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Properly discontinuous actions on homotopy surfaces

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Term paper #3: Properly discontinuous actions on homotopy surfaces*

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Abstract

Fujii has observed that a finite group G acts freely on a surface M other than the 2-sphere or the projective plane if and only if there exists a group extension

$$1 \to \pi_1(M) \to \pi_1(N) \to G \to 1,$$

where N is a surface such that $\chi(M) = |G|\chi(N)$. We discuss an extension of this result to the case of properly discontinuous and cellular actions of arbitrary groups on homotopy surfaces, with the key ingredient in our approach being the notion of cohomological dimension of a group.

Among applications, we prove that the family of finite groups which act freely on a surface M is precisely the same as the family of finite groups which act freely and cellularly on homotopy M's. We also deduce that a torsionfree group with infinite cohomological dimension cannot act properly discontinuously and cellularly on any homotopy surface, with four possible exceptions.

Special attention is given to M_2, the orientable surface of genus 2. In particular, we provide a complete list of groups which act properly discontinuously on $M_2 \times \mathbb{R}$.

1 Introduction

1.1. Given a closed manifold M, a question that is often of interest is the following: which finite groups act freely on M? One can also be inclined to consider its 'homotopic' counterpart: which finite groups act freely and cellularly on finite CW-complexes homotopy equivalent to M? Looking simultaneously at both versions of the question proved to be very fruitful; the solution of the spherical space form problem springs to mind (see, for example, [5]).

Having settled the issue for finite groups, a more general problem emerges, namely that of providing a description of groups which act properly discontinuously on $M \times \mathbb{R}^m$, or, more generally, on finite-dimensional CW-complexes homotopy equivalent to M. This sort of question has been studied especially extensively in the case when M is a sphere (see [4], [10], [11], [17], [18]). Since

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spheres are the simplest manifolds homologically, it seems only natural to consider a similar problem for the nicest possible spaces in the sense of homotopy, i.e., for aspherical manifolds. With this idea in mind, we investigate properly discontinuous and cellular transformation groups of finite-dimensional CW-complexes homotopy equivalent to a surface M, for M other than the 2-sphere or the projective plane. Considering that the homotopy type of an aspherical CW-complex is determined by its fundamental group, it is perhaps not surprising that this problem can be reduced to a question about the existence of a specific group extension.

1.2. Recall the following theorem due to Fujii.

Theorem 1.1 ([8, Theorem 1.6]). *A finite group G acts freely on a surface M other than the 2-sphere S^2 or the projective plane $\mathbb{R}P^2$ if and only if there exists a group extension*

$$1 \to \pi_1(M) \to \pi_1(N) \to G \to 1,$$

where N is a surface such that $\chi(M) = |G|\chi(N)$.

This result gives a reasonably efficient method of determining which finite groups act freely a surface M, especially if M is of low genus. For example, Fujii has been able to show that a finite group acts freely on the Klein bottle if and only if it cyclic of order either $2m+1$ or $4m+2$, $m \geq 0$ ([8, Theorem 1.8]).

We discuss a generalization of this result to the case of properly discontinuous and cellular actions of arbitrary groups on homotopy surfaces, with the key ingredient in our approach being the notion of cohomological dimension for groups (Proposition 3.1). This turns out to have some interesting consequences. It allows us to show that the family of finite groups which act freely on a surface M is precisely the same as the family of finite groups which act freely and cellularly on homotopy M's (Corollary 3.4). We also derive a lower bound on the dimension of a homotopy surface which admits a properly discontinuous and cellular action of a group with finite virtual cohomological dimension; this extends results obtained by Golasiński–Gonçalves–Jimenéz [11, Proposition 3.4] and Lee [15, Theorem 5.2] (Theorem 4.2). As another application, we prove that a torsionfree group with infinite cohomological dimension cannot act properly discontinuously and cellularly on any homotopy M, with possible exceptions when M is either the 2-sphere, the torus, the projective plane, or the Klein bottle (Theorem 5.2).

One notable feature of our approach is that it makes the task of showing that certain groups act properly discontinuously and cellularly on homotopy surfaces particularly easy. Most of Section 6 is dedicated to exhibiting this sentiment. For example, we show that the general linear group $GL(n, \mathbb{Z})$ acts in such a way on homotopy orientable surfaces of genus $(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1}) + 1$, where $p \geq 3$ is a prime number (Corollary 6.2).

