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The topological spherical space form problem

Praca semestralna nr 2

(semestr zimowy 2010/11)

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UNIA EUROPEJSKA
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FUNDUSZ SPOŁECZNY

Term paper #2:
The topological spherical space form problem*

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1 Introduction

The aim of this note is to discuss the history and solution of the following problem:

Suppose X is a finite CW*-complex homotopy equivalent to a sphere and π is a finite group acting freely and cellularly on X.*[1] *What restrictions does this condition impose on π? Precisely which groups can behave in this way?*

This is the most general statement of the problem we will be interested in; often the emphasis is on the 'geometric' case where X is a sphere itself. There are some subtle differences between the two.

The history of the problem goes back to Hopf's [11] and his question about the classification of closed manifolds with universal cover a sphere. (These are precisely *spherical space forms*: quotients of S^n by a finite group acting freely.) The real development began, however, some 30 years later when Cartan and Eilenberg shown that a necessary condition for a finite group to act freely on a sphere is periodicity of its cohomology. Shortly thereafter Swan proved that this is also sufficient, albeit only in the topological category: the topological counterpart of an algebraic phenomenon of periodicity of π is the existence of a finite free π-complex homotopy equivalent to a sphere. On the other hand, by that time it was already known, thanks to Milnor, that there exist finite periodic groups which cannot act freely on an actual sphere. Finally, in 1976, Madsen, Thomas and Wall provided the final answer in the smooth category. Historically, this was one of the first triumphs of surgery theory.

The solution turned out to be interesting not only in its own right; perhaps more importantly, it stimulated growth of new fields. Davis and Milgram wrote in [7, Preface]:

*Date: January 30, 2011; *Revised:*

2010 *Mathematics Subject Classification*: Primary 57-02; Secondary 57S17, 57S25

Key words and phrases: finite group, free action, periodic cohomology, pq-condition, sphere, spherical space form, surgery, Swan complex

[1] Such a space X is often called a *free π-complex*.

Swan's results were one of the major beginnings of algebraic K-theory, while Milnor's results led in large to the work of C. T. C. Wall on the foundations of non-simply connected surgery theory.

Furthermore, the 'obvious' extension of the problem – taking X to be a product of spheres – currently receives quite some attention.

The paper is organized as follows. Section 2 contains a few basic examples which both elucidate the statement of the problem and set up the stage for the introduction of cohomological periodicity. Section 3 is mostly about the description and classification of finite periodic groups. We have also included some of their group-theoretic properties, though proofs are suppressed to the references. The real action starts with Section 4, where we introduce the notion and estabilish the existence of Swan complexes for finite periodic groups. We also briefly discuss Milnor's $2p$-condition. Section 5 contains a (sketch) proof of the celebrated Madsen–Thomas–Wall result, with emphasis on the case of soluble groups. Finally, in Section 6 we discuss the extension of the original problem.

Notation. We employ standard notation throughout: S^n denotes the n-dimensional sphere; $\mathbb{Z}$ the infinite cyclic group; $\mathbb{Z}_m$ the cyclic group of order m; $\mathbb{R}$, $\mathbb{C}$ fields of real and complex numbers, respectively; $\mathbb{H}$ the skew field of quaternions. For π a finite group, $\#\pi$ stands for the order of π. Given a CW-complex X, we will write $\mathcal{C}_*(X)$ for its cellular chain complex and $\tilde{X}$ for its universal covering. $\hat{H}$ denotes for the Tate cohomology functor.

We assume familiarity with surgery theory, as given in [17].

2 Examples and preliminary remarks

We begin with an observation that the stated problem is interesting only for odd-dimensional spheres. We then present the solution for S^1 and also give some examples for higher-dimensional spheres. In particular, we introduce the notion of pq-condition in Example 2.10.

2.1 Even-dimensional spheres

Consider the following example:

Example 2.1. The formula $(-1, x) \mapsto -x$, $x \in S^n$, defines a $\mathbb{Z}_2$-action on S^n.

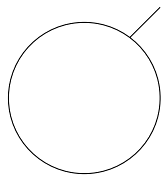
If we assume n is even, this is, in fact, everything we can afford:

Theorem 2.2. *Suppose X is a finite-dimensional* CW*-complex homotopy equivalent to an even-dimensional sphere and π is a nontrivial finite group acting freely on X. Then $\pi \cong \mathbb{Z}_2$.*

Proof. For X a finite CW-complex it is an easy consequence of the Lefschetz Fixed Point Theorem. Also see [9, Proposition 2.29] for an elementary proof.

