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On free $\mathcal{A}_d$-actions on products of spheres

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Term paper #1: On free $\mathcal{A}_d$-actions on products of spheres*

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Abstract

We investigate the problem of existence of free alternating group actions on products of equidimensional spheres. We begin with a discussion of classical results obtained by R. Oliver, then present some new developments. Most notably, we prove that for a prime number $p \geq 7$, the alternating group $\mathcal{A}_{p+1}$ cannot act freely on a finite-dimensional CW-complex with the integral cohomology ring isomorphic to that of $(S^n)^p$ for any positive integer n. This extends previous results due to L. P. Plakhta.

1 Introduction

The study of free actions of alternating groups on products of spheres tracks back to Oliver's [9]. A result estabilished therein states that $\mathcal{A}_4$, the alternating group on four letters, cannot act freely on any finite CW-complex with the integral cohomology ring isomorphic to that of $S^n \times S^n$ for any positive integer n. This yields an interesting problem:

Given a finite group G, determine the minimal number $k = k(G)$ such that G acts freely on a finite CW*-complex homotopy equivalent to $(S^n)^k$ for some n.*

As observed by Oliver, such a number always exists; this is explained in the Appendix. For example, $k(\mathcal{A}_4) = 3$; a free $\mathcal{A}_4$-action on $(S^3)^3$ was constructed by Plakhta in [10]. We discuss this in detail in Section 3.

The aim of this paper is to investigate k for other alternating groups. We note that $k(\mathcal{A}_d) > d - 2$ for any integer $d \geq 7$ (Proposition 4.1), then focus on improving this bound. Our main result is:

Theorem 4.3 *Let $p \geq 7$ be a prime number. The alternating group $\mathcal{A}_{p+1}$ cannot act freely on any finite-dimensional* CW*-complex X such that the cohomology rings $H^*(X;\mathbb{Z})$ and $H^*((S^n)^p;\mathbb{Z})$ are isomorphic, where n is any positive integer.*

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* This paper is a more expository-oriented version of [3].

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In other words, for $p \geq 7$ a prime number, $k(\mathcal{A}_{p+1}) > p$. A version of this theorem for $p = 5$ was obtained by Plakhta. Our work stems from his: a careful examination of his ideas and a subtle change in arguments give rise to a more general result. In particular, we do not rely on elusive information regarding the number of ideal classes in certain Dedekind domains (cf. [8, Remark 3.4]).

We also deduce an analogous phenomenon for the groups $\mathcal{A}_p$, $p \geq 5$ (Theorem 4.2). In fact, in this case the reasoning is much more straightforward.

Our proofs are based on representation theory of alternating groups and integral representation theory of cyclic groups of prime order. Essentially, we investigate representations which arise from the $\mathcal{A}_d$-module structure on $H^n(X;\mathbb{Z})$; comparing the outcome with Adem's results from [1] leads to a contradiction. Theorems: 3.2, 4.2 and 4.3 arise as variations on this theme.

Note that we could *a priori* assume that n, the dimension of spheres in the product, is odd: a finite group acting freely on a finite-dimensional CW-complex with integral cohomology groups of a product of even-dimensional spheres is necessarily a 2-group (see [11, Theorem 1]). Our considerations are, however, independent of the dimension of involved spheres.

Notation. We employ a standard notation throughout: S^n denotes the n-dimensional sphere; $\mathbb{Z}$ the infinite cyclic group; $\mathbb{Z}_m$ the cyclic group of order m; finally, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ fields of rational, real and complex numbers, respectively. If ρ is a representation of a group G and H is a subgroup of G, then $\mathrm{Res}_H\rho$ stands for the restriction of ρ to H. We will also write $X \sim (S^n)^k$ whenever the integral cohomology rings of X and $(S^n)^k$ are isomorphic.

2 Preliminaries

2.1 On representation theory of alternating groups

Let $d \geq 7$ be an integer. The following is a classical result of representation theory of symmetric and alternating groups.

Proposition 2.1 ([6, Chapters 4, 5])**.** *The only (up to isomorphism) non-trivial $(d-1)$-dimensional complex representation of the alternating group $\mathcal{A}_d$ comes from the restriction of the standard representation of the symmetric group $\mathcal{S}_d$ [which in turn corresponds to the partition $(d-1,1)$]. This is the only non-trivial representation of $\mathcal{A}_d$ of dimension less than d.*

Assume additionally that d is odd. Let $\{e_{t_1}, e_{t_2}, \ldots, e_{t_d}\}$ be the standard basis of the Specht module associated with the partition $(d, 1)$ of $d+1$, with the standard $(d, 1)$-tableaux $t_1, t_2, \ldots, t_d$ ordered in the usual way (consult [7, Sections 2 – 4, 8] for related definitions). Denote the corresponding representation of the group $\mathcal{A}_{d+1}$ by φ.

