = 1_L \wedge c \sqcap a = 0_L \wedge d \sqcap b = 0_L))$$

The next fact is an easy observation (see, e.g. [14, Theorem 2.2]).

Fact 4.5. *L is normal if and only if wL is Hausdorff.*

Proof. (only if): Let u, v be distinct ultrafilters in wL. Then there are $a, b \in L$ that $a \in u$, $b \in v$ and $a \sqcap b = 0_L$. Hence, there are $c, d \in L$ as in the definition of normality. Sets $wL \setminus \bar{c}$, $wL \setminus \bar{d}$ are open disjoint, and separate u and v in wL.

(if): Let $a, b \in L$, $a \sqcap b = 0_L$. Take ultrafilters $u, v \in wL$ such that $a \in u$, $b \in v$. Obviously $u \neq v$ so there are disjoint open sets U, V separating u, v. We can take

U, V as complements of basic closed sets, so $U = wL \setminus \bar{c}$, $V = wL \setminus \bar{d}$ for some $c, d \in L$. Then c, d satisfy $c \sqcup d = 1_L \wedge c \sqcap a = 0_L \wedge d \sqcap b = 0_L$. □

Remark 4.6. A sublattice L' of L yields the continuous surjection $q\colon wL \to wL'$, given by $q(u) = u \cap L'$.

4.2. **Proof of Theorem 4.1.**

The proof is rather simple, but it uses some set-theoretic framework. We deal with some inner model of (large enough fragment of) ZFC and its countable elementary submodel.

Our strategy is to bring the non-metric case to the metric one: Suppose X is a non-metric continuum. We will find a countable sublattice $L \subseteq 2^X$ such that wL is a metric continuum, $\dim wL = \dim X$ and $\dim C(wL) = \dim C(X)$. Also wL is hereditarily indecomposable if and only if such is X.

We apply the technique used in [1] to find the sublattice L.

For an infinite cardinal κ, $H(\kappa)$ is the set of all sets x, such that $|TC(x)| < \kappa$. (TC is the transitive closure, i.e. $TC(x) = x \cup \bigcup x \cup \bigcup\bigcup x \cup \ldots$). If κ is regular then $H(\kappa)$ is a model of ZFC minus the Power Set Axiom (see [8, p. 162]). But if κ is large enough there are power sets in $H(\kappa)$ for all sets we need.

Let X be a (non-metric) continuum. Fix a suitably large regular cardinal κ (it is enough if $\mathcal{P}(\mathcal{P}(X)) \in H(\kappa)$). Take a countable elementary submodel $\mathcal{M} \prec H(\kappa)$, which contains X (use the Löwenheim-Skolem theorem). Then $\mathcal{M}$ also models a lot of ZFC and every finite subset of $\mathcal{M}$ belongs to $\mathcal{M}$. Denote $L = 2^X \cap \mathcal{M}$. By elementarity, L is a countable sublattice of 2^X.

Fact 4.7. *wL is a metric continuum.*

Proof. The lattice 2^X is normal, so L is normal by elementarity. By Fact 4.5, wL is T_2. It follows from Remark 4.6 that wL is a continuous image of $w2^X$ which is homeomorphic to X (Fact 4.3). Hence wL is a continuum. Since $\mathcal{M}$ is countable, wL admits a countable basis, so it is metrizable. □

We fix the countable open base $\mathcal{B} = \{wL \setminus \bar{F}\colon\ F \in L\}$ for wL. Notice that $\mathcal{B}$ is closed under finite unions and intersections.

Proposition 4.8. $\forall n \in \mathbb{N}\ (\dim X \leq n \ \Leftrightarrow \dim wL \leq n)$.

Proof. The idea of the proof resembles the one of Propositions 2.1 and 2.2 in [1], where the chainability was considered in a similar context.