In this more general setting, consider the equality $\chi(X) = \#\pi\chi(X/\pi)$, where χ denotes the Euler characteristic. (See [23, Theorem 1] for details.) □

Example 2.3. (1) Not every X as above admits a free action of $\mathbb{Z}_2$. Consider the following example:



One easily sees that only the trivial group can act freely on this space. (The dimension of the sphere in the wedge is irrelevant.)

(2) The assumption of π being finite is crucial: $\mathbb{Z}$ acts freely on $S^n \times \mathbb{R}$ by translations of the second coordinate.

2.2 The 1-sphere

The case of odd-dimensional spheres is much more complicated; we will take a look at S^1 to begin with.

Example 2.4. $\mathbb{Z}_m \subseteq S^1$ yields a free $\mathbb{Z}_m$-action on S^1.

This actually depletes the supply of finite groups which can act freely on the 1-sphere.

Theorem 2.5. *Suppose X is a finite-dimensional* CW*-complex homotopy equivalent to S^1 and π is a finite group acting freely and cellularly on X. Then π is necessarily cyclic.*

Proof. Consider the fibration $X \to X/\pi \to K(\pi, 1)$. Inspection of its long exact sequence of homotopy groups reveals that:

(1) X/π is an Eilenberg–MacLane space $K(\pi_1(X/\pi), 1)$.

(2) $\pi_1(X/\pi)$ is given by the extension $0 \to \mathbb{Z} \to \pi_1(X/\pi) \to \pi \to 1$, and therefore is virtually cyclic[2].

[2] A group is said to be *virtually cyclic* if it contains a cyclic subgroup of finite index.

Since X/π is a finite-dimensional CW-complex, the group $\pi_1(X/\pi)$ has to be torsionfree. The only nontrivial torsionfree virtually cyclic group is $\mathbb{Z}$ by [12, Lemma 3.2]. Consequently, we have an extension

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \pi \to 1$$

and the conlusion follows. □

Remark 2.6. If one is willing to restrict attention to groups acting freely on an actual S^1, there is a neat way of proving the above assertion using only the theory of covering spaces. I will elaborate on this and related ideas in a forthcoming paper.

2.3 'Higher-dimensional' examples

So far so good. Let's take a look at some higher-dimensional examples.

Example 2.7. Recall the *generalized quaternion group*:

$$\mathcal{Q}_{4n} = \langle x, y \mid x^n = y^2, xyx = y \rangle,\ n \geq 2.$$

This group can be realized as a subgroup of $S^3 \subseteq \mathbb{H}$ by setting $x = e^{\pi i/n}$, $y = j$. It follows that $\mathcal{Q}_{4n}$ acts freely on S^3 by multiplication.

Example 2.8. The action generated by the rotation $z \mapsto e^{2\pi i/m}z$ provides an example of a free action of $\mathbb{Z}_m$ on $S^{2n-1} \subseteq \mathbb{R}^{2n} \subseteq \mathbb{C}^n$. More generally, given an integer $m > 1$ and integers $l_1, l_2, \ldots, l_n$ relatively prime to m, there is a free action of $\mathbb{Z}_m$ on S^{2n-1} generated by

$$(z_1, z_2, \ldots, z_n) \mapsto (e^{2\pi l_1 i/m}z_1, e^{2\pi l_2 i/m}z_2, \ldots, e^{2\pi l_n i/m}z_n).$$

The difference between the two is that in the second example, each coordinate is rotated by a different angle. The assumption of $l_1, l_2, \ldots, l_n$ being relatively prime to m is needed in order to assure freeness of the action.

Notice that orbit spaces of the last family of actions are the well-known *lens spaces* $L_m(l_1, l_2, \ldots, l_n)$.

Remark 2.9. The last example yields another problem: classification of orbit types. We confine ourselves to stating that suprisingly lot is known about this; for example, there are always only finitely many of them. Consult [8] and subsequent papers for more details.

Let p, q be (not necessarily distinct) prime numbers. Recall that a finite group is said to satisfy the *pq-condition* if and only if all of its subgroups of order pq are cyclic.

