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Aksjomatyzacja modelu Mathiasa w terminach gier

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Axiomatization of the Mathias model in terms of games

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Streszczenie

Przedmiotem badań zawartych w rozprawie jest jeden z najważniejszych modeli rozważanych w teorii mnogości, mianowicie model Mathiasa. Mimo że jest on otrzymany za pomocą iteracji długości ω_2 forcingu Mathiasa, w niniejszej rozprawie jego kombinatoryczne własności ujęto w języku deskryptywnej teorii mnogości. Wzorując się na aksjomacie CPA dla modelu Sacksa (zob. [7]), wprowadzono szereg aksjomatów, które coraz dokładniej odzwierciedlają kombinatoryczną strukturę rozpatrywanego modelu. Wykazano ich niesprzeczność oraz zbadano konsekwencje. W tym celu przeformułowano iterację forcingu Mathiasa, charakteryzując ją w terminach deskryptywnej teorii mnogości. Aksjomaty wyrażono przy użyciu zbiorów i funkcji borelowskich, ideałów na przestrzeniach polskich oraz gier i strategii. Opracowano w ten sposób aksjomatyzację modelu Mathiasa, która pozwala spojrzeć na jego strukturę od strony topologiczno-desktyptywnej, co znacznie ułatwia dalsze badania nad jego własnościami i prowadzi do nowych wyników. Dodatkowo uzyskano kilka wniosków dotyczących V-ultrafiltrów[1] indukowanych przez liczby rzeczywiste z modelu Mathiasa.

Rozdział 1 wprowadza do tematyki rozprawy i nakreśla tło historyczne opisanych w niej badań. Zawiera skrócony opis poszczególnych rozdziałów, uwagi o możliwości zastosowania przedstawionych narzędzi do badania innych modeli, a także podziękowania. W Rodziale 2 zebrano wstępne, z reguły powszechnie znane fakty dotyczące rozważanych pojęć. Ma to w szczególności na celu ustalenie notacji oraz interpretacji symboli czy zwrotów. W Rozdziale 3 podano deskryptywną charakteryzację iterowanego forcingu Mathiasa, przy użyciu zbiorów borelowskich i ideałów na przestrzeniach polskich. Wykorzystano w tym celu metodę zastowaną w [7], opierając się na ideach z [21].

[1] filtrów maksymalnych na $\mathcal{P}(\omega) \cap V$, gdzie V jest modelem wyjściowym

W kolejnych rozdziałach wprowadzano poszczególne aksjomaty. W każdym przypadku wykazano, że rozpatrywany aksjomat jest spełniony w modelu Mathiasa, oraz rozważano jego konsekwencje. Rozdział 4 opisuje „podstawowy" aksjomat, CPA, analogiczny do przedstawionego w [7]. Implikuje on równość $\mathrm{cov}(\mathcal{J}) = \mathfrak{x} = \omega_1$, gdzie $\mathcal{J}$ oznacza σ-ideał zbiorów I kategorii Baire'a lub zbiorów miary zero na prostej, natomiast $\mathfrak{x}$ - dystrybutywność algebry $\mathrm{r.o.}(\mathbb{R}^*, \subseteq^*)$ lub $\mathrm{r.o.}(c_0 \backslash \ell^1, \leq^*)$ (zob. [9], [11]). Jego wzmocnienie, sformułowany w Rozdziale 5 aksjomat CPAs, dowodzi takich własności jak:

- $\mathfrak{h} > \omega_1$, gdzie $\mathfrak{h}$ oznacza dystrybutywność algebry $(\mathcal{P}(\omega)/\,\mathrm{fin})$,
- Hipoteza Borela (Borel Conjecture, zob. [2]),
- brak ultrafiltrów typu „rapid" na ω (zob. [18]),
- brak dalekich punktów rozspajających $\mathbb{I}_{\mathcal{U}}$ dla $\mathcal{U} \in \omega^*$ (zob. [10]).

Ten ostatni wynik jest nową własnością modelu Mathiasa, która dotychczas znana była dla modelu Lavera (zob. [10]).

W Rozdziale 6 przedstawiono aksjomat SCPA^-, będący słabszą, mniej techniczną, „taktyczną" wersją aksjomatu SCPA (omówionego w Rozdziale 8). Jako wniosek z SCPA^- uzyskano równość $\mathfrak{h}(2) = \omega_1$, gdzie $\mathfrak{h}(2)$ oznacza dystrybutywność algebry $(\mathcal{P}(\omega)/\,\mathrm{fin}) \times (\mathcal{P}(\omega)/\,\mathrm{fin})$. Wynik Shelaha i Spinasa z [21], mówiący, że równość ta zachodzi w modelu Mathiasa, jest główną motywacją do poszukiwania silniejszych wersji aksjomatu CPA. Ponadto SCPA^- dowodzi, że dystrybutywność $((\omega)^\omega, \leq^*)$ wynosi ω_1. ($(\omega)^\omega$ to rodzina nieskończonych partycji zbioru ω, z porządkiem $X \leq^* Y$, gdy prawie wszystkie[2] elementy partycji X są sumami elementów partycji Y, zob. [22].)

Rozdział 7 poświęcony jest aksjomatowi $\diamondsuit$CPA, który oddaje strukturę modelu iteracyjnego związaną z aksjomatem $\diamondsuit$, występującym w pośrednich rozszerzeniach o kofinalności ω_1. W Rozdziale 8 opisano zależności między rozważanymi dotychczas aksjomatami oraz podano ich uogólnienia, w szczególności wspomniany aksjomat SCPA, w jego pełnej, „strategicznej" wersji, a także aksjomat $\diamondsuit$SCPAs, implikujący wszystkie poprzednie.

Pozostałe rozdziały należy traktować jako dodatek do głównego tematu rozprawy. W Rozdziale 9 omówiono pewne własności V-ultrafiltrów w mo-

[2] poza skończoną ilością

delu Mathiasa. W szczególności podano elementarne dowody uogólnień głównych lematów z [21]. (Rozumowania w [21] wykorzystują zaawansowane i nader techniczne narzędzia, co znacznie utrudnia ich śledzenie.) Rozdział 10 opisuje aksjomat $\Diamond$mCPA – modyfikację $\Diamond$CPA, która implikuje kombinatoryczną zasadę $\clubsuit$. Z uwagi na techniczny charakter $\Diamond$mCPA rozdział ten stanowi jedynie uzupełnienie opracowanej akjomatyzacji.

Abstract

The objective of this thesis is one of the most significant models considered in set theory, namely the Mathias model, which is cnstructed via the technique of iterated forcing. Although it is obtained by iteration of length ω_2 of Mathias forcing, we study its combinatorial structure using the framework of descriptive set theory. We present a series of axioms, modeled on the CPA axiom from [7], that describe the combinatorial core of the model. We prove their consistency and study their consequences. To this end, we give a descriptive set theoretical characterization of the iterated Mathias forcing. Our axioms are formulated in terms of Borel sets and functions, σ-ideals on Polish spaces, games and strategies. In this way we develope an axiomatization of the Mathias model, which gives a descriptive set theoretic insight into its structure, makes it more approachable, and leads to new results. As a byproduct, we obtain a few facts about V-ultrafilters[3] induced by reals from the generi extension via the iterated Mathias forcing.

Chapter 1 is an itroduction to the topic, gives some background and history. It contains an overview of the thesis, remarks about possible application of developed methods for investigating other models, and acknowledgement. In Chapter 2 we gather the neccesary preliminaries as well as fix the notation and interpretation of symbols and phrases. In Chapter 3 we reformulate the iteration of Mathias forcing. This approach was considered in [7] and its adaptation to our case uses ideas of [21].

The following chapters describe the axioms. In each case, we prove that the considered axiom holds true in the model as well as discuss its consequences. In Chapter 4 we present the basic axiom, CPA, which is analogous to the one from [7]. It implies that $\mathrm{cov}(\mathcal{J}) = \mathfrak{r} = \omega_1$, where $\mathcal{J}$ is the σ-ideal of meager, or null sets, and $\mathfrak{r}$ denotes the distributivity of Boolean alge-

[3]maximal filters on $\mathcal{P}(\omega) \cap V$, where V is the ground model

bras r.o.$(\mathbb{R}^*, \subseteq^*)$ or r.o.$(c_0 \backslash \ell^1, \leq^*)$ (see [9], [11]). Its modificatiom, the axiom CPAs, introduced in Chapter 5, proves such assertions as

- $\mathfrak{h} > \omega_1$, where $\mathfrak{h}$ is the distributivity of $(\mathcal{P}(\omega)/\,\mathrm{fin})$,
- Borel Conjecture (see [2]),
- the lack of rapid ultrafilters (see [18]),
- the lack of far cut points in $\mathbb{I}_{\mathcal{U}}$ for $\mathcal{U} \in \omega^*$ (see [10]).

The latter statement is a new property of the Mathias model, so far it was known for the Laver model.

The motivation to search for a stronger version of CPA stems from the result of Shelah and Spinas from [21], which says that $\mathfrak{h}(2) = \omega_1$ in the Mathias model. The axiom SCPA$^-$, which implies this equality, is formulated in Chapter 6. It is a weaker, "tactic" version of the axiom SCPA (introduced in Chapter 8) but its expression is much less technical. The axiom SCPA$^-$ also proves that the distributivity of $((\omega)^\omega, \leq^*)$ equals ω_1. ($(\omega)^\omega$ is the set of all infinite partitions of ω with the ordering $X \leq^* Y$ iff all but finitely many elements of X are unions of elements of Y, see [22].)

Chapter 7 is devoted to the axiom $\Diamond$CPA, which is a natural modifaction of CPA capturing some combinatorics provided by the principle $\Diamond$, which holds in the intermediate generic extensions of cofinality ω_1. In Chapter 8 we discuss implications between the axioms introduced so far and formulate their generalizations. In particular, we present the full, "strategic" version of SCPA as well as the strongest axiom, called $\Diamond$SCPAs, which implies all the previously stated.

Two following chapters can be considered as an appendix to the main topic. In Chapter 9 we present some results concerning V-ultrafilters in the Mathias model. In particular, we give elementary proofs of two main Propositions from [21] and, in fact, we generalize them. (Original proofs are based on difficult and technically complicated methods, which make them hard to follow.) In Chapter 10 we modify $\Diamond$CPA obtaining the axiom $\Diamond$mCPA, which is strong enough to imply the $\clubsuit$ principle. This modification is quite technical. We present it for for the sake of the completeness of the research.

Contents

Chapter 1

Introduction

The theory of proper forcing turned out to be one of the most important topics in modern set theory. Properness is a desirable attribute of forcing notions. Models obtained as generic extensions via the countable support iteration of proper forcing serve as the main source of consistency results concerning the cardinal characteristics of the continuum (see [5]). They play a significant role not only in set theory but also in the related areas.

The Mathias model is an example of such an extension. It is obtain via the countable support iteration of length ω_2 of Mathias forcing. Its combinatorial properties proved to be useful in many fields of mathematics. Besides set theory one should mention Boolean algebras, Ramsey theory, measure theory and set-theoretic topology. Having the Laver property, Mathias forcing shares many features with Laver forcing. For instance, the Borel Conjecture, which says that every strong measure zero set is countable, holds in the Mathias model. Also, the iterated Mathias forcing increases many cardinal characteristics (they become $\omega_2 = \mathfrak{c}$ in the generic extension). On the other hand, the distributivity of some Boolean algebras remains small (equal to ω_1). Using the Mathias model, Shelah and Spinas obtained the consistency of $\mathfrak{h} < \mathfrak{h}(2)$, where $\mathfrak{h}$ is the distributivity of the algebra $\mathcal{P}(\omega)/\operatorname{fin}$ and $\mathfrak{h}(2)$ that of its square ([21]). This result is considered to be very significant.

The main purpose of this thesis is to develope a comprehensive, descriptive set theoretic axiomatization of the Mathias model. This is done by investigating a series of axioms, which capture its combinatorial core. The axioms do not refere to the model itself and they are formulated in terms of games and strategies rather than forcing. In this way they shed a new light on the structure of the Mathias model and make it more approachable.

1.1 Background

The concept of descriptive set theoretic axioms which reflect the combinatorial structure of some models obtained by the iterated proper forcing was started by Ciesielski and Pawlikowski. In [7] they introduced the Covering Property Axiom (CPA), which captures the combinatorial core of the Sacks model. Using CPA one can eliminate forcing and, instead of working with the iterated perfect set model, deduce directly from the axiom in the desctiptive set theoretic framework.

Similar axioms can be stated for other models. They were rephrased in terms of ideals by Zapletal in [23]: if $\mathcal{I}$ is a "nicely definable" σ-ideal on an uncountable Polish space X, then the axiom CPA($\mathcal{I}$) holds in the model obtained by the countable support iteration of the forcing $(Bor(X) \setminus \mathcal{I}, \subseteq)$. Zapletal also proved a kind of completeness theorem: for a *very tame* cardinal invariant $\mathfrak{x}$, if $\mathfrak{x} < \mathrm{cov}(\mathcal{I})$ is true in some generic extension, then it must be a consequence of CPA($\mathcal{I}$).

Zapletal's approach relies on the Fubini powers of the σ-ideal $\mathcal{I}$. In order to have the necessary absoluteness, $\mathcal{I}$ must be *iterable* (see [23, Definition 5.1.3]). The assumptions on $\mathcal{I}$ are mild enough to catch the σ-ideals of meager, null, and countable sets, as well as the σ-ideal on the Baire space ω^ω generated by σ-compact sets. These σ-ideals correspond to Cohen, radom, Sacks, and Miller forcings respectively.

Mathias forcing correspondes to the σ-ideal of Ramsey null sets, which is not iterable, and Zapletal's approach fails. Indeed, according to the result of Sabok from [19], the set of codes for Ramsey positive analytic sets is $\mathbf{\Sigma^1_2}$-complete. It follows that, without extra large cardinal assumptions, the countable support iteration of Mathias forcing can not be expressed using the Fubini powers of the σ-ideal of Ramsey null sets. Fortunatelly, an approach similar to that of [7] and using ideas of [21] works.

1.2 Overview

Throughout the thesis we introduce a series of axioms (CPA, SCPA, $\diamondsuit$CPA, CPAs, $\diamondsuit$SCPAs etc.). Although they are descriptive set theoretic statements, rather than forcing ones, they reflect combinatorial properties of the Mathias model. Each time we introduce an axiom we prove that it is true in the Mathias model and then study its consequences.

The axioms are formulated in terms of games and strategies (or tactics, which are particular types of strategies). A typical application of an axiom needs a certain *density lemma*, which describes a strategy (or tactic) of one of the players. Usually, the lemma can be stated (and proved) in terms of forcing, as well as within the framework of descriptive set theory (both proofs would use the same argument expressed in different languages). Such a density lemma can always be considered as the black-box principle - its proof is irrelevant for the rest of the argument.

The dissertation is organized as follows:

Chapter 2 gathers some background and fixes the notation. The necessary preliminaries about the theory of forcing are given and the notion of distributivity of a Boolean algebra (or a partially ordered set) is introduced and characterized. In particular, the distributivity game is described. We will refere to it a number of times throughout the thesis, when calculating cardinal characteristics from our axioms. Finally, the Mathias forcing is introduced, together with a bundle of preliminary facts and the definition of the Mathias model.

In Chapter 3 we present a descriptive set theoretic reformulation of the iterated Mathias forcing. This approach was developed by Shelah and Spinas in [21], and, in contrast to the one elaborated by Zapletal ([23]), it does not involve the Fubini powers. Anyway, this part is quite technical.

Section 3.1 is devoted to the iteration of countable lenght. For $\alpha < \omega_1$, a σ-ideal $\mathcal{I}^\alpha$ on the Polish space $([\omega]^\omega)^\alpha$ is defined. Each $\mathcal{I}^\alpha$ is connected to the iteration of the Mathias forcing of lenght α. These σ-ideals are necessary to state our axioms. Section 3.2 describes in details the reformulation of the two-step iteration. Section 3.3 gives a glance at the limit step of the iteration. In Section 3.4 we give the full reformulation of the iteration of arbitrary lenght. This is needed to prove the consistency of our axioms. Therefore, the Reader who is interested just in applying the axioms, not in the consistency proofs, may simply skip this part after the end of Section 3.1 (Definition 3.10). The only exception is Theorem 10.3 from Chapter 10.

Chapter 4 is dedicated to the Covering Property Axiom (CPA), which is the "Mathias analoque" of the axiom stated for the Sacks model in [7]. Although it is the weakest of the axioms considered in the thesis, it implies that $\mathfrak{x} = \omega_1$, where $\mathfrak{x}$ denotes the covering number of meager, or null sets, or the distributivity of Boolean algebras r.o.$(\mathbb{R}^*, \subseteq^*)$ or r.o.$(c_0 \setminus \ell^1, \leq^*)$ (see Sections 4.2, 4.3).

In Chapter 5 we present CPAs, a modification of CPA, strong enough to imply such assertions as $\mathfrak{h} > \omega_1$, Borel Conjecture, as well as the lack of rapid ultrafilters and far cut points in $\mathbb{I}_{\mathcal{U}}$ for $\mathcal{U} \in \omega^*$ (see Section 5.3).

The main motivation to search for a stronger version of the "basic" axiom CPA is the result of Shelah and Spinas from [21], which says that $\mathfrak{h}(2) = \omega_1$ in the Mathias model. We want to formulate an axiom which implies this equality. This is done in Chapter 6, which is devoted to the tactic version of the Strong Covering Property Axiom (SCPA$^-$). We also consider the result of Spinas from [22], which says that $\mathfrak{h}(((\omega)^\omega, \leq^*)) = \omega_1$ in the Mathias model (see Section 6.3).

Although the $\diamondsuit$ principle fails in the Mathias model, it is well known that it holds at the intermediate generic extensions of cofinality ω_1. In Chapter 7 we introduce $\diamondsuit$CPA, which is a natural modifaction of CPA capturing some combinatorics provided by $\diamondsuit$.

In Chapter 8 we discuss implications between the axioms introduced so far and formulate their generalizations. In particular we present the full (strategic) version of the Strong Covering Property Axiom (SCPA). Moreover, we introduce the strongest axiom, called $\diamondsuit$SCPAs, which implies all the previously stated.

Two following chapters can be considered as an appendix to the main topic. Chapter 9 is devoted to the ground model ultrafilters on ω, induced by reals from the generic extension via the Mathias forcing (iterated with another notion with the Laver property). In particular, we give elementary proofs of Propositions 2.3, 2.4 from [21]. The main result of [21] follows from these Propositions quite easily. However, their original proofs are based on difficult and technically complicated methods, which make them hard to follow. In fact, we generalize these results using just the combinatorial structure of conditions in the Mathias forcing. Finally, we discuss possible generalizations to the ground model ultrafilters added by $\mathcal{P}(\omega)/\operatorname{fin}$.

In Chapter 10 we modify $\diamondsuit$CPA obtaining the axiom $\diamondsuit$mCPA, which is strong enough to imply the $\clubsuit$ principle. This modification is quite technical. We present it for for the sake of completeness of the research.

1.3 Remarks

We would like to point out that the axiomatization we developed for the Mathias model can be transfered to some other models obtained by the iteration of "simply definable" proper forcing. Indeed, it is not difficult to repeat the descriptive set theoretic reformulation of the iteration (Chapter 3) as well as to state the axioms and prove their consistency for Cohen, Sacks, Miller, Laver, or random forcings.

It is natural to ask if any two of the axioms we introduce are equivalent. Although we heuristically conjecture the negative answer, we put no effort to find any models discerning them. The motivation was to catch the combinatorial core of the Mathias model by a series of descriptive set theoretic statements. Searching for models distinguishing them is a separate problem, though, and may be interesting in itself.

1.4 Acknowledgement

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Chapter 2

Preliminaries

A forcing notion is a partially ordered set $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ with a maximal element. Stronger condition is the smaller one. Conditions $p, q \in \mathbb{P}$ are compatible iff there exists $r \in \mathbb{P}$ such that $r \leq_{\mathbb{P}} p$ and $r \leq_{\mathbb{P}} q$. Otherwise they are incompatible, which is denoted by $p \perp_{\mathbb{P}} q$ or simply by $p \perp q$.

Let V be a countable transitive model of ZFC* with $\mathbb{P} \in V$ (ZFC* is a finite fragment of ZFC containing all the relevant axioms). A filter G on $\mathbb{P}$ is $\mathbb{P}$-generic over V iff $G \cap D \neq \emptyset$ for every open dense $D \subseteq \mathbb{P}$ such that $D \in V$. It is easy to show that for every $p \in \mathbb{P}$ there exists a $\mathbb{P}$-generic filter over V which contains p (Rasiowa-Sikorski Lemma). If G is $\mathbb{P}$-generic then $V[G]$, the generic extension of the ground model V, is the minimal transitive model of ZFC* such that $V \subseteq V[G]$ and $G \in V[G]$. For a $\mathbb{P}$-name $\dot{x}$ and a $\mathbb{P}$-generic filter G over V the evaluation of $\dot{x}$ with G in $V[G]$ is denoted by $\dot{x}/G$. The $\mathbb{P}$-generic filter G is considered as an object "outside of the universe V". We refere to [20, I,§1] or [15, VII,§1] for a comprehensive discussion.

Two forcing notions $(\mathbb{P}, \leq_{\mathbb{P}})$ and $(\mathbb{P}', \leq_{\mathbb{P}'})$ are forcingwise equivalent iff for every $\mathbb{P}$-generic filter G over V there is $\mathbb{P}'$-generic filter $G' \in V[G]$ such that $V[G] = V[G']$ and vice versa (see [20, I,Definition 5.2,p. 30]). The forcingwise equivalence is denoted by $\cong$.

A function $\pi\colon \mathbb{P} \to \mathbb{P}'$ is a dense embedding if

- π preserves the ordering, i.e. $p_0 \leq_{\mathbb{P}} p_1 \;\Rightarrow\; \pi(p_0) \leq_{\mathbb{P}'} \pi(p_1)$,
- π preserves the incompatibility, i.e. $p_0 \perp_{\mathbb{P}} p_1 \;\Rightarrow\; \pi(p_0) \perp_{\mathbb{P}'} \pi(p_1)$,
- the image of π is dense, i.e. $(\forall p' \in \mathbb{P}')(\exists p \in \mathbb{P})\ \pi(p) \leq_{\mathbb{P}'} p'$.

If there exists a dense embedding $\pi: \mathbb{P} \to \mathbb{P}'$ then $\mathbb{P} \cong \mathbb{P}'$. (see e.g. [15, VII, Theorem 7.11]).

For an ordinal ε, by ε-iteration of a forcing notion we mean the countable support iteration of length ε. We refere to [14, Definition 16.29] for the general definition. The countable support means that we take inverse limits at limit steps of cofinality ω and direct limits at other limit steps. Equivalently, if $\mathbb{P}_\varepsilon$ is an ε-iteration (and $\dot{P}_\xi$ are names for iterands) and $p \in \mathbb{P}_\varepsilon$ then for all but countably many $\xi < \varepsilon$ the condition $p{\restriction}\xi \in \mathbb{P}_\varepsilon{\restriction}\xi$ forces that $p(\xi)$ is the maximal element of $\dot{P}_\xi$.

The structure H_κ consists of sets x such that the cardinality of the transitive closure of x is smaller than κ. Whenever H_κ appears, we always assume that $\kappa > \omega_1$ is regular and H_κ contains everything relevant, in particular the forcing notion we consider, all its subsets etc.

Let $M \subseteq H_\kappa$ be a model of ZFC* (not necessarily transitive). For a Borel set B we say that $B \in M$ if the code $\#B$ of B (which can be seen as a real, the usual $\mathbf{\Pi}^1_1$ coding is used, see [14, p. 504]) is in M. Whenever M is a fixed model and $\#B \in M$, B^M is the set computed in M from the code $\#B$; we often drop the upperscript M if context permits. Note that if $V[G]$ is a generic extension of V and $B \in V$ then B^V and $B^{V[G]}$ may be different sets (but $B^V = B^{V[G]} \cap V$). Similarily, if f is a Borel function, then by $f \in M$ we mean that the code $\#f$ of f is in M. Moreover, when we work in M and $\#f \in M$, then we often simply write f for the function f^M computed in M from the code $\#f$.

If $N \prec H_\kappa$ is countable (note that N is non-transitive) and $\mathbb{P} \in N$, then we say that a filter G on $\mathbb{P} \cap N$ is $\mathbb{P}$-generic over N iff $G \cap D \neq \emptyset$ for any open dense set $D \subseteq \mathbb{P}$ such that $D \in N$. As in [23], the expression $N[G]$ denotes the generic extension of the transitive collapse N^* of N by the collapsed image G^* of G. If $\dot{x} \in N$ is a $\mathbb{P}$-name, then $\dot{x}/G$ is the G^* evaluation of $\dot{x}^*$, the image of $\dot{x}$ under the collapsing function.

If $\dot{x} \in N$ is a $\mathbb{P}$-name for an element of ω^ω then the set

$$A = \{y \in \omega^\omega \colon (\exists G \subseteq \mathbb{P} \cap M)\ G \text{ is } \mathbb{P}\text{-generic over } N \ \&\ y = \dot{x}/G\}$$

is Borel (see [23, Fact 1.4.8]). Moreover, the space $\mathcal{G}$ of all filters on $\mathbb{P} \cap N$ which are $\mathbb{P}$-generic over N is Polish[1] and the function $e\colon \mathcal{G} \to A$ given by

[1] with the topology generated by the family $\{\mathcal{G}_p \colon p \in \mathbb{P} \cap N\}$ as the base for closed sets, where $\mathcal{G}_p = \{G \in \mathcal{G} \colon p \in G\}$

$e(G) = \dot{x}/G$ is a Borel bijection. In particular, for every $y \in A$ the $\mathbb{P}$-generic filter G over N such that $\dot{x}/G = y$ is unique.

If G is a filter on $\mathbb{P}$ (not necessarily contained in N), then by saying that G is $\mathbb{P}$-generic over N we mean that $G \cap N$ is $\mathbb{P}$-generic over N. We simply write $N[G]$ and $\dot{x}/G$ in place of $N[G \cap N]$ and $\dot{x}/(G \cap N)$.