Special attention is given to M_2, the orientable surface of genus 2. In particular, we determine all groups that act properly discontinuously on $M_2 \times \mathbb{R}$ (Proposition 7.1). In fact, there are only four of them: the cyclic group of order 2, the infinite cyclic group, the direct product $\mathbb{Z} \oplus \mathbb{Z}_2$ and the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$.

1.3. Throughout this paper, the term *surface* is taken to mean a closed and connected 2-dimensional manifold.

Since there does not seem to be a generally accepted definition of the term ‘properly discontinuous action’, we shall state explicitly what we mean by this. An action $G \times X \to X$ is said to be *properly discontinuous* if for any $x \in X$ there exists its neighbourhood U such that $gU \cap U = \emptyset$ for any $g \in G$, $g \neq 1$. The bottom line is that we want the natural projection $X \to X/G$ to be a covering.

Apart from these, we employ the standard notation.

1.4. Quite a few results presented during the course of this paper can be promoted to higher-dimensional aspherical manifolds. The emphasis, however, is not on the greatest possible generality, but rather on specific calculations regarding homotopy surfaces.

2 Cohomological dimension

2.1. Since our approach relies heavily on the notion and properties of cohomological dimension, we will take some time to recall the related facts.

The *cohomological dimension* of a group Γ, written $\operatorname{cd}\Gamma$, is defined to be the projective dimension of $\mathbb{Z}$ over the integral group ring $\mathbb{Z}\Gamma$. It is not hard to see that

$$\begin{aligned} \operatorname{cd}\Gamma &= \inf\{n \mid H^i(\Gamma, -) = 0 \text{ for } i > n\} \\ &= \sup\{n \mid H^n(\Gamma, M) \neq 0 \text{ for a } \Gamma\text{-module } M\}, \end{aligned}$$

where $H^*(\Gamma, -)$ stands for the cohomology group of Γ.

The topological analogue of $\operatorname{cd}\Gamma$, the *geometric dimension* of Γ ($\operatorname{gd}\Gamma$), is defined to be the minimal dimension of a classifying space for Γ. By results of Eilenberg–Ganea [6], Stallings [20] and Swan [21], $\operatorname{cd}\Gamma = \operatorname{gd}\Gamma$ whenever $\operatorname{cd}\Gamma \neq 2$. In the case when $\operatorname{cd}\Gamma = 2$, it is still true that $\operatorname{gd}\Gamma \leq 3$.

It is well-known that a group Γ with torsion necessarily has $\operatorname{cd}\Gamma = \infty$. Because of this the following notion has been introduced: a group Γ is said to have finite *virtual cohomological dimension* if there exists a subgroup $\Gamma' \subseteq \Gamma$ with $\operatorname{cd}\Gamma' < \infty$ and of finite index; we then set $\operatorname{vcd}\Gamma = \operatorname{cd}\Gamma'$. By Serre's theorem [3, Chapter VII, Theorem 3.1], $\operatorname{vcd}\Gamma$ does not depend on the choice of Γ'. Observe that if $\operatorname{vcd}\Gamma < \infty$, we may choose $\Gamma' \subseteq \Gamma$ to be normal; it suffices to consider the normal core of any such Γ'.

We will repeatedly make use of the following facts:

(1) *If Γ is a virtually torsionfree group and $\Gamma' \subseteq \Gamma$ is any subgroup, then $\operatorname{vcd}\Gamma' \leq \operatorname{vcd}\Gamma$, with equality if the index $[\Gamma : \Gamma']$ is finite.*

(2) *Given a group extension $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ with $\operatorname{cd}\Gamma' < \infty$ and $\operatorname{vcd}\Gamma'' < \infty$, the inequality $\operatorname{vcd}\Gamma \leq \operatorname{cd}\Gamma' + \operatorname{vcd}\Gamma''$ holds.*

Both of these properties are best proved by looking at the torsionfree case first: the former then follows from the fact that a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ can be regarded as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma'$, the latter is a straightforward consequence of the Lyndon–Hochschild–Serre spectral sequence.