Example 2.10. The special orthogonal group $SO(2n)$ acts on S^{2n-1} in the obvious way. The action is not free if $n > 1$, but it is interesting to ask which finite subgroups of $SO(2n)$ act freely on S^{2n-1} via the whole group. (Note the relationship with Example 2.4.) This was adressed by Wolf in [21]: if a finite group acts freely and orthogonally on a sphere, it satisfies all *pq*-conditions. The converse is also true, albeit only for soluble groups.

The bottom line is that there seems to be a rich supply of groups which act freely on spheres. Our immediate goal is to find their common feature.

3 Finite periodic groups

We will now introduce the notion of a periodic group, then concentrate on their classification and properties. Most of the arguments in the latter part of this section are suppressed to the references.

3.1 Introducing periodicity

Throughout this section $n - 1$ is an odd integer.

Suppose a finite group π acts freely and cellularly on a finite $(n-1)$-dimensional CW-complex X homotopy equivalent to S^{n-1}. Then the induced π-action on $H_{n-1}(X) \cong \mathbb{Z}$ is trivial by the Lefschetz Fixed Theorem, and thus we have the following exact sequence of π-modules:

$$0 \to \mathbb{Z} \xrightarrow{\omega} \mathcal{C}_{n-1}(X) \to \mathcal{C}_{n-2}(X) \to \cdots \to \mathcal{C}_1(X) \to \mathcal{C}_0(X) \xrightarrow{\eta} \mathbb{Z} \to 0.$$

Since all the $\mathcal{C}_i(X)$'s are free as $\mathbb{Z}\pi$-modules, splicing together an infinite number of copies of the above sequence via the map $\eta\omega$ yields a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$. Consequently, cohomology of π has the following property:

$$H^k(\pi; \mathbb{Z}) \cong H^{k+n}(\pi; \mathbb{Z}) \text{ for any } k > 0.$$

Such a group π is called *periodic*; the smallest number n for which cohomology of π exhibits the above phenomenon is called the *cohomological period*. We have thus proven the following theorem:

Theorem 3.1 ([7, Proposition 0.2])**.** *If π is a finite group acting freely and cellularly on a finite $(n-1)$-dimensional* CW*-complex homotopy equivalent to S^{n-1}, then π is periodic with cohomological period dividing n.*

It turns out that periodicity is exactly the property we are looking for: **the topological counterpart of an algebraic phenomenon of periodicity of π is the existence of a finite free π-complex homotopy equivalent to a sphere**. We will prove this statement in Section 4. For now we concentrate on a complete classification of finite periodic groups, originally obtained by Suzuki and Zassenhaus.

3.2 Classification

Since Aut $\mathcal{Q}_8 \cong \mathcal{S}_4$, the symmetric group on four letters, the action $\mathbb{Z}_3 \subseteq$ Aut $\mathcal{Q}_8$ gives rise to the semi-direct product $\mathcal{Q}_8 \rtimes \mathbb{Z}_3 = T^*$, the binary tetrahedral group. Let O^* denote the binary octahedral group. Then there is an extension

$$1 \to T^* \to O^* \to \mathbb{Z}_2 \to 0.$$

More generally, for $i \geq 1$ set $T_i = \mathcal{Q}_8 \rtimes \mathbb{Z}_{3^i}$ with the action given by $\mathbb{Z}_{3^i} \to \mathbb{Z}_3 \subseteq$ Aut $\mathcal{Q}_8$. For each i, $\mathbb{Z}_2$ then admits an extension

$$1 \to T_i \to O_i^* \to \mathbb{Z}_2 \to 0.$$

Finally, given a prime number p and the special linear group $SL_2(\mathbb{F}_p)$, there exists an extension

$$1 \ \to SL_2(\mathbb{F}_p) \to TL_2(\mathbb{F}_p) \to \mathbb{Z}_2 \to 0.$$

Consult [3, Chapter IV, Section 6] or [7, Section 1] for more information about these groups.