A forcing notion $\mathbb{P}$ is proper if for each countable $N \prec H_\kappa$ such that $\mathbb{P} \in N$ and every condition $p \in \mathbb{P} \cap N$ there exists $q \leq_{\mathbb{P}} p$ which is N-master, i.e. for every open dense $D \subseteq \mathbb{P}$ such that $D \in N$ we have

$$q \Vdash \dot{G} \cap \check{D} \cap \check{N} \neq \emptyset.$$

Properness is preserved under countable support iteration (see [20]). If $\mathbb{P}$ is proper then $\mathbb{P}$ preserves ω_1, i.e. $\omega_1^{V[G]} = \omega_1^V$ whenever G is $\mathbb{P}$-generic over V.

A forcing notion $\mathbb{P}$ has the Laver property if for any $p \in \mathbb{P}$, any $\mathbb{P}$-name $\dot{g}$ for an element of ω^ω, and any collection $\langle S(n) \colon n \in \omega \rangle \in V$ such that

$$(\forall n \in \omega)\ |S(n)| \leq 2^n \quad \text{and} \quad p \Vdash (\forall n \in \omega)\ \dot{g}(n) \in S(n)$$

there exists $q \leq_{\mathbb{P}} p$ and $\langle T(n) \colon n \in \omega \rangle \in V$ such that

$$(\forall n \in \omega)\,(T(n) \subseteq S(n)\ \&\ |T(n)| \leq n) \quad \text{and} \quad q \Vdash (\forall n \in \omega)\ \dot{g}(n) \in T(n).$$

The Laver property is preserved under countable support iteration (see [2]).

2.1 Distributivity

Recall that a complete Boolean algebra B is κ-distributive if

$$\prod_{\alpha<\kappa} \sum_{i \in I_\alpha} u_{\alpha,i} = \sum_{f \in \prod_{\alpha<\kappa} I_\alpha} \prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}$$

for any $\langle u_{\alpha,i} \colon \alpha < \kappa, i \in I_\alpha \rangle \subseteq B$ (see [14, p. 85]). For any partially ordered set $\mathbb{P}$ there is a unique complete Boolean algebra r.o.($\mathbb{P}$) such that the separative quotient of $\mathbb{P}$ can be densely embedded into r.o.($\mathbb{P}$). The distributivity $\mathfrak{h}(\mathbb{P})$ of $\mathbb{P}$ is defined as the minimal κ such that r.o.($\mathbb{P}$) is not κ-distributive. The symbol $\mathfrak{h}$ denotes the distributivity of the algebra $\mathcal{P}(\omega)/\operatorname{fin}$ and $\mathfrak{h}(2)$ is the distributivity of its square $(\mathcal{P}(\omega)/\operatorname{fin}) \times (\mathcal{P}(\omega)/\operatorname{fin})$.

We introduce the distributivity game $G(\mathbb{P}, \alpha)$ to characterize $\mathfrak{h}(\mathbb{P})$ (see [21, Introduction]).

Definition 2.1. Let $\mathbb{P}$ be a partial order and let α be an ordinal. Consider the game $G(\mathbb{P}, \alpha)$ between two players PI and PII. The game lasts α rounds. PII begins with some initial $p_{in} \in \mathbb{P}$. Then PI chooses some $p_0^I \leq_{\mathbb{P}} p_{in}$, PII chooses $p_0^{II} \leq_{\mathbb{P}} p_0^I$, and so on. At step γ, PI plays (if possible) $p_\gamma^I \in \mathbb{P}$ which is below p_β^{II} for every $\beta < \gamma$. Then PII responds with $p_\gamma^{II} \leq_{\mathbb{P}} p_\gamma^I$. PI wins iff he can always pick a legal p_γ^I during the play and there is a $p \in \mathbb{P}$ such that $p \leq p_\beta^{II}$ for all $\beta < \alpha$.

If r.o.($\mathbb{P}$) is homogenous, the initial move p_{in} of PII may be skipped.

Fact 2.2 ([21]). *Suppose that $\mathbb{P}$ is a partial order such that* r.o.($\mathbb{P}$) *is homogenous. The following conditions are equivalent:*

1. $\mathbb{P}$ *is κ-distributive.*
2. *The intersection of any family $\{\mathcal{D}_\gamma : \gamma < \kappa\}$ consisting of open dense sets*[2] *in $\mathbb{P}$ is dense.*
3. *The intersection of any decreasing family $\{\mathcal{D}_\gamma : \gamma < \kappa\}$ consisting of open dense sets*[3] *in $\mathbb{P}$ is dense.*
4. PII *has no winning strategy in the distributivity game $G(\mathbb{P}, \kappa)$.*

2.2 Mathias forcing and Mathias model

Denote the sets of finite and infinite subsets of ω by $[\omega]^{<\omega}$ and $[\omega]^\omega$ respectively. We consider $[\omega]^\omega$ as a topological subspace of the Cantor set 2^ω. Recall the Mathias forcing

$$Q = \{\langle s, A\rangle : s \in [\omega]^{<\omega}, A \in [\omega]^\omega \ \& \ \max(s) < \min(A)\}$$

with the ordering

$$\langle s, A\rangle \leq_Q \langle t, B\rangle \quad \Leftrightarrow \quad t \subseteq s \subseteq t \cup B \quad \& \quad A \subseteq B.$$

It is well known that Q is proper and has the Laver property. The topology on Q is inherited from $[\omega]^{<\omega} \times [\omega]^\omega$, where $[\omega]^{<\omega}$ is discrete. For $q \in Q$ let

[2] If $\mathbb{P}$ is not separative then we assume that each $\mathcal{D}_\gamma$ is closed under the "separation" equivalence: i.e. if $x \in \mathcal{D}_\gamma$ and $y \in \mathbb{P}$ satisfies $(\forall z \in \mathbb{P})(z \perp x \Leftrightarrow z \perp y)$, then $y \in \mathcal{D}_\gamma$.

[3] As above.

$s^q \in [\omega]^{<\omega}$ and $A^q \in [\omega]^\omega$ denote its finite and infinite part respectively, i.e. $q = \langle s^q, A^q \rangle$. Let G be Q-generic filter over V. The generic real associated with G is given by the formula

$$r = \bigcup \{s \in [\omega]^{<\omega} \colon (\exists A \in [\omega]^\omega \cap V)\ \langle s, A \rangle \in G\}.$$

Note that $V[r] = V[G]$, since $G = \{\langle s, A \rangle \in Q \cap V \colon s \subseteq r \subseteq s \cup A\}$.

Moreover, any real $x \in [\omega]^\omega$ induces a filter

$$G_x = \{\langle s, A \rangle \in Q \colon s \subseteq x \subseteq s \cup A\}$$

on Q. We say that x is Mathias over N if G_x is Q-generic over N. The following result is due to Mathias.

Theorem 2.3 ([14, Corollary 26.38])**.** *If $x \in [\omega]^\omega$ is a Mathias real over N and $y \subseteq x$ is infinite, then y is Mathias over N.*

We shall fix some further notation. If $s \in [\omega]^{<\omega}$ and $n > \max(s)$ (we agree on $\min(\emptyset) = \max(\emptyset) = 0$) then $s^\frown n = s \cup \{n\}$. Moreover, if $s, t \in [\omega]^{<\omega}$ are such that $\max(s) < \min(t)$, or one of them is empty, then $s^\frown t = s \cup t$. If $A \in [\omega]^\omega$ and $n \in \omega$ then by A/n we mean $A \setminus (n+1)$.

Recall the following well-known lemma:

Lemma 2.4 ([2, Lemma 7.4.6])**.** *Suppose that $\langle s, A \rangle \in Q$ and ψ is a sentence in the forcing language. There exists an infinite $B \subseteq A$ such that either $\langle s, B \rangle \Vdash_Q \psi$ or $\langle s, B \rangle \Vdash_Q \neg\psi$.*

Let Q_ε be the ε-iteration of Q. We will identify Q_1 with Q (note that formally $Q_1 = \{\langle q \rangle \colon q \in Q\}$). Let Q_0 denote the trivial poset $\{\emptyset\}$. For any ε the notion Q_ε is proper and has the Laver property. Recall that the Mathias model is obtained as a Q_{ω_2}-generic extension $V[G]$ of V.

Fix ε and $\xi < \varepsilon$. Let G_ε denote a Q_ε-generic filter over V. Then $G_\xi = G_\varepsilon \restriction \xi = \{p \restriction \xi \colon p \in G_\varepsilon\}$ is Q_ξ-generic over V. By $\dot{r}_\xi$ we denote the cannonical $Q_{\xi+1}$-name for the ξ-th generic real and by r_ξ its evaluation in the generic extension $V[G_{\xi+1}] \subseteq V[G_\varepsilon]$. For $x \in [\omega]^\omega$ let $\underline{x} \in \omega^\omega$ be the increasing ennumeration of x. We use $\underline{\dot{r}}_\xi$ to denote the cannonical $Q_{\xi+1}$-name for $\underline{r}_\xi$. For $p \in Q_\varepsilon$ let $\mathrm{cl}(p) \subseteq \varepsilon$ be any countable set of ordinals such that p depends only on $\langle \dot{r}_\zeta \colon \zeta \in \mathrm{cl}(p) \rangle$ (for example, find a countable $N \prec H_\kappa$, where $\varepsilon, p \in N$ and take $\mathrm{cl}(p) = N \cap \varepsilon$).[4]

[4]Note that $\mathrm{cl}(p)$ is not necessarily equal to the support of p, but it must contain it.

Sometimes we will deal with forcing notions of the form $\mathbb{P} = Q * \mathbb{P}'$, where $\mathbb{P}'$ has the Laver property. Then r, $\dot{r}$, $\underline{r}$ and $\dot{\underline{r}}$ denote the generic real added by Q, its cannonical name, its ennumeration, and the cannonical name for the ennumeration respectively.

Chapter 3

Iteration

The reformulation of iterated forcing in terms of ideals and descriptive set theory developed by Zapletal (see [23, Chapter 5]) does not catch Mathias forcing unless large cardinals are used to guarantee enough absoluteness. Fortunatelly, a ZFC-only approach, similar to that of [7], is possible using ideas of Shelah and Spinas from [21]. We present it in this chapter.

For any ordinal ε, we will define a forcing notion $P_\varepsilon \cong Q_\varepsilon$ that provides the necessary absoluteness. This is quite technical and we will often cite [21]. However, we sketch enough details to give, hopefully, the flavor of the method. In particular, we present a detailed proof that $P_2 \cong Q_2$ (Section 3.2).

First we will deal with iterations of countable length.

3.1 Iteration of countable length

Definition 3.1. For $\alpha < \omega_1$ we put $p \in P_\alpha$ if and only if:

(i) p is a function and $\mathrm{dom}(p) = \alpha$,

(ii) for $\beta < \alpha$, $p(\beta)$ is a code for a Borel function, $p(\beta)\colon ([\omega]^\omega)^\beta \to Q$.

The idea of Definition 3.1 is that $p(\beta)$ assignes in a Borel way a Mathias condition to any sequence of reals of length β.

To define $\leq_{P_\alpha}$ we first assign a Borel set $B_p \subseteq ([\omega]^\omega)^\alpha$ to each $p \in P_\alpha$. The definition proceeds by induction on α:

- $\alpha = 0$: $P_0 = \{\emptyset\}$ and $B_\emptyset = ([\omega]^\omega)^0 = \{\emptyset\}$.

- $\alpha = \beta+1$: $\langle x_\gamma : \gamma < \alpha\rangle \in B_p$ if $\langle x_\gamma : \gamma < \beta\rangle \in B_{p\restriction\beta}$ (note that $p\restriction\beta \in P_\beta$) and $s^q \subseteq x_\beta \subseteq s^q \cup A^q$, where $q = p(\beta)\left(\langle x_\gamma : \gamma < \beta\rangle\right)$, (i.e. q is in the filter on Q induced by x_β).
- $\alpha = \bigcup \alpha$: $\langle x_\gamma : \gamma < \alpha\rangle \in B_p$ if for all $\beta < \alpha$ we have $\langle x_\gamma : \gamma < \beta\rangle \in B_{p\restriction\beta}$.

A sequence from B_p is said to *satisfy* p.

For $p, p' \in P_\alpha$ we define

$$p \leq_{P_\alpha} p' \Leftrightarrow B_p \subseteq B_{p'}.$$

One can easily show that $p \leq_{P_\alpha} p'$ iff for every $\langle x_\gamma : \gamma < \alpha\rangle \in B_p$ and for each $\beta < \alpha$

$$p(\beta)\left(\langle x_\gamma : \gamma < \beta\rangle\right) \leq_Q p'(\beta)\left(\langle x_\gamma : \gamma < \beta\rangle\right).$$

Theorem 3.2 (cf. [21, Corollary 3.7])**.** *For any $\alpha < \omega_1$ we have $P_\alpha \cong Q_\alpha$.*

The complexity of $(P_\alpha, \leq_{P_\alpha})$ is $\mathbf{\Pi}^1_1$ so its definition is absolute between transitive models of large enough finite fragment of ZFC. Therefore, we have an absolute description of the iteration of Mathias forcing of countable length.

The proof of Theorem 3.2 boils down to showing a dense embedding of P_α into Q_α ([15, VII, Definition 7.7]). The following definition and lemma play the key role.

Definition 3.3. Define the function $\pi : P_\alpha \to Q_\alpha$ as follows: $\pi(p) = q$, where

- $q(0) = p(0)(\emptyset)$,
- $q(\gamma)$ is a Q_γ-name such that $Q_\gamma \Vdash q(\gamma) = p(\gamma)(\langle \dot r_\beta : \beta < \gamma\rangle)$.

Lemma 3.4 ([21, Lemma 3.8])**.** *Let $\alpha < \omega_1$. Suppose that $N \prec H_\kappa$ is countable, $\alpha \in N$, and $p \in P_\alpha \cap N$. There exists $p' \leq_{P_\alpha} p$ such that every sequence $\bar x \in B_{p'}$ is Q_α-generic over N, i.e. there is a (necessarily unique) filter $G_{\bar x} \subseteq Q_\alpha \cap N$ which is Q_α-generic over N, and*

$$\bar x = \langle \dot r_\gamma : \gamma < \alpha\rangle / G_{\bar x} = \langle \dot r_\gamma / G_{\bar x} : \gamma < \alpha\rangle.$$

If G_α is Q_α-generic, then $\{p \in P_\alpha : \pi(p) \in G_\alpha\}$ is P_α-generic. We will also denote it with G_α (and its cannonical P_α-name with $\dot G_\alpha$). It should make no confusion. An inductive argument gives the following fact (c.f. Fact 3.26).

Fact 3.5. $Q_\alpha \Vdash \Big(\pi(p) \in \dot{G}_\alpha \ \leftrightarrow \ \langle \dot{r}_\gamma\colon \gamma < \alpha\rangle \in B_p\Big)$, *for every* $p \in P_\alpha$. $\square$

It allows us to treat $\dot{r}_\gamma$ as a P_α-name (in fact $P_{\gamma+1}$-name) in the following way: $\langle \dot{r}_\gamma\colon \gamma < \alpha\rangle$ is the P_α-name for the unique sequence in $([\omega]^\omega)^\alpha$, which satisfies every $p \in \dot{G}_\alpha$. Moreover, for every $p \in P_\alpha$,

$$P_\alpha \Vdash \Big(p \in \dot{G}_\alpha \ \leftrightarrow \ \langle \dot{r}_\gamma\colon \gamma < \alpha\rangle \in B_p\Big).$$

Fact 3.6. *Let* $B \subseteq ([\omega]^\omega)^\alpha$ *be a Borel set. For every* $p \in P_\alpha$ *there exists* $p' \leq_{P_\alpha} p$ *such that either* $B_{p'} \subseteq B$ *or* $B_{p'} \cap B = \emptyset$.

Proof. There exists $p'' \leq_{P_\alpha} p$ such that either $p'' \Vdash_{P_\alpha} \langle \dot{r}_\gamma\colon \gamma < \alpha\rangle \in B$ or $p'' \Vdash_{P_\alpha} \langle \dot{r}_\gamma\colon \gamma < \alpha\rangle \in B^c$. Without lost of generality assume the former. Let $N \prec H_\kappa$ be a countable structure, containing α, B and p''. By Lemma 3.4 there is $p' \leq_{P_\alpha} p''$ such that for every sequence $\bar{x} \in B_{p'}$ the filter $\langle G_{\bar{x}(\xi)}\colon \xi < \alpha\rangle$ is Q_α-generic over N. Fix such $\bar{x} \in B_{p'}$. By genericity $N[\bar{x}] \models \bar{x} \in B$ and by absoluteness $\bar{x} \in B$. It follows that $B_{p'} \subseteq B$. $\square$

Definition 3.7. For each $\alpha < \omega_1$ let $\mathcal{I}^\alpha$ be the σ-ideal of subsets of $([\omega]^\omega)^\alpha$ generated by Borel sets B such that

$$(\forall p \in P_\alpha)(\exists p' \leq_{P_\alpha} p)\ B_{p'} \cap B = \emptyset.$$

Remark 3.8. For Borel sets being $\mathcal{I}^\alpha$-positive is a $\mathbf{\Sigma^1_2}$ statement.

As a corollary from Facts 3.5 and 3.6 we get:

Corollary 3.9. *For a Borel set* $B \subseteq ([\omega]^\omega)^\alpha$ *we have:*

(i) $B \notin \mathcal{I}^\alpha \ \Leftrightarrow \ (\exists p \in P_\alpha)\ B_p \subseteq B$

(ii) $B \in \mathcal{I}^\alpha \ \Leftrightarrow \ P_\alpha \Vdash \langle \dot{r}_\gamma\colon \gamma < \alpha\rangle \notin B \ \Leftrightarrow (\forall p \in P_\alpha)(\exists p' \leq_{P_\alpha} p) B_{p'} \cap B = \emptyset.$

$\square$

Definition 3.10. Define $Q'_\alpha = Bor(([\omega]^\omega)^\alpha) \setminus \mathcal{I}^\alpha$. Arguing as in the proof of Fact 3.6 one can show that the function $\iota\colon (P_\alpha, \leq_{P_\alpha}) \to (Q'_\alpha, \subseteq)$ given by $\iota(p) = B_p$ is a dense embedding. Hence, the ordering $(Q'_\alpha, \subseteq)$ is forcingwise equivalent to Q_α.

We will often treat $\dot{r}_\gamma$ as the Q'_α-name ($Q'_{\gamma+1}$-name): the γ-th coordinate of the unique member in the intersection of conditions in Q'_α-generic filter.

3.2 Two-step iteration

In this section we consider the two-step iteration and give a proof Theorem 3.2 when $\alpha = 2$. The complete proof in [21] is rather technical, the two step case gives a good insight into its ideas.

According to our notation Q_2 is the "classical" two-step iteration of the Mathias forcing. For $q \in Q_2$ and $i = 0, 1$ let us denote $q_i = q(i)$ (formally q is a sequence of length 2). So, for $\langle q_0, \dot{q}_1 \rangle, \langle q'_0, \dot{q}'_1 \rangle$ in Q_2 we have

$$\langle q_0, \dot{q}_1 \rangle \leq_{Q_2} \langle q'_0, \dot{q}'_1 \rangle \quad \Leftrightarrow \quad q_0 \leq_Q q'_0 \quad \& \quad q_0 \Vdash_Q \dot{q}_1 \leq_{\dot{Q}} \dot{q}'_1.$$

Recall that $p \in P_2$ iff $p = \langle p(0), p(1) \rangle$, where $p(i)\colon ([\omega]^\omega)^i \to Q$ is a Borel function for $i = 0, 1$. Let us denote $p(i)$ by p_i. Note that formally $p_0\colon \{\emptyset\} \to Q$. However, it seems ridiculous to keep this notation, so we will treat p_0 simply as a condition from Q, writing p_0 rather than $p_0(\emptyset)$. Hence, $p = \langle p_0, p_1 \rangle \in P_2$ iff $p_0 \in Q$ and $p_1\colon [\omega]^\omega \to Q$ is a Borel function.

We associate with $p = q \in Q = P$ the Borel (in fact closed) set

$$B_p = \{x \in [\omega]^\omega \colon s^q \subseteq x \subseteq s^q \cup A^q\},$$

and with $p = \langle p_0, p_1 \rangle \in P_2$ the Borel set

$$B_p = \{\langle x_0, x_1 \rangle \in ([\omega]^\omega)^2 : x_0 \in B_{p_0} \;\; \& \;\; s^{p_1(r_0)} \subseteq x_1 \subseteq s^{p_1(r_0)} \cup A^{p_1(r_0)}\}.$$

For $p, p' \in P_2$ we have

$$p \leq_{P_2} p' \;\; \Leftrightarrow \;\; B_p \subseteq B_{p'} \;\; \Leftrightarrow \;\; p_0 \leq_Q p'_0 \;\; \& \;\; (\forall x \in B_{p_0})\; p_1(x) \leq_Q p'_1(x).$$

There is the cannonical function

$$\pi\colon (P_2, \leq_{P_2}) \to (Q_2, \leq_{Q_2}),$$

given by $\pi(p) = \langle p_0, p_1(\dot{r}_0) \rangle$, where $p = \langle p_0, p_1 \rangle$ and $\dot{r}_0$ is the cannonical name for the Mathias real, added by the first iterand. (By $p_1(\dot{r}_0)$ we mean any Q-name $\dot{q}$ such that $Q \Vdash \dot{q} = p_1(\dot{r}_0)$.) Note that π is not one-to-one. For $p = \langle p_0, p_1 \rangle, p' = \langle p'_0, p'_1 \rangle$ in P_2 we have

$$\pi(p) \leq_{Q_2} \pi(p') \quad \Leftrightarrow \quad p_0 \leq_Q p'_0 \quad \& \quad p_0 \Vdash_Q p_1(\dot{r}_0) \leq_{\dot{Q}} p'_1(\dot{r}_0).$$

Fact 3.11. *Let $p = \langle p_0, p_1 \rangle \in P_2$. Then $Q_2 \Vdash \pi(p) \in \dot{G}_2 \leftrightarrow \langle \dot{r}_0, \dot{r}_1 \rangle \in B_p$.*

Proof. Suppose that $q = \langle q_0, \dot{q}_1\rangle \in Q_2$ and $q \Vdash_{Q_2} \pi(p) \in \dot{G}_2$. Then

$$q_0 \leq_Q p_0 \quad \& \quad q_0 \Vdash_Q \dot{q}_1 \leq_Q p_1(\dot{r}_0).$$

Therefore, $q_0 \Vdash_Q \dot{r}_0 \in B_{p_0}$ and $q \Vdash_{Q_2} s^{p_1(\dot{r}_0)} \subseteq \dot{r}_1 \subseteq A^{p_1(\dot{r}_0)}$. It follows that $q \Vdash_{Q_2} \langle \dot{r}_0, \dot{r}_1\rangle \in B_p$.

On the other hand, suppose that the condition $q = \langle q_0, \dot{q}_1\rangle \in Q_2$ forces that $\langle \dot{r}_0, \dot{r}_1\rangle \in B_p$. Then

$$q \Vdash_{Q_2} \left(\dot{r}_0 \in B_{p_0} \ \wedge \ s^{p_1(\dot{r}_0)} \subseteq \dot{r}_1 \subseteq A^{p_1(\dot{r}_0)}\right).$$

Since $\dot{r}_0$ is a Q-name, we have $q_0 \Vdash_Q \dot{r}_0 \in B_{p_0}$. But it means that $q_0 \leq_Q p_0$. Moreover, $q_0 \Vdash_Q q_1 \leq_Q p_1(\dot{r}_0)$. Hence, $q \leq_{Q_2} \pi(p)$, so $q \Vdash_{Q_2} \pi(p) \in \dot{G}_2$. □

Corollary 3.12. *The function π is order preserving, i.e. for every $p, p' \in P_2$, if $p \leq_{P_2} p'$ then $\pi(p) \leq_{Q_2} \pi(p')$.*

Proof. Suppose that $p \leq_{P_2} p'$. By Fact 3.11, $\pi(p) \Vdash_{Q_2} \langle \dot{r}_0, \dot{r}_1\rangle \in B_p$. Since $B_p \subseteq B_{p'}$, we have $\pi(p) \Vdash_{Q_2} \langle \dot{r}_0, \dot{r}_1\rangle \in B_{p'}$. Using the fact again we get $\pi(p) \Vdash_{Q_2} \pi(p') \in \dot{G}_2$, hence $\pi(p) \leq_{Q_2} \pi(p')$. □

Observe that the converse is not true, i.e. there exist $p, p' \in P_2$ such that $\pi(p) \leq_{Q_2} \pi(p')$ holds, but $p \leq_{P_2} p'$ fails (take any p, p' such that $p_0 \leq_Q p'_0$ and the set $\{x \in B_{p_0} \colon \neg p_1(x) \leq_Q p'_1(x)\}$ is non-empty and Ramsey null).

Lemma 3.13. *The image of π is dense in Q_2, i.e. for each $q = \langle q_0, \dot{q}_1\rangle \in Q_2$ there exists $p = \langle p_0, p_1\rangle \in P_2$ such that $\pi(p) \leq_{Q_2} q$.*

Proof. Let N be a countable elementary submodel of H_κ with $q \in N$. Take $p_0 = q_0$. If x is Mathias over N such that $q_0 \in G_x$ then define $p_1(x) = \dot{q}_1/G_x$, i.e. the evaluation of $\dot{q}_1$ in the model $N[G_x]$. Otherwise let $p_1(x) = \langle \emptyset, \omega\rangle$. It is easy to verify that $\pi(p) \leq_{Q_2} q$. □

Lemma 3.14. *Suppose that $p, p' \in P_2$ and $\pi(p) \leq_{Q_2} \pi(p')$. There exists $p'' \in P_2$ such that $p'' \leq_{P_2} p$ and $p'' \leq_{P_2} p'$.*

Proof. Let $N \prec H_\kappa$ with $p, p' \in N$. The following Claim is a special case of Lemma 3.4, where $\alpha = 1$.

Claim 3.15. *There is p''_0 below p_0 such that every $x \in B_{p''_0}$ is Mathias over N. In particular p'' is an N-master condition.*

Proof. Let y be Mathias over N, such that $p_0 \in G_x$. Define $p''_0 = \langle s^{p_0}, y \setminus s^{p_0} \rangle$. Then every x from $B_{p''_0}$ is a subset of y, so by Theorem 2.3 it is Mathias over N. □

Now let $p''_1 = p_1$. Define $p'' = \langle p''_0, p''_1 \rangle$. Clearly $p'' \leq_{P_2} p$. We claim that $p'' \leq_{Q_2} p'$. Obviously $p''_0 \leq_Q p'_0$. Fix $x \in B_{p''_0}$. The filter G_x is Q-generic over N. Since

$$p''_0 \leq_Q p_0 \Vdash_Q p_1(\dot{r}_0) \leq_{\dot{Q}} p'_1(\dot{r}_0)$$

we have

$$N[G_x] \models p''_1(x) = p_1(x) \leq_Q p'_1(x)$$

(the evaluation of $\dot{r}_0$ in $N[G_x]$ is x). By the absoluteness of $\leq_Q$ we obtain $p''_1(x) \leq_Q p'_1(x)$. Since $x \in B_{p''_0}$ was arbitrary, we have $p'' \leq_{P_2} p'$. □

Corollary 3.16. $(P_2, \leq_{P_2}) \cong (Q_2, \leq_{Q_2})$.