2.2. Recall that any group extension $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ gives rise to a homomorphism $\Gamma'' \to \operatorname{Out}(\Gamma')$, where Out stands for the outer automorphism group; consult [3, Chapter IV, Section 6] for details. What we need is a sort of converse:

Proposition 2.1 ([3, Chapter IV, Corollary 6.8]). *If Γ' has trivial center then there is exactly one (up to equivalence) extension of Γ'' by Γ' corresponding to any homomorphism $\varphi\colon \Gamma'' \to \mathrm{Out}(\Gamma')$.*

In the above situation, the unique extension of Γ'' by Γ' can be described as the fibered product of the diagram

$$\begin{array}{ccc} & & \Gamma'' \\ & & \big\downarrow \varphi \\ \mathrm{Aut}(\Gamma') & \longrightarrow & \mathrm{Out}(\Gamma'). \end{array}$$

3 A generalization of Fujii's theorem

Let M^n be an n-dimensional manifold. A *homotopy manifold* of type M^n (or a *homotopy* M^n) is a finite-dimensional CW-complex X homotopy equivalent to M^n. Whenever $n = 2$ and M is closed and connected, we say that X is a *homotopy surface*.

Our first goal is to obtain a version of Theorem 1.1 for homotopy surfaces; this is achieved in Proposition 3.1 below. As we have already mentioned, the key ingredient in our approach is the notion of cohomological dimension.

In what follows, the fundamental group of a surface is frequently called a *surface group*.

Proposition 3.1. (1) *A group G acts properly discontinuously and cellularly on a homotopy surface X of type $M \neq S^2$, $\mathbb{R}P^2$ if and only if there exists a group extension*

$$1 \to \pi_1(M) \to \Gamma \to G \to 1,$$

where Γ is a group with $\operatorname{cd}\Gamma < \infty$. In this case, the orbit space X/G is a finite-dimensional classifying space for the group Γ.

(2) *In particular, a finite group G acts freely and cellularly on a homotopy surface of type $M \neq S^2$, $\mathbb{R}P^2$ if and only if there exists a group extension*

$$1 \to \pi_1(M) \to \pi_1(N) \to G \to 1,$$

where N is a surface such that $\chi(M) = |G|\chi(N)$.

Proof. (1) Suppose a group G acts properly discontinuously and cellularly on a homotopy surface X of type $M \neq S^2$, $\mathbb{R}P^2$. Consider the fibration $G \to X \to X/G$; inspection of its long exact sequence of homotopy groups reveals that the orbit space X/G is aspherical and that $\pi_1(X/G)$ fits into the short exact sequence

$$1 \to \pi_1(M) \to \pi_1(X/G) \to G \to 1.$$

Moreover, since X/G is a finite-dimensional classifying space for $\pi_1(X/G)$, it follows that $\operatorname{cd}\pi_1(X/G) < \infty$.

Conversely, consider a group extension

$$1 \to \pi_1(M) \to \Gamma \to G \to 1,$$

where Γ is a group with $\operatorname{cd}\Gamma < \infty$. Let X be a finite-dimensional classifying space for Γ and let $\tilde{X}$ be the covering space of X corresponding to the subgroup $\pi_1(M) \subseteq \Gamma$. Then $G \cong \Gamma/\pi_1(M)$ acts properly discontinuously and cellularly on $\tilde{X}$. Since $\tilde{X}$ is a finite-dimensional aspherical CW-complex with $\pi_1(\tilde{X}) \cong \pi_1(M)$, we have $\tilde{X} \simeq M$ and the conclusion follows.

(2) Proceed along the same lines as above. For the 'only if' part, recall that a group with finite cd is torsionfree. By a theorem of Zieschang [23], [24], a torsionfree extension of a finite group by a surface group is again a surface group, hence $\pi_1(X/G) \cong \pi_1(N)$ for a surface N. The asserted equality is a consequence of the general behaviour of χ; see [3, Chapter IX, Section 6]. □

Remark 3.2. (1) The above proof in fact yields one more piece of information: if $\operatorname{gd}\Gamma = n$, then G acts properly discontinuously and cellularly on an n-dimensional homotopy surface of type M.

(2) Note that the assumption of cellularity of the action is imposed in order to ensure that the orbit space is a finite-dimensional CW-complex. In the case when the homotopy surface in question is a manifold itself, the orbit space under a properly discontinuous action is again a manifold, thus cellularity can be neglected. More often than not we will state a result only in the cellular version, understanding that it can be carried over to the geometric case.