Theorem 3.2 ([3, Chapter IV, Theorem 6.15]). *A complete list of finite periodic groups is given by:*

FAMILY	DEFINITION	CONDITIONS
I	$\mathbb{Z}_a \rtimes \mathbb{Z}_b$	$(a, b) = 1$
II	$\mathbb{Z}_a \rtimes (\mathbb{Z}_b \times \mathcal{Q}_{2^i})$	$(a, b) = (ab, 2) = 1$
III	$\mathbb{Z}_a \rtimes (\mathbb{Z}_b \times T_i)$	$(a, b) = (ab, 6) = 1$
IV	$\mathbb{Z}_a \rtimes (\mathbb{Z}_b \times O_i^*)$	$(a, b) = (ab, 6) = 1$
V	$(\mathbb{Z}_a \rtimes \mathbb{Z}_b) \times SL_2(\mathbb{F}_p)$	$(a, b) = \big(a, p(p^2 - 1)\big) = 1$
VI	$\mathbb{Z}_a \rtimes \big(\mathbb{Z}_b \times TL_2(\mathbb{F}_p)\big)$	$(a, b) = \big(ab, p(p^2 - 1)\big) = 1$

Remark 3.3. (1) Let $O(\pi)$ denote the maximal normal subgroup of odd order in π. The six families above come from whether $\pi/O(\pi)$ is I. cyclic; II. generalized quaternion; III. T^*; IV. O^*; V. $SL_2(\mathbb{F}_p)$ for $p \geq 5$; VI. $TL_2(\mathbb{F}_p)$ for $p \geq 5$.

(2) A group which is given by a soluble-by-soluble extension is again soluble, thus groups of types I – IV are soluble. The other ones are not.

We will now present some group-theoretic properties of finite periodic groups that will come in handy in later sections of this paper.

Proposition 3.4 ([19, Hypothesis 3.7]). *Every finite soluble periodic group π contains a subgroup τ such that:*

(1) *τ contains a Sylow 2-subgroup of π,*

(2) *the only prime divisors of $\#\tau$ are 2 and 3,*

(3) *the restriction homomorphism $H^2(\pi;\mathbb{Z}_2) \to H^2(\tau;\mathbb{Z}_2)$ is an isomorphism,*

(4) *there exists a free orthogonal action of τ on a sphere.*

The following theorem will be crucial to our considerations.

Theorem 3.5 ([18, Theorem B]). *Let π be a finite periodic group with cohomological period q. Then π has a periodic free resolution of period dq, where d is the greatest common divisor of $\#\pi$ and $\varphi(\#\pi)$, φ being the Euler function.*

We conclude this section with a theorem that further elucidates the nature of periodic groups. It will be of some use in Section 6.

Theorem 3.6 ([6, Chapter VI, Theorem 9.5]). *A finite group is periodic if and only if every of its abelian subgroups is cyclic.*

4 Swan complexes

A *Swan complex* is a n-dimensional CW-complex X such that $\pi_1(X) \cong \pi$ and $\tilde{X} \simeq S^n$. We already know that necessary condition for a Swan complex to exist is periodicity of π. We will now show that this is also sufficient.

4.1 Preliminaries

The 'shifting process'. Let Λ be an arbitrary ring, not necessarily commutative. Given two Λ-modules A and B, we will write $A \sim B$ whenever $A \oplus F \cong B \oplus F'$ for F, F' free Λ-modules.[3]

The following is an easy consequence of the Schanuel Lemma:

Lemma 4.1 ([18, Corollary 1.1]). *Suppose we have two exact sequences of Λ-modules:*

$$0 \to B \to P_m \to \cdots \to P_1 \to P_0 \to A \to 0,$$
$$0 \to B' \to Q_m \to \cdots \to Q_1 \to Q_0 \to A' \to 0.$$

If all the P_i's and Q_j's are free, and $A \sim A'$, then $B \sim B'$.

We will also make use of:

[3]Our notation slightly differs from the one originally used by Swan.

Lemma 4.2 ([18, Lemma 2.1]). *Let π be a finite group. Suppose we have an exact sequence of $\mathbb{Z}\pi$-modules of the form*

$$0 \to A \xrightarrow{\mu} X_{k-1} \to X_{k-2} \to \cdots \to X_1 \to X_0 \to 0$$

where μ splits as a $\mathbb{Z}$-map. Suppose that there exist a module A' and projective modules P and P' such that $P \oplus A \cong P' \oplus A'$. Then there is an exact sequence of the form

$$0 \to A' \to X_{k-1} \oplus P \to X_{k-2} \oplus P' \to X_{k-3} \to \cdots \to X_1 \to X_0 \to 0$$

where all maps from X_{k-3} onwards are the same as the original ones.