Proof. By Corollary 3.12 and Lemma 3.13 π is order preserving and its image is dense in $(Q_2, \leq_{Q_2})$. By Lemmas 3.14 and 3.13, π preserves incompatibility. Therefore, π is a dense embedding. □

We end this section proving Lemma 3.4 for $\alpha = 2$. We would need it to show that $P_3 \cong Q_3$.

Lemma 3.17. *Suppose that $N \prec H_\kappa$ is countable and $p = \langle p_0, p_1 \rangle \in P_2 \cap N$. There exists $p' = \langle p'_0, p'_1 \rangle \leq_{P_2} p$ such every $\langle x_0, x_1 \rangle \in B_{p'}$ is Q_2-generic over N, i.e. x_0 is Mathias over N and x_1 is Mathias over $N[G_{x_0}]$.*

Proof. Use Claim 3.15 to find $p'_0 \leq_Q p_0$ such that each real in $B_{p'_0}$ is Mathias over N. It remains to define the Borel function $p'_1 \colon [\omega]^\omega \to Q$. If $x \notin B_{p'_0}$ let $p'_1(x) = p_1(x)$. For $x \in B_{p'_0}$ define $p'_1(x)$ in the following way:

Let $\langle \dot{D}'_\xi \colon \xi < \nu \rangle \in N$ be a Q-name for the sequence of all open dense subsets of $\dot{Q}$. Let $\langle \dot{D}_n \colon n \in \omega \rangle \in V$ be an ennumeration of $\{\dot{D}'_\xi \colon \xi \in \nu \cap N\}$. Then for any $x \in B_{p'_0}$ the sequence $\langle \dot{D}_n / G_x \colon n \in \omega \rangle \in V$ lists all dense subsets of Q from the model $N[G_x]$. Now for each $n \in \omega$ let $\langle \dot{q}^n_\zeta \colon \zeta < \mu \rangle \in N$ be a Q-name for a sequence of all elements of $\dot{D}_n$. For $x \in B_{p'_0}$ we have a "matrix" $\langle \dot{q}^n_\zeta / G_x \colon n \in \omega, \zeta \in \mu \cap N \rangle$ such that its n-th row ennumerates the set $(\dot{D}_n / G_x) \cap N[G_x]$. The assignment $x \mapsto \dot{g}^n_\zeta / G_x$ is Borel for each $n \in \omega$ and $\zeta \in \mu \cap N$.

We will treat the countable well-ordered set $A = \mu \cap N$ as an alphabet and for each $x \in B_{p'_0}$ we will construct a subtree T_x of $A^{<\omega}$. Fix $x \in B_{p'_0}$. The tree T_x is obtain by the following procedure:

- the empty sequence belongs to T_x,
- $\langle \zeta_0 \rangle \in T_x \iff \dot{q}^0_{\zeta_0}/G_x \leq_Q p_1(x)$,
- if $\langle \zeta_0, \ldots, \zeta_i \rangle \in T_x$, then

$$\langle \zeta_0, \ldots, \zeta_i, \zeta_{i+1} \rangle \in T_x \iff \dot{q}^{i+1}_{\zeta_{i+1}}/G_x \leq_Q \dot{q}^i_{\zeta_i}/G_x.$$

The function $x \mapsto T_x \in 2^{(A^{<\omega})}$ is Borel. Since Q forces that each $\dot{D}_n$ is open dense, the tree T_x has no terminals for each $x \in B_{p'_0}$. We consider the function $f\colon B_{p'_0} \to A^\omega$ such that for $x \in B_{p'_0}$, $f(x)$ is the left-most branch of T_x. Note that f is Borel. Indeed, since T_x has no terminals, being its left-most branch is expressible with quantifiers over $A^{<\omega}$ only.

Let $y_x = \bigcup\{s^{q^n_x}\colon n \in \omega\}$, where $q^n_x = \dot{q}^n_{\zeta_{f(x)(n)}}/G_x$. Then y_x is Mathias over $N[G_x]$. Finally, we define $p'_1(x) = \langle s^{p_1(x)}, y_x \setminus s^{p_1(x)} \rangle$. $\square$

3.3 Iteration of length ω

Now we will sketch an argument that $P_\omega \cong Q_\omega$ to give a brief glance at the limit step of iteration.

If $p \in P_\omega$ then $p = \langle p(i)\colon i < \omega \rangle$ is such that each $p(i)\colon ([\omega]^\omega)^i \to Q$ is Borel. As before, we denote $q(i)$ by q_i, $p(i)$ by p_i, and we treat p_0 simply as a condition in Q rather than a function. Recall that we have defined the Borel set

$$B_p = \{\bar{x} \in ([\omega]^\omega)^\omega\colon (\forall i < \omega)\ \bar{x}{\restriction}i \in B_{p_i}\},$$

where the Borel sets $B_{p\restriction i} \subseteq ([\omega]^\omega)^i$, $i < \omega$, are given by:

- $B_{p\restriction 0} = \{\emptyset\}$
- $\langle x_0, x_1, \ldots, x_i \rangle \in B_{p\restriction i+1}$ iff $\langle x_0, x_1, \ldots, x_{i-1} \rangle \in B_{p\restriction i}$ and

$$s^q \subseteq x_i \subseteq s^q \cup A^q,$$

where $q = p_i(\langle x_0, x_1, \ldots, x_{i-1} \rangle)$.

For $p = \langle p_i\colon i < \omega\rangle, p' = \langle p'_i\colon i < \omega\rangle$ from P_ω we have

$$p \leq_{P_\omega} p' \quad \Leftrightarrow \quad B_p \subseteq B_{p'} \quad \Leftrightarrow \quad (\forall \bar{x} \in B_p)(\forall i < \omega)\ p_i(\bar{x}{\restriction}i) \leq_Q p'_i(\bar{x}{\restriction}i).$$

Recall that if $q \in Q_\omega$ then $q = \langle q_i\colon i < \omega\rangle$ is such that $q_0 \in Q$ and for each $i < \omega$

$$\langle q_0, q_1, \ldots q_i\rangle \Vdash_{Q_i} q_{i+1} \in Q.$$

The function $\pi\colon P_\omega \to Q_\omega$ is given as follows:

$$\pi(\langle p_i\colon i < \omega\rangle) = \langle q_i\colon i < \omega\rangle,$$

where $q_0 = p_0$ and $Q_{i+1} \Vdash q_{i+1} = p_{i+1}(\langle \dot{r}_j\colon j \leq i\rangle)$ for $i < \omega$.

To show that $P_\omega \cong Q_\omega$ we need to prove analogous facts to those from the previous section.

Fact 3.18. $\pi\colon (P_\omega, \leq_{P_\omega}) \to (Q_\omega, \leq_{Q_\omega})$ *is order preserving.*

Proof. One can show that for $p \in P_\omega$, $Q_\omega \Vdash \pi(p) \in \dot{G}_\omega \leftrightarrow \langle \dot{r}_i\colon i \in \omega\rangle \in B_p$ (c.f. Fact 3.11). Hence, if $p \leq_{P_\omega} p'$, then $\pi(p) \leq_{Q_\omega} \pi(p')$. □

Lemma 3.19. *The image $\pi[P_\omega]$ is dense in $(Q_\omega, \leq_{Q_\omega})$, i.e. for each $q \in Q_\omega$ there exists $p = \langle p_i\colon i < \omega\rangle \in P_\omega$ such that $\pi(p) \leq_{Q_\omega} q$.*

Proof. Take a countable $N \prec H_\kappa$ with $q = \langle q_i\colon i < \omega\rangle \in N$. Suppose that $\bar{x} = \langle x_i\colon i < \omega\rangle \in ([\omega]^\omega)^\omega$ is Q_ω-generic over N, i.e. $\langle G_{x_i}\colon i < \omega\rangle$ is a Q_ω-generic filter over N. Moreover, let $q \in \langle G_{x_i}\colon i < \omega\rangle$. For $j < \omega$ define $p_j(\bar{x}{\restriction}j) = q_j/\langle G_{x_i}\colon i < \omega\rangle$, i.e. the evaluation of q_j in the model $N[\langle G_{x_i}\colon i < \omega\rangle]$. Note that if $\bar{x}' = \langle x'_i\colon i < \omega\rangle \in ([\omega]^\omega)^\omega$ is Q_ω-generic over N such that $\bar{x}'{\restriction}j = \bar{x}{\restriction}j$ then $q_j/\langle G_{x_i}\colon i < \omega\rangle = q_j/\langle G_{x'_i}\colon i < \omega\rangle$, since q_j is a Q_j-name.

If $\langle x_0, x_1, \ldots, x_{i-1}\rangle$ can not be extended to an N-generic sequence with q in its induced filter, put $p_j(\langle x_0, x_1, \ldots, x_{j-1}\rangle) = \langle \emptyset, \omega\rangle$. □

Now we present the analogue of Lemma 3.14.

Lemma 3.20. *Suppose that $p, p' \in P_\omega$ and $\pi(p) \leq_{Q_\omega} \pi(p')$. There exists $p'' \in P_\omega$ such that $p'' \leq_{P_\omega} p$ and $p'' \leq_{P_\omega} p'$.*

Proof. Let $p, p' \in N$, where $N \prec H_\kappa$ is countable. By Lemma 3.4 (with $\alpha = \omega$), there exists $p'' \in P_\omega$ such that each $\bar{x} \in B_{p''}$ is Q_ω-generic over N. Then, similarily as in the proof of Lemma 3.14, one can show that $p'' \leq_{P_\omega} p$ and $p'' \leq_{P_\omega} p'$. □

Arguing as in the Corollary 3.16 we get

Corollary 3.21. $(P_\omega, \leq_{P_\omega}) \cong (Q_\omega, \leq_{Q_\omega})$.

□

3.4 Iteration of arbitrary length

Now we will generalize the concept from Section 3.1 to arbitrarily long iterations of the Mathias forcing.

Let us introduce the notation. If $a \subseteq a'$ are countable and $B \subseteq ([\omega]^\omega)^{a'}$, then $B{\restriction}a$ is the projection of B onto $([\omega]^\omega)^a$, i.e. $B{\restriction}a = \{\bar{x}{\restriction}a : \bar{x} \in B\}$. If $i\colon a \to a'$ is a bijection and $B \subseteq ([\omega]^\omega)^{a'}$, then $B \bullet i = \{\bar{x} \circ i : \bar{x} \in B\} \subseteq ([\omega]^\omega)^a$.

Fix an ordinal ε. Take $a \in [\varepsilon]^{\leq\omega}$. We define a set P_a as follows: $p \in P_a$ iff p is a function, $\mathrm{dom}(p) = a$ and for each $\xi \in a$ the value $p(\xi)$ is a code for a Borel function $p(\xi)\colon ([\omega]^\omega)^{a\cap\xi} \to Q$.

For $p \in P_a$ we define the Borel set $B_p \subseteq ([\omega]^\omega)^a$ of sequences which satisfy p. The definition proceeds by induction on ε (at the ε-th step of induction we deal with all possible P_a, such that $\sup(a) = \varepsilon$).

- Suppose that $\varepsilon = \max(a)$. In this case

  $$\langle x_\xi : \xi \in a \rangle \in B_p \iff \langle x_\xi : \xi \in a \cap \varepsilon \rangle \in B_{p{\restriction}a\cap\varepsilon} \quad \& \quad s \subseteq x_\varepsilon \subseteq s \cup A,$$

  where $\langle s, A \rangle = p(\beta)(\langle x_\xi : \xi \in a \cap \varepsilon \rangle)$.

- Suppose that $\varepsilon = \sup(a)$ is limit. Then

  $$\langle x_\xi : \xi \in a \rangle \in B_p \iff (\forall \gamma \in a)\ \langle x_\xi : \xi \in a \cap \gamma \rangle \in B_{p{\restriction}a\cap\gamma}.$$

Definition 3.22. For $\varepsilon \geq \omega_1$ let $P_\varepsilon = \bigcup\{P_a : a \in [\varepsilon]^{\leq\omega}\}$ with the ordering

$$p' \leq_{P_\varepsilon} p \iff \mathrm{dom}(p) \subseteq \mathrm{dom}(p') \quad \& \quad B_{p'}{\restriction}\mathrm{dom}(p) \subseteq B_p.$$

Note that if we use this definition for $\varepsilon = \alpha < \omega_1$ then the resulting poset is not equal to the one from Definition 3.1. However, the "old" one is a dense subset of the "new" one. We shall rather use the old one, i.e. for $\alpha < \omega_1$ we shall tacitly assume that $p \in P_\alpha$ implies $\mathrm{dom}(p) = \alpha$.

Note also that for $a \in [\varepsilon]^{\leq\omega}$, P_a itself can be considered as a forcing notion (with the ordering $p' \leq_{P_a} p$ iff $B'_p \subseteq B_p$). Then P_a is isomorphic to P_α, where $\alpha = \mathrm{o.t.}(a)$. Although $P_a \subseteq P_\varepsilon$, it is not a complete subposet unless $a = \alpha$.

Example 3.23. $P_{\{1\}}$ is not a complete subposet of $P_{\{0,1\}}$. Let $A_0, A_1, \ldots$ be pairwise disjoint infinite subsets of ω. Let $\mathcal{A}'$ be a maximal almost disjoint family of infinite subsets of ω with $\{A_i : i \in \omega\} \subseteq \mathcal{A}'$. Define

$$\mathcal{A} = \{A \in [\omega]^\omega : (\exists A' \in \mathcal{A}')\ |A \bigtriangleup A'| < \omega\}.$$

For $A \in \mathcal{A}$ let p_A be such that $p(1)(\emptyset) = \langle \emptyset, A\rangle$. Then $\{p_A : A \in \mathcal{A}\}$ is predense in $P_{\{1\}}$. Let $p \in P_{\{0,1\}}$ be such that $p(0)(\emptyset) = \langle \emptyset, \omega\rangle$ and $p(1)(x) = \langle \emptyset, A_x\rangle$, where A_x is obtained as follows: By induction we define natural numbers n_i for $i \in \omega$. Let $n_0 = \min(A_0/\underline{x}(0))$ (recall that $\underline{x} \in \omega^\omega$ is the ennumeration of x) and $n_{i+1} = \min(A_{i+1}/\max\{\underline{x}(i+1), n_i\})$. Define $A_x = \{n_i : i \in \omega\}$.

We claim that for every $A \in \mathcal{A}$, p_A is incompatible with p in $P_{\{0,1\}}$. Indeed, suppose that we have $p' \in P_{\{0,1\}}$ and $A \in \mathcal{A}$ such that $B_{p'} \subseteq B_p$ and $B_{p'}\restriction\{1\} \subseteq B_{p_A}$. We have two cases: either

$$(\exists i \in \omega)\ |A \bigtriangleup A_i| < \omega$$

or

$$(\forall i \in \omega)\ |A \cap A_i| < \omega.$$

The first one is impossible: For every $x \in [\omega]^\omega$ we have $|A_x \cap A_i| = 1$, hence $A_x \cap A$ is finite. However, if $\langle x, y\rangle \in B_p$ then $y \subseteq A_x$, so $y \notin B_{p_A}$.

Therefore, we must be in the second case. Let $f \in \omega^\omega$ be given by $f(i) = \max(A \cap A_i)$. Take $\langle x, y\rangle \in B_{p'}$ such that $\underline{x}$ dominates f. Again, we have that $A_x \cap A$ is finite, so $y \notin B_{p_A}$. A contradiction. $\square$

Let $a \in [\varepsilon]^{\leq\omega}$, let $\alpha = \mathrm{o.t.}(a)$ and let $i_a : \alpha \to a$ be the increasing ennumeration of a. The cannonical isomorphism $\iota_a : P_\alpha \to P_a$ is given by $\iota_a(p) = \tilde{p}$, where $\tilde{p}$ is a function on a, such that for each $\beta < \alpha$, $p(\beta) = \tilde{p} \circ (i_a\restriction\beta)$.

Observe that for $p \in P_\alpha$

$$B_p = B_{\iota_a(p)} \bullet i_a = \{x \circ i_a : x \in B_{\iota_a(p)}\}.$$

Fact 3.24. *If $a \subseteq a' \in [\varepsilon]^{\leq\omega}$ and $p' \in P_{a'}$ then there exists $p \in P_a$ such that $B_p \subseteq B_{p'}{\restriction}a$.*

Proof. Define p as follows: For $\xi \in a$ let

$$p(\xi)(\langle x_\zeta\colon \zeta \in a \cap \xi\rangle) = p'(\xi)(\langle x'_\zeta\colon \zeta \in a' \cap \xi\rangle),$$

where x'_ζ are defined by induction on $\zeta \in a'$ in the following way: Suppose that we have x'_η for every $\eta \in a' \cap \zeta$. If $\zeta \in a$ then $x'_\zeta = x_\zeta$. If $\zeta \notin a$ then $x'_\zeta = s^q \cup A^q$ where $q = p'(\zeta)(\langle x'_\eta\colon \eta \in a' \cap \zeta\rangle)$. It is easy to verify that $B_p \subseteq B_{p'}{\restriction}a$. □

Definition 3.25. Define $\pi\colon P_\varepsilon \to Q_\varepsilon$, $\pi(p) = q$ such that:

- if $0 \notin \mathrm{dom}(p)$ then $q(0) = \langle \emptyset, \omega\rangle$,
- if $0 \in \mathrm{dom}(p)$ then $q(0) = p(0)(\emptyset)$,
- if $0 < \gamma \notin \mathrm{dom}(p)$ then $q(\gamma)$ is a Q_γ-name for $\langle \emptyset, \omega\rangle$,
- if $0 < \gamma \in \mathrm{dom}(p)$ then $q(\gamma)$ is a Q_γ-name such that

$$Q_\gamma \Vdash q(\gamma) = p(\gamma)(\langle \dot r_\xi\colon \xi \in \mathrm{dom}(p) \cap \gamma\rangle).$$

A straightforward argument by induction on ε gives the next fact. The successor step is similar to Fact 3.11, the limit step is trivial.

Fact 3.26.

$$Q_\varepsilon \Vdash \pi(p) \in \dot G_\varepsilon \leftrightarrow \langle \dot r_\xi\colon \xi \in \mathrm{dom}(p)\rangle \in B_p.$$

□

Arguing as in Corollary 3.12 we get

Corollary 3.27. *π preserves the ordering, i.e. for every $p, p' \in P_\varepsilon$*

$$p \leq_{P_\varepsilon} p' \;\Rightarrow\; \pi(p) \leq_{Q_\varepsilon} \pi(p').$$

□

The next lemma is a modification of [21, Lemma 3.8]. We present it without proof. One can get it following the argument from [21], which in fact resembles the proof of preservation of properness under countable support iteration. We just point out that one can repeat the argument from Lemma 3.17 to deal with a successor step.

Lemma 3.28 (c.f. [21, Lemma 3.8])**.** *Suppose that $N \prec H_\kappa$ is countable, $\varepsilon \in N$ is an ordinal, and $q \in Q_\varepsilon \cap N$. Let $a = \varepsilon \cap N$. Then there exists $p_q \in P_a$ such that each $\bar{x} \in B_{p_q}$ is Q_ε-generic over N with q in its induced filter, i.e. there is a (necessarily unique) $G_{\bar{x}} \subseteq Q_\varepsilon \cap N$ such that Q_ε-generic over N, $q \in G_{\bar{x}}$, and*

$$\bar{x} = \langle \dot{r}_\xi \colon \xi < \varepsilon \rangle / G_{\bar{x}} = \langle \dot{r}_\xi / G_{\bar{x}} \colon \xi \in a \rangle.$$

In particular $\pi(p_q) \leq_{Q_\varepsilon} q$ is N-master. □

Using this lemma we get:

Lemma 3.29. *$\pi \colon P_\varepsilon \to Q_\varepsilon$ is a dense embedding.*

Proof. To show that the image of π is dense fix $q \in Q_\varepsilon$ and take a countable $N \prec H_\kappa$ with $\varepsilon, q \in N$. Denote $a = \varepsilon \cap N$ and let $p_q \in P_a$ be given by Lemma 3.28. Then $\pi(p_q) \leq_{Q_\varepsilon} q$.

We need to show that π preserves incompatibility. To this end it is enough to prove that for any $p, p' \in P_\varepsilon$, if $\pi(p) \leq_{Q_\varepsilon} \pi(p')$ then there exists $p'' \leq_{P_\varepsilon} p, p'$. Fix $p, p' \in P_\varepsilon$ such that $\pi(p) \leq_{Q_\varepsilon} \pi(p')$. Let $N \prec H_\kappa$ be countable, ε, p, p', $\pi(p)$, $\pi(p') \in N$ and let $a = \varepsilon \cap N$. Define $q = \pi(p)$ and take $p'' = p_q$ as in Lemma 3.28.

We will show that $p_q \leq_{P_\alpha} p'$. Let $\bar{x} \in B_{p_q}$. Take $G_{\bar{x}}$ as in Lemma 3.28 (so $\bar{x} = \langle \dot{r}_\xi / G_{\bar{x}} \colon \xi \in a \rangle$ and $q \in G_{\bar{x}}$). Since $q \leq_{Q_\varepsilon} \pi(p')$ we have $\pi(p') \in G_{\bar{x}}$. By Fact 3.26, $N[G_{\bar{x}}] \models x \in B_{p'}$ and by absoluteness, $x \in B_{p'}$. Hence, $B_{p_q} \subseteq B_{p'}$.

Similarily, one can show that $p_q \leq_{P_\alpha} p$. □

Corollary 3.30. *For any ordinal ε we have $P_\varepsilon \cong Q_\varepsilon$.*

□

From now on, we will rather use P_ε than Q_ε and we will treat $\dot{G}_\varepsilon$, $\dot{r}_\xi$ as P_ε-names. Note that P_ε forces that $\langle \xi, \dot{r}_\xi \rangle$ is the only member in the intersection of all $B_p \restriction \{\xi\}$, such that p is in the generic filter and $\xi \in \mathrm{dom}(p)$.

By Fact 3.26 we get:

Corollary 3.31. *For any ordinal ε we have*

$$P_\varepsilon \Vdash \left(p \in \dot{G}_\varepsilon \leftrightarrow \langle \dot{r}_\xi \colon \xi \in \mathrm{dom}(p) \rangle \in B_p \right).$$

□

The following is an analogue of Lemma 3.6.

Lemma 3.32. *Let $a \in [\varepsilon]^{\leq\omega}$ and $B \subseteq ([\omega]^{\omega})^a$ be Borel. Then for every $p \in P_\varepsilon$ there exists $p' \leq_{P_\varepsilon} p$ such that $a \subseteq \mathrm{dom}(p')$ and either $(B_{p'}{\restriction}a) \subseteq B$ or $(B_{p'}{\restriction}a) \cap B = \emptyset$.*

Proof. Take $p'' \leq_{P_\alpha} p$ be such that $a \subseteq \mathrm{dom}(p'')$ and $p'' \Vdash_{P_\varepsilon} \langle \dot{r}_\xi : \xi \in a\rangle \in B$ or $p'' \Vdash_{P_\varepsilon} \langle \dot{r}_\xi : \xi \in a\rangle \in B^c$. Without loss of generality assume the former. Denote $q = \pi(p'')$. Take countable $N \prec H_\kappa$, such that $\varepsilon, p'', q, a, B \in N$. Let $a' = \varepsilon \cap N$. So, $a \subseteq a'$. Using Lemma 3.28 find p_q, such that $\mathrm{dom}(p_q) = a'$ and B_{p_q} consists solely of $(Q_\varepsilon)^N$-generics over N. Define $p' = p_q$. Arguing as in the proof of Lemma 3.29, we get that $p' \leq_{P_\alpha} p$ and $B_{p'}{\restriction}a \subseteq B$. □

The next lemma is needed to prove the consistency of axioms, which will be introduced in forthcoming chapters.

Lemma 3.33. *Suppose that ε is an ordinal, $a \in [\varepsilon]^{\leq\omega}$, $\alpha = \mathrm{o.t.}(a)$, $i_a\colon \alpha \to a$ is the increasing ennumeration of a, $\iota_a\colon P_\alpha \to P_a$ is the cannonical isomorphism, $p \in P_a$, and $B \subseteq ([\omega]^{\omega})^a$ is a Borel set. Then*

$$(p \Vdash_{P_\varepsilon} \langle \dot{r}_\xi : \xi \in a\rangle \in B) \;\Leftrightarrow\; \left(\iota_a^{-1}(p) \Vdash_{P_\alpha} \langle \dot{r}_\gamma : \gamma < \alpha\rangle \in B \bullet i_a\right).$$

Proof. We will show the right-to-left implication only; the opposite one is similar. Suppose that

$$\iota_a^{-1}(p) \Vdash_{P_\alpha} \langle \dot{r}_\gamma : \gamma < \alpha\rangle \in B \bullet i_a.$$

Fix $p' \leq_{P_\varepsilon} p$. By Lemma 3.32 there exists $p'' \leq_{P_\varepsilon} p'$, such that either $B_{p''}{\restriction}a \subseteq B$ or $B_{p''}{\restriction}a \cap B = \emptyset$. If $B_{p''}{\restriction}a \subseteq B$, then by Corollary 3.31, $p'' \Vdash_{P_\varepsilon} \langle \dot{r}_\xi : \xi \in a\rangle \in B$. We will show that the other case is impossible.