($\Rightarrow$): Let $\mathcal{U}$ be a finite open cover of wL. We want to find a finite open refinement $\mathcal{V} \preceq \mathcal{U}$ of order $\leq n$ which covers wL. We can find a finite open cover $\mathcal{U}' = \{U_1, U_2, \ldots, U_m\} \preceq \mathcal{U}$ of wL, which consists of sets from the base $\mathcal{B}$. For $i \leq m$ let $F_i \in L$ be such that $U_i = wL \setminus \bar{F}_i$ and $\mathcal{F}' = \{F_1, F_2, \ldots, F_m\}$. Since $\mathcal{F}' \subseteq \mathcal{M}$ and $\mathcal{F}'$ is finite, $\mathcal{F}' \in \mathcal{M}$. Moreover, $\mathcal{U}'$ being a cover, gives $F_1 \cap F_2 \cap \ldots \cap F_m = \emptyset$.

The spaces X and $w2^X$ are homeomorphic, thus $H(\kappa) \models \dim w2^X \leq\ n$, i.e. it is true in $H(\kappa)$ that each finite open cover of $w2^X$ has a finite open refining cover of order $\leq n$. In terms of lattices it means that $H(\kappa)$ models the following sentence φ:

(φ)

For every finite subset $F \subseteq 2^X$ such that $\bigcap \mathcal{F} = \emptyset$ there exists $\mathcal{G}$ which satisfies:

1. $\mathcal{G}$ is a finite subset of 2^X,
2. $\bigcap \mathcal{G} = \emptyset$,
3. $\forall G \in \mathcal{G}\ \exists F \in \mathcal{F}\ (F \subseteq G)$,
4. For each pairwise distinct sets $G_1, G_2, \ldots, G_{n+2} \in \mathcal{G}$ we have $G_1 \cup G_2 \cup \ldots \cup G_{n+2} = X$.

(This is just the definition of dim $\leq n$, translated to closed sets.) As an elementary substructure of $H(\kappa)$, the model $\mathcal{M}$ also satisfies φ. Take $\mathcal{F}'$ for $\mathcal{F}$ in φ. Since $\mathcal{F}' \in \mathcal{M}$, there is $\mathcal{G} \in \mathcal{M}$ which satisfies (1), (2), (4) and (3) with $\mathcal{F}'$ in place of $\mathcal{F}$. Then $\mathcal{V} = \{wL \setminus \bar{G} \colon G \in \mathcal{G}\}$ is a finite cover of wL, which refines $\mathcal{U}$ and has order $\leq n$.

($\Leftarrow$): Suppose $\dim X > n$. Then $H(\kappa) \models \neg\varphi$, where φ is the sentence from the previous part. Since $\mathcal{M} \prec H(\kappa)$, $\mathcal{M} \models \neg\varphi$. That means there exists $\mathcal{F} \in \mathcal{M}$, a finite subset of 2^X, such that no $\mathcal{G} \in \mathcal{M}$ satisfies (1 - 4).

We claim $\mathcal{U} = \{wL \setminus \bar{F} \colon F \in \mathcal{F}\}$ is a finite open cover of wL, without a finite refining open cover $\mathcal{V}$ of order $\leq n$. It is clear that no such $\mathcal{V}$ can be contained in the base $\mathcal{B}$, since it would immediately produce $\mathcal{G} \in \mathcal{M}$ satisfying (1 - 4). But we must show that there is no such cover at all.

Suppose to the contrary that there is such $\mathcal{V} = \{V_1, V_2, \ldots, V_m\}$. We can shrink each V_i to a closed set $C_i \subseteq V_i$ so that $C_1 \cup C_2 \cup \ldots \cup C_m = wL$. Each C_i is compact, so it can be covered by sets $B_1, B_2, \ldots, B_{j(i)} \in \mathcal{B}$, which are contained in V_i. Let $V_i' = B_1 \cup B_2 \cup \ldots \cup B_{j(i)}$. Then $\mathcal{V}' = \{V_1', V_2', \ldots, V_m'\}$ is a cover of wL which refines $\mathcal{U}$ and has order $\leq n$. But each V_i' belongs to $\mathcal{B}$ as a finite union of sets from $\mathcal{B}$. A contradiction. □

Corollary 4.9. $\dim X = \dim wL$. □

Remark 4.10. The proof of Proposition 4.8 can be easily adopted to get the following more general fact: *Let K^* be a lattice in $\mathcal{M}$ and $K = K^* \cap \mathcal{M}$. Then* $\dim wK^* = \dim wK$.