Milnor's construction. At the topological level, we will need:

Proposition 4.3 ([18, Lemma 3.1]). *Suppose a finite group π acts freely on a simply-connected* CW*-complex X. Assume that $H_i(X;\mathbb{Z}) = 0$ for $1 \leq i \leq m-2$. Let F be a free $\mathbb{Z}\pi$-module and $f\colon F \to \mathcal{Z}_{m-1}(X)$ a $\mathbb{Z}\pi$-map. Then we can attach m-cells to X, getting a* CW*-complex X' such that:*

(1) *π acts freely on X',*

(2) *$\mathcal{C}_m(X') \cong \mathcal{C}_m(X) \oplus F$,*

(3) *$\partial_m(c, x) = \partial_m(c) + f(x)$ for all $c \in \mathcal{C}_m(X)$ and $x \in F$.*

4.2 Existence of Swan complexes

Fix a finite periodic group π with period $q \geq 4$. Note that this assumption is not restrictive at all from our point of view: groups with cohomological period equal to 2 are cyclic, and these act freely on any odd-dimensional sphere.

Theorem 4.4 ([18, Theorem A]). *There exists a finite $(dq-1)$-dimensional* CW*-complex X such that $\pi_1(X) \cong \pi$ and $\tilde{X} \simeq S^{dq-1}$, where d is the greatest common divisor of $\#\pi$ and $\varphi(\#\pi)$.*

Proof. Let K be a finite 2-dimensional CW-complex with $\pi_1(K) \cong \pi$. Then π acts freely on the cellular chain complex of $\tilde{K}$. Consequently, we obtain an exact sequence

$$\mathcal{C}_2(\tilde{K}) \to \mathcal{C}_1(\tilde{K}) \to \mathcal{C}_0(\tilde{K}) \to \mathbb{Z} \to 0.$$

By choosing a resolution for $\mathcal{Z}_2(\tilde{K})$, the cycles of $\mathcal{C}_2(\tilde{K})$, we can extend this to an exact sequence of the form

$$0 \to A \to W_{dq-1} \to W_{dq-2} \to \cdots \to W_1 \to W_0 \to \mathbb{Z} \to 0,$$

with $W_2 \to W_1 \to W_0 \to \mathbb{Z} \to 0$ being the same as in the original sequence.

Now, by Theorem 3.5, π has a periodic free resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$ of period dq. Therefore $A \sim \mathbb{Z}$ over $\mathbb{Z}\pi$ by Lemma 4.1. Applying Lemma 4.2 yields a periodic resolution

$$0 \to \mathbb{Z} \to W'_{dq-1} \to W'_{dq-2} \to W_{dq-3} \to W_{dq-4} \to \cdots \to W_1 \to W_0 \to \mathbb{Z} \to 0.$$

By a slight abuse of notation, set $W_{dq-1} := W'_{dq-1}$ and $W_{dq-2} := W'_{dq-2}$.

We will now show that for $m = 2, 3, \ldots, dq-1$ there exists a simply-connected m-dimensional free π-complex X^m whose cellular chain complex (over $\mathbb{Z}\pi$) has the form

$$W_m \to W_{m-1} \to \cdots \to W_2 \to W_1 \to W_0.$$

If $m = 2$, the complex $\tilde{K}$ will do for X^2. If X^{m-1} has been found, applying Milnor's construction gives the desired X^m. Clearly, the complex X^{dq-1} has required properties.

Since a homology sphere is homotopy equivalent to an honest sphere, the conlusion follows. □

Remark 4.5. (1) Swan also proved a 'mod $\mathcal{C}$' version of the above theorem. See [18, Theorem A'] for details.

(2) One can always produce a finite-dimensional CW-complex X such that $\pi_1(X) \cong \pi$ and $\tilde{X} \simeq S^{q-1}$.

4.3 Milnor's $2p$-condition

There is (at least) one more question that begs to be asked: does a finite periodic group act freely on a sphere? The answer is no.