Assume a contrario that $(B_{p''}{\restriction}a) \cap B = \emptyset$. By Fact 3.24 get $p^+ \in P_a$, such that $B_{p^+} \subseteq B_{p''}{\restriction}a$. We have $B_{p^+} \cap B = \emptyset$ and $B_{p^+} \subseteq B_p$, so $p^+ \leq_{P_a} p$. Therefore, $\iota_a^{-1}(p^+) \leq_{P_\alpha} \iota_a^{-1}(p)$ and $B_{\iota_a^{-1}(p^+)} \cap (B \bullet i_a) = \emptyset$. By the (remark below) Lemma 3.5 we have

$$\iota_a^{-1}(p^+) \Vdash_{P_\alpha} \langle \dot{r}_\gamma : \gamma < \alpha\rangle \notin B \bullet i_a.$$

A contradiction. □

Definition 3.34. For $a \in [\varepsilon]^{\leq\omega}$ let $\mathcal{I}^a$ be the σ-ideal on $([\omega]^\omega)^a$ generated by Borel sets B, such that

$$P_\varepsilon \Vdash \langle \dot{r}_\xi \colon\ \xi \in a\rangle \notin B.$$

Observe that, by Lemma 3.33, $\mathcal{I}^a$ is isomorphic to $\mathcal{I}^{\text{o.t.}(a)}$. Precilesly, if ε is an ordinal, $a \in [\varepsilon]^{\leq\omega}$, $\alpha = \text{o.t.}(a)$ and $i_a \colon \alpha \to a$ is the increasing ennumeration of a, then

$$(\forall X \subseteq ([\omega]^\omega)^a)\ \ (X \in \mathcal{I}^a \ \leftrightarrow\ X \bullet i_a \in \mathcal{I}^\alpha)\,.$$

Thus, $\mathcal{I}^a$ does not depend on ε.

By Lemma 3.32, Corollary 3.31 and Fact 3.24 we get:

Corollary 3.35. *For an ordinal ε, $a \in [\varepsilon]^{\leq\omega}$ and a Borel set $B \subseteq ([\omega]^\omega)^a$ we have:*

(i) $B \notin \mathcal{I}^a \ \Leftrightarrow\ (\exists p \in P_\varepsilon)(a \subseteq \text{dom}(p)\ \&\ B_p \subseteq B)$
$B \notin \mathcal{I}^a \ \Leftrightarrow\ (\exists p \in P_\varepsilon)(a = \text{dom}(p)\ \&\ B_p \subseteq B)$,

(ii) $B \in \mathcal{I}^a \ \Leftrightarrow\ P_\varepsilon \Vdash \langle \dot{r}_\xi \colon \xi \in a\rangle \notin B$
$B \in \mathcal{I}^a \ \Leftrightarrow\ (\forall p \in P_\varepsilon)(\exists p' \leq_{P_\varepsilon} p)\,(a \subseteq \text{dom}(p')\ \&\ (B_{p'}{\restriction}a) \cap B = \emptyset)$.

□

The next theorem describes the Borel Reading of Names. We formulate it in a little bit more general way than its analogue in [23] (c.f. [23, Proposition 2.3.1]).

Theorem 3.36. *Suppose that ε is an ordinal, $p \in P_\varepsilon$, and for every $n \in \omega$, $\dot{\tau}_n$ is a P_ε-name for an element of 2. Let $N \prec H_\kappa$ be a countable structure with $\varepsilon, p \in N$ and $\{\dot{\tau}_n \colon n \in \omega\} \subseteq N$ (note $\subseteq$ here, not $\in$). Let $a = \varepsilon \cap N$. There exist an N-master $p' \leq_{P_\varepsilon} p$ with $\text{dom}(p') = a$ and $\{f_n \colon n \in \omega\}$ such that for each $n \in \omega$, $f_n \colon B_{p'} \to 2$ is a Borel function and*

$$p' \Vdash_{P_\varepsilon} f_n(\langle \dot{r}_\xi \colon \xi \in a\rangle) = \dot{\tau}_n.$$

Proof. Let $q = \pi(p) \in Q_\varepsilon$ and $p' = p_q$, where p_q is given by Lemma 3.28. Then $p' \leq_{P_\varepsilon} p$. For $n \in \omega$ and $\bar{x} \in B_{p'}$ let $f_n(\bar{x}) = \dot{\tau}_n / G_{\bar{x}}$, where $G_{\bar{x}} \subseteq Q_\varepsilon \cap N$ is the unique Q_ε-generic filter over N such that $q \in G_{\bar{x}}$ and

$$\bar{x} = \langle \dot{r}_\xi / G_{\bar{x}} \colon \xi \in a\rangle.$$

These definitions work. Indeed, for $n \in \omega$ let $\mathcal{A}_n^0, \mathcal{A}_n^1 \in N$ be such that $\mathcal{A}_n^0 \cup \mathcal{A}_n^1$ is a maximal antichain in P_ε and

$$p'' \in \mathcal{A}_n^i \;\Rightarrow\; p'' \Vdash_{P_\varepsilon} \dot{\tau}_n = i.$$

By Fact 3.26 applied in N, for $\bar{x} \in B_{p'}$, $G_{\bar{x}}$ picks the unique $p'' \in (\mathcal{A}_n^0 \cup \mathcal{A}_n^1) \cap N$ for which $\bar{x} \in B_{p''}$. For $\bar{x} \in B_{p'}$, $f_n(\bar{x}) = i$ iff this p'' belongs to $\mathcal{A}_n^i$. So we have

$$(*) \qquad (\forall \bar{x} \in B_{p'}) \left((\exists p'' \in \mathcal{A}_n^i)(\bar{x} \in B_{p''}) \;\Leftrightarrow\; (f_n(\bar{x}) = i) \right)$$

Since N is countable f_n is Borel. Now let G be any P_ε-generic over V containing p' and let $\bar{y} = \langle \dot{r}_\xi / G \colon \xi \in a \rangle$. By Corollary 3.31, $\bar{y} \in B_{p'}$. Since, by absolutness $(*)$ holds in $V[G]$, we get

$$f_n(\bar{y}) = i \;\Leftrightarrow\; (\exists p'' \in \mathcal{A}_n^i \cap N)\; \bar{y} \in B_{p''} \;\Leftrightarrow\; \mathcal{A}_n^i \cap N \cap G \neq \emptyset$$

which is equivalent to $V[G] \models \dot{\tau}_n / G = i$. $\square$

Chapter 4

Axiom CPA

Definition 4.1. Let us introduce the CPA game. Two players, Adam and Eve, play ω_1 rounds. Adam starts each round $\gamma < \omega_1$ playing a triple $(\alpha_\gamma, A_\gamma, f_\gamma)$, where:

- α_γ is a countable ordinal[1],
- $A_\gamma \in Q'_{\alpha_\gamma}$,
- $f_\gamma\colon A_\gamma \to 2^\omega$ is a Borel function.

Then Eve responds with some $E_\gamma \subseteq A_\gamma$ from Q'_{α_γ}. Adam wins iff

$$\bigcup_{\gamma<\omega_1} f_\gamma[E_\gamma] = 2^\omega.$$

The axiom CPA says that CH fails and Eve has no winning strategy in the above game.

It is clear that when we replace 2^ω by any other uncountable Polish space, we get the equivalent axiom. We shall often do this while deriving consequences of CPA.

The next theorem says that CPA is true in the Mathias models. The argument resembles the one for the Sacks forcing (cf. [7, Theorem 7.2.1], [23, Proposition 6.1.3]).

Theorem 4.2. $P_{\omega_2} \Vdash$ CPA.

[1]Note that if $\alpha_\gamma = 0$ then Q'_{α_γ} contains exactly one element, namely $\{\emptyset\}$.

Proof. Since $V[G_{\omega_1}] \models \mathrm{CH}$ and $P_{\omega_2} \cong P_{\omega_1} * P_{\omega_2}$ without loss of generality we may assume that $V \models \mathrm{CH}$. Now, for each $\alpha < \omega_2$ we have $V[G_\alpha] \models \mathrm{CH}$ and $V[G_{\omega_2}] \models |2^\omega| = \omega_2$.

Suppose that $\sigma \in V[G_{\omega_2}]$ is a strategy of Eve for the CPA game. We will show that σ is not winning. A standard closure argument gives $C \in V$ which is an ω_1-club on ω_2 such that for every $\beta \in C$ we have $\sigma\restriction V[G_\beta] \in V[G_\beta]$. Taking $V[G_\beta]$ for some $\beta \in C$ as the ground model, we may assume that $\sigma\restriction V \in V$.

Let $\langle(\alpha_\gamma, A_\gamma, f_\gamma)\colon \gamma < \omega_1\rangle \in V$ ennumerate all triples (α, A, f) which are legal moves of Adam from V. (Note that by Remark 3.8 and by Schoenfield's Theorem on absoluteness, we have $V \models A \notin \mathcal{I}^\alpha$ iff $V[G_{\omega_2}] \models A \notin \mathcal{I}^\alpha$.) We claim that this is a winning counterplay of Adam against σ.

Work in V. Let $\langle E_\gamma\colon \gamma < \omega_1\rangle$ be the sequence of responses of Eve dictated by σ (by absoluteness, $V \models E_\gamma \notin \mathcal{I}^{\alpha_\gamma}$). We need to show:

$$V[G_{\omega_2}] \models \bigcup_{\gamma<\omega_1} f_\gamma[E_\gamma] = 2^\omega.$$

This is done by the following density argument. Suppose that $p \in P_{\omega_2}$ and $p \Vdash_{P_{\omega_2}} \dot{x} \in 2^\omega$. Find a countable $N \prec H_\kappa$, such that $\omega_2, p, \dot{x} \in N$. Let $a = \omega_2 \cap N$. By Theorem 3.36 (taking $\dot{\tau}_n = \dot{x}(n)$) there are $p_0 \leq_{P_{\omega_2}} p$ and f such that $\mathrm{dom}(p_0) = a$, $f\colon B_{p_0} \to 2^\omega$ is a Borel function and

$$p_0 \Vdash_{P_{\omega_2}} \dot{x} = f(\langle \dot{r}_\xi\colon \xi \in a\rangle).$$

Denote $\alpha = \mathrm{o.t.}(a)$. Let $i_a\colon \alpha \to a$ be the increasing ennumeration of a and let $\iota_a\colon P_\alpha \to P_a$ be the cannonical isomorphism. The triple $(\alpha, B_{\iota_a^{-1}(p_2)}, f \circ i_a)$ is a legal move of Adam. Hence, it is equal to $(\alpha_\gamma, A_\gamma, f_\gamma)$ for some $\gamma < \omega_1$. Since E_γ is $\mathcal{I}^\alpha$-positive, by Corollary 3.9, there is $p' \in P_\alpha$ such that $B_{p'} \subseteq E_\gamma$. Define $p_1 = \iota_a(p')$. By Lemma 3.33, we have

$$p_1 \Vdash_{P_{\omega_2}} \langle \dot{r}_\xi\colon\ \xi \in a\rangle \in E_\gamma \bullet i_a^{-1}.$$

Note that $E_\gamma \bullet i_a^{-1} = \{\bar{x} \circ i_a^{-1}\colon \bar{x} \in E_\gamma\} \subseteq B_{p_0}$. Hence $p_1 \leq_{P_{\omega_2}} p_0$ and $p_1 \Vdash_{P_{\omega_2}} \dot{x} \in f[E_\gamma \bullet i_a^{-1}]$. But $f[E_\gamma \bullet i_a^{-1}] = f_\gamma[E_\gamma]$. □

4.1 Covering of meager and null ideals

Let us start the applications with a very simple consequence of CPA.

Fact 4.3. CPA $\vdash \mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \omega_1$, *where* $\mathcal{M}$ *and* $\mathcal{N}$ *denote the* σ*-ideal of meager and null subsets of* 2^ω *respectively.*

Proof. Let us describe the strategy of Eve. Suppose that at the γ-th round Adam plays $(\alpha_\gamma, A_\gamma, f_\gamma)$. Then Eve plays $E_\gamma \subseteq A_\gamma$ from Q'_{α_γ} such that $f_\gamma[E_\gamma]$ is meager [null]. This is possible, otherwise A_γ would force $f_\gamma(\langle \dot r_\xi \colon \xi < \alpha_\gamma\rangle)$ to be a Cohen [random] real. But since Q'_{α_γ} has the Laver property, it adds no Cohen [random] reals (see [2, Lemma 7.2.3]).

By CPA, fix a play of the game such that Adam wins and Eve plays according to this strategy. The play provides the collection $\{f_\gamma[E_\gamma] \colon \gamma < \omega_1\}$ of meager [null] subsets of 2^ω such that $\bigcup_{\gamma<\omega_1} f_\gamma[E_\gamma] = 2^\omega$. □

4.2 Distributivity of r.o.($\mathbb{R}^*$)

In this section, for $A, B \subseteq \mathbb{R}$, $A \subseteq^* B$ means that $A \setminus B$ is bounded.

Dow proved in [9] that $Q_{\omega_2} \Vdash \mathfrak{h}(\mathrm{r.o.}(\mathbb{R}^*)) = \omega_1$, where r.o.($\mathbb{R}^*$) is the algebra of regular open subsets of $\mathbb{R}$ modulo the ideal of bounded sets. We will show that this result follows from CPA. The argument uses the following Lemma:

Lemma 4.4 ([9, Lemma 2.4])**.** *Suppose that* $\mathbb{P}$ *is a poset with the Laver property and* G *is* $\mathbb{P}$*-generic. Let* $W \in V[G]$ *be an unbounded subset of* $\mathbb{R}$*. Then there is a dense open subset* $U \in V$ *of* $\mathbb{R}$ *such that* $W \setminus U$ *is unbounded.*

Theorem 4.5. CPA $\vdash \mathfrak{h}(\mathrm{r.o.}(\mathbb{R}^*)) = \omega_1$.

Proof. Let X be the Polish space consisting of (codes for) open unbounded subsets of $\mathbb{R}$. We use the CPA game in which functions f_γ value in X. Consider the following strategy of Eve. Suppose that at the round γ Adam plays $(\alpha_\gamma, A_\gamma, f_\gamma)$. Denote $f_\gamma(\langle \dot r_\xi \colon \xi < \alpha_\gamma\rangle)$ by $\dot W_\gamma$.[2] By Lemma 4.4, there are $E'_\gamma \in Q'_{\alpha_\gamma}$ and an open dense set $U_\gamma \subseteq \mathbb{R}$ such that

$$E'_\gamma \Vdash_{Q'_{\alpha_\gamma}} \dot W_\gamma \setminus U_\gamma \text{ is unbounded.}$$

Let

$$E_\gamma = \{\bar x \in E'_\gamma \colon\ f_\gamma(\bar x) \setminus U_\gamma \text{ is unbounded}\}.$$

[2]Precisely, take a Q'_{α_γ}-name $\dot W_\gamma$ such that $Q'_{\alpha_\gamma} \Vdash \dot W_\gamma = f_\gamma(\langle \dot r_\xi \colon \xi < \alpha_\gamma\rangle)$.

Then E_γ is Borel, since f_γ is a Borel function. Moreover, E_γ is $\mathcal{I}^\alpha$-positive, since $E'_\gamma \setminus E_\gamma \in \mathcal{I}^\alpha$. Eve plays this set.

By CPA, Adam has a winning counterplay $\langle(\alpha_\gamma, A_\gamma, f_\gamma)\colon \gamma < \omega_1\rangle$ against this strategy, so that $\bigcup_{\gamma<\omega_1} f_\gamma[E_\gamma] = X$. Moreover, the play provides the collection $\langle U_\gamma\colon \gamma < \omega_1\rangle$ of open dense subsets of $\mathbb{R}$.

Now we define the following strategy of PII for the distributivity game $G(\text{r.o.}(\mathbb{R}^*), \omega_1)$. At each step $\gamma < \omega_1$, PI plays some regular open unbounded $O^{I}_\gamma \subseteq \mathbb{R}$, such that $O^{I}_\gamma \subseteq^* O^{II}_\beta$ for every $\beta < \gamma$. PII responds with some regular open unbounded $O^{II}_\gamma \subseteq O^{I}_\gamma \cap U_\gamma$. This is possible since $U_\gamma \subseteq \mathbb{R}$ is open and dense.

We claim that this strategy of PII is winning. Indeed, suppose that PII loses. Then there exists regular open unbounded $O \subseteq \mathbb{R}$ such that $O \subseteq^* O^{II}_\gamma$ for every $\gamma < \omega_1$. Since Adam wins the play of the CPA game, we have $O \in f_\gamma[E_\gamma]$ for some $\gamma < \omega_1$. It means that $O\setminus U_\gamma$ is unbounded. On the other hand, $O^{II}_\gamma \subseteq U_\gamma$, by the strategy of PII. Thus, $O \subseteq^* U_\gamma$. A contradiction.

By Fact 2.2, $\mathfrak{h}(\text{r.o.}(\mathbb{R}^*)) \leq \omega_1$. The opposite inequality is trivially true in ZFC. □

4.3 Distributivity of $(c_0\setminus\ell^1, \leq^*)$

In this section we deal with the distributivity of $(c_0\setminus\ell^1, \leq^*)$, where

$$c_0\setminus\ell^1 = \{\bar{c} = \langle c_n\colon n \in \omega\rangle \in \mathbb{R}^\omega_+\colon \lim c_n = 0 \ \wedge \ \sum c_n = \infty\}$$

and $\bar{d} \leq^* \bar{c}$ iff $d_n \leq c_n$ for all but finitely many n. It was shown in [11] that $\mathfrak{h}((c_0\setminus\ell^1, \leq^*)) = \omega_1$ in the Mathias model. We will get this equality from CPA.

For $H \in [\omega]^\omega$ we define the open dense $\mathcal{D}_H \subseteq c_0\setminus\ell^1$:

$$\mathcal{D}_H = \{\bar{a} \in c_0\setminus\ell^1\colon \sum_{n\in H} a_n < \infty \text{ or } \sum_{n\in\omega\setminus H} a_n < \infty\}.$$

The following lemma was essentially proved in [11].

Lemma 4.6. *Suppose that $\mathbb{P}$ satisfies the Laver property. Let $\bar{b} \in c_0\setminus\ell^1$ and suppose that $p \in \mathbb{P}$ forces $\dot{x} \leq^* \bar{b}$, where $\dot{x}$ is a $\mathbb{P}$-name for a sequence in $c_0\setminus\ell^1$. Then there is $H \in [\omega]^\omega \cap V$ and $q \leq_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}} \dot{x} \notin \mathcal{D}_H$.*

Theorem 4.7. CPA $\vdash \mathfrak{h}((c_0\backslash\ell^1,\leq^*))=\omega_1$.

Proof. Let $\bar{b}=\langle 1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots\rangle$ and $X=\{\bar{a}\in c_0\backslash\ell^1\colon \bar{a}\leq^*\bar{b}\}$. We describe a strategy of Eve. Suppose that for $\gamma<\omega_1$ Adam plays $(\alpha_\gamma,A_\gamma,f_\gamma)$, where $f_\gamma\colon A_\gamma\to X$. Denote $f_\gamma(\langle\dot{r}_\xi\colon\xi<\alpha_\gamma\rangle)$ by $\dot{x}_\gamma$. Using Lemma 4.6, Eve finds $H_\gamma\in[\omega]^\omega$ and a Borel, $\mathcal{I}^{\alpha_\gamma}$-positive set $E'_\gamma\subseteq A_\gamma$ such that

$$E'_\gamma\Vdash_{Q'_{\alpha_\gamma}}\dot{x}_\gamma\notin\mathcal{D}_{H_\gamma}.$$

Then she plays

$$E_\gamma=\{\bar{x}\in E'_\gamma\colon f_\gamma(\bar{x})\notin\mathcal{D}_{H_\gamma}\}.$$

Let $\{H_\gamma\in[\omega]^\omega\colon\gamma<\omega_1\}$ be the collection produced by this strategy during the play in which Adam wins. A winning strategy for PII in the distributivity game $G(c_0\backslash\ell^1,\omega_1)$ is the following: PII begins with the initial $p_{in}=\bar{b}$. For each $\gamma<\omega_1$ he plays $p^{II}_\gamma\leq^* p^{I}_\gamma$ such that $p^{II}_\gamma\in\mathcal{D}_{H_\gamma}$. It is clear that this strategy is winning. Indeed, suppose that there is $p\in c_0\backslash\ell^1$ such that $p\leq^* p^{II}_\gamma$ for all $\gamma<\omega_1$. Since Adam wins the CPA game, there is $\gamma<\omega_1$ such that $p\in f_\gamma[E_\gamma]$. Thus, $p\notin\mathcal{D}_{H_\gamma}$. On the other hand, $p\leq^* p^{II}_\gamma\in D_{H_\gamma}$. A contradiction.

By Fact 2.2 $\mathfrak{h}((c_0\backslash\ell^1,\leq^*))\leq\omega_1$. The opposite inequality is obviously true in ZFC. □

Chapter 5

Axiom CPAs

In this chapter we present a little stronger version of the axiom, which we call CPAs (CPA with sections). We show four consequences.

Definition 5.1. Consider the modification of the CPA game, which makes it harder for Adam to win. Legal moves of players are the same. However, in the modified game Adam wins if there exists $r \in [\omega]^\omega$ such that

$$\bigcup_{\gamma\in\omega_1} f_\gamma[(E_\gamma)_r] = 2^\omega,$$

where $(E_\gamma)_r$ is the section of E_γ at r, i.e. $(E_\gamma)_r$ consists of those sequences from E_γ whose initial segment is $\langle r\rangle$.[1]

Let us call this modification the CPAs game. The axiom CPAs says that CH fails and Eve has no winning strategy in the CPAs game.

One can obtain the next theorem modifying slightly the proof of Theorem 4.2: If σ is a fixed strategy of Eve and V is the ground model (without loss of generality $V \models \mathrm{CH}$ and $\sigma{\restriction}V \in V$), then the first Mathias real $r_0 \in V[G_{\omega_2}]$ is a good candidate for the section point r.

Theorem 5.2. $P_{\omega_2} \Vdash$ CPAs. □

5.1 Distributivity of $\mathcal{P}(\omega)/\operatorname{fin}$

From now on, $\subseteq^*$ denotes the almost inclusion relation on $\mathcal{P}(\omega)$.

[1]If $\alpha_\gamma = 0$ then $E_\gamma = \{\emptyset\}$ and $(E_\gamma)_r = \emptyset$.

It is folklore that $\mathfrak{h}(\mathcal{P}(\omega)/\operatorname{fin}) = \omega_2$ in the Mathias model (see e.g. [11] or [21]). We will get the essential inequality as a consequence of CPAs.

Fact 5.3. $\mathrm{CPAs} \vdash \mathfrak{h}(\mathcal{P}(\omega)/\operatorname{fin}) > \omega_1$.

Proof. Let $\{\mathcal{D}_\gamma : \gamma < \omega_1\}$ be a decreasing family of open dense subsets of $(\mathcal{P}(\omega), \subseteq^*)$. By the homogenity and by Fact 2.2 it is enough to show that

$$\bigcap_{\gamma<\omega_1} \mathcal{D}_\gamma \neq \emptyset.$$

We ask Eve to use the following strategy: Let $(\alpha_\gamma, A_\gamma, f_\gamma)$ be Adam's move at step γ. Eve plays $E_\gamma \subseteq A_\gamma$ for which there exists $d_\gamma \in \mathcal{D}_\gamma$ such that:

(i) for each sequence $\bar{x} \in E_\gamma$ we have $\bar{x}(0) \subseteq^* d_\gamma$,

(ii) the image $f_\gamma[E_\gamma]$ is meager in 2^ω.

Such d_γ, E_γ exist since $\mathcal{D}_\gamma$ is open dense and Q'_{α_γ} adds no Cohen reals.

By CPAs, Eve loses to some counterplay of Adam. Therefore, there is $r \in 2^\omega$ such that $\bigcup_{\gamma<\omega_1} f_\gamma[(E_\gamma)_r] = 2^\omega$. But then $r \subseteq^* d_\gamma$ (so $r \in \mathcal{D}_\gamma$) for uncountably many $\gamma < \omega_1$ (otherwise 2^ω would be a union of countably many meager sets). Since $\{\mathcal{D}_\gamma : \gamma < \omega_1\}$ is decreasing we have $r \in \bigcap_{\gamma<\omega_1} \mathcal{D}_\gamma$. □

Notice that we do not need the full strength of CPAs in this proof. We just use that $\bigcup_{\gamma<\omega_1} f_\gamma[(E_\gamma)_r]$ is non-meager.

5.2 Borel Conjecture

Recall the definition of a strong measure zero (strong null) set and the Borel Conjecture BC:

Definition 5.4. $X \subseteq 2^\omega$ has strong measure zero iff for each $f \in \omega^\omega$ there exists $g\colon \omega \to 2^{<\omega}$ such that $(\forall n \in \omega)\ g(n) \in 2^{f(n)}$ and

$$(\forall x \in X)(\exists n \in \omega)\ g(n) \subseteq x.$$

It is clear that countable sets are strong null. The Borel Conjecture (BC) says the opposite (see [2]):

$$\text{BC} \Leftrightarrow \text{every strong null set is countable}$$

It is easy to show that BC fails under CH (see [2, Lemma 8.2.3]). Laver proved its consistency obtaining BC in the generic extension via the iteratated Laver forcing. A similar argument gives that the Borel Conjecture holds in the Mathias model (see [3]). Now we will derive BC from CPAs.

Theorem 5.5. CPAs $\vdash$ BC.

Proof. Fix an uncountable set $X \subseteq 2^\omega$. We describe a CPAs strategy of Eve. At the step $\gamma < \omega_1$, let Adam play $(\alpha_\gamma, A_\gamma, f_\gamma)$, where $f_\gamma\colon A_\gamma \to (2^{<\omega})^\omega$. Denote $\dot{g}_\gamma = f_\gamma(\langle \dot{r}_\xi\colon \xi < \alpha_\gamma\rangle)$.