We have specified properties of the lattice 2^X which are responsible for the dimension of $w2^X$. These properties are reflected by the elementary sublattice L. Hence, wL keeps the dimension of X. We will do the same with hereditary indecomposability in order to prove the second part of Theorem 4.1. We will use the following characterization by Krasinkiewicz and Minc:

Theorem 4.11 ([9])**.** *A compact space X is hereditarily indecomposable if and only if for every F_1, F_2 closed disjoint in X, for every $U_1, U_2 \subseteq X$ open disjoint such that $F_i \subseteq U_i$, there exist closed sets $X_1, X_2, X' \subseteq X$ such that $X_1 \cup X_2 \cup X' = X$, $F_i \subseteq X_i$, $X_i \cap X' \subseteq U_i$ and $X_1 \cap X_2 = \emptyset$.* □

Let us call such (F_1, F_2, U_1, U_2) a *proper quadruple* and (X_1, X_2, X') a *fold* in X for (F_1, F_2, U_1, U_2).

Theorem 4.11 is used to obtain the following well-known fact (proved also in [6]):

Proposition 4.12. *A continuum X is hereditarily indecomposable if and only if wL is.*

Proof. (if): Suppose X is not hereditarily indecomposable. Hence the following sentence ψ is true in $H(\kappa)$

(ψ) There exist connected $F, G \in 2^X$ such that $F \cap G \neq \emptyset, F \cap G \neq F$ and $F \cap G \neq G$.

(The connectedness can be expressed as follows: $F \in 2^X$ is connected iff for no $F_1, F_2 \in 2^X$ such that $F_1 \cup F_2 = F$ we have $F_1 \cap F_2 = \emptyset$.)

By elementarity $\mathcal{M} \models \psi$. Pick $F, G \in \mathcal{M}$ as in ψ. Then $\bar{F}, \bar{G}$ witness that wL is not hereditarily indecomposable.

(only if): Suppose X is hereditarily indecomposable. We want to show that each proper quadruple in wL has a fold. Then, by Theorem 4.11 wL is heredirarily indecomposable. It is easy to express the sentence *each proper quadruple in X has a fold* in terms of lattices.

By elementarity this sentence is true in $\mathcal{M}$. That means a proper quadruple (F_1, F_2, U_1, U_2) in wL has a fold, but only when U_1, U_2 belong to $\mathcal{B}$ and F_1, F_2 are complements of sets from $\mathcal{B}$. (Recall $\mathcal{B} = \{wL \setminus \bar{F}\colon\ F \in L\}$ is the base of wL.)

Take any proper quadruple (F_1, F_2, U_1, U_2) in wL. We will find $F_i' \supseteq F_i$ and $U_i' \subseteq U_i$ such that $wL \setminus F_i', U_i' \in \mathcal{B}$ and (F_1', F_2', U_1', U_2') is a proper quadruple. So it has a fold which is also a fold for (F_1, F_2, U_1, U_2).

The set $F_1 \subseteq U_1$ is closed so it is an intersection of (infinitely many) complements of sets from $\mathcal{B}$. Since wL is compact and U_1 is open we can choose finitely many of these complements, whose intersection is contained in U_1. This intersection is our F_1'. The base $\mathcal{B}$ is closed under finite unions and intersections, thus F_1' is itself a complement of a set from $\mathcal{B}$. By the compactness we can cover F_1' by finitely many sets from $\mathcal{B}$ which are contained in U_1. Again, their union is an element of $\mathcal{B}$ and this is our U_1'. Sets F_2' and U_2' are obtained in the same way. □

Now we know that $\dim wL = \dim X$ and wL is hereditarily indecomposable if and only if such is X. We will show that $\dim C(X) = \dim C(wL)$. Let us outline an idea. Starting with a continuum X, we easily define $C(X)$. Similarily, having the lattice 2^X we can define a lattice $K^* \in \mathcal{M}$, such that wK^* is homeomorphic to $C(X)$. Taking $K = K^* \cap \mathcal{M}$ we will show that wK is homeomorphic to $C(wL)$.