Example 3.3. Given a surface $M \neq S^2, \mathbb{R}P^2$ and a group G with $\operatorname{cd} G < \infty$, there exists a homotopy surface of type M which admits a properly discontinuous and cellular G-action. To see this, set $\Gamma = \pi_1(M) \times G$ in Proposition 3.1; since both $\pi_1(M)$ and G have finite cohomological dimension, this is also the case for Γ.

The geometric counterpart of this phenomenon can be obtained by taking the product $EG \times M$, EG being the universal cover of a finite-dimensional classifying space for G, with the usual G-action on the first summand and the trivial one on the second.

Let us record some immediate consequences of Proposition 3.1.

Corollary 3.4. (1) *A finite group acts freely and cellularly on a homotopy surface of type M if and only if it acts freely on M.*

(2) *A finite group acts freely on a surface M if and only if it acts freely and cellularly on M.*

Proof. (1) Combine Theorem 1.1 with the second part of Proposition 3.1.

(2) The 'if' part is trivial. For the 'only if' part, apply Theorem 1.1 to obtain an appropriate group extension. Now proceed as in the proof of Proposition 3.1, with the following improvement: take X to be the surface N itself. Then $\tilde{X}$ is again a surface and, since surfaces are classified by their fundamental groups up to homeomorphism, $\tilde{X} \approx M$. □

Recall that a group Γ is said to be of *type FL* if $\mathbb{Z}$ admits a finite free resolution over $\mathbb{Z}\Gamma$. If, additionally, Γ is finitely presented, this property amounts to the existence of a finite classifying space for Γ ([3, Chapter VIII, Theorem 7.1]).

Corollary 3.5. *Suppose a group G acts properly discontinuously and cellularly on a homotopy surface X.*

(1) *The orbit space X/G is a homotopy surface if and only if G is finite.*

(2) *If X/G has the homotopy type of a finite CW-complex, then G is finitely generated. Conversely, if G is finitely presented and of type FL, then X/G has the homotopy type of a finite CW-complex.*

Proof. (1) This is a straightforward consequence of Proposition 3.1 and the subgroup structure of surface groups: any subgroup of finite index of a surface group is again a surface group, while any subgroup of infinite index is free. See, for example, [1, Theorem 2.1].

(2) If X/G has the homotopy type of a finite CW-complex, then $\pi_1(X/G)$ is finitely generated. Due to the existence of an epimorphism $\pi_1(X/G) \to G$, so is G.

The key observation for the converse is that $\pi_1(X/G)$ is finitely presented and of type FL. This holds because both $\pi_1(M)$ and G are so, and these two properties are well-known to be closed under extension. By the preceding remark, X/G has the homotopy type of a finite CW-complex. □

4 The dimension of a homotopy surface

A natural question to consider is the following: assuming a group G acts properly discontinuously and cellularly on a homotopy surface X of a given type, what is the minimal dimension of X as a CW-complex? Clearly, $\dim X \geq 2$, but it is reasonable to expect a better bound in terms of cohomological dimension of G, vide [11, Proposition 3.4] or [15, Theorem 5.2]. This is indeed the case, at least for homotopy surfaces of an orientable type, and we now set out to make this assertion precise.

Proposition 4.1. *Let $M = M^n$ be an aspherical, closed, orientable manifold and G a group with $\operatorname{vcd} G = m < \infty$. Given an extension*

$$1 \to \pi_1(M) \to \Gamma \to G \to 1,$$

the equality $\operatorname{vcd}\Gamma = \operatorname{cd}\pi_1(M) + \operatorname{vcd} G$ holds.

Proof. We can assume without loss of generality that G has finite cohomological dimension, for otherwise we can pass to its torsionfree subgroup of finite index.

Write π for $\pi_1(M)$. Choose a G-module A such that $H^m(G, A) \neq 0$ and consider the Lyndon–Hochschild–Serre spectral sequence of the above extension:

$$E_2^{p,q} \cong H^p\big(G, H^q(\pi, A)\big) \Rightarrow H^{p+q}(\Gamma, A).$$

By the usual spectral sequence argument, $H^{m+n}(\Gamma, A) \cong H^m\big(G, H^n(\pi, A)\big)$. The Universal Coefficient Theorem gives an extension of G-modules

$$0 \to \operatorname{Ext}\big(H_{n-1}(\pi, \mathbb{Z}), A\big) \to H^n(\pi, A) \to \operatorname{Hom}\big(H_n(\pi, \mathbb{Z}), A\big) \to 0.$$