Following ideas of Plakhta, we will analyze the form of certain automorphisms produced by φ. This constitutes the tedious part of the proof of Theorem 4.3.

Consider the d-cycle $\sigma = (1\ 2\ \ldots\ d) \in \mathcal{A}_{d+1}$. As one can easily verify, the following equalities hold:

$$\begin{cases} \sigma e_{t_k} = e_{t_{k+1}} - e_{t_1}, \ k = 1, 2, \ldots, d-2, \\ \sigma e_{t_{d-1}} = -e_{t_1}, \\ \sigma e_{t_d} = e_{t_d} - e_{t_1}. \end{cases}$$

Consequently, the matrix $\varphi_\sigma^{\mathcal{B}}$ of the automorphism $\varphi(\sigma)$ with respect to the basis $\mathcal{B} = \{e_{t_d} - e_{t_1}, e_{t_d} - e_{t_2}, \ldots, e_{t_d} - e_{t_{d-1}}, e_{t_d}\}$ assumes the form

$$\begin{bmatrix} \Theta & 1 \\ E_{d-1} & \Theta \end{bmatrix}$$

where E_{d-1} stands for the $(d-1)$-dimensional identity matrix and the Θ's are appropriate zero matrices.

Now let $\tau = (1\ d+1\ d) \in \mathcal{A}_{d+1}$. Similarly as above, the equalities:

$$\begin{cases} \tau e_{t_k} = e_{t_k} - e_{t_d}, \ k = 1, 2, \ldots, d-2, \\ \tau e_{t_{d-1}} = -e_{t_d}, \\ \tau e_{t_d} = e_{t_{d-1}} - e_{t_d}, \end{cases}$$

imply that the matrix $\varphi_\tau^{\mathcal{B}}$ has the form

$$\begin{bmatrix} E_{d-2} & \multicolumn{2}{c}{\Theta} \\ \multicolumn{3}{c}{-\mathbf{1}} \\ \Theta & 1 & 0 \end{bmatrix}$$

where $-\mathbf{1}$ is a $1 \times d$ matrix consisting of -1's.

2.2 On integral representations of $\mathbb{Z}_p$

We will now recall a complete description of integral representations of cyclic groups of prime order. This result is attributed to Diederichsen and Reiner; a standard reference is [5, §74]. Throughout this section p denotes an arbitrary prime number.

Let $g \in \mathbb{Z}_p$ be a generator, $\zeta \in \mathbb{C}$ a primitive p-th root of unity, and let R denote the ring of algebraic integers in $K = \mathbb{Q}(\zeta)$. Suppose A is a (fractional) ideal in K. We may turn A into a $\mathbb{Z}_p$-module, again denoted by A, by setting:

$$ga = \zeta a, \ a \in A.$$

In this way, every ideal in K becomes a $\mathbb{Z}_p$-module of $\mathbb{Z}$-rank equal to $p-1$.

Another $\mathbb{Z}_p$-module can be constructed as follows. Let A be an ideal in K and $a_0 \in A$ a fixed element. Consider the direct sum $A \oplus \mathbb{Z}x$ with the $\mathbb{Z}_p$-action given by:

$$\begin{cases} ga = \zeta a, \ a \in A, \\ gx = a_0 + x. \end{cases}$$

This defines a $\mathbb{Z}_p$-module of $\mathbb{Z}$-rank equal to p, which is usually denoted by (A, a_0).

It turns out that these are the only possibilities for the non-trivial indecomposable $\mathbb{Z}_p$-modules with a finite $\mathbb{Z}$-basis. More specifically, we have:

Theorem 2.2 ([5, Theorem 74.4]). *Every $\mathbb{Z}_p$-module with a finite $\mathbb{Z}$-basis is isomorphic to a direct sum*

$$A_1 \oplus A_2 \oplus \cdots \oplus A_k \oplus (A_{k+1}, a_{k+1}) \oplus (A_{k+2}, a_{k+2}) \oplus \cdots \oplus (A_m, a_m) \oplus Y,$$

where the A_i's are ideals in K, a_i's are chosen so that $a_i \notin (\zeta - 1)A_i$, and $Y \cong \mathbb{Z}^n$ is a trivial $\mathbb{Z}_p$-module.