$C(X)$ is defined (as a set) only in terms of 2^X:

$$C(X) = \{F \in 2^X\colon \neg(\exists G_1, G_2 \in 2^X)(G_1 \cup G_2 = F \wedge G_1 \cap G_2 = \emptyset)\}.$$

Define K^* as a sublattice of $(\mathcal{P}(C(X)), \cup, \cap, \emptyset, C(X))$ generated by the family $\{\mathcal{F}^*\colon \mathcal{F} \in [2^X]^{<\omega}\}$, where

$$\mathcal{F}^* = C(X) \setminus \{G \in C(X)\colon G \cap \bigcap \mathcal{F} = \emptyset \wedge (\forall F \in \mathcal{F})(F \cup G \neq X)\}.$$

In other words K^* is the closure under finite unions and intersections of the family of sets $\mathcal{F}^*$ for all finite $\mathcal{F} \subseteq 2^X$. It is easy to verify that sets $\mathcal{F}^*$ form a closed base for $C(X)$. Hence, K^* is a closed base and a lattice simultaneously. By Fact 4.3, $C(X)$ is homeomorphic to wK^*.

Since $X \in \mathcal{M}$, it follows directly by the definition of K^* that $K^* \in \mathcal{M}$. Take $K = K^* \cap \mathcal{M}$. Then $\dim wK = \dim wK^* = \dim C(X)$ (see Remark 4.10). The only thing we still lack is:

Proposition 4.13. *wK is homeomorphic to $C(wL)$.*

Proof. We define the homeomorphism $\tilde{h}\colon wK \to C(wL)$:

$$wK \ni u \mapsto \tilde{h}(u) = q[\overline{h^{-1}(u^*)}] \in C(wL)$$

where:

- $u^* \in wK^*$ is any ultrafilter extending u ($u^* \cap K = u$),
- $h\colon C(X) \to wK^*$ is the homeomorphism defined in Fact 4.3,
- $\overline{F} = \{u \in w2^X : F \in U\}$ for any $F \in 2^X$,
- $q\colon w2^X \to wL$ is the surjection from Remark 4.6.

We will show that this is a well-defined homeomorphism. Recall that a basic closed set $\mathcal{C}$ in $C(wL)$ is determined by a family $\mathcal{F} \in [L]^{<\omega}$ via the formula

$$\mathcal{C} = C(wL) \setminus \{C \in C(wL)\colon C \cap \bigcap\{\bar{F}\colon F \in \mathcal{F}\} = \emptyset \wedge (\forall F \in \mathcal{F})(\bar{F} \cup C \neq wL)\}.$$

(a) $\tilde{h}(u)$ *does not depend on the choice of* $u^* \in wK^*$*:*

Suppose $u_1^*, u_2^* \in wK^*$ both contain u and $C_1 = q[\overline{h^{-1}(u_1^*)}] \neq q[\overline{h^{-1}(u_2^*)}] = C_2$. There exists a basic closed set $\mathcal{C} \subseteq C(wL)$ such that $C_1 \in \mathcal{C}$, $C_2 \notin \mathcal{C}$. But $\mathcal{C}$ is determined by some $\mathcal{F} = \{F_1, F_2, \ldots, F_m\} \subseteq L$, so $\mathcal{F}^* \in K$. It is easy to check that $\mathcal{F}^* \in u_1^*$, $\mathcal{F}^* \notin u_2^*$ which gives a contradiction.