Now apply the associated long exact sequence of cohomology groups to obtain a surjection $H^m\big(G, H^n(\pi, A)\big) \to H^m(G, A)$. (We can always assume the induced action $G \to \operatorname{Aut}\big(H_n(\pi, \mathbb{Z})\big)$ is trivial, hence there is an isomorphism $\operatorname{Hom}\big(H_n(\pi, \mathbb{Z}), A\big) \cong A$ of G-modules.) Consequently, $H^{m+n}(\Gamma, A) \neq 0$. □

We are now ready to consider an extension of both [11, Proposition 3.4] and [15, Theorem 5.2].

Theorem 4.2. *Let $M = M^n$ be an aspherical, closed and orientable manifold and G a group with $\operatorname{vcd} G < \infty$.*

(1) *If G acts properly discontinuously and cellularly on a homotopy manifold X of type M, then $\dim X \geq \operatorname{vcd} G + n$.*

(2) *If G acts properly discontinuously on $M \times \mathbb{R}^m$, then $\operatorname{vcd} G \leq m$, with equality if and only if $M \times \mathbb{R}^m/G$ is closed.*

Proof. (1) Just as in the proof of Proposition 3.1, the orbit space X/G is a finite-dimensional aspherical CW-complex and its fundamental group is given by the extension

$$1 \to \pi_1(M) \to \pi_1(X/G) \to G \to 1.$$

By Proposition 4.1, $\operatorname{cd} \pi_1(X/G) = \operatorname{cd} \pi_1(M) + \operatorname{cd} G$. Furthermore,

$$\dim X = \dim X/G \geq \operatorname{gd} \pi_1(X/G) \geq \operatorname{cd} \pi_1(X/G),$$

and the conclusion follows.

(2) The above proof works equally well for the first part of (2), only that now $M \times \mathbb{R}^m/G$ is an aspherical manifold and dim stands for the dimension of a space as a manifold.

For the second part, recall that whenever a space Y is simultaneously a classifying space for a group Γ and a d-dimensional manifold, then $\operatorname{cd} \Gamma \leq d$, with equality if and only if Y is closed ([3, Chapter VIII, Proposition 8.1]). This and a similar string of (in)equalities as above imply the claimed result. □

Clearly, $\operatorname{vcd} \Gamma = 0$ if and only if Γ is a finite group. By results of Stallings [20] and Swan [21], $\operatorname{vcd} \Gamma = 1$ if and only if Γ is a virtually free group. Consequently, we have:

Corollary 4.3. *Suppose a group G with $\operatorname{vcd} G < \infty$ acts properly discontinuously and cellularly on a homotopy surface X of an orientable type.*

(1) *If $\dim X = 2$, then G is finite.*

(2) *If $\dim X = 3$, then G is either finite or virtually free.*

5 The non-existence of actions of torsionfree groups with infinite cd

We now turn attention to the problem of determining which groups act properly discontinuously and cellularly on homotopy surfaces. Example 3.3 shows that we need not worry about groups with finite cohomological dimension, as this case is trivial in the sense that 'everything acts on everything'. This leaves us in the realm of groups with infinite cohomological dimension, and the first question that springs to mind is whether a torsionfree group with infinite cohomological dimension can act properly discontinuously and cellularly on a homotopy surface. We answer this question in the negative in Theorem 5.2, albeit with four possible exceptions.

Proposition 5.1. *The outer automorphism group of the fundamental group of any surface has finite virtual cohomological dimension.*

Proof. It is well-known that the outer automorphism group of a surface group is the extended mapping class group of that surface. (For orientable surfaces, this is the celebrated Dehn–Nielsen–Baer Theorem.)

(Orientable surfaces) Harer [13, Theorem 4.1] has shown that the mapping class group of an orientable surface has finite virtual cohomological dimension. Since the mapping class group of an orientable surface M is an index 2 subgroup of the extended mapping class group of M, the conclusion follows.

(Nonorientable surfaces) A result of Hope–Tillmann [14, Lemma 2.1] states that the mapping class group of a nonorientable surface of genus g is a subgroup of the mapping class group of an orientable surface of genus $g-1$, which gives the desired upper bound. □

Theorem 5.2. *Let M be a surface other than the 2-sphere, the torus, the projective plane, or the Klein bottle.*

A torsionfree group with infinite cohomological dimension cannot act properly discontinuously and cellularly on any homotopy surface of type M.