Theorem 4.6 ([15, Corollary 1])**.** *If a group G acts freely on a sphere then every involution in G is central.*

Proof. Let $g \in G$ be an involution. Suppose $gh \neq hg$ for an $h \in G$. Then $(gh)x \neq (hg)x$ for every $x \in S^n$. This, however, would imply that h has even degree by [15, Theorem 1]. □

Example 4.7. Even though $\mathcal{S}_3$ has period 4 by [3, Lemma 6.3], it cannot act freely on any sphere. In fact, all but certain groups of type I satisfy Milnor's condition (see [3, Chapter IV, Remark 6.16]).

5 The Madsen–Thomas–Wall result

We will need a refinement of Theorem 4.4. This is, in fact, the key idea of the original proof by Madsen, Thomas and Wall.

Theorem 5.1 ([13, Lemma 2.1]). *X in Theorem 4.4 can be chosen to satisfy the following additional condition: for any subgroup $\rho \subseteq \pi$ which can act freely and orthogonally on a sphere, the covering space $X(\rho)$ of X corresponding to the subgroup ρ is homotopy equivalent to a closed manifold.*

Before we move on to the proof of this assertion, we need a slight generalization of the notion of a Swan complex. A CW-complex X, dominated by a finite CW-complex, is called a *(π, n)-polarised complex* if we are given an isomorphism $\pi_1(X) \to \pi$ and a homotopy equivalence $\tilde{X} \to S^{n-1}$, $n \geq 3$. One easily sees that a (π, n)-polarised complex is a Poincaré complex.

Lemma 5.2 ([19, Theorem 2.2]). *If π is a finite periodic group with cohomological period q, the equivalence classes of (π, q)-polarised complexes correspond bijectively to generators of $\hat{H}^q(\pi; \mathbb{Z})$.*

We can now move on to:

Proof of Theorem 5.1. By Theorem 4.4, there exists a finite (π, m)-polarised complex X', corresponding to, say, $h \in H^m(\pi; \mathbb{Z})$. Let r denote the exponent of the multiplicative group of units of $\mathbb{Z}_{\#\pi}$. Then join of r copies of $\tilde{X}'$ is homotopy equivalent to S^{rm-1}, and we inherit a free action of π on it. Write $g = h^r$, $n = rm$ and X for the corresponding orbit space.

Since π acts cellularly on the join, X is a finite CW-complex. Note that any generator of $H^m(\pi; \mathbb{Z})$ is of the form uh, u being a unit of $\mathbb{Z}_{\#\pi}$, and

$$(uh)^r = u^r h^r = h^r = g.$$

Thus g is the only generator of $H^n(\pi; \mathbb{Z})$ which is an r-th power. Exactly the same argument shows that the restriction of g to τ is the only generator of $H^n(\tau; \mathbb{Z})$ which is an r-th power.

Let $\rho \subseteq \pi$ be such that it can act freely and orthogonally on a sphere, with the action arising from a representation, say, χ. The direct sum of r copies of χ corresponds to the join construction described above. The action of ρ on a sphere is smooth, so has the orbit space a smooth manifold. □

Proposition 5.3 ([13, Lemma 3.2]). *Let π be a finite soluble periodic group and $\tau \subseteq \pi$ be the subgroup of Proposition 3.4. Let X be a finite Swan complex chosen as in Theorem 5.1. Then any (topological) normal invariant for $X(\tau)$ extends to one for X.*

Let p be a prime number. Recall that a group π is called *p-hyperelementary*, if it is given by a split extension

$$0 \to \mathbb{Z}_m \to \pi \to \mathcal{P} \to 1,$$

where m is not divisible by p and $\mathcal{P}$ is a p-group.

Theorem 5.4 ([13, Theorem 4.1]). *Let M be a closed manifold, Y a finite Poincaré complex of formal dimension at least 5 with a finite fundamental group π and $\Phi\colon M \to Y$ a normal map of degree 1. Then surgery on Φ to obtain a homotopy equivalence is possible if and only if:*

(1) *For each 2-hyperelementary subgroup $\rho \subseteq \pi$, the covering space $Y(\rho)$ is homotopy equivalent to a manifold.*

(2) *Surgery is possible for the covering normal map $\tilde{\Phi}\colon X(\pi_2) \to Y(\pi_2)$, π_2 being the Sylow 2-subgroup of π.*

Finally, we arrive at:

Corollary 5.5 ([13, Theorem 0.5]). *A finite group acts freely on a sphere if and only if it is periodic and has at most one element of order 2.*

Proof. Suppose π is soluble; let $\tau \subseteq \pi$ be the subgroup of Proposition 3.4. Choose X as in Theorem 5.1. By the choice of X and τ, the covering space $X(\tau)$ of X corresponding to the subgroup τ is homotopy equivalent to a closed manifold, say M. By Proposition 5.3, the normal invariant defined by M extends to a normal invariant for X. We will now verify the hypotheses of Theorem 5.4.