The isomorphism class of any such $\mathbb{Z}_p$-module is determined by the integers k, m and n, and the ideal class $A_1 A_2 \cdots A_m$.

With this theorem in mind, recall that a $\mathbb{Z}_p$-module M is said to be of the *type* (r, s, t) if there exists a decomposition

$$M \cong \left(\bigoplus_{i=1}^{r} A_i \right) \oplus \left(\bigoplus_{j=1}^{s} (A_j, a_j) \right) \oplus \left(\bigoplus_{k=1}^{t} \mathbb{Z} \right).$$

Let X be a finitistic $\mathbb{Z}_p$-space such that $X \sim (S^n)^k$ for some positive integers k, n. We conclude this section with a result due to Adem, which relates the $\mathbb{Z}_p$-module structure of $H^n(X;\mathbb{Z})$ and the nature of the action's fixed point set.

Theorem 2.3 ([1, Theorems 4.5, 4.6]). *For X as above, the following hold:*

(1) *If $H^n(X;\mathbb{Z})$ is of the type $(0, s, 0)$, then the fixed point set F of the action is non-empty. Furthermore, F has the cohomology ring of $(S^n)^s$ with $\mathbb{Z}_p$ coefficients.*

(2) *If p is odd and $H^n(X;\mathbb{Z})$ is of the type $(r, 0, 0)$, then the fixed point set F of the action is non-empty. Furthermore, $H^*(F;\mathbb{Z}_p)$ is torsion-free, trivial in odd dimensions, and of rank p^r.*

3 On Oliver's result

Theorem 3.1 below was the starting point for Plakhta's work; it provides the backbone for our considerations.

For arbitrary positive integers k, n, we have:

Theorem 3.1 ([9, Theorem 1])**.** *There is no free action of the alternating group $\mathcal{A}_4$ on any finite-dimensional* CW*-complex X such that the cohomology rings $H^*(X;\mathbb{Z}_2)$ and $H^*((S^n)^k;\mathbb{Z}_2)$ are isomorphic, with $\mathcal{A}_4$ acting trivially on cohomology.*

As a corollary, we deduce:

Theorem 3.2. *The alternating group $\mathcal{A}_4$ cannot act freely on any finite-dimensional* CW*-complex X such that $X \sim S^n \times S^n$, where n is any positive integer.*

Proof. Suppose $\mathcal{A}_4$ acts freely on X as above. Then $H^n(X;\mathbb{Z})$ becomes an $\mathcal{A}_4$-module; denote the corresponding representation by ρ.

In view of Theorem 3.1, ρ is non-trivial. Thus there exists an element of order 3 in $\mathcal{A}_4$ which turns $H^n(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ into a non-trivial $\mathbb{Z}_3$-module. The only possibility is that $H^n(X;\mathbb{Z})$ is of the type (1, 0, 0). Since finite-dimensional CW-complexes are finitistic, Theorem 2.3 (2) implies that the $\mathcal{A}_4$-action on X cannot be free. □

Theorem 3.2 was originally obtained by Oliver for finite CW-complexes, by means of the Lefschetz Fixed Point Theorem (cf. [9, Theorem 2]).

Remark 3.3. As observed in [2, Section 5], the alternating group $\mathcal{A}_4$ is a subgroup of any finite rank-two simple group. (The *rank* of a finite group is the largest number among the ranks of all of its elementary abelian subgroups.) Consequently, such a group cannot act freely on any finite-dimensional CW-complex X such that $X \sim S^n \times S^n$.

We will now present an example of a free $\mathcal{A}_4$-action on $(S^3)^3$. The construction was carried out by Plakhta. Together with Theorem 3.2, this shows that $k(\mathcal{A}_4) = 3$.

Example 3.4 ([10, Example 1])**.** As a preliminary remark, recall that $\mathcal{A}_4$ can be described twofolds: either by the presentation

$$\langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle$$

or by the extension

$$0 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \mathcal{A}_4 \xrightarrow{\epsilon} \mathbb{Z}_3 \longrightarrow 0.$$

We will make use of both.