Proof. Suppose a torsionfree group G with $\operatorname{cd} G = \infty$ acts properly discontinuously and cellularly on a homotopy surface X of type M. Similarly as in the proof of Proposition 3.1, the fibration $G \to X \to X/G$ yields the extension

$$1 \to \pi_1(M) \to \Gamma \to G \to 1,$$

which in turn gives rise to a homomorphism $\varphi\colon G \to \operatorname{Out}\bigl(\pi_1(M)\bigr)$. Since the center of $\pi_1(M)$ is trivial by a result of Griffiths [12, Theorem 4.4], it follows from Proposition 2.1 that Γ is uniquely determined by φ. More precisely, there is an isomorphism

$$\Gamma \cong \operatorname{Aut}\bigl(\pi_1(M)\bigr) \times_{\operatorname{Out}(\pi_1(M))} G = \bigl\{(\sigma, g) \in \operatorname{Aut}\bigl(\pi_1(M)\bigr) \times G \bigm| \eta(\sigma) = \varphi(g)\bigr\},$$

where $\eta\colon \operatorname{Aut}\bigl(\pi_1(M)\bigr) \to \operatorname{Out}\bigl(\pi_1(M)\bigr)$ is the natural projection.

The extension

$$1 \to \ker\varphi \to G \to \operatorname{im}\varphi \to 1$$

gives the inequality $\operatorname{cd} G \leq \operatorname{cd}\ker\varphi + \operatorname{vcd}\operatorname{im}\varphi$. Since $\operatorname{vcd}\operatorname{im}\varphi < \infty$ by Proposition 5.1 and $\operatorname{cd} G = \infty$ by hypothesis, it follows that $\operatorname{cd}\ker\varphi = \infty$. To conclude the proof, observe that $\pi_1(M) \times \ker\varphi$ is a subgroup of Γ, hence $\operatorname{cd}\Gamma = \infty$. This yields a contradiction with Proposition 3.1. □

A careful examination of the proof of Theorem 5.2 allows one to claim:

Corollary 5.3. *If a finite group G acts freely on a homotopy surface of type M as above, then G is a subgroup of* $\operatorname{Out}\bigl(\pi_1(M)\bigr)$.

Proof. Similarly as above, we arrive at a homomorphism $\varphi\colon G \to \operatorname{Out}\bigl(\pi_1(M)\bigr)$ and an explicit description of Γ. If $\ker\varphi$ were not trivial, it would amount to torsion in Γ, thus making its cohomological dimension infinite. □

Example 5.4. It is certainly possible, in general, for a torsionfree group of infinite cohomological dimension to act properly discontinuously and cellularly on a finite-dimensional CW-complex.

(1) Let F denote a free group on a countably infinite number of generators. Then $\operatorname{cd} F = 1$ and, since the commutator subgroup $[F,F] \subseteq F$ is also free, $\operatorname{cd}[F,F] = 1$. Hence we obtain the extension

$$1 \to [F,F] \to F \to \bigoplus_{i=1}^{\infty} \mathbb{Z} \to 0,$$

which yields a properly discontinuous and cellular action of $\bigoplus_{i=1}^{\infty} \mathbb{Z}$ on a 1-dimensional CW-complex homotopy equivalent to a bouquet of certain number of circles.

(2) The above construction shows that, in fact, *any* group G acts properly discontinuously and cellularly on *some* 1-dimensional homotopy bouquet of circles in such a way that the orbit space is again a 1-dimensional homotopy bouquet of circles. Too see this, simply take any free resolution of G of length 1.

Remark 5.5. As $\mathbb{Z}^n \subseteq \bigoplus_{i=1}^{\infty} \mathbb{Z}$ for any $n \geq 0$, it follows from Theorem 4.2 that $\bigoplus_{i=1}^{\infty} \mathbb{Z}$ cannot act properly discontinuously and cellularly on a homotopy manifold of type M for M aspherical, closed and orientable. A spectral sequence argument similar to that of Proposition 4.1 shows that this also cannot happen when M is a nonorientable surface. There are, however, examples of torsionfree groups with infinite cohomological dimension and no infinitely generated free abelian subgroup.