Claim 5.6. *There is $E'_\gamma \in Q'_{\alpha_\gamma}$ stronger than A_γ which forces the formula*

$$(\forall n \in \omega)\left(\dot{g}_\gamma(n) \in 2^{\dot{r}_0(n)}\right) \;\to\; (\exists y \in \check{X})(\forall n \in \omega)(\dot{g}_\gamma(n) \nsubseteq y).$$

Proof. It is well known (see [2, Theorem 8.3.2]) that the ennumeration of Mathias real $\underline{r}_0$ witnesses that no uncountable subset of reals from the ground model is strong null. This fact remains true in any further extension by a poset with the Laver property (see [2, Theorem 8.3.1]). □

Let Eve play

$$E_\gamma = \{\bar{x} \in E'_\gamma\colon\ \varphi(\bar{x}) \;\Rightarrow\; \psi(\bar{x})\},$$

where $\varphi(\bar{x}) \;=\; (\forall n \in \omega)\ f_\gamma(\bar{x})(n) \in 2^{\underline{\bar{x}(0)}(n)}$

and $\psi(\bar{x}) \;=\; (\exists y \in X)(\forall n \in \omega)\ f_\gamma(\bar{x})(n) \nsubseteq y.$

By CPAs Adam has a winning counterplay. So there exists $r \in 2^\omega$, such that $\bigcup_{\gamma<\omega_1} f_\gamma[(E_\gamma)_r] = (2^{<\omega})^\omega$. Its ennumeration $\underline{r}$ witnesses that X is not a strong measure zero set. Indeed, take any $g\colon \omega \to 2^{<\omega}$ such that $g(n) \in 2^{\underline{r}(n)}$ for all $n \in \omega$. There exists $\gamma < \omega_1$ such that $g \in f_\gamma[(E_\gamma)_r]$. Therefore,

$$(\exists y \in X)(\forall n \in \omega)\ g(n) \nsubseteq y.$$

□

5.3 Sub-cut points in $\mathbb{R}^*$

In this section $\mathbb{R}^* = \beta\mathbb{R} \setminus \mathbb{R}$ denotes the Čech-Stone remainder of the real line. We refer to [12] for a comprehensive introduction to the topic. Recall that for a connected topological space X, $x \in X$ is a cut point of X if $X \setminus \{x\}$

is not connected. The space $\mathbb{R}^*$ is an indecomposable continuum. Hence, it does not contain cut points. Nevertheless, $\mathbb{R}^*$ contains sub-cut points, i.e. points which cut its subcontinua.

Let $\mathbb{I} = [0, 1]$ and $\mathbb{M} = \bigcup_{n\in\omega}\{n\} \times \mathbb{I}$. Consider the Čech-Stone remainder $\beta\mathbb{M}$ of the space $\mathbb{M}$. The space of all non-principal ultrafilters on ω is denoted by ω^*. Suppose that $\mathcal{U} \in \omega^*$. Define

$$\mathbb{I}_{\mathcal{U}} = \bigcap_{U\in\mathcal{U}} \mathrm{cl}_{\beta\mathbb{M}} \left(\bigcup_{n\in U} \{n\} \times \mathbb{I} \right).$$

Spaces $\mathbb{I}_{\mathcal{U}}$ are crucial for research in subcontinua of $\mathbb{R}^*$. Indeed, each nontrivial subcontinuum of $\mathbb{R}^*$ contains a subcontinuum homeomorphic to $\mathbb{I}_{\mathcal{U}}$, for some ultrafilter $\mathcal{U}$. Therefore investigating sub-cut points of $\mathbb{R}^*$ is reduced to investigate cut points of $\mathbb{I}_{\mathcal{U}}$.

Definition 5.7. Recall that $x \in \mathbb{I}_{\mathcal{U}}$ if x is an ultrafilter on the lattice $2^{\mathbb{M}}$ (which consists of all closed subsets of $\mathbb{M}$) and for all $U \in \mathcal{U}$

$$\bigcup_{n\in U} \{n\} \times \mathbb{I} \in x.$$

(i) A point $x \in \mathbb{I}_{\mathcal{U}}$ is *near* if there exists a closed and discrete $D \subseteq \mathbb{M}$ such that $D \in x$.

(ii) A point $x \in \mathbb{I}_{\mathcal{U}}$ is *far* if it is not near.

(iii) A point $x \in \mathbb{I}_{\mathcal{U}}$ is *remote* if no closed nowhere dense subset of $\mathbb{M}$ belongs to x.

Constructions of a far cut point are known under MA or CH ([1], [24]). Zhu proved in [25] that in the Laver model there are no remote cut points of $\mathbb{I}_{\mathcal{U}}$ for any $u \in \omega^*$. He remarked that the proof works for the Mathias model, as well. Dow and Hart ([10]) generalized his result showing that in the Laver model there are no far cut points. In this section we will show that the same holds in the Mathias model, deriving the assertion from CPAs. Our argument resembles the one from [10].

Theorem 5.8. CPAs $\vdash$ *"For any $\mathcal{U} \in \omega^*$ there are no far cut points in $\mathbb{I}_{\mathcal{U}}$"*.

For $f, g \in \omega^\omega$ we say $g < f$ if for all $n \in \omega$ we have $g(n) < f(n)$. We will use the following lemma.

Lemma 5.9 ([10, Lemma 1.3]). *Let $\mathcal{U} \in \omega^*$. A point $x \in \mathbb{I}_{\mathcal{U}}$ is a cut point of $\mathbb{I}_{\mathcal{U}}$ if and only if for every $f \in \omega^\omega$ with $f > 0$ there exists $g \in \omega^\omega$ such that $g < f$ and*

$$\bigcup_{n\in\omega}\left(\{n\} \times \left[\frac{g(n)}{f(n)}, \frac{g(n)+1}{f(n)}\right]\right) \in x.$$

Now let us state the density lemma.

Lemma 5.10. *Suppose that $\mathbb{P} = Q * \mathbb{P}'$, where $\mathbb{P}'$ has the Laver property. Let G be $\mathbb{P}$-generic over V and let $\underline{r} \in \omega^\omega$ be the ennumeration of the Mathias real added by Q. Assume that $\mathcal{U} \in \omega^* \cap V$ and $x \in \mathbb{I}_u \cap V$. If x is a far point of $\mathbb{I}_{\mathcal{U}}$ in V, then for every $g \in \omega^\omega \cap V[G]$ such that $g < \underline{r}$ there exists $X \in x \cap V$ such that*

$$\bigcup_{n\in\omega}\left(\{n\} \times \left[\frac{g(n)}{\underline{r}(n)}, \frac{g(n)+1}{\underline{r}(n)}\right]\right) \cap X = \emptyset.$$

Proof. Let $\mathbb{P} \Vdash \dot{g} \in \omega^\omega \wedge \dot{g} < \dot{\underline{r}}$. Suppose that $\langle s, A\rangle * p_0 \in \mathbb{P}$. Since $\mathbb{P}'$ has the Laver property, there exist p_1 and Q-names $\langle \dot{g}_i(n) \colon i \leq n < \omega\rangle$ such that $Q \Vdash p_1 \leq_{\mathbb{P}'} p_0$ and

$$\langle s, A\rangle * p_1 \Vdash (\forall n \in \omega)\ \dot{g}(n) \in \{\dot{g}_0(n), \dot{g}_1(n), \dots, \dot{g}_n(n)\}.$$

We will find $B \subseteq A$ and $X \in x$ such that

$$\langle s, B\rangle * p_1 \Vdash_{\mathbb{P}} \bigcup_{n\in\omega}\left(\{n\} \times \left[\frac{\dot{g}(n)}{\dot{\underline{r}}(n)}, \frac{\dot{g}(n)+1}{\dot{\underline{r}}(n)}\right]\right) \cap X = \emptyset.$$

For simplicity let us assume that $s = \emptyset$. We loose no generality. Now we involve the standard technique of *Mathias diagonalization.*

Step 1: We find $A' \subseteq A$ and $\langle g_i^s \colon s \subseteq A', |s| < \omega, i < |s|\rangle$ such that for each non-empty finite $s \subseteq A'$ we have $g_i^s \leq \max(s)$ and

$$\langle s, A'/\max(s)\rangle \Vdash_Q \dot{g}_i(|s|-1) = g_i^s.$$

To this end we construct a decreasing sequence $A_0 \supseteq A_1 \supseteq \dots$ of infinite subsets of A and natural numbers $n_0 < n_1 < \dots$. Let $A_0 = A$ and $n_0 = \min(A_0)$. Suppose that A_k and n_k are constructed. Let $s_1, \dots, s_{2^k}$ ennumerate all subsets of $\{n_0, \dots, n_k\}$ containing n_k. By induction on $j \leq 2^k$

we find infinite $A_k^1 \supseteq \ldots \supseteq A_k^j \supseteq \ldots \supseteq A_k^{2^k}$ subsets of A_k/n_k such that $\langle s_j, A_k^j \rangle$ decides the value of $\dot{g}_i(|s_j|-1)$ for every $i < |s_j|$ (note that

$$Q \Vdash \dot{g}_i(|s_j|-1) < \dot{\underline{r}}(|s_j|-1) = \max(s_j)$$

and use Lemma 2.4 repeatedly). Let g_i^s be this value. Put $A_{k+1} = A_k^{2^k}$ and $n_{k+1} = \min(A_{k+1})$. Finally, we define $A' = \{n_0, n_1, n_2, \ldots\}$.

Step 2: We find $A'' \subseteq A'$ and $\langle a_i^l(s)\colon\ s \subseteq A'',\ |s| < \omega,\ l \geq |s|,\ i < l\rangle$, such that

$$a_i^{|s|}(s) = \lim_{\substack{m \in A'' \\ m > \max(s)}} \frac{1}{m} \cdot g_i^{s^\frown m} \quad \text{(note that } \langle s^\frown m, A''/m \rangle \Vdash_Q \dot{\underline{r}}(|s|) = m)$$

$$\text{and} \quad a_i^l(s) = \lim_{\substack{m \in A'' \\ m > \max(s)}} a_i^l(s^\frown m) \text{ for } s < l.$$

Again, we construct a decreasing sequence $A_0 \supseteq A_1 \supseteq \ldots$ of infinite subsets of A' and natural numbers $n_0 < n_1 < \ldots$. Let $A_0 = A'$ and $n_0 = \min(A_0)$. Suppose that we have A_k and n_k. Let $s_1, \ldots, s_{2^{k+1}}$ ennumerate all subsets of $\{n_0, \ldots, n_k\}$. By induction on $j \leq 2^k$ we find infinite $A_k^1 \supseteq \ldots \supseteq A_k^j \supseteq \ldots \supseteq A_k^{2^{k+1}}$ subsets of A_k.

Let us describe how to find A_k^j having A_k^{j-1} (we define A_k^0 as A_k). Suppose that $\max(s_j) = n_{k'}$ (if $s_j = \emptyset$ put $k' = 0$). Now we have to care about $a_i^{|s_j|+k-k'}(s)$ for all $i < |s_j| + k - k'$.

We will define sets $A_k^{j-1} = B_0 \supseteq B_1 \supseteq \ldots \supseteq B_{k-k'+1} = A_k^j$. Let B_1 be such that for each $i \leq |s_j| + k - k'$ and for each $s' \subseteq B_1$ satisfying $|s'| = k - k'$ the sequence $\langle \frac{1}{m} \cdot g_i^{s_j{}^\frown s'^\frown m}\colon\ m \in B_1/\max s' \rangle$ is convergent.

The set B_1 is obtained by a construction similar to the one from Step 1. We will find $C_0 \supseteq C_1 \supseteq \ldots$ and $y_0 < y_1 < \ldots$. We start with $C_0 = B_0$ and $y_0 = \min C_0$. Suppose that we have C_n and y_n.

To get C_{n+1} shrink the set C_n/y_n finitely many times so that for each $i \leq |s_j| + k - k'$ and for each $(k-k')$-element subset s' of $\{y_0, y_1, \ldots, y_n\}$ the sequence

$$\langle \frac{1}{m} \cdot g_i^{s_j{}^\frown s'^\frown m}\colon\ m \in C_{n+1}/\max s' \rangle$$

is convergent and define $a_i^{|s_j|+k-k'}(s_j{}^\frown s')$ as its limit. Note that it may already be defined, but the definitions agree anyway. Let $y_{n+1} = \min C_{n+1}$ and $B_1 = \{y_0, y_1, \ldots\}$.

By similar construction we find $B_2 \subseteq B_1$ such that for each $i \leq |s|+k-k'$ and for each $(k-k'-1)$-element $s' \subseteq B_2$ the sequence

$$\langle a_i^{|s_j|+k-k'}(s_j{}^\frown s'{}^\frown m): m \in B_2/\max s'\rangle$$

is convergent (note that now $s_j{}^\frown s'{}^\frown m$ has one element less than before). Again, we define $a_i^{|s_j|+k-k'}(s_j{}^\frown s')$ as its limit.

Repeating this procedure we find B_n for every $n \leq k-k'+1$. Together with B_n we define $a_i^{|s_j|+k-k'}(s_j{}^\frown s')$ for $s' \subseteq B_n$ such that $|s'| = k-k'-n+1$ and $i \leq |s_j|+k-k'$.

To finish step 2 we put $A_k^j = B_{k-k'+1}$, $A_{k+1} = A_k^{2^k}$, $n_{k+1} = \min A_{k+1}$, and $A'' = \{n_0, n_1, \ldots\}$.

Step 3: Let

$$D = \bigcup_{n\in\omega} \left(\{n\} \times \{a_i^n(s): i < n,\ s \subseteq A'' \cap n\}\right).$$

Then $D \subseteq M$ is discrete. Since $x \in \mathbb{I}_u$ is far, there is $X \in x$ such that $X \cap D = \emptyset$. It is enough to find $B \subseteq A''$ such that for any non-empty finite $s \subseteq B$ we have

$$\langle s, B/\max(s)\rangle * q' \Vdash_Q \left(\{|s|-1\} \times \left[\frac{\dot g(|s|-1)}{\underline{\dot r}(|s|-1)}, \frac{\dot g(|s|-1)+1}{\underline{\dot r}(|s|-1)}\right]\right) \cap X = \emptyset.$$

To get B we repeat the procedure from step 2, but insted of taking convergent subsequence of the remaining, we cut off its large enough initial fragment. Therefore, we care that for any finite $s \subseteq B$ and $m \in B$ greater than $\max(s)$ the value $\frac{1}{m} \cdot q_i^{s^\frown m}$ is "close" to $a_i^{|s|}(t)$, where t is the maximal initial segment of s such that $\max(t) < |s|$. By "close" we mean

$$\{|s|\} \times \left[\frac{q_i^{s^\frown m}}{m}, \frac{q_i^{s^\frown m}+1}{m}\right] \cap X = \emptyset.$$

□

Proof of Theorem 5.8. Fix $\mathcal{U} \in \omega^*$ and $x \in \mathbb{I}_{\mathcal{U}}$. Let Eve play the CPAs game keeping the strategy described below.

At the step $\gamma < \omega_1$ Adam plays α_γ, A_γ and $f_\gamma: A_\gamma \to \omega^\omega$. Denote $f_\gamma(\langle \dot r_\xi: \xi < \alpha_\gamma\rangle)$ by $\dot g_\gamma$. By Lemma 5.10, there exists $E'_\gamma \subseteq A_\gamma$ which forces that if $\dot g_\gamma < \underline{\dot r}_0$ then

$$(\exists X \in \check x) \bigcup_{n\in\omega} \left(\{n\} \times \left[\frac{\dot g_\gamma(n)}{\underline{\dot r}_0(n)}, \frac{\dot g_\gamma(n)+1}{\underline{\dot r}_0(n)}\right]\right) \cap X = \emptyset.$$

Eve plays

$$E_\gamma = \left\{\bar{x} \in E'_\gamma \colon f_\gamma(\bar{x}) < \underline{\bar{x}(0)} \Rightarrow \psi(\bar{x})\right\},$$

where

$$\psi(\bar{x}) = (\exists X \in x) \bigcup_{n\in\omega} \left(\{n\} \times \left[\frac{f_\gamma(\bar{x})(n)}{\underline{\bar{x}(0)}(n)}, \frac{f_\gamma(\bar{x})(n)+1}{\underline{\bar{x}(0)}(n)}\right]\right) \cap X = \emptyset.$$

By CPAs Adam has a winning counterplay. So there exists $r \in [\omega]^\omega$ such that $\bigcup_{\gamma<\omega_1} f_\gamma[(E_\gamma)_r] = \omega^\omega$. It means that for any $g \in \omega^\omega$ such that $g < \underline{r}$ we have

$$\bigcup_{n\in\omega} \left(\{n\} \times \left[\frac{g(n)}{\underline{r}(n)}, \frac{g(n)+1}{\underline{r}(n)}\right]\right) \notin x.$$

By Lemma 5.9, $\underline{r}$ witnesses that x is not a cut point of $\mathbb{I}_{\mathcal{U}}$. □

5.4 Rapid ultrafilters

Definition 5.11. Let $\mathcal{U} \in \omega^*$.

(*i*) $\mathcal{U}$ is a Q-point iff for every finite-to-one function $f \in \omega^\omega$ there exists $X \in \mathcal{U}$ such that $f \restriction X$ is one-to-one,

(*ii*) $\mathcal{U}$ is *rapid* (semi-Q-point) iff $(\forall f \in \omega^\omega)(\exists X \in \mathcal{U})(\forall i \in \omega)\ f(i) < \underline{X}(i)$, i.e $\{\underline{X} \colon X \in \mathcal{U}\}$ is a dominating family in $(\omega^\omega, \leq^*)$ (recall that $\underline{X} \in \omega^\omega$ is the increasing ennumeration of $X \in [\omega]^\omega$).

One can easily show that $\mathcal{U} \in \omega^*$ is rapid iff for every partition $\{X_i \colon i \in \omega\}$ of ω into finite sets there exists $X \in \mathcal{U}$ such that $(\forall i \in \omega)\ |X \cap X_i| \leq i$, and that $\mathcal{U}$ is a Q-point iff for every partition $\{X_i \colon i \in \omega\}$ of ω into finite sets there exists $X \in \mathcal{U}$ such that $(\forall i \in \omega)\ |X \cap X_i| \leq 1$. Therefore, a Q-point is rapid.

Miller showed in [18] that there are no rapid ultrafilters in the Laver model. He remarked that a similar result is true in the Mathias model. In this section we show that in fact this follows from CPAs. The following density lemma is needed.

Lemma 5.12. *Suppose that* $\mathbb{P} = Q * \mathbb{P}'$, *where* $\mathbb{P}'$ *has the Laver property and*

$$\mathbb{P} \Vdash \dot{X} \subseteq \omega \ \&\ \forall i\ |\dot{X} \cap \underline{\dot{r}}(i)| \leq i.$$

Let $\mathcal{U} \in \omega^*$. For any $p \in \mathbb{P}$ there exists $p' \leq_{\mathbb{P}} p$ and $Y \in [\omega]^\omega \setminus \mathcal{U}$ such that $p' \Vdash_{\mathbb{P}'} \dot{X} \subseteq^* Y$.

Since the Lemma follows from the combinatorial arguments we use in Chapter 6, we will give its proof later, in Section 6.2.

Theorem 5.13. CPAs $\vdash$ *"No non-principal ultrafilter on ω is rapid."*

Proof. Let $\mathcal{U} \in \omega^*$. Consider the following strategy of Eve for the CPAs game. Suppose that at the γ-th step Adam plays α_γ, A_γ and $f_\gamma\colon A_\gamma \to [\omega]^\omega$. Denote $f_\gamma(\langle \dot{r}_\xi\colon \xi < \alpha_\gamma\rangle)$ by $\dot{X}_\gamma$. Note that for $X \in [\omega]^\omega$

$$(\forall i \in \omega)(\underline{r}_0(i) < \underline{X}(i)) \;\Rightarrow\; (\forall i \in \omega)(|X \cap \underline{r}_0(i)| \leq i).$$

Therefore, by Lemma 5.12 there exists $E'_\gamma \leq_{Q'_{\alpha_\gamma}} A_\gamma$ and $Y_\gamma \in [\omega]^\omega \setminus \mathcal{U}$ such that

$$E'_\gamma \Vdash_{Q'_{\alpha_\gamma}} (\forall i \in \omega)(\dot{\underline{r}}_0(i) < \dot{\underline{X}}(i)) \Rightarrow \dot{X}_\gamma \subseteq^* Y_\gamma.$$

Eve plays

$$E_\gamma = \{\bar{x} \in E'_\gamma\colon (\forall i \in \omega)(\underline{\bar{x}}(0)(i) < f_\gamma(\bar{x})(i)) \Rightarrow f_\gamma(\bar{x}) \subseteq^* Y_\gamma\}.$$

By CPAs, Adam wins this play, so there exists $r \in [\omega]^\omega$ such that $\bigcup_{\gamma<\omega_1} f_\gamma[(E_\gamma)_r] = [\omega]^\omega$. This r witnesses that $\mathcal{U}$ is not rapid. Indeed, suppose a contrario that some $X \in \mathcal{U}$ satisfies $(\forall i \in \omega)\ f(i) < \underline{X}(i)$. There exists $\gamma < \omega_1$ such that $X \in f_\gamma[(E_\gamma)_r]$. Then, by the definition of E_γ, $X \subseteq^* Y_\gamma$. But $Y_\gamma \notin \mathcal{U}$. A contradiction. □

Chapter 6

Axiom SCPA^-

In this chapter we formulate the tactic version of the Strong Covering Property Axiom (SCPA^-) and discuss its consequences.

By a CPA tactic of Eve we mean a "naive" strategy, which does not depent on the history of the play, nor on the number of the round. Let us define this precisely. Denote the set of all possible moves of Adam in the CPA game by $\mathcal{A}$. A CPA tactic of Eve is a function $\sigma\colon \mathcal{A} \to \bigcup_{\gamma<\omega_1} Q'_\gamma$ such that $E \subseteq A$ whenever $(\alpha, A, f) \in \mathcal{A}$ and $\sigma((\alpha, A, f)) = E$.

Note that the history of the play (in particular the moves of Eve) does not influence the variety of possible moves of Adam for the forthcoming rounds. In other words, in the CPA game rounds are "independent".

Definition 6.1. Fix a CPA tactic σ of Eve. Let us describe the $\mathrm{SCPA}^-(\sigma)$ game. Again, the game lasts ω_1 rounds. At step $\gamma < \omega_1$ Adam plays a triple $(\alpha_\gamma, A_\gamma, f_\gamma)$ just like in the CPA game.

Let $E_\gamma = \sigma((\alpha_\gamma, A_\gamma, f_\gamma))$ be the move of Eve given by σ. In the $\mathrm{SCPA}^-(\sigma)$ game Eve plays $\widetilde{E}_\gamma \in Q'_{\alpha_\gamma}$, a subset of A_γ, such that $\widetilde{E}_\gamma \triangle E_\gamma \in \mathcal{I}^{\alpha_\gamma}$. Adam wins iff

$$\bigcup_{\gamma<\omega_1} f_\gamma[\widetilde{E}_\gamma] = 2^\omega.$$

SCPA^- says that CH fails and for any CPA tactic σ of Eve Adam has a winning strategy in the $\mathrm{SCPA}^-(\sigma)$ game.

Obviously SCPA^- implies CPA^-, which says that CH fails and Eve has no winning tactic in the CPA game. Note that all but one strategies of Eve, which we used so far to derive consequences of CPA and CPAs, were in fact tactics. The only exception is the one from Fact 5.3, where Eve paid attention to the number of the actual round. In Chapter 8 we give more comments on tactic and strategic versions of axioms.

Theorem 6.2. $P_{\omega_2} \Vdash \mathrm{SCPA}^-$.

Proof. Suppose that σ is a tactic of Eve in the CPA game. Like in the proof of Theorem 4.2, without loss of generality we assume that $V \models \mathrm{CH}$. Work in the P_{ω_2}-generic extension $V[G_{\omega_2}]$ of V. As before, let $C \in V$ be an ω_1-club on ω_2 such that if $\beta \in C$ then $\sigma \restriction V[G_\beta] \in V[G_\beta]$. For $\beta \in C$ let

$$\langle a^\beta_\delta = (\alpha^\beta_\delta, A^\beta_\delta, f^\beta_\delta) \colon \delta \in \omega_1 \rangle$$

be a sequence in $V[G_\beta]$ that ennumerates all possible CPA moves of Adam from $V[G_\beta]$. By Remark 3.8, for Borel sets being $\mathcal{I}^{\alpha_\gamma}$-positive is absolute between $V[G_\beta]$ and $V[G_{\beta'}]$, $\beta' \le \omega_2$. Hence, being a legal move of Adam or being a legal respond of Eve is also absolute between these models.

Fix in V a surjection $e = (e_0, e_1) \colon \omega_1 \to \omega_1 \times \omega_1$ such that $e_0(\gamma) \le \gamma$ for $\gamma < \omega_1$. Define a strategy for Adam in the SCPA$^-(\sigma)$ game as follows. At the step $\gamma < \omega_1$ he should play some $a_\gamma = (\alpha_\gamma, A_\gamma, f_\gamma)$. He finds $\beta_\gamma \in C$ such that $\beta_\gamma > \sup_{\delta<\gamma} \beta_\delta$ and $a_\delta, \widetilde{E}_\delta \in V[G_{\beta_\gamma}]$ for $\delta < \gamma$, and plays $a_\gamma = a^{\beta_{e_0(\gamma)}}_{e_1(\gamma)}$.

We claim that this strategy of Adam is winning. Let $\beta = \sup_{\gamma<\omega_1} \beta_\gamma$. Note that $\mathrm{cf}(\beta) = \omega_1$, so each (code for an) Adam's possible CPA move from $V[G_\beta]$ is actually in $V[G_{\beta_\gamma}]$ for some $\gamma < \omega_1$. Moreover, all moves a_γ Adam played in the game as well as Eve's responses $\widetilde{E}_\gamma$ belong to $V[G_\beta]$. (Note, however, that we do not claim that $\langle a_\gamma \colon \gamma < \omega_1 \rangle$ is in $V[G_\beta]$.)

The sequence $\langle a^\beta_\delta = (\alpha^\beta_\delta, A^\beta_\delta, f^\beta_\delta) \colon \delta \in \omega_1 \rangle$ ennumerates all possible moves of Adam from $V[G_\beta]$ and it belongs to $V[G_\beta]$ as well. Since Adam used all moves from $\{a^\beta_\delta \colon \delta < \omega_1\}$ during the play, it is enough to show that

$$V[G_{\omega_2}] \models \bigcup_{\delta<\omega_1} f^\beta_\delta[E'_\delta] = 2^\omega,$$

whenever $E'_\delta \subseteq A^\beta_\delta$ are coded in $V[G_\beta]$ Borel sets such that $E'_\delta \triangle \sigma(a^\beta_\delta) \in \mathcal{I}^{\alpha^\beta_\delta}$. (Note that we do not assume that $\langle E'_\delta \colon \delta < \omega_1 \rangle \in V[G_\beta]$.)

To this end treat $V[G_\beta]$ as the ground model, so that now $V[G_{\omega_2}]$ is P_{ω_2}-generic extension of $V[G_\beta]$. Suppose that $p \in P_{\omega_2}$ and $p \Vdash_{P_{\omega_2}} \dot{x} \in 2^\omega$.