Let $F_2 = \langle a, b \rangle$ be the free group on two generators. Define a F_2-action on $S^3 \times S^3$ by setting:

$$\begin{cases} a(x,y) = (-x,y) \\ b(x,y) = (y, y^{-1}x^{-1}) \end{cases} \quad \text{for } x, y \in S^3.$$

(Think of S^3 as a subgroup of $\mathbb{H}^*$, the multiplicative group of the skew field of quaternions.) This action is trivial while restricted to the normal closure of a^2, b^3 and $(ab)^3$, and thus induces an $\mathcal{A}_4$-action on $S^3 \times S^3$, with the subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle a, bab^2 \rangle \subseteq \mathcal{A}_4$ acting freely.

Now take any free action of $\mathbb{Z}_3$ on S^3 (for example the one generated by the rotation $x \mapsto e^{2\pi i/3}x$) and extend it to the action of $\mathcal{A}_4$ by means of the epimorphism ϵ. One easily verifies that the product of these two actions gives rise to a free $\mathcal{A}_4$-action on $(S^3)^3$.

4 Main results

A combination of Proposition 2.1 and Theorem 3.1 gives:

Proposition 4.1. *Let $d \geq 7$ be an integer. The alternating group $\mathcal{A}_d$ cannot act freely on any finite-dimensional* CW-*complex X such that $X \sim (S^n)^{d-2}$, where n is any positive integer.*

Consequently, $k(\mathcal{A}_d) > d - 2$ for any integer $d \geq 7$. Our immediate goal is to improve this bound.

First of all, observe that by modifying the proof of Theorem 3.2 we arrive at:

Theorem 4.2. *Let $p \geq 5$ be a prime number. The alternating group $\mathcal{A}_p$ cannot act freely on any finite-dimensional* CW-*complex X such that $X \sim (S^n)^{p-1}$, where n is any positive integer.*

Proof. Suppose $\mathcal{A}_p$ acts freely on X as above. Similarly as before, let ρ denote the corresponding representation of $\mathcal{A}_p$ in $H^n(X;\mathbb{Z})$. Since $\mathcal{A}_p$ is a simple group, ρ is either trivial or faithful.

The first possibility is excluded by Theorem 3.1. It follows that for any non-trivial element $\pi \in \mathcal{A}_p$, the representation $\mathrm{Res}_{\langle \pi \rangle}\rho$ is non-trivial. In particular, for π an element of order p, the $\mathbb{Z}_p$-module $H^n(X;\mathbb{Z}) \cong \mathbb{Z}^{p-1}$ has to be of the type (1, 0, 0). Theorem 2.3 (2) now yields a contradiction. □

We will now move on to what we consider the main result of this paper.

Theorem 4.3. *Let $p \geq 7$ be a prime number. The alternating group $\mathcal{A}_{p+1}$ cannot act freely on any finite-dimensional* CW-*complex X such that $X \sim (S^n)^p$, where n is any positive integer.*

Proof. Suppose $\mathcal{A}_{p+1}$ acts freely on X as in the hypothesis. Let ρ be the corresponding representation of $\mathcal{A}_{p+1}$ in $H^n(X;\mathbb{Z})$. Observe that:

(1) In view of Proposition 2.1 and Theorem 3.1, ρ is isomorphic (over $\mathbb{C}$) to φ, the only non-trivial p-dimensional complex representation of $\mathcal{A}_{p+1}$ (see Section 2.1).

(2) Let $\sigma = (1\ 2\ \ldots\ p) \in \mathcal{A}_{p+1}$. Theorem 2.2 implies that the $\mathbb{Z}_p$-module determined by $\mathrm{Res}_{\langle\sigma\rangle}\rho$ is either of the type $(1,\ 0,\ 1)$ or $(0,\ 1,\ 0)$ [or $(0, 0, p)$, but this case is readily excluded by simplicity of $\mathcal{A}_{p+1}$].

Assume the first possibility, i.e., $H^n(X;\mathbb{Z})$ is isomorphic (as a $\mathbb{Z}_p$-module) to $A \oplus \mathbb{Z}$, where A is an ideal in $\mathbb{Q}(\zeta)$ and $\mathbb{Z}$ is a trivial $\mathbb{Z}_p$-module. Consequently, there exists a $\mathbb{Z}$-basis $\mathcal{B}'$ of $H^n(X;\mathbb{Z})$ such that the representation $\mathrm{Res}_{\langle\sigma\rangle}\rho$ is given by the matrix

$$D = \begin{bmatrix} D_1 & \Theta \\ \Theta & 1 \end{bmatrix}$$

By (1), there exists an invertible matrix $T = [t_{ij}]$ such that

$$T\varphi_\pi^{\mathcal{B}}T^{-1} = \rho_\pi^{\mathcal{B}'} \text{ for every } \pi \in \mathcal{A}_{p+1}.$$