Let us prove in details that $\mathcal{F}^* \in u_1^*$. Recall that for any $u^* \in wK^*$ we have the preimage $h^{-1}(u^*)$ is the only element in $\bigcap u^*$. By the definition of $\mathcal{F}^*$, to obtain $\mathcal{F}^* \in u_1^*$ we need to show that the following is false:

$$h^{-1}(u_1^*) \cap \bigcap \mathcal{F} = \emptyset \wedge (\forall F \in \mathcal{F})(F \cup h^{-1}(u_1^*) \neq X).$$

Suppose it is true. Then also

$$C_1 \cap \bigcap\{\bar{F}\colon F \in \mathcal{F}\} = \emptyset \wedge (\forall F \in \mathcal{F})(\bar{F} \cup C_1 \neq wL).$$

But then $C_1 \notin \mathcal{C}$ by the definition of $\mathcal{C}$, a contradiction. Similarily we check that $\mathcal{F}^* \notin u_2^*$.

(b) $\tilde{h}$ *is one-to-one:*

Take $u_1, u_2 \in wK$, $u_1 \neq u_2$. We can assume that there is $\mathcal{G}^* \in \mathcal{M}$ such that $\mathcal{G}^* \in u_1$, $\mathcal{G}^* \notin u_2$. By the definition of K^* there are $\mathcal{F}_{1,1}, \ldots, \mathcal{F}_{1,\xi(1)}, \mathcal{F}_{2,1}, \ldots,$ $\mathcal{F}_{2,\xi(2)}, \ldots, \mathcal{F}_{m,1}, \ldots, \mathcal{F}_{m,\xi(m)} \in [2^X]^{<\omega}$, such that

$$\mathcal{G}^* = \bigcap_{i<m} \bigcup_{j<\xi(i)} \mathcal{F}_{i,j}^*.$$

Since $\mathcal{G}^* \in \mathcal{M}$ we may assume $\mathcal{F}_{i,j} \in \mathcal{M}$, hence $\mathcal{F}_{i,j} \subseteq L$ for all i, j. So each $\mathcal{F}_{i,j}$ determines basic closed set $\mathcal{C}_{i,j} \subseteq C(wL)$. It follows that (details as in (a)):

$$\tilde{h}(u_1) \in \bigcap_{i<m} \bigcup_{j<\xi(i)} \mathcal{C}_{i,j} \quad \text{and} \quad \tilde{h}(u_2) \notin \bigcap_{i<m} \bigcup_{j<\xi(i)} \mathcal{C}_{i,j}$$

so $\tilde{h}(u_1) \neq \tilde{h}(u_2)$.

(c) $\tilde{h}$ *is continuous:*

Let $\mathcal{C} \subseteq C(wL)$ be a basic closed set. So $\mathcal{C}$ is determined by some family $\mathcal{F} =$

$\{F_1, F_2, \ldots, F_m\} \subseteq L$. Hence $\mathcal{F}^* \in K$ and $\tilde{h}^{-1}[\mathcal{C}] = \{u \in wK : \mathcal{F}^* \in u\}$ is closed in wK.

(d) $\tilde{h}$ *is onto:*

Take $C \in C(wL)$. Consider the family:

$$\mathcal{U} = \{\mathcal{F} \in [L]^{<\omega} : \text{ the basic closed set determined by } \mathcal{F} \text{ contains } C\}.$$

Then $\{\mathcal{F}^* : \mathcal{F} \in \mathcal{U}\}$ generates an ultrafilter $u \in wK$, such that $\tilde{h}(u) = C$. □

Corollary 4.14. dim $C(wL) =$ dim $C(X)$. □

Now we have all ingredients to prove Theorem 4.1.
Proof of Theorem 4.1.
(i) Suppose that we have a continuum X, $\dim X \geq 2$ and $\dim C(X) < \infty$. By Fact 4.7, wL is a metric continuum. By Corollary 4.9, $\dim wL = \dim X \geq 2$ and, by Corollary 4.14, $\dim C(wL) = \dim C(X) < \infty$ which contradicts the result of M. Levin and Y. Sternfeld [13].
(ii) Suppose X is a 1-dimensional hereditarily indecomposable continuum and $\dim C(X) \notin \{2, \infty\}$. As in (i), the space wL is a 1-dimensional metric continuum with $\dim C(wL) = \dim C(X)$. By Proposition 4.12, wL is hereditaraily indecomposable. This gives a contradiction with [3], where it is shown that for a metric contiuum Y we have $\dim C(Y)$ is either 2 or ∞. □