Use Lemma 3.36 to find $p_0 \le_{P_{\omega_2}} p$ and a Borel function $f\colon B_{p_0} \to 2^\omega$ such that

$$p_0 \Vdash_{P_{\omega_2}} \dot{x} = f(\langle \dot{r}_\xi \colon \xi \in a\rangle),$$

where $a = \mathrm{dom}(p_0)$. Let $\alpha = \mathrm{o.t.}(a)$, let $i_a\colon \alpha \to a$ be the increasing ennumeration of a, and $\iota_a\colon P_\alpha \to P_a$ the cannonical isomorphism. There exists $\delta < \omega_1$ such that $(\alpha, B_{\iota_a^{-1}(p_0)}, f \circ i_a) = a^\beta_\delta$. Since $\sigma(a^\beta_\delta) \notin \mathcal{I}^\alpha$, by Corollary 3.9, there is $p' \in P_\alpha$ such that $B_{p'} \subseteq \sigma(a^\beta_\delta)$. Let $p_1 = \iota_a(p')$. Then $p_1 \le_{P_{\omega_2}} p_0$ and

$$p_1 \Vdash_{P_{\omega_2}} \langle \dot{r}_\xi \colon \xi \in a\rangle \in \sigma(a^\beta_\delta) \bullet i_a^{-1}.$$

By genericity, we may assume that $p_1 \in G_{\omega_2}/G_\beta$. Since

$$(E'_\delta \triangle \sigma(a^\beta_\delta)) \bullet i_a^{-1} \in \mathcal{I}^a,$$

by Corollary 3.35, we have

$$p_1 \Vdash_{P_{\omega_2}} \langle \dot{r}_\xi \colon \xi \in a\rangle \notin (E'_\delta \bullet i_a^{-1}) \triangle (\sigma(a^\beta_\delta) \bullet i_a^{-1}).$$

It follows that

$$p_1 \Vdash_{P_{\omega_2}} \langle \dot{r}_\xi \colon \xi \in a\rangle \in E'_\delta \bullet i_a^{-1},$$

so

$$p_1 \Vdash_{P_{\omega_2}} \dot{x} \in f^\beta_\delta[E'_\delta].$$

□

6.1 Combinatorics of Mathias forcing

Further in this chapter we will show two consequences of SCPA$^-$. Both need a density lemma to describe the strategy of Eve. First, we will present a combinatorial proposition, which is a simplification of the main component of these lemmas. The proof uses the technique of Mathias diagonalization.

Let us start with introducing the notation, which is used throughout this chapter. Suppose that $\langle s, A\rangle \in Q$ and $n \in \omega$. Let $n^- = \max(s \cup (A \cap n))$ ($\max(\emptyset) = 0$) and $n^+ = \min(A/n)$. This depends on $\langle s, A\rangle$. We shall use this notation in the remainig part of this chapter, always with s and A fixed.

Proposition 6.3. *Suppose that*

$$Q \Vdash \dot{x} \subseteq \omega \ \wedge \ (\forall i \in \omega)\ |\dot{x} \cap \underline{\dot{r}}(i)| \leq i.$$

Then for all $\langle s, A' \rangle \in Q$ there exists an infinite $A \subseteq A'$ such that for every $n \in A$

$$\langle s^\frown n, A/n \rangle \Vdash \dot{x} \cap n \subseteq \max(s)^+ \cup [n^-, n).$$

Proof. First, we find $A'' \subseteq A'$ and $\langle t_n \subseteq n : n \in A'' \rangle$ such that for each $n \in A''$ we have $\langle s^\frown n, A''/n \rangle \Vdash \dot{x} \cap n = t_n$. To this end we define infinite sets $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ and natural numbers $n_0 < n_1 < n_2 < \ldots$; then we set $A'' = \{n_i : i \in \omega\}$. Let $A_0 = A'$ and $n_0 = \min(A_0)$. Suppose that we have A_i and n_i. By Lemma 2.4 there exists $A_{i+1} \subseteq A_i/n_i$ and t_{n_i} such that $\langle s^\frown n_i, A_{i+1} \rangle \Vdash \dot{x} \cap n_i = t_{n_i}$. Then we put $n_{i+1} = \min(A_{i+1})$.

Next, identifying the power set of ω with 2^ω, we pick an infinite $A''' \subseteq A''$ such that

$$\lim_{n \in A'''} t_n = t$$

for some $t \in 2^\omega$. Moreover, we demand the sequence to converge "rapidly", i.e. $t \cap n = t_m \cap n$, where $n, m \in A'''$ and $n < m$.

Since $Q \Vdash \forall i \in \omega\ |\dot{x} \cap \underline{\dot{r}}(i)| \leq i$, for every $n \in A'''$ we have

$$\langle s^\frown n, A'''/n \rangle \Vdash |\dot{x} \cap n| \leq |s|,$$

so $|t_n| \leq |s|$ and thus $|t \cap n| \leq |s|$. Hence, there is $k \in A'''$ such that $t \subseteq t_k$ and then $t_l \cap m = t$ for any $l, m \in A'''$ such that $k < m < l$.

Finally we define $A = A'''/k$. □

6.2 Distributivity of $(\mathcal{P}(\omega)/\operatorname{fin})^2$

The main result of [21] asserts that $\mathfrak{h}(2) = \omega_1$ in the Mathias model. In [8], Dow gave another proof of this theorem. It is based on combinatorics of $[\omega]^\omega$ only, without involving any advanced methods of proper forcing. Following his argument we want to derive this equality from SCPA$^-$. We need the following density lemma.

Lemma 6.4. *Suppose that $\mathbb{P} = Q * \mathbb{P}'$ where $\mathbb{P}'$ has the Laver property and*

$$\mathbb{P} \Vdash \dot{x} \subseteq \omega \ \&\ \forall i\ |\dot{x} \cap \underline{\dot{r}}(i)| \leq i.$$

For any $\langle s, A' \rangle * p_0 \in \mathbb{P}$ there is $A \subseteq A'$ and p_1 which satisfy:

- $\langle s, A\rangle * p_1 \le_{\mathbb{P}} \langle s, A'\rangle * p_0$,
- *for all* $C', D' \in [\omega]^\omega$ *there exist* $C, D \in [\omega]^\omega$ *such that* $C \subseteq C', D \subseteq D'$ *and*
$$\langle s, A\rangle * p_1 \Vdash_{\mathbb{P}} |C \cap \dot{x}| < \omega \ \vee \ |D \cap \dot{x}| < \omega.$$

Proof. Since the notion $\mathbb{P}'$ has the Laver property, there are p_1 and Q-names $\langle \dot{x}^k_i : i \le k < \omega\rangle$ such that $Q \Vdash p_1 \le_{\mathbb{P}'} p_0$ and

$$\langle s, A'\rangle * p_1 \Vdash_{\mathbb{P}} (\forall k \in \omega)\ \dot{x} \cap \dot{\underline{r}}(k) \in \{\dot{x}^0_k, x^1_k \ldots, \dot{x}^k_k\}.$$

We will find an infinite $A \subseteq A'$ with the following property: For every $t \in [\omega]^{<\omega}$ and $n \in \omega$ such that $s \subseteq t \subseteq t^\frown n \subseteq s \cup A$ there holds

$$\langle t^\frown n, A/n\rangle * p_1 \Vdash_{\mathbb{P}} \dot{x} \cap n \subseteq \max(t)^+ \cup [n^-, n).$$

To this end we construct a sequence of infinite sets $B_0 \supseteq B_1 \supseteq B_2 \supseteq \ldots$ and an increasing sequence of natural numbers $n_0 < n_1 < n_2 < \ldots$. To get B_0 we modify the argument from the Proposition 6.3. Let us point out the differences. Instead of $\dot{x} \cap n$ we have $\{\dot{x}^{\,0}_{|s|}, \dot{x}^{\,1}_{|s|}, \ldots, \dot{x}^{|s|}_{|s|}\}$ (note that $\langle s^\frown n, A'/n\rangle \Vdash_Q \dot{\underline{r}}(|s|) = n \ \&\ \forall j \le |s|\ \dot{x}^{\,j}_{|s|} \subseteq n$). Hence, in place of t_n we have $(t^0_n, t^1_n, \ldots, t^{|s|}_n) \in [2^n]^{|s|}$. We take a subsequence "rapidly" converging to the limit $(t^0, t^1, \ldots, t^{|s|}) \in [2^\omega]^{|s|}$. Having B_0 we define n_0 as the minimum of B_0.

Suppose that we have B_i and n_i. Let $\{s_j : j < 2^i\}$ ennumerate all subsets of $\{n_0, n_1, \ldots, n_i\}$ which contain n_i. By induction on $j \le 2^i$ we will find sets $B^0_i \supseteq B^1_i \supseteq \ldots \supseteq B^{2^i}_i$. Let $B^0_i = B_i$. Suppose that B^j_i is already defined. To get B^{j+1}_i we repeat the argument from Proposition 6.3 modified in the way as above with $\langle s \cup s_j, B^j_i\rangle$ as the initial condition (note that now we have $\{t^0_n, t^1_n, \ldots, t^{|s\cup s_j|}_n\} \in [2^n]^{\le|s\cup s_j|}$ converging to a limit in $[2^\omega]^{\le|s\cup s_j|}$). We define $B_{i+1} = B^{2^i}_i$ and $n_{i+1} = \min(B_{i+1})$. Finally, let $A = \{n_i : i \in \omega\}$.

Now let us show that A satisfies the thesis of the Lemma.

Claim 6.5. *For every* $\langle t, B\rangle \le_Q \langle s, A\rangle$,

$$\langle t, B\rangle * p_1 \Vdash_{\mathbb{P}} \dot{x} \subseteq^* \bigcup_{n\in B} [n^-, n^+).$$

Proof. Fix $\langle t, B\rangle \le_Q \langle s, A\rangle$. Suppose that $k \ge \min(B)$ and some condition $\langle t', B'\rangle * p_2 \le_{\mathbb{P}} \langle t, B\rangle * p_1$ forces $k \in \dot{x}$. Without loss of generality $k < \max(t')$.

Let $m < n$ be the consecutive elements of t' such that $k \in [m, n)$. Let $s' = t' \cap [0, m]$. Since

$$\langle s'^\frown n, A/n\rangle * p_1 \Vdash_{\mathbb{P}} \dot{x} \cap n \subseteq \max(s')^+ \cup [n^-, n)$$

and

$$\langle s'^\frown n, A/n\rangle * p_1 \geq_{\mathbb{P}} \langle t', B'\rangle * p_2 \Vdash k \in \dot{x}$$

we get that k must really be in $[m, m^+) \cup [n^-, n)$. The Claim is satisfied. □

Fix C' and D' in $[\omega]^\omega$. Let $\gamma_0 < \gamma_1 < \ldots$ and $\delta_0 < \delta_1 < \ldots$ be subsequences of C' and D' respectively such that

$$(\forall k \in \omega)\left(\gamma_k^{++} \leq \delta_k \ \wedge \ \delta_k^{++} \leq \gamma_{k+1}\right).$$

Define $C = \{\gamma_k : k \in \omega\}$ and $D = \{\delta_k : k \in \omega\}$. Now take any $\langle t, B\rangle * p_2$ below $\langle s, A\rangle * p_1$. Find infinite $B' \subseteq B$ such that

$$B' \subseteq \bigcup_{k\in\omega}(\gamma_k^+, \delta_k^+] \qquad \text{or} \qquad B' \subseteq \bigcup_{k\in\omega}(\delta_k^+, \gamma_{k+1}^+].$$

Without loss of generality assume the former. Notice that if $n \in A \cap (\gamma_k^+, \delta_k^+]$, then $[n^-, n^+) \cap C = \emptyset$, since

$$\gamma_k < \gamma_k^+ \leq n^- < n < n^+ \leq \delta_k^{++} \leq \gamma_{k+1}.$$

By the Claim, it follows that

$$\langle t, B'\rangle * p_2 \Vdash_{\mathbb{P}} \dot{x} \subseteq^* \omega \setminus C.$$

□

Theorem 6.6. SCPA$^- \vdash \mathfrak{h}(2) = \omega_1$.

Proof. It is enough to show $\mathfrak{h}(2) \leq \omega_1$, the other inequality is obvious.

Let us denote $[\omega]^\omega$ by $\mathbb{R}$.

The above lemma provides the CPA tactic σ of Eve. We will show that the winning SCPA$^-(\sigma)$ strategy of Adam produces the winning strategy of PII in the distributivity game $G((\mathcal{P}(\omega), \subseteq^*)^2, \omega_1)$. Then, by Fact 2.2, $\mathfrak{h}(2) \leq \omega_1$ follows.

First, let us describe the tactic σ. Suppose that $(\alpha_\gamma, A_\gamma, f_\gamma)$ is a move of Adam in the CPA game and $f_\gamma\colon A_\gamma \to \mathbb{R}^2$. Let $f_\gamma = (f^0_\gamma, f^1_\gamma)$. Denote

$$\dot y_\gamma = (\dot y^0_\gamma, \dot y^1_\gamma) = f_\gamma(\langle \dot r_\xi\colon \xi < \alpha_\gamma\rangle).$$

Let $\dot x_\gamma$ be a Q'_{α_γ}-name such that A_γ forces that

$$\dot x_\gamma \subseteq \omega \quad \& \quad (\forall i<\omega)\ |\dot x_\gamma \cap \underline{\dot r}_0(i)| \le i$$

as well as

$$\{\underline{\dot x}_\gamma(2i)\colon i<\omega\} \subseteq \dot y_\gamma(0) \quad \& \quad \{\underline{\dot x}_\gamma(2i+1)\colon i<\omega\} \subseteq \dot y_\gamma(1).$$

By Corollary 3.9 and Theorem 3.36, there is $\widetilde{A}_\gamma \in Q'_{\alpha_\gamma}$, a subset of A_γ, and a Borel function $g_\gamma\colon \widetilde{A}_\gamma \to \mathbb{R}$ such that $\widetilde{A}_\gamma \Vdash \dot x_\gamma = g_\gamma(\langle \dot r_\xi\colon \xi<\alpha_\gamma\rangle)$.

Now we use Lemma 6.4. Put Q'_{α_γ}, $\widetilde{A}_\gamma$, $\dot x_\gamma$ and $\Vdash_{Q'_{\alpha_\gamma}}$ in place of $\mathbb{P}$, p, $\dot x$ and $\Vdash_{\mathbb{P}}$, respectively. The lemma gives $E_\gamma \subseteq \widetilde{A}_\gamma$ such that for all $C', D' \in \mathbb{R}$ there exist $C, D \in \mathbb{R}$ such that $C \subseteq C'$, $D \subseteq D'$ and

$$(*) \qquad E_\gamma \Vdash \dot x_\gamma {\subseteq^*} (\omega\setminus C)\ \vee\ \dot x_\gamma {\subseteq^*} (\omega \setminus D).$$

Then $\sigma((\alpha_\gamma, A_\gamma, f_\gamma)) = E_\gamma$. It is easy to observe that σ depends neither on γ nor on the history of the play. Note also that σ provides with $\dot x_\gamma$ and g_γ.

Suppose now that $G((\mathcal{P}(\omega), \subseteq^*)^2, \omega_1)$ is played simultaneously with $\mathrm{SCPA}^-(\sigma)$ and, moreover, Adam follows his winning strategy in the latter game. We describe the strategy of PII for $G((\mathcal{P}(\omega), \subseteq^*)^2, \omega_1)$ as well as the moves of Eve in $\mathrm{SCPA}^-(\sigma)$, which help PII to win. For $\gamma < \omega_1$, let $(\alpha_\gamma, A_\gamma, f_\gamma)$ be the move of Adam, let E_γ be the set dictated by σ, and let $p^I_\gamma = (p^I_\gamma(0), p^I_\gamma(1)) \in \mathbb{R}^2$ be the γ-th move of PI in the distributivity game. Define the γ-th move $\widetilde{E}_\gamma$ of Eve and $p^{II}_\gamma = (p^{II}_\gamma(0), p^{II}_\gamma(1))$, the one of PII, as follows. There are $C \subseteq p^I_\gamma(0)$ and $D \subseteq p^I_\gamma(1)$ such that $(*)$ holds. Then PII plays $(p^{II}_\gamma(0), p^{II}_\gamma(1)) = (C, D)$ and Eve plays

$$\widetilde{E}_\gamma = \{\bar x \in E_\gamma\colon \varphi(\bar x)\ \wedge\ \psi(\bar x)\ \wedge\ \theta(\bar x)\},$$

where

$$\varphi(\bar x) \;=\; g_\gamma(\bar x) \subseteq \omega\ \wedge\ (\forall i<\omega)\ |g_\gamma(\bar x) \cap \underline{\bar x}(0)(i)| \le i,$$

$$\psi(\bar x) \;=\; \{\underline{g_\gamma(\bar x)}(2i)\colon i<\omega\} \subseteq f^0_\gamma(\bar x)\ \wedge\ \{\underline{g_\gamma(\bar x)}(2i+1)\colon i<\omega\} \subseteq f^1_\gamma(\bar x),$$

$$\theta(\bar{x}) \;=\; g_\gamma(\bar{x}) \subseteq^* (\omega \setminus p^{II}_\gamma(0)) \;\vee\; g_\gamma(\bar{x}) \subseteq^* (\omega \setminus p^{II}_\gamma(1)).$$

It is easy to verify that this strategy is winning for PII. Indeed, suppose that we have $X \subseteq^* p^{II}_\gamma(0)$ and $Y \subseteq^* p^{II}_\gamma(1)$ for all $\gamma < \omega_1$. Since Adam won the SCPA$^-(\sigma)$ game, there is $\gamma < \omega_1$ and $\bar{x} \in \widetilde{E}_\gamma$ such that $f^0_\gamma(\bar{x}) = X$ and $f^1_\gamma(\bar{x}) = Y$. Let $z = g_\gamma(\bar{x})$. Then both sets $z \cap X, z \cap Y$ are infinite (by $\psi(\bar{x})$) and one of $z \cap p^{II}_\gamma(0), z \cap p^{II}_\gamma(1)$ is finite (by $\theta(\bar{x})$). But $X \subseteq^* p^{II}_\gamma(0)$ and $Y \subseteq^* p^{II}_\gamma(1)$. A contradiction. □

At the end of this section we give the proof of the density lemma about ultrafilters from Section 5.4.

Proof of Lemma 5.12. Let $p = \langle s, A'\rangle * p_0$. Arguing as in the proof of Lemma 6.4 up to Claim 6.5 we get an infinite $A \subseteq A'$ and p_1 such that $\langle s, A\rangle * p_1 \leq_{\mathbb{P}} \langle s, A'\rangle * p_0$ and

$$\forall B \in [A]^\omega \quad \langle s, B\rangle * p_1 \Vdash_{\mathbb{P}} \dot{X} \subseteq^* \bigcup_{n \in B} [n^-, n^+).$$

(Note that for $n \in B$ the values n^-, n^+ are computed with respect to A.) Denote by $\langle \gamma_i \colon i \in \omega\rangle$ the increasing ennumeration of A. Let

$$Y_0 = \bigcup_{i \in \omega} [\gamma^-_{2i}, \gamma^+_{2i}) \quad \text{and} \quad Y_1 = \bigcup_{i \in \omega} [\gamma^-_{2i+1}, \gamma^+_{2i+1}).$$

Then $Y_0 \cap Y_1 = \emptyset$, so $\{Y_0, Y_1\} \not\subseteq \mathcal{U}$. Without loss of generality assume that $Y = Y_0 \in [\omega]^\omega \setminus \mathcal{U}$. Let $B = \{\gamma_{2i} \colon i \in \omega\}$. Define $p' = \langle s, B\rangle * p_1$. Then $p' \Vdash_{\mathbb{P}} \dot{X} \subseteq^* Y$. □

6.3 Distributivity of $((\omega)^\omega, \leq^*)$

In this section infinite partitions of ω are considered. We use the notation from [22] mostly.

Let (ω) be the set of all partitions of ω and let $(\omega)^\omega$ be the set of infinite partitions of ω. By $\mathbf{0} \in (\omega)$ we denote the trivial partition onto one piece. For $X, Y \in (\omega)$, we say that $X \leq Y$ if every piece of X is a union of elements from Y. We define leaders$(X) = \{\min(a) \colon a \in X\}$. Thus, there is an ordering of X which is induced by the natural order of leaders(X). For $n \in \omega$ we say that $X \leq_n Y$ if, from the n-th piece on, every piece of X is a union of pieces of Y (so $X \leq_0 Y$ iff $X \leq Y$). We define $X \leq^* Y$ if $X \leq_n Y$ for some $n \in \omega$. We say that X *rejects* Y, if no piece of Y is a union of pieces of X. Similarily, for $\mathcal{R} \subseteq (\omega)$ we say that X *rejects* $\mathcal{R}$, if X rejects every $Y \in \mathcal{R}$. We will freely confuse X with the equivalence relation on ω given by X, i.e. two numbers are equivalent iff they are in the same piece of X. Thus we can write $X \subseteq \omega \times \omega$ or $X \in 2^{\omega\times\omega}$. By this identification we have $(\omega)^\omega \subseteq (\omega) \subseteq 2^{\omega\times\omega}$. Notice that (ω) is closed in $2^{\omega\times\omega}$, hence it is a Polish space.

Spinas proved in [22] that $\mathfrak{h}(((\omega)^\omega, \leq^*)) = \omega_1$ in the Mathias model. We will derive this result from SCPA$^-$.

Lemma 6.7. *Assume that $\mathbb{P} = Q * \mathbb{P}'$, where $\mathbb{P}'$ has the Laver property and $\mathbb{P} \Vdash \dot{X} \in (\omega) \setminus \{\mathbf{0}\}$. Then for any $\langle s, A'\rangle * p_0 \in \mathbb{P}$ there is $A \subseteq A'$, p_1 and a countable $\mathcal{R} \subseteq (\omega) \setminus \{\mathbf{0}\}$ such that $\langle s, A\rangle * p_1 \leq_{\mathbb{P}} \langle s, A'\rangle * p_0$ and for every $Y \in (\omega)$ which rejects $\mathcal{R}$ we have $\langle s, A\rangle * p_1 \Vdash_{\mathbb{P}} Y$* rejects *$\dot{X}$.*

Proof. For simplicity let us assume that $\dot{x}$ is a Q-name. We will comment on it later. We perform a construction similar to the one from Lemma 6.4. Therefore, we get $A \subseteq A'$ such that for every finite t satisfying $s \subseteq t \subseteq s \cup A$ and for every $n \in A/\max(t)$ there is $\sigma^t_n \in 2^{n\times n}$ which is an equivalence relation on n such that

$$\langle t^\frown n, A/n\rangle \Vdash_Q \dot{X}\restriction n \times n = \sigma^t_n$$

and the sequence $\langle \sigma^t_n \colon n \in A/\max(t)\rangle$ converges "rapidly" to some X^t from $2^{\omega\times\omega}$. Since (ω) is closed in $2^{\omega\times\omega}$ we have $X^t \in (\omega)$. We define

$$\mathcal{R} = \{X^t \colon s \subseteq t \subseteq s \cup A\} \setminus \{\mathbf{0}\}$$

(it may happen that $X^t = \mathbf{0}$ for some t).

Now take any $Y \in (\omega)$ which rejects $\mathcal{R}$ and suppose a contrario that there is $\langle t, B\rangle \leq_Q \langle s, A\rangle$ which forces that Y does not reject $\dot{X}$. Without loss of generality there are natural numbers $u < v < \max(t)$ such that $\langle t, B\rangle \Vdash_Q (u, v) \notin \dot{X}$ (since $Q \Vdash_Q \dot{X} \neq \mathbf{0}$). Hence, $(u, v) \notin \sigma^t_n$ for $n \in B$, so $X^t \neq \mathbf{0}$. We may also assume that there are a Q-name $\dot{b}$ and $k < \max(t)$ such that

$$\langle t, B\rangle \Vdash_Q \dot{b} \text{ is a piece of } \dot{X} \ \wedge\ \dot{b} \text{ is a union of pieces of } Y \ \wedge\ \min(\dot{b}) = k.$$

Let c be the piece of X^t such that $k \in c$. Since Y rejects X^t, there is $d \in Y$ such that

$$d \cap c \neq \emptyset \neq d \cap (\omega \setminus c).$$

Let $m \in B$ be so large that

$$d \cap c \cap m \neq \emptyset \neq d \cap (\omega \setminus c) \cap m,$$

let $m' = \min(B/m)$ and let $p = \langle t^\frown m', B/m' \rangle$. Then

$$p \Vdash_Q \dot{X} \restriction m \times m = X^t \restriction m \times m = \sigma^t_{m'} \restriction m \times m,$$

since $\langle \sigma^t_n : n \in B \rangle$ converges "rapidly" to X^t. In particular, since $k \in c$ and $p \Vdash_Q k \in \dot{b}$, we have $p \Vdash_Q \dot{b} \cap m = c \cap m$. So

$$p \Vdash_Q d \cap \dot{b} \cap m \neq \emptyset \neq d \cap (\omega \setminus \dot{b}) \cap m.$$

But d is a piece of Y, so p forces that $\dot{b}$ is not a union of pieces of Y. On the other hand, $p \leq_Q \langle t, B \rangle$. A contradiction.

The difference between the construction for Q and $Q * \mathbb{P}'$ is similar to the one we pointed out in the proof of Lemma 6.4. Instead of one X^t for each t we would get a finite collection of such partitions (its cardinality would depend only on $|t|$). and R would be their union over all possible t, without the trivial partition $\{\mathbf{0}\}$. □

The following combinatorial lemma was proved in [22].

Lemma 6.8 ([22, Lemma 4.3])**.** *Let $\mathcal{R} \subseteq (\omega) \setminus \{\mathbf{0}\}$ be countable and $Y \in (\omega)^\omega$. There exists $Z \in (\omega)^\omega$ such that $Z \leq Y$ and Z rejects $\mathcal{R}$.*

Theorem 6.9. SCPA$^-$ $\vdash \mathfrak{h}(((\omega)^\omega, \leq^*)) = \omega_1$.

Proof. We will show the nontrivial inequality $\mathfrak{h}(((\omega)^\omega, \leq^*)) \leq \omega_1$.

As before, we define a CPA tactic σ of Eve using the density lemma. Then, as in the proof of Theorem 6.6, the distributivity game $G(((\omega)^\omega, \leq^*)), \omega_1)$ and the SCPA$^-(\sigma)$ game are played simultaneously. In the latter, Adam keeps his winning strategy and Eve helps PII to win the distributivity game.

Let us describe σ. Suppose that at the γ-th step Adam plays a triple $(\alpha_\gamma, A_\gamma, f_\gamma)$, where $f_\gamma : A_\gamma \to (\omega)$. Denote $\dot{X}_\gamma = f_\gamma(\langle \dot{r}_\xi : \xi < \alpha_\gamma \rangle)$. The tactic σ gives the move of Eve together with a countable subset $\mathcal{R}_\gamma$ of $(\omega) \setminus \{\mathbf{0}\}$.