The relation $T\varphi_\sigma^{\mathcal{B}}T^{-1} = D$ implies that:

$$\begin{cases} t_{p1} = t_{p2} = \cdots = t_{pp}, \\ s_{1p} = s_{2p} = \cdots = s_{pp}, \\ pt_{pp}s_{pp} = 1, \end{cases}$$

where $T^{-1} = [s_{ij}]$. Let $\tau = (1\ p+1\ p) \in \mathcal{A}_{p+1}$. Since $T\varphi_\tau^{\mathcal{B}}T^{-1} = \rho_\tau^{\mathcal{B}'}$, the bottom right element of $\rho_\tau^{\mathcal{B}'}$ is equal to $-t_{pp}s_{pp} = -1/p$. This contradicts the fact that ρ is an integral representation.

It follows that $\mathrm{Res}_{\langle\sigma\rangle}\rho$ has to be of the type $(0,1,0)$. By Theorem 2.3 (1), the $\mathcal{A}_{p+1}$-action on X cannot be free. □

As noted in the introduction, the analogue of the last result for $p = 5$ was obtained by Plakhta ([10, Theorem 1]). Observe the little discrepancy in the number of spheres in comparison with Theorem 3.2, which deals with the case $p = 3$. This is a consequence of the fact that $\mathcal{A}_4$ is not simple (cf. Example 3.4).

Appendix

We will now briefly discuss the well-posedness and statement of the initial problem.

Theorem A.1 ([9, p. 547])**.** *Every finite group acts freely on a product of spheres.*

Proof. Let G be a finite group, $H \subseteq G$ a subgroup and X an H-space. Then the space $\mathrm{Map}_H(G, X)$ of all H-equivariant maps $G \to X$, endowed with the compact-open topology, is a G-space in the obvious way. Futhermore

$$\mathrm{Map}_H(G, X) \approx X^{[G:H]},$$

where $[G : H]$ denotes the index of H in G.

For any non-trivial element $g \in G$, the cyclic group $\langle g \rangle$ acts freely on S^1 (in fact, on any odd-dimensional sphere; on any sphere, if g is of order 2). Then G acts on

$$M_g = \mathrm{Map}_{\langle g \rangle}(G, S^1) \approx (S^1)^{[G:\langle g \rangle]},$$

with the subgroup $\langle g \rangle \subseteq G$ acting freely. The product $\prod_{g \in G} M_g$ with a component-wise G-action is a free G-space. □

Observe, however, that the outlined construction is very inefficient; the number of spheres in the resulting product is *very* unlikely to be the minimum.

Remark A.2. In fact, Oliver proves a more general statement: a compact Lie group acts freely and smoothly on a product of equidimensional spheres if and only if it does not contain a copy of $SO(3)$, the 3-dimensional special orthogonal group, as a subgroup ([9, Theorem 5]).

Finally, note that the requirement of equidimensionality of spheres in the product is actually restrictive.

Example A.3 ([9, p. 543])**.** We will show that $\mathcal{A}_4$ acts freely on $S^2 \times S^3$. To see this, consider the twisted product $SO(3) \times_{S^1} S^3$, with $S^1 \cong SO(2)$ acting as a subgroup on both $SO(3)$ and S^3. This, as usual, is a fiber bundle over $SO(3)/SO(2) \approx S^2$, with fiber S^3 and structure group S^1.

Observe that the S^1-action on S^3 is contained in the group action of S^3, and consequently $SO(3) \times_{S^1} S^3$ can be thought of as a principal S^3-bundle. Since

$$\pi_2(BS^3) \cong \pi_1(S^3) = 0,$$

the bundle is trivial, thus $SO(3) \times_{S^1} S^3 \approx S^2 \times S^3$. The conclusion follows from the fact that $\mathcal{A}_4$ is a subgroup of $SO(3)$. (We have shown, more generally, that every subgroup of $SO(3)$ acts freely on $S^2 \times S^3$.)

Consequently, all results of Sections 3 and 4 are not true if we admit arbitrary products of spheres. It has been conjectured by Benson and Carlson that in this general case, k is equal to the rank of G ([4, Conjecture 5.2]).

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