(1) Any 2-hyperelementary subgroup $\rho \subseteq \pi$ is soluble (because π is). Since every odd-order subgroup of ρ is cyclic and, by our main hypothesis, every subgroup of ρ of order $2p$ is also cyclic, ρ satisfies all pq-conditions. It follows that ρ acts freely and orthogonally on a sphere (see Example 2.10). Consequently, $X(\rho)$ is homotopy equivalent to a manifold.

(2) Surgery on the covering map corresponding to τ yields a homotopy equivalence $M \to X(\tau)$ by construction. Since τ contains a Sylow 2-subgroup π_2, the same is true for the covering corresponding to π_2.

The conclusion now follows from Theorem 5.4. (The proof goes along the same lines for the nonsoluble case, although one needs somewhat more delicate argument in place of Proposition 3.4. See [13, Lemma 3.3].) □

Remark 5.6. The above reasoning can be refined in a way which guarantees that obtained actions are actually smooth with respect to the standard differentiable structure. See [14] for details.

6 Products of spheres

Consider the following extension of the original problem:

Suppose X is a finite CW*-complex homotopy equivalent to a product of spheres and π is a finite group acting freely and cellularly on X. What restrictions does this condition impose on π? Precisely which groups can behave in this way?*

Here the situation is much more complicated: there seems to be a consensus among the experts in the subject that a complete solution is out of reach at this point. What follows is a selection of famous theorems and conjectures.

6.1 The Benson-Carlson conjecture

For a finite group π and a prime number p define:

$$\begin{aligned} r_p(\pi) &= \max\{n \mid (\mathbb{Z}_p)^n \subseteq \pi\}, \\ r(\pi) &= \max_{p \mid \#G} r_p(\pi), \\ h(\pi) &= \min\left\{k \mid \pi \text{ acts freely on a finite CW-complex } X \simeq \prod_{i=1}^{k} S^{n_i}\right\}. \end{aligned}$$

Remark 6.1. $h(\pi)$ is well-defined: every finite group acts freely on a product of spheres. The original source for this statement is [16, p. 547]. It was also explained in the author's previous term paper; see Theorem A.1.

Conjecture 6.2 ([5, Conjecture 5.2])**.** *For every finite group π, $r(\pi) = h(\pi)$.*

Remark 6.3. Theorems: 3.6 and 4.4 imply this result for rank-one groups.

A version of this conjecture has been extensively studied especially in the case of elementary abelian p-groups. Consider the following question:

If $(\mathbb{Z}_p)^r$ acts freely on $\prod_{i=1}^{k} S^{n_i}$, is it true that $r \leq k$?

The current state of this problem is as follows:

- Heller [10] answered the question affirmatively for $k = 2$.
- Adem–Browder [1], building on the previous work by Carlsson, settled affirmatively the equidimensional case ($n_1 = n_2 = \cdots = n_k$), except for $p = 2$ and the dimension of the spheres equal to 1, 3 or 7.
- Yalçin [22] settled the case of $\mathbb{Z}_2$-actions on $(S^1)^k$.

Other cases remain intriguingly open.

6.2 Rank-two p-group actions

On the other hand, the problem amounts to constructing actions. This was virtually untouched territory until recently.

Theorem 6.4 ([4, Theorem 3.2]). *Let π be a finite group. Suppose X is a simply-connected finite π-complex with all isotropy subgroups periodic. Then there exists a finite free π-complex Y such that Y is homotopy equivalent to $S^n \times X$.*

Since a finite rank-two p-group acts on a sphere with rank-one isotropy subgroups by [4, Corollary 4.10], this yields:

Theorem 6.5 ([4, Theorem 4.12]). *A finite p-group π acts freely on a finite* CW-*complex homotopy equivalent to $S^m \times S^n$ if and only if $r(\pi) \leq 2$.*

Remark 6.6. A lot of work has been put into promoting this result to actions on actual products of spheres. The first major step towards this was the paper of Adem–Davis–Ünlü [2], where it was obtained for primes greater than 3. Recently Ünlü–Yalçin [20] announced the result in full generality.