We have two cases.
Case 1: $A_\gamma \nVdash \dot{X}_\gamma \in (\omega)^\omega$. Then σ indicates $E_\gamma \subseteq A_\gamma$ which forces that $\dot{X}_\gamma \notin (\omega)^\omega$ and $\mathcal{R}_\gamma = \emptyset$.
Case 2: $A_\gamma \Vdash \dot{X}_\gamma \in (\omega)^\omega$. Then E_γ and $\mathcal{R}_\gamma$ are given by Lemma 6.7. So for any Y which rejects $\mathcal{R}_\gamma$, $E_\gamma \Vdash Y$ rejects $\dot{X}_\gamma$.

Now we describe the strategy of PII for $G(((\omega)^\omega, \leq^*), \omega_1)$. For $\gamma < \omega_1$ let $(\alpha_\gamma, A_\gamma, f_\gamma)$ be the move of Adam given by his winning SCPA$^-(\sigma)$ strategy, let E_γ be the set dictated by σ, and let $p^{I}_\gamma \in (\omega)^\omega$ be the γ-th move of PII in the distributivity game. We define the γ-th moves of Eve and PII as follows.
Case 1: $E_\gamma \Vdash \dot{X}_\gamma \notin (\omega)^\omega$ and $\mathcal{R}_\gamma = \emptyset$. Then PII plays $p^{II}_\gamma = p^{I}_\gamma$ and Eve plays

$$\widetilde{E}_\gamma = \{\bar{x} \in E_\gamma \colon f_\gamma(\bar{x}) \notin (\omega)^\omega\}.$$

Case 2: $E_\gamma \Vdash \dot{X}_\gamma \in (\omega)^\omega$ and $\mathcal{R}_\gamma \neq \emptyset$. Using Lemma 6.8, PII finds $p^{II}_\gamma \leq p^{I}_\gamma$ which rejects $\mathcal{R}_\gamma$. Then Eve plays

$$\widetilde{E}_\gamma = \{\bar{x} \in E_\gamma \colon f_\gamma(\bar{x}) \in (\omega)^\omega \ \&\ p^{II}_\gamma \text{ rejects } f_\gamma(\bar{x})\}.$$

We claim that playing like this PII wins. Indeed, suppose that there is $X \in (\omega)^\omega$ such that $X \leq^* p^{II}_\gamma$ for all $\gamma < \omega_1$. Adam wins the SCPA$^-(\sigma)$ game, so there is γ such that $X \in f_\gamma[\widetilde{E}_\gamma]$. Since $X \in (\omega)^\omega$, we must have been in Case 2. But then p^{II}_γ rejects X. A contradiction.

By Fact 2.2, $\mathfrak{h}(((\omega)^\omega, \leq^*)) \leq \omega_1$. □

Chapter 7

Axiom $\diamondsuit$CPA

In this chapter we present a natural modification of the CPA game obtaining an axiom called $\diamondsuit$CPA. It captures some combinatorics provided by the principle $\diamondsuit$. Its further technical modification gives an axiom that implies the principle $\clubsuit$ (see Chapter 10).

Recall that a $\diamondsuit$-sequence is a sequence $\langle X_\alpha : \alpha < \omega_1\rangle$ such that for all $\alpha < \omega_1$ we have $X_\alpha \subseteq \alpha$ and for all $X \subset \omega_1$ the set $\{\alpha : X \cap \alpha = X_\alpha\}$ is stationary in ω_1. The principle $\diamondsuit$ says that there exists a diamond sequence. It was introduced by Jensen, who used it to construct a Suslin tree.

Definition 7.1. In the $\diamondsuit$CPA game at each step $\gamma < \omega_1$ Adam plays $(\alpha_\gamma, A_\gamma, f_\gamma)$ as in the CPA game, but now

$$f_\gamma : A_\gamma \to 2^\gamma,$$

i.e. f_γ is a Borel function into the space of binary sequences of length γ. Again, Eve responds with some $E_\gamma \subseteq A_\gamma$ from Q'_{α_γ}. Adam wins iff for every $t \in 2^{\omega_1}$ the set

$$\{\gamma < \omega_1 : t\restriction\gamma \in f_\gamma[E_\gamma]\}$$

is stationary in ω_1. The axiom $\diamondsuit$CPA says that CH fails and Eve has no winning strategy in the $\diamondsuit$CPA game.

It is easy to observe the following:

Fact 7.2. $\diamondsuit\text{CPA} \Rightarrow \text{CPA}$.

Proof. Suppose that σ is a winning strategy of Eve in the CPA game. We will find a winning strategy for Eve in the ◊CPA game. Suppose that at the γ-th step of the latter game Adam plays $(\alpha_\gamma, A_\gamma, f_\gamma)$. For $\beta \le \gamma$ let $f'_\beta\colon A_\beta \to 2^\omega$ be the composition $f'_\beta = \pi^\beta_\omega \circ f_\beta$, where $\pi^\beta_\omega\colon 2^\beta \to 2^\omega$ is the projection onto the first ω coordinates.[1] Let

$$E_\gamma = \sigma(\langle(\alpha_\beta, A_\beta, f'_\beta)\colon \beta \le \gamma\rangle).$$

It is easy to verify that playing like this Eve wins. □

Theorem 7.3. $P_{\omega_2} \Vdash \Diamond\mathrm{CPA}$

Proof. Suppose that $\sigma \in V[G_{\omega_2}]$ is a strategy of Eve in the ◊CPA game. Let $C \in V$ be an ω_1-club on ω_2 such that for every $\beta \in C$, $\mathrm{cf}(\beta) = \omega_1$ and $\sigma\restriction V[G_\beta] \in V[G_\beta]$. It is well-known that the ◊-principle holds at the intermediate steps of cofinality ω_1, so we have $V[G_\beta] \models \Diamond$ for $\beta \in C$. Taking one such $V[G_\beta]$ as the ground model we may in fact assume that $\sigma\restriction V \in V$ and that $V \models \Diamond$.

In $V[G_{\omega_2}]$ we will find a winning counterplay of Adam. Work in V. Fix a large enough regular κ so that H_κ contains everything relevant. Let $\langle \varepsilon_\gamma \subseteq \gamma\times\gamma\colon\ \gamma < \omega_1\rangle$ be a ◊-sequence which predicts subsets of $\omega_1 \times \omega_1$. At the round γ Adam attempts to perform the following procedure. If ε_γ is an extensional and well-founded relation on γ, $\pi\colon(\gamma, \varepsilon_\gamma) \to (M^*_\gamma, \in)$ is the Mostowski collapse, and $\gamma = (\omega_1)^{M^*}$, then Adam tries to find an elementary embedding $e\colon (M^*_\gamma, \in) \to (H_\kappa, \in)$ such that $(e\circ\pi)(0) \in P_{\omega_2}$ and $e\circ\pi(1)$ is a P_{ω_2}-name for an element of 2^{ω_1}.

If he fails he plays anything legal. Suppose that he succeds. Denote $M_\gamma = e[M^*_\gamma] \prec H_\kappa$, $p_\gamma = (e\circ\pi)(0)$ and $\dot r^\gamma_\beta = (e\circ\pi)(1)(\beta)$ for $\beta < \gamma$. By Theorem 3.36, there exists $p'_\gamma \in P_{\omega_2}$ and Borel functions $f^\gamma_\beta\colon B_{p'_\gamma} \to 2$, for $\beta < \gamma$, such that $a_\gamma = \mathrm{dom}(p'_\gamma) = \omega_2 \cap M_\gamma$ and

$$p'_\gamma \Vdash_{P_{\omega_2}} f^\gamma_\beta(\langle \dot r_\xi\colon \xi \in a_\gamma\rangle) = \dot r^\gamma_\beta.$$

Let $\alpha_\gamma = \mathrm{o.t.}(a_\gamma)$, let $i_{a_\gamma}\colon \alpha_\gamma \to a_\gamma$ be the increasing ennumeration of a_γ, and let $\iota_{a_\gamma}\colon P_{\alpha_\gamma} \to P_{a_\gamma}$ be the cannonical isomorphism. Then Adam plays $(\alpha_\gamma, A_\gamma, f_\gamma)$, where

$$A_\gamma = B_{p'_\gamma} \bullet i_{a_\gamma} = \{\bar x \circ i_{a_\gamma}\colon \bar x \in B_{p'_\gamma}\} = B_{\iota^{-1}_{a_\gamma}(p'_\gamma)}$$

[1] if $\beta < \omega$ let $\pi^\beta_\omega((x_0, x_1, \ldots, x_{\beta-1})) = ((x_0, x_1, \ldots, x_{\beta-1}, 0, 0, \ldots))$

and

$$f_\gamma : A_\gamma \to 2^\gamma, \quad f_\gamma = \langle f^\gamma_\beta \circ i_{a_\gamma} : \beta < \gamma \rangle.$$

We claim that playing like this Adam wins. Let E_γ be the move of Eve given by σ at the γ-th step. We will show that

$$P_{\omega_2} \Vdash (\forall t \in 2^{\omega_1}) \left(\{\gamma < \omega_1 : t{\restriction}\gamma \in f_\gamma[E_\gamma]\} \text{ is stationary} \right).$$

To this end fix a condition $p \in P_{\omega_2}$, a P_{ω_2}-name $\dot D$ for a club on ω_1, and a P_{ω_2}-name $\dot t$ for an element of 2^{ω_1}. We identify $\dot t$ with a sequence of P_{ω_2}-names $\langle \dot\tau_\delta : \delta < \omega_1 \rangle$ for elements of 2. Find an elementary submodel $(M, \in) \prec (H_\kappa, \in)$ of size ω_1 such that $\omega_1 \subseteq M$, $\{\dot\tau_\delta : \delta < \omega_1\} \subseteq M$ and $p, \dot D, \dot t \in M$. Fix a bijection $\psi : \omega_1 \to M$ such that

$$\psi(0) = p, \quad \psi(1) = \dot t, \quad \psi(2) = \dot D.$$

For club many $\gamma < \omega_1$ we have

$$(*) \qquad \psi[\gamma] \cap \omega_1 = \gamma, \quad \{\dot\tau_\delta : \delta < \gamma\} \subseteq \psi[\gamma] \quad \text{and} \quad (\psi[\gamma], \in) \prec (H_\kappa, \in).$$

Define

$$\varepsilon = \{(\xi, \zeta) \in \omega_1 \times \omega_1 : \psi(\xi) \in \psi(\zeta)\}.$$

Find $\gamma < \omega_1$ which satisfies $(*)$ and $\varepsilon_\gamma = \varepsilon \cap (\gamma \times \gamma)$. In particular $\psi[\gamma] = M_\gamma$, $p = p_\gamma$; let $p' = p'_\gamma$.

Since $E_\gamma \notin \mathcal{I}^{\alpha_\gamma}$ there exists $p^* \in P_{\alpha_\gamma}$ such that $B_{p^*} \subseteq E_\gamma$. By Fact 3.5,

$$p^* \Vdash_{P_{\alpha_\gamma}} \langle \dot r_\beta : \beta < \alpha_\gamma \rangle \in E_\gamma.$$

Let $p'' = \iota_{a_\gamma}(p^*)$. Then $p'' \le_{P_{\omega_2}} p'$ and, by Lemma 3.33,

$$p'' \Vdash_{P_{\omega_2}} \langle \dot r_\xi : \xi \in a_\gamma \rangle \in E_\gamma \bullet i^{-1}_{a_\gamma}.$$

Therefore $p'' \Vdash_{P_{\omega_2}} \dot t \in f_\gamma[E_\gamma]$. Moreover, since $\dot D \in M_\gamma$, $\omega_1 \cap M_\gamma = \gamma$ and p' is M_γ-master, we have $p' \Vdash_{P_{\omega_2}} \gamma \in \dot D$. □

We remark that the "tactic version" of $\Diamond$CPA makes no sense. Although we can consider strategies of Eve, which do not depend on the history of the play, the number of the actual round is always indicated by the move of Adam. His possible moves depend on the number of the round. Therefore, the notion of $\Diamond$CPA tactic of Eve is redundant.

Chapter 8

Generalizations

In this chapter we want to emphase relations between the axioms introduced throughout the thesis and state their stronger versions.

We will start with a brief review. All axioms we consider say that CH fails, so we do not bother to mention this each time. So far we have introduced the following axioms: $\mathrm{CPA}, \mathrm{CPAs}, \diamondsuit\mathrm{CPA}$ and SCPA^-. The former three say that Eve has no winning strategy in the appropriate game. Both CPAs and $\diamondsuit$CPA are stronger than CPA. The last one SCPA^- claims something about tactics of Eve, rather than arbitrary strategies. Thus it does not seem to be a strenghtening of CPA (at least we can not prove it). However, we can formulate CPA^-, which says that Eve has no winning tactic in the CPA game. Similarily, the axiom CPAs^- would say that Eve has no winning tactic in the CPAs game. Then SCPA^- and CPAs^- are both stronger than CPA^-, but we see no relation between them. The axiom $\diamondsuit$CPA does not have a reasonable tactic version. This is because of the rules of the $\diamondsuit$CPA game. Possible moves of Adam change at every round and none of them is adequate for two different ones.

In most of the applications we invoked some tactic of Eve. A strategy which was not a tactic was used only once: in the proof that CPAs implies $\mathfrak{h} > \omega_1$ (Fact 5.3). Other consequences of CPA and CPAs shown in the thesis follow in fact from CPA^- and CPAs^-.

Although we have no particular applications, it is natural to ask if we can state a "strategic" version of SCPA^-. Moreover, is there one axiom which implies all the other we have considered so far? In this chapter we answer both questions positively.

8.1 Axiom $\diamondsuit$CPAs

We start with formulating a common strengthening of $\diamondsuit$CPA and CPAs.

Definition 8.1. Consider the $\diamondsuit$CPAs game, which merges both $\diamondsuit$CPA and CPAs games. Moves of players are exactly the same as in the $\diamondsuit$CPA game. (Hence, a $\diamondsuit$CPA strategy of a player is his or her $\diamondsuit$CPAs strategy as well and vice versa.) However, Adam wins if there is a sequence $\bar{r} \in ([\omega]^\omega)^{\omega_1}$ such that for every $t \in 2^{\omega_1}$ there are stationary many $\gamma < \omega_1$ such that

$$t\restriction\gamma \in f_\gamma[(E_\gamma)_{\bar{r}\restriction\gamma}],$$

where $(E_\gamma)_{\bar{r}\restriction\gamma}$ is the section of E_γ determined by $\bar{r}\restriction\gamma$, i.e. $(E_\gamma)_{\bar{r}\restriction\gamma}$ consists of those sequences from E_γ whose initial segment is $\bar{r}\restriction\gamma$.[1] (Note that the "section condition" is stronger than in CPAs.)

The axiom $\diamondsuit$CPAs says that CH fails and Eve has no winning strategy in the $\diamondsuit$CPAs game.

Theorem 8.2. *$P_{\omega_2} \Vdash \diamondsuit$CPAs.*

Proof. Repeat the proof of Theorem 7.3 with $\bar{r} \in ([\omega]^\omega)^{\omega_1}$ defined by taking $\bar{r}(\zeta)$ as the ζ-th Mathias real over the V from this proof for which $V \models \diamondsuit$ and $\sigma\restriction V \in V$. □

8.2 Axiom SCPA

Definition 8.3. Fix a CPA strategy σ of Eve. Let us describe the SCPA(σ) game. As usually, Adam and Eve play ω_1-many rounds. At the γ-th round Adam plays

$$\bar{a}_\gamma = \langle a^\gamma_\delta : \delta \le \lambda(\gamma)\rangle = \langle (\alpha^\gamma_\delta, A^\gamma_\delta, f^\gamma_\delta) : \delta \le \lambda(\gamma)\rangle,$$

where $\lambda(\gamma) < \omega_1$ and $\bar{a}_\gamma$ is a sequence of CPA moves of Adam. We can treat $\bar{a}_\gamma$ as a partial CPA play, in which Eve follows σ. Let E_γ be the move of Eve prescribed by σ at the $\lambda(\gamma)$-th step of the CPA game, in which moves of Adam are $\langle a^\gamma_\delta : \delta \le \lambda(\gamma)\rangle$ (we write $E_\gamma = \sigma(\bar{a}_\gamma)$).

[1] Therefore, if $\alpha_\gamma < \gamma$ then $(E_\gamma)_{\bar{r}\restriction\gamma} = \emptyset$.

In the SCPA(σ) game Eve responds to $\bar{a}_\gamma$ by shrinking the set $A^\gamma_{\lambda(\gamma)}$ to some $\widetilde{E}_\gamma \in Q'_{\alpha^\gamma_{\lambda(\gamma)}}$ such that $\widetilde{E}_\gamma \triangle E_\gamma \in \mathcal{I}^{\alpha^\gamma_{\lambda(\gamma)}}$.

Adam wins if he can put together some of the rounds of the SCPA(σ) play, to get a winning CPA play. Precisely, Adam wins if there exists an increasing function $\eta\colon \omega_1 \to \omega_1$ such that

- $(\forall \gamma < \omega_1)\ (\lambda(\eta(\gamma)) = \gamma)$,
- $(\forall \beta < \gamma < \omega_1)\ (\bar{a}_{\eta(\gamma)} \restriction (\beta+1) = \bar{a}_{\eta(\beta)})$ and

$$\bigcup_{\gamma<\omega_1} f^{\eta(\gamma)}_\gamma[\widetilde{E}_{\eta(\gamma)}] = 2^\omega$$

The axiom SCPA says that CH fails and for each CPA strategy σ of Eve, Adam has a winning SCPA(σ) strategy.

It is easy to deduce both CPA and SCPA$^-$ from SCPA:

Fact 8.4.

(*i*) SCPA $\Rightarrow$ SCPA$^-$

(*ii*) SCPA $\Rightarrow$ CPA

Proof. (*i*) Fix a CPA tactic σ of Eve and a winning SCPA(σ) strategy of Adam. We want to define his winning SCPA$^-$(σ) strategy. We prescribe his SCPA$^-$(σ) moves a_γ together with moves $\bar{a}_\gamma$ from the SCPA(σ) game. Suppose that we are at the γ-th round and $\langle a_\delta, E_\delta\colon \delta < \gamma\rangle$ is the partial play. Let $\bar{a}_\gamma$ be the SCPA(σ) move indicated by Adam's winning SCPA(σ) strategy, when the so far SCPA(σ) play equals $\langle \bar{a}_\delta, E_\delta\colon \delta < \gamma\rangle$. In the SCPA$^-$($\sigma$) play, he plays $a_\gamma = a^\gamma_{\lambda(\gamma)}$. It is easy to verify that playing like this Adam wins.

(*ii*): Suppose a contrario that σ is a winning CPA strategy of Eve. Then she would win the SCPA(σ) game whenever she would play $\widetilde{E}_\gamma = E_\gamma$ for each $\gamma < \omega_1$. □

Theorem 8.5. *$P_{\omega_2} \Vdash$ SCPA.*

Proof. The proof resembles the one of Theorem 6.2. Suppose that σ is a CPA strategy of Eve. Without loss of generality we assume that $V \models$ CH. Let $C \in V$ be an ω_1-club on ω_2 such that if $\beta \in C$ then $\sigma \restriction V[G_\beta] \in V[G_\beta]$. For $\beta \in C$ let $\langle \bar{a}^\beta_\delta\colon \delta \in \omega_1\rangle \in V[G_\beta]$ ennumerate all possible SCPA(σ) moves of Adam from $V[G_\beta]$. Fix in V a surjection $e = (e_0, e_1)\colon \omega_1 \to \omega_1 \times \omega_1$ such that $e_0(\gamma) \le \gamma$ for $\gamma < \omega_1$.

The winning SCPA(σ) strategy of Adam is defined as follows. At the γ-th round he finds $\beta_\gamma \in C$ such that $\beta_\gamma > \sup_{\delta<\gamma} \beta_\delta$ and $\bar{a}_\delta, \widetilde{E}_\delta \in V[G_{\beta_\gamma}]$ for $\delta < \gamma$. Then he plays $\bar{a}_\gamma = \bar{a}^{\beta_{e_0(\gamma)}}_{e_1(\gamma)}$.

We claim that this strategy of Adam is winning. Let $\beta = \sup_{\gamma<\omega_1} \beta_\gamma$. Then $\beta \in C$ and all moves $\bar{a}_\gamma$ Adam played in the game, as well as responses $\widetilde{E}_\gamma$ of Eve, are in $V[G_\beta]$. Fix an ennumeration $\langle a_\delta \colon \delta < \omega_1\rangle \in V[G_\beta]$ of all CPA moves of Adam from $V[G_\beta]$. Note that every SCPA(σ) move of Adam from $V[G_\beta]$ was used by him cofinally many times during the play. Hence, there is an increasing function $\eta\colon \omega_1 \to \omega_1$ such that $\bar{a}_{\eta(\gamma)} = \langle a_\delta \colon \delta \le \gamma\rangle$, for $\gamma < \omega_1$. Then, by the argument from Theorem 6.2 we have

$$V[G_{\omega_2}] \models \bigcup_{\gamma<\omega_1} f^{\eta(\gamma)}_\gamma[E_{\eta(\gamma)}] = 2^\omega.$$

□

8.3 Axiom ♢SCPAs

Now we will define one axiom, which implies all the previously considered. Its relation to ♢CPAs is analogical to that of SCPA to CPA. Thus, we call it ♢SCPAs.

Definition 8.6. Let us define the ♢SCPAs(σ) game. Fix a ♢CPA strategy σ of Eve. (Recall that σ is a ♢CPAs strategy as well.) At the γ-th round Adam plays a sequence

$$\bar{a}_\gamma = \langle a^\gamma_\delta \colon \delta \le \lambda(\gamma)\rangle = \langle (\alpha^\gamma_\delta, A^\gamma_\delta, f^\gamma_\delta) \colon \delta \le \lambda(\gamma)\rangle$$

which together with moves of Eve given by σ forms a ♢CPA (so also the ♢CPAs) partial play. Let $E_\gamma = \sigma(\bar{a}_\gamma)$ be the $\lambda(\gamma)$-th move of Eve in this play. In the ♢SCPAs(σ) game she shrinks $A^\gamma_{\lambda(\gamma)}$ to $\widetilde{E}_\gamma \in Q'_{\alpha^\gamma_{\lambda(\gamma)}}$ such that $\widetilde{E}_\gamma \triangle E_\gamma \in \mathcal{I}^{\alpha^\gamma_{\lambda(\gamma)}}$.

Adam wins if there exists an increasing function $\eta\colon \omega_1 \to \omega_1$ such that

- for all $\gamma < \omega_1$ we have $\lambda(\eta(\gamma)) = \gamma$,

- for all $\beta < \gamma < \omega_1$ we have $\bar{a}_{\eta(\gamma)}\restriction\beta + 1 = \bar{a}_{\eta(\beta)}$
- there is $\bar{r} \in (2^\omega)^{\omega_1}$ such that for all $t \in 2^{\omega_1}$ the set

$$\{\gamma: t\restriction\gamma \in f_\gamma^{\eta(\gamma)}[(\widetilde{E}_{\eta(\gamma)})_{\bar{r}\restriction\gamma}]\}$$

  is stationary.

  (Again, the section $(\widetilde{E}_{\eta(\gamma)})_{\bar{r}\restriction\gamma}$ of $\widetilde{E}_{\eta(\gamma)}$ determined by $\bar{r}\restriction\gamma$ consists of those sequences from $\widetilde{E}_{\eta(\gamma)}$ whose initial segment is $\bar{r}\restriction\gamma$.)

The axiom $\diamondsuit$SCPAs says that CH fails and for every $\diamondsuit$CPAs strategy σ of Eve, Adam has a winning strategy in the $\diamondsuit$SCPAs(σ) game.

Theorem 8.7. $Q_{\omega_2} \Vdash \diamondsuit$SCPAs

Proof. To get the theorem we will mix the proofs of Theorems 7.3 and 8.5.

We assume that $V \models$ CH. Suppose that σ is a $\diamondsuit$CPA strategy of Eve. As before, we fix an ω_1-club $C \in V$ on ω_2 such that if $\beta \in C$ then $\sigma\restriction V[G_\beta] \in V[G_\beta]$. For $\beta \in C$ let $\langle \bar{a}_\delta^\beta : \delta \in \omega_1\rangle \in V[G_\beta]$ be an enumeration of all possible $\diamondsuit$SCPAs(σ) moves of Adam from $V[G_\beta]$ and let $e = (e_0, e_1): \omega_1 \to \omega_1 \times \omega_1$ be a surjection from V such that $e_0(\gamma) \leq \gamma$ for $\gamma < \omega_1$.

The winning $\diamondsuit$SCPAs(σ) strategy of Adam is defined like the one in the proof of Theorem 8.5. Suppose that we are at the γ-th round of the play. Let $\beta_\gamma \in C$ be such that $\beta_\gamma > \sup_{\delta<\gamma} \beta_\delta$ and $\bar{a}_\delta, \widetilde{E}_\delta \in V[G_{\beta_\gamma}]$ for $\delta < \gamma$. Adam plays $\bar{a}_\gamma = \bar{a}_{e_1(\gamma)}^{\beta_{e_0(\gamma)}}$.

This strategy is winning. Let $\beta = \sup_{\gamma<\omega_1} \beta_\gamma$. Since $\mathrm{cf}(\beta) = \omega_1$, we have $V[G_\beta] \models \diamondsuit$. Let $\langle a_\delta : \delta < \omega_1\rangle \in V[G_\beta]$ be the sequence of Adam's $\diamondsuit$CPA moves given by some fixed $\diamondsuit$-sequence from $V[G_\beta]$, just like in the proof of Theorem 7.3.

Again, we can find an increasing $\eta: \omega_1 \to \omega_1$ such that $\bar{a}_{\eta(\gamma)} = \langle a_\delta : \delta \leq \gamma\rangle$. We define $\bar{r} \in (2^\omega)^{\omega_1}$ by taking $\bar{r}(\zeta)$ as the $(\beta + \zeta)$-th Mathias real. Arguing as in Theorems 7.3 and 8.5 we obtain that for each $t \in 2^{\omega_1}$ the set

$$\{\gamma: t\restriction\gamma \in f_\gamma^{\eta(\gamma)}[(\widetilde{E}_{\eta(\gamma)})_{\bar{r}\restriction\gamma}]\}$$

is stationary. □

Chapter 9

V-ultrafilters

The main result of [21] was obtained as a corollary to two propositions, which are very interesting in themselves (see [4]). The first one ([21, Proposition 2.3]) says that if G_{ω_2} is Q_{ω_2}-generic over V, then any diagonalizable V-Ramsey ultrafilter in $V[G_{\omega_2}]$ is in fact diagonalizable in the first intermediate extension $V[G_0]$. The second one ([21, Proposition 2.4]) concerns the extension given just by the single Mathias forcing. It says that any diagonalizable V-Ramsey ultrafilter from $V[G_0]$ is isomorphic with the cannonical one $G_0' = \{X \in [\omega]^\omega \cap V : r_0 \subseteq^* X\}$ via a ground model bijection.