4.3. **Open questions.**

The method of the proof of Theorem 4.1 can be used to generalize some results on metric continua to non-metric ones. Consider a theorem of the form: *if X has property P then $C(X)$ has property Q.* Suppose the theorem is true for metric continua, and we can prove that properties $P, \neg Q$ of a continuum wM are reflected by $w(M \cap \mathcal{M})$, where $M \in \mathcal{M}$ is a lattice. Then the theorem is true in general, since a non-metric counterexample would produce a metric one. Proposition 4.13 plays the key role to link X with $C(X)$ in terms of lattices.

Let us state the following question:
Question 4.15.

- (i) *Suppose X is a non-metric continuum and* $\dim X \geq 2$. *Can $C(X)$ be a C-space?*
- (ii) *Suppose X is a hereditarily indecomposable non-metric continuum with* $\dim X = 1$ *and* $\dim C(X) > 2$. *Can $C(X)$ be a C-space?*

If property C is elementarily reflected, the answer for both questions would be *no*, since the answers are negative in the metric cases. Unfortunately, so far we are able to get only the opposite:

Proposition 4.16. *If X is not a C-space, then neither is wL.*

Proof. This is similar to the right-to-left implication of Proposition 4.8. For a continuum X, being a non-C-space means that there exists a sequence $(\mathcal{U}_i)_{i<\omega}$ of finite open covers of X, such that for every $m \in \omega$ and finite families of open disjoint sets $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_m$ which satisfy $\mathcal{V}_i \prec \mathcal{U}_i$, their union $\mathcal{V}_0 \cup \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_m$ is not a cover of X. We can make every open cover finite because of compactness. Translating it into the lattice 2^X we have that $H(\kappa)$ models the following sentence φ:

(φ) There exists $(\mathcal{F}_i)_{i<\omega}$ - a sequence of finite subsets of 2^X such that for each $i < \omega$ the intersection $\bigcap \mathcal{F}_i$ is empty and for every $m \in \omega$ and $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_m \subseteq 2^X$ the following holds:

$(*)$ If for each $j \leq m$ and $G \in \mathcal{G}_j$ there exists $F \in \mathcal{F}_j$ such that $F \subseteq G$ and for each $G, G' \in \mathcal{G}_j$ we have $G \cup G' = X$ then $\bigcap(\mathcal{G}_0 \cup \mathcal{G}_1 \cup \ldots \cup \mathcal{G}_m) \neq \emptyset$

$\mathcal{M} \models \varphi$ by elementarity. So there is a sequence $(\mathcal{F}_i)_{i<\omega} \in \mathcal{M}$ as in φ, such that $(*)$ holds for every $m < \omega$ and $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_m \in \mathcal{M}$.

$(\mathcal{F}_i)_{i<\omega}$ gives rise to a sequence $(\mathcal{U}_i)_{i<\omega}$ of open covers of wL (namely $\mathcal{U}_i = \{wL \setminus \bar{F} \colon F \in \mathcal{F}_i\}$), which witnesses that wL is not a C-space. Indeed, suppose we have a finite sequence $\mathcal{V}'_0, \mathcal{V}'_1, \ldots, \mathcal{V}'_m$ of finite families of open disjoint sets, $\mathcal{V}'_i \prec \mathcal{U}_i$ and their union is a cover of wL. We produce $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_m$, which are additionally contained in the base $\mathcal{B}$. This can be done exactly as at the end of the proof of Proposition 4.8. But then there are $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_m \in \mathcal{M}$ which do not satisfy $(*)$. □

Question 4.17. *Is the converse to Proposition 4.16 true?*

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