Perhaps the most interesting fact is that – in great constrast with the original problem – thus far there are known no restrictions imposed by geometry of the product, no analogues of Milnor's condition.

Acknowledgements. Most of this work has been carried out under supervision of Professor K. Pawałowski during the author's residency at the Faculty of Mathematics and Computer Science of the Adam Mickiewicz University.

The author is supported by the joint PhD programme 'Środowiskowe Studia Doktoranckie z Nauk Matematycznych'.

References

[1] A. ADEM and W. BROWDER, The free rank of symmetry of $(S^n)^k$, *Invent. Math.* 92 (1988), 431–440.

[2] A. ADEM, J. DAVIS, and O. UNLU, Fixity and free group actions on products of spheres, *Comment. Math. Helv.* 79 (2004), 758–778.

[3] A. ADEM and R. J. MILGRAM, *Cohomology of Finite Groups*, Springer-Verlag Grundlehren 309, Springer-Verlag, Berlin, 1994.

[4] A. ADEM and J. H. SMITH, Periodic complexes and group actions, *Ann. of Math.* 154 (2001), 407–435.

[5] D. BENSON and J. CARLSON, Complexity and Multiple Complexes, *Math. Zeitschrift* 195 (1987), 221–238.

[6] K. S. BROWN, *Cohomology of Groups*, Graduate Texts in Mathematics 87, Springer-Verlag, New York, 1982.

[7] J. DAVIS and R. J. MILGRAM, *A Survey of the Spherical Space Form Problem*, Mathematical Reports 2, Harwood Academic Publishers, 1984.

[8] M. GOLASIŃSKI and D. L. GONCALVES, Spherical space forms – a numerical bound for number of homotopy types, *Hiroshima Math. J.* 31 (2001), 107–116.

[9] A. HATCHER, *Algebraic Topology*, Cambridge University Press, 2001.

[10] A. HELLER, A note on spaces with operators, *Illinois J. Math* 3 (1959), 98–100.

[11] H. HOPF, Zum Clifford-Kleinschen Raumproblem, *Math. Ann.* 95 (1926), 313–329.

[12] D. MACPHERSON, Permutation groups whose subgroups have just finitely many orbits, in: *Ordered Groups and Infinite Permutation Groups*, Mathematics and Its Applications, Kluwer Academic Publishers, 1996, pp. 221–230.

[13] I. MADSEN, C. B. THOMAS, and C. T. C. WALL, The topological spherical space form problem II: Existence of free actions, *Topology* 15 (1976), 375–382.

[14] I. MADSEN, C. B. THOMAS, and C. T. C. WALL, The topological spherical space form problem III: Dimensional bounds and smoothing, *Pacific J. Math.* 106 (1983), 135–143.

[15] J. MILNOR, Groups which act on S^n without fixed points, *Amer. J. Math.* 79 (1957), 623–630.

[16] R. OLIVER, Free compact group actions on products of spheres, in: *Algebraic Topology: Aarhus, Denmark 1978*, Lecture Notes in Mathematics 763, Springer-Verlag, Berlin, 1979, pp. 539–548.

[17] A. RANICKI, *Algebraic and Geometric Surgery*, Oxford Mathematical Monographs, Oxford University Press, London, 2002.

[18] R. G. SWAN, Periodic resolutions for finite groups, *Ann. of Math.* 72 (1960), 267–291.

[19] C. B. THOMAS and C. T. C. WALL, The topological spherical space form problem I, *Compositio Math.* 23 (1971), 101–114.

[20] O. UNLU and E. YALCIN, Fusion systems and constructing free actions on products of spheres, *preprint*.

[21] J. WOLF, *Spaces of Constant Curvature*, Publish or Perish, 1984.

[22] E. YALCIN, Group actions and group extensions, *Trans. Amer. Math. Soc.* 352 (200), 2689–2700.

[23] A. V. ZARELUA, On finite groups of transformations, in: *Proceedings of the International Symposium on Topology and its Applications,* Herzeg-Novi, 1968, pp. 334–339.

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