We found these results very smart and quite surprising. However, the original arguments in [21] are hard to follow. In this chapter we show their elementary proofs, using just the combinatorial structure of Q. Moreover, we generalize them to any diagonalizable V-Q-points (not necesserily V-Ramsey) added by any iteration $Q * \mathbb{P}'$, where $\mathbb{P}'$ has the Laver property.

Recall the Rudin-Keisler ordering. Let $\mathcal{U}, \mathcal{U}' \in \omega^*$. We define $\mathcal{U} \leq_{RK} \mathcal{U}'$ iff there exist a function $f : \omega \to \omega$ such that $\mathcal{U} = f_*(\mathcal{U}')$, where

$$f_*(\mathcal{U}') = \{X \subseteq \omega : f^{-1}[X] \in \mathcal{U}'\}.$$

Ultrafilters $\mathcal{U}$ and $\mathcal{U}'$ are isomorphic (RK-equivalent) if $\mathcal{U} \leq_{RK} \mathcal{U}'$ and $\mathcal{U}' \leq_{RK} \mathcal{U}$. In this case, there exists a bijection $f : \omega \to \omega$ such that $\mathcal{U} = f_*(\mathcal{U}')$ (see [14, Exercise 7.11]) and we say that $\mathcal{U}$ and $\mathcal{U}'$ are isomorphic (RK-equivalent) via f.

Recall that $\mathcal{U} \in \omega^*$ is a Q-point iff for every finite-to-one function $f \in \omega^\omega$ there exists $X \in \mathcal{U}$ such that $f \restriction X$ is one-to-one. Moreover, $\mathcal{U}$ is Ramsey iff for every $f \in \omega^\omega$ there exists $X \in \mathcal{U}$ such that $f \restriction X$ is either one-to-one or constant (this is one of many equivalent definitions). Clearly a Ramsey ultrafilter is a Q-point.

Working in a $\mathbb{P}$-generic extension $V[G]$ we say that $\mathcal{U} \subseteq [\omega]^\omega \cap V$ is a V-ultrafilter iff $\mathcal{U}$ is a filter and for every $X \subseteq \omega$ from V either $X \in \mathcal{U}$ or $\omega \setminus X \in \mathcal{U}$. If $\mathcal{U}$ is a V-ultrafilter then:

1. $\mathcal{U}$ is a V-Q-point iff for every finite-to-one function $f \in \omega^\omega \cap V$ there exists $X \in \mathcal{U}$ such that $f \restriction X$ is one-to-one,
2. $\mathcal{U}$ is V-Ramsey if each $f \in \omega^\omega \cap V$ is one-to-one or constant on some element of $\mathcal{U}$ (hence, V-Ramsey ultrafilter is a V-Q-point),
3. $\mathcal{U}$ is diagonalizable in $V[G]$ iff there exists $x \in [\omega]^\omega \cap V[G]$ such that $\mathcal{U} = \{X \in [\omega]^\omega \cap V : x \subseteq^* X\}$.

If $x \in [\omega]^\omega \cap V[G]$ then we define the filter $\mathcal{U}_x$ on $[\omega]^\omega \cap V$ generated by x as follows:

$$\mathcal{U}_x = \{X \in [\omega]^\omega \cap V : x \subseteq^* X\}.$$

If $\dot{x}$ is a name for x then let $\mathcal{U}_{\dot{x}}$ be a name for $\mathcal{U}_x$, i.e.

$$\mathbb{P} \Vdash \mathcal{U}_{\dot{x}} = \{X \in [\omega]^\omega \cap V : \dot{x} \subseteq^* X\}.$$

In particular $\mathcal{U}_{\dot{r}}$ is the Q-name for the cannonical V-ultrafilter added by Q and $\mathcal{U}_r = \mathcal{U}_{\dot{r}}/G$ is its evaluation in $V[G]$. A simple density argument shows that $\mathcal{U}_r$ is V-Ramsey.

Proposition 9.1. *Suppose that G is Q-generic over V and r is the associated Mathias real. Let $x \in [\omega]^\omega \cap V[G]$. If $\mathcal{U}_x$ is a V-ultrafilter, then there is $f \in \omega^\omega \cap V$ such that $U_r = f_*(U_x)$.*

The proof resembles the ones of Proposition 6.3 and Lemma 6.4. Let us recall the notation. When $\langle s, A\rangle \in Q$ is fixed and $n \in \omega$ we denote $n^- = \max(s \cup (A \cap n))$ ($\max(\emptyset) = 0$) and $n^+ = \min(A/n)$.

Proof. We assume that

$$Q \Vdash \dot{x} \in [\omega]^\omega \ \&\ \mathcal{U}_{\dot{x}} \text{ is a } V\text{-ultrafilter}.$$

Without loss of generality we may assume that $Q \Vdash (\forall i < \omega)\ |\dot{x} \cap \underline{\dot{r}}(i)| < i$. Indeed, there is a Q-name $\dot{x}'$ such that

$$Q \Vdash \dot{x}' \in [\omega]^\omega \quad \& \quad \dot{x}' \subseteq \dot{x} \quad \& \quad |\dot{x}' \cap \underline{\dot{r}}(i)| < i.$$

In particular Q forces that $\mathcal{U}_{\dot{x}} = \mathcal{U}_{\dot{x}'}$. So we can replace $\dot{x}$ with $\dot{x}'$.

Arguing as in the proof of Lemma 6.4 (up to Claim 6.5) we get

Claim 9.2. *For all $\langle s, A'\rangle \in Q$ there is an infinite $A \subseteq A'$ such that for every $\langle t, B\rangle \leq_Q \langle s, A\rangle$*

$$\langle t, B\rangle \Vdash_Q \dot{x} \subseteq^* \bigcup_{n\in B} [n^-, n^+).$$

□

Note that this statement is equivalent to

$$\langle s, A\rangle \Vdash_Q (\forall B \in \mathcal{U}_{\dot{r}})\ \dot{x} \subseteq^* \bigcup_{n\in B} [n^-, n^+).$$

Let $\langle a_i : i < \omega\rangle$ be the increasing ennumeration of A and let $C = \{a_{2i}: i < \omega\}$. Then

$$\langle s, C\rangle \Vdash_Q (\forall B \in \mathcal{U}_{\dot{r}})\ \dot{x} \subseteq^* \bigcup_{i: a_{2i+1}\in B} [a_{2i}, a_{2i+2}).$$

Define $f: \omega \to \omega$ so that f maps $[0, a_2)$ to a_1 and $[a_{2i}, a_{2i+2})$ to a_{2i+1}. Then clearly

$$\langle s, C\rangle \Vdash_Q (\forall B \in \mathcal{U}_{\dot{r}})\ f^{-1}[B] \in \mathcal{U}_{\dot{x}},$$

so $\langle s, C\rangle$ forces that $\mathcal{U}_{\dot{r}} \subseteq \{X \subseteq \omega: f^{-1}[X] \in U_{\dot{x}}\}$. But since it forces that $\mathcal{U}_{\dot{r}}, \mathcal{U}_{\dot{x}}$ are V-ultrafilters, it must force the equality. Hence

$$\langle s, C\rangle \Vdash_Q \mathcal{U}_{\dot{r}} = f_*(\mathcal{U}_{\dot{x}}).$$

□

Corollary 9.3 (cf. [21, Proposition 2.4])**.** *Let G, r and x be as above. Moreover, suppose that the V-ultrafilter $\mathcal{U}_x$ is a V-Q-point. Then there is a bijection $g \in \omega^\omega \cap V$ such that $\mathcal{U}_{\dot{r}} = g_*(\mathcal{U}_x)$, i.e. $\mathcal{U}_{\dot{r}}$ and $\mathcal{U}_x$ are isomorphic via a ground model bijection.*

Proof. Get back to the proof of the Proposition 9.1. We have a finite-to-one function $f \in \omega^\omega$ and $\langle s, C\rangle \in Q$ such that $\langle s, C\rangle \Vdash_Q \mathcal{U}_{\dot{r}} = f_*(\mathcal{U}_{\dot{x}})$. Now, since $\langle s, C\rangle$ forces that $\mathcal{U}_{\dot{x}}$ is V-Q-point, there is $\langle s', C'\rangle \leq_Q \langle s, C\rangle$ and $D \in [\omega]^\omega$ such that $\omega \setminus D, \omega \setminus f[D]$ are infinite, $f{\restriction}D$ is one-to-one and $\langle s', C'\rangle \Vdash_Q D \in \mathcal{U}_{\dot{x}}$. Let $g{\restriction}D = f{\restriction}D$ and extend it to a bijection $g: \omega \to \omega$. □

Corollary 9.3 is a strenghtening of [21, Proposition 2.4]. Proposition 9.1 as well as the corollary remain true if $\dot{x}$ is a $\mathbb{P}$-name, where $\mathbb{P} = Q * \mathbb{P}'$ and $\mathbb{P}'$ has the Laver property. The proof can be easily modified to deal with such notions. Therefore, we get:

Corollary 9.4. *Let $\langle G, H\rangle$ be $\mathbb{P} = Q * \mathbb{P}'$-generic over V, where $\mathbb{P}'$ has the Laver property. Then any diagonalizable V-Q-point in $V[G][H]$ is isomorphic via a bijection from V with the V-ultrafilter $\mathcal{U}_r$ generated by the Mathias real r added by the first iterand.* □

Corollary 9.5 (cf. [21, Proposition 2.3])**.** *Let $\langle G, H\rangle$ be $\mathbb{P} = Q * \mathbb{P}'$-generic over V, where $\mathbb{P}'$ has the Laver property. Let $x \in [\omega]^\omega \cap V[G][H]$ be such that $\mathcal{U}_x$ is a V-Q-point. Then there exists $y \in [\omega]^\omega \cap V[G]$, such that $\mathcal{U}_x = \mathcal{U}_y$. In particular, for $\mathbb{P} = Q_{\omega_2}$ we get Proposition 2.3 of [21].*

Proof. Take a bijection $g \in \omega^\omega \cap V$ such that $\mathcal{U}_r = g_*(\mathcal{U}_x)$. Define $y = g^{-1}[r]$. □

Remark 9.6. A natural question arises: Is the assertion of Corollary 9.4 true for $\mathbb{P} = (\mathcal{P}(\omega)/\operatorname{fin}) * \mathbb{P}'$? More precisely, suppose that $\mathbb{P}'$ has the Laver property. Is every diagonalizable V-Q-point added by $\mathcal{P}(\omega)/\operatorname{fin} * \mathbb{P}'$ isomorphic via a ground model bijection with the generic ultrafilter on V added by $\mathcal{P}(\omega)/\operatorname{fin}$? It turns out that this is not the case.

For a filter $\mathcal{F}$ on ω, which contains all co-finite sets, let $Q(\mathcal{F})$ be the Mathias forcing relativized to $\mathcal{F}$, i.e. $Q(\mathcal{F}) = \{\langle s, A\rangle \in Q : A \in \mathcal{F}\}$ with the ordering inherited from Q. It was essentially proved in [6] that if $\mathcal{F}$ is not a Ramsey ultrafilter then $Q(\mathcal{F})$ adds a Cohen real, so it can not have the Laver property. On the other hand if $\mathcal{U}$ is a Ramsey ultrafilter, one can show that $Q(\mathcal{U})$ has the Laver property, repeating the argument for Q.

Denote $\mathcal{P}(\omega)/\operatorname{fin}$ by Q'. It is well known that $Q \cong Q' * Q(\dot{G}')$, where $\dot{G}'$ is the cannonical name for the Q'-generic ultrafilter.

Let $\langle G'_0, G'_1\rangle$ be $Q' \times Q'$-generic over V. Note that since Q' adds no reals, we have $Q' \times Q' \cong Q' * Q'$, so we can write $V[\langle G'_0, G'_1\rangle] = V[G'_0][G'_1]$. A density argument shows that G'_0, G'_1 are not isomorphic in $V[G'_0][G'_1]$. Let H_1 be $Q(G'_1)$-generic over $V[G'_0][G'_1]$. Then in $V[G'_0][G'_1][H_1]$ we have two V-ultrafilters G'_0, G'_1 (the latter is diagonalizable) which are not isomorphic by any bijection from V. The extension from V to $V[G'_0][G'_1][H_1]$ is via $(Q' \times Q') * Q(\dot{G}_1) \cong Q' * Q$.

Moreover, in $V[G_0'][G_1'][H_1]$ the filter G_0' can not be extended to any Ramsey ultrafilter. Indeed, suppose that a Ramsey ultrafiler $\mathcal{U} \in V[G_0'][G_1'][H_1]$ extends G_0'. Let H_0 be $Q(\mathcal{U})$-generic over $V[G_0'][G_1'][H_1]$. The extension from $V[G_0]$ to $V[G_0'][G_1'][H_1][H_0]$ is via $Q * Q(\mathcal{U})$ and $Q(\mathcal{U})$ has the Laver property. By Corollary 9.3, G_0', G_1' are isomorphic via some bijection $f \in V[G_0']$. But since $\omega^\omega \cap V[G_0'] = \omega^\omega \cap V$, we have $f \in V$. A contradiction.

Chapter 10

Axiom $\Diamond$mCPA

We want to find a $\Diamond$-like version of CPA, which is strong enough to imply the principle $\clubsuit$ introduced by Ostaszewski. Chapter 7 was dedicated to $\Diamond$CPA – a natural strenghtening of CPA gathering some combinatorics of $\Diamond$. We were unable, however, to construct a $\clubsuit$-sequence from $\Diamond$CPA. Thus we developed a modification called $\Diamond$mCPA.

Recall that a $\clubsuit$-sequence is a sequence $\langle X_\alpha \colon \alpha \in C\rangle$ such that C is a club on ω_1, each X_α is a cofinal subset of α, and for every uncountable $X \subset \omega_1$ the set $\{\alpha \in C \colon X_\alpha \subseteq X\}$ is non-empty (equivalently, stationary in ω_1). The principle $\clubsuit$ says that there exists a $\clubsuit$-sequence.

It is easy to deduce both CH and $\clubsuit$ from $\Diamond$. In fact $\Diamond \Leftrightarrow \clubsuit +$ CH ([20, p. 42]). Hrušák proved in [13] that $\clubsuit$ is forced by the ω_2-iteration of Sacks forcing. In [17] Mildenberger generalized this to any forcing $\mathbb{P}$ whose conditions carry "countable" information and which satisfies the Axiom A with respect to orderings $\leq_n$ such that $p \leq_n q$ if p keeps some finite fragment of q. These assumptions catch many natural cases, e.g. Cohen, Sacks, Miller, Laver and Mathias forcings. In particular $\clubsuit$ holds in the Mathias model. We will set the proof from [17] into the $\Diamond$mCPA framework.

Definition 10.1. In the $\Diamond$mCPA game at each step $\gamma < \omega_1$ Adam plays (if possible) $(M_\gamma, \alpha_\gamma, A_\gamma, f_\gamma)$, where

(i) $M_\gamma \prec H_\kappa$ is a countable structure such that $\omega_1 \cap M_\gamma = \gamma$,

(ii) $\alpha_\gamma = \text{o.t.}(\omega_2 \cap M_\gamma)$,

(iii) $A_\gamma \in Q'_{\alpha_\gamma}$ and there is $p_\gamma \in P_{\omega_2} \cap M_\gamma$ such that

$$A_\gamma \bullet i_{a_\gamma}^{-1} = \{\langle \dot{r}_\xi / H \colon \xi \in a_\gamma \rangle \colon H \text{ is } P_{\omega_2}\text{-generic over } M_\gamma \text{ and } p_\gamma \in H\},$$

where $a_\gamma = \omega_2 \cap M_\gamma$ and $i_{a_\gamma} \colon \alpha_\gamma \to a_\gamma$ is the increasing ennumeration of a_γ,

(iv) $f_\gamma \colon A_\gamma \to 2^\gamma$ is a Borel function and there is $\dot{t}_\gamma \in M_\gamma$ such that $\dot{t}_\gamma$ is in M_γ a P_{ω_2}-name for an element of 2^{ω_1} and for $\bar{r} \in A_\gamma$ we have $f_\gamma(\bar{r}) = \dot{t}_\gamma / H_{\bar{r}}$, where $H_{\bar{r}}$ is the unique P_{ω_2}-generic filter over M_γ determined by $\bar{r}$, i.e., such that $\bar{r} \circ i_{a_\gamma}^{-1} = \langle \dot{r}_\xi / H_{\bar{r}} \colon \xi \in a_\gamma \rangle$.

Then Eve responds with some $E_\gamma \subseteq A_\gamma$ from Q'_{α_γ} (or $E_\gamma = \emptyset$ if failed to play anything correct). Adam wins iff for every $t \in 2^{\omega_1}$ the set

$$\{\gamma < \omega_1 \colon t \restriction \gamma \in f_\gamma[E_\gamma]\}$$

is stationary in ω_1. The axiom $\Diamond$mCPA says that CH fails and Eve has no winning strategy in the $\Diamond$CPA game.

Obviously $\Diamond$mCPA $\Rightarrow$ $\Diamond$CPA. Repeating the proof of Theorem 7.3 we get:

Theorem 10.2. $Q_{\omega_2} \Vdash \Diamond$mCPA.

Proof. We just need to make the following modification to the proof of Theorem 7.3 (in the description of the winning counterplay of Adam).

At the γ-th step, when the model M_γ is already set, and in the $\Diamond$CPA game Adam playes $A_\gamma = B_{p'_\gamma} \bullet i_{a_\gamma}$, in the $\Diamond$mCPA game he should now play $A_\gamma = A'_\gamma \bullet i_{a_\gamma}$, where A'_γ consists exactly of sequences $\bar{x}$ for which a (necessarily unique) P_{ω_2}-generic filter $G_{\bar{x}}$ over M_γ exists such that $p_\gamma \in G_{\bar{x}}$ and $\bar{x} = \langle \dot{r}_\xi / G_{\bar{x}} \colon \xi \in a_\gamma \rangle$. (Then $B_{p'_\gamma} \subseteq A'_\gamma$, so A_γ is $\mathcal{I}^{\alpha_\gamma}$-positive.) Moreover, the function $f_\gamma \colon A_\gamma \to 2^\gamma$ Adam should play now is given by

$$f_\gamma(\bar{x} \circ i_{a_\gamma}) = \langle \dot{\tau}_\delta / G_{\bar{x}} \colon \delta < \gamma \rangle,$$

where $\dot{t}_\gamma = \langle \dot{\tau}_\delta \colon \delta < \gamma \rangle = e \circ \pi(1)$. □

Theorem 10.3. $\Diamond$mCPA $\vdash \clubsuit$.

We need some of the technical notions and the crucial lemma from [17].

First we recall the fusion technique for iterated forcing. Let $n \in \omega$. For $\langle s, A\rangle, \langle t, B\rangle \in Q$, let $\langle s, A\rangle \leq_n \langle t, B\rangle$ iff $s = t$, $A \subseteq B$, and the first n elements of B are in A. Now fix an ordinal ε. For $q, q' \in Q_\varepsilon$ and a finite $F \subseteq \varepsilon$ let

$$q \leq_{(F,n)} q' \quad \Leftrightarrow \quad q \leq_{Q_\varepsilon} q' \;\;\&\;\; (\forall \beta \in F)(q \restriction \beta \Vdash_{Q_\beta} q(\beta) \leq_n q'(\beta)).$$

Suppose that $\langle F_i : i < \omega\rangle$, $\langle q_i : i < \omega\rangle$, and $a \in [\varepsilon]^{\leq\omega}$ satisfy

- $a = \bigcup_{i\in\omega} \mathrm{cl}(q_i)$,
- F_i is finite, $F_i \subseteq F_{i+1}$ for each $i < \omega$, and $\bigcup_{i<\omega} F_i = a$,
- $q_i \in Q_\varepsilon$ and $q_{i+1} \leq_{(F_i,i)} q_i$ for each $i < \omega$.

Then there is $q \in Q_\varepsilon$ such that $\mathrm{cl}(q) = a$ and $q \leq_{Q_\varepsilon} p_i$ for each $i < \omega$ (in fact $q \leq_{(F_i,i)} q_i$; see [3, Lemma 7.2]).

Fix $q \in Q_\alpha$, $n < \omega$, a finite set $F \subseteq \alpha$ and $\dot{x}$ such that $Q_\alpha \Vdash \dot{x} \in 2^{\omega_1}$, let

$$A_{F,n}(q, \dot{x}) = \{\gamma < \omega_1 : (\exists q' \in Q_\alpha)(q' \leq_{(F,n)} q \quad \& \quad q' \Vdash_{Q_\alpha} \dot{x}(\gamma) = 1)\}.$$

In [17] the technical notion of q being $(\dot{x}, F, n)$-good is introduced. From [17] we need two things about this notion. One is the fact that if q is $(\dot{x}, F, n)$-good then

$$(\forall q' \leq_{(F,n)} q)\; |A_{F,n}(q', \dot{x})| = \omega_1.$$

The other is the following lemma.

Lemma 10.4 ([17, Lemma 3.5]). *Suppose that $q \in Q_\alpha$, $F \subseteq F'$ are finite subsetes of α, $\dot{x}$ is a Q_α-name for an element of 2^{ω_1} and $n, m < \omega$. If q is $(\dot{x}, F, n)$-good then there exists $q' \in Q_\alpha$ such that $q' \leq_{(F,n)} q$ and q' is $(\dot{x}, F', m)$-good.*

Now we are ready to show that $\diamondsuit$mCPA implies $\clubsuit$.

Proof of Theorem 10.3. We describe a $\diamondsuit$mCPA strategy of Eve. If at the γ-th step Adam plays a correct tuple $(M_\gamma, \alpha_\gamma, A_\gamma, f_\gamma)$ then Eve performs the following construction.

Let $p_\gamma \in P_{\alpha_\gamma}$ be such that

$$A_\gamma \bullet i_{a_\gamma}^{-1} = \{\langle \dot{r}_\xi / H \colon \xi \in a_\gamma \rangle \colon H \text{ is } P_{\omega_2}\text{-generic over } M_\gamma \text{ and } p_\gamma \in H\},$$

where $a_\gamma = \omega_2 \cap M_\gamma$ and $i_{a_\gamma} \colon \alpha_\gamma \to a_\gamma$ is the increasing ennumeration of a_γ. Fix an increasing sequence of ordinals $\langle \zeta_\gamma^i \colon i < \omega \rangle$ such that $\sup_{i<\omega} \zeta_\gamma^i = \gamma$ and let $\langle \mathcal{D}_\gamma^i \colon i < \omega \rangle$ be an ennumeration of all dense subsets of Q_{ω_2} (not of P_{ω_2}) from M_γ. Eve constructs a sequence $\langle (q_\gamma^i, F_\gamma^i, \beta_\gamma^i) \colon i < \omega \rangle$ satisfying:

- for $i < \omega$ the set F_γ^i is finite, $F_\gamma^i \subseteq F_\gamma^{i+1}$ and $\bigcup_{i<\omega} F_\gamma^i = a_\gamma$,
- $q_\gamma^0 \leq \pi(p_\gamma)$ (recall that $\pi \colon P_{\omega_2} \to Q_{\omega_2}$ is a dense embedding)
- $q_\gamma^{i+1} \leq_{(F_\gamma^i, i)} q_\gamma^i$ for $i < \omega$,
- $q_\gamma^i \in \mathcal{D}_\gamma^i \cap M_\gamma$,
- $M_\gamma \models q_\gamma^i$ is $(\dot{x}_\gamma, F_\gamma^i, i)$-good,
- $\zeta_\gamma^i < \beta_\gamma^i < \alpha_\gamma$ and $q_\gamma^i \Vdash \dot{t}_\gamma(\beta_\gamma^i) = 1$ ($\dot{t}_\gamma$ can be considered as a Q_{ω_2}-name).

This construction is possible by Lemma 10.4. Let $X_\gamma = \langle \beta_\gamma^i \colon i < \omega \rangle$. Let q_γ be a fusion of $\langle q_\gamma^i \colon i < \omega \rangle$, i.e. $q_\gamma \leq_{Q_{\omega_2}} q_\gamma^i$ for every $i < \omega$ and $\mathrm{cl}(q_\gamma) = a_\gamma$. Since π is a dense embedding and by Fact 3.24, there exists $p'_\gamma \leq_{P_{\omega_2}} p_\gamma$ such that $\mathrm{dom}(p'_\gamma) = a_\gamma$ and $\pi(p'_\gamma) \leq_{Q_{\omega_2}} q_\gamma$. Note that $p'_\gamma \Vdash_{P_{\omega_2}} \dot{t}_\gamma(\beta_\gamma^i) = 1$, for every $i \in \omega$. Let $E_\gamma = A_\gamma \cap (B_{p'} \bullet i_{a_\gamma})$.

Suppose that Adam follows his winning counterplay against the strategy of Eve described above. Let $C \subseteq \omega_1$ be the club of $\gamma < \omega_1$ such that Adam played a correct tuple at the γ-th step. It is easy to verify that $\langle X_\gamma \colon \gamma \in C \rangle$ is a $\clubsuit$-sequence. Indeed, suppose that $X \subseteq \omega_1$ is uncountable. Let $\mathbb{1}_X \in 2^{\omega_1}$ be the characteristic function of X. Since Adam won the game, there exists $\gamma \in C$ such that $\mathbb{1}_X \restriction \gamma \in f_\gamma[E_\gamma]$. Let $\bar{x} \in E_\gamma$ be such that $f_\gamma(\bar{x}) = \mathbb{1}_X \restriction \gamma$. Take the P_{ω_2}-generic filter H over M_γ such that

$$\bar{x} \circ i_{a_\gamma}^{-1} = \langle \dot{r}_\xi / H \colon \xi \in a_\gamma \rangle.$$

Then $M_\gamma[H] \models X_\gamma \subseteq X$. By absoluteness, $X_\gamma \subseteq X$. $\square$

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