6 Actions of groups with finite vcd

6.1. Having settled the issue for torsionfree groups with infinite cohomological dimension, we will now consider the behaviour of groups with torsion, especially those with finite virtual cohomological dimension. After series of very general results and examples, we concentrate on actions on homotopy surfaces of a specified type.

To begin with, consider the following generalization of the phenomenon described in Example 3.3.

Proposition 6.1. *Let G be a group with $\operatorname{vcd} G < \infty$ and $N \subseteq G$ a normal torsionfree subgroup of finite index. If G/N acts freely on a surface M, then G acts properly discontinuously and cellularly on a $\big([G:N]\operatorname{gd} G + 2\big)$-dimensional homotopy surface of type M.*

Proof. Using Corollary 3.4, we can assume without loss of generality that G/N acts freely and cellularly on M. Then G acts cellularly, although not properly discontinuously, on M via the homomorphism $G \to G/N \to \operatorname{Homeo}(M)$. On the other hand, by [3, Chapter VIII, Theorem 11.1], there exists a contractible and proper[1] G-complex X of dimension $[G:N]\operatorname{gd} N$. [Take X to be the space of N-equivariant maps $G \to EN$, endowed with the compact-open topology. One then easily sees that $X \approx (EN)^{[G:N]}$.] Clearly, X is a free N-complex. It is routine to verify that the action $G \times (X \times M) \to X \times M$ given by

$$g(x,m) = \big(gx, (gN)m\big)$$

[1]Recall that a G-complex X is said to be *proper* if the isotropy group G_x of any $x \in X$ is finite and any $x \in X$ has a neighbourhood U such that $gU \cap U = \emptyset$, $g \in G \setminus G_x$.

is properly discontinuous and cellular. □

Corollary 6.2. *The general linear group $GL(n,\mathbb{Z})$, $n \geq 1$, acts properly discontinunously and cellularly on a homotopy surface of type M_{g_p}, where M_{g_p} is the orientable surface of genus $g_p = (p^n-1)(p^n-p)\cdots(p^n-p^{n-1})+1$.*

Proof. Recall that the general linear group $GL(n,\mathbb{Z}_p)$, where $n \geq 1$ and $p \geq 3$ is a prime number, acts freely on the orientable surface of genus g_p. Indeed, by Waterhouse's [22], $GL(n,\mathbb{Z}_p)$ is generated by two elements, say A_1, A_2, thus there is a projection

$$\pi_1(M_2) = \big\langle x_1, x_2, y_1, y_2 \;\big|\; [x_1,y_1][x_2,y_2] = 1 \big\rangle \to GL(n,\mathbb{Z}_p)$$

given by $x_i \mapsto A_i$, $y_i \mapsto 1$, $i = 1, 2$. As already mentioned, the kernel of this map is the surface group $\pi_1(M_g)$ of the orientable surface M_g. The equality $\chi(M_g) = |GL(n,\mathbb{Z}_p)|\chi(M_2)$ gives $g = g_p$ and the conlusion follows.

It is well-known that $GL(n,\mathbb{Z})$ has finite virtual cohomological dimension. Furthermore, the kernel of the projection $GL(n,\mathbb{Z}) \to GL(n,\mathbb{Z}_p)$, the so called principal congruence subgroup, is torsionfree, hence the group $GL(n,\mathbb{Z})$ acts properly discontinuously and cellularly on a homotopy surface of type M_{g_p} by Proposition 6.1. □

Let G be a one-relator group with torsion. Then $\operatorname{vcd} G \leq 2$, as explained in [3, Chapter XIII, Section 11, Example 3]. Fischer–Karrass–Solitar have shown that a normal torsionfree subgroup $N \subseteq G$ of finite index is free if and only if G is the free product of a free group and a finite cyclic group ([7, Theorem 3]). As a consequence, we deduce:

Corollary 6.3. *Let F be a free group. If the cyclic group $\mathbb{Z}_n$ acts freely on a surface M, then the free product $F * \mathbb{Z}_n$ acts properly discontinuously and cellularly on an $(n+2)$-dimensional homotopy surface of type M.*

Proof. Write R for the generator of $\mathbb{Z}_n \subseteq F * \mathbb{Z}_n$. Consider the short exact sequence

$$1 \to N \to F * \mathbb{Z}_n \xrightarrow{p} \mathbb{Z}_n \to 0,$$

where p is the obvious projection. Since the elements of finite order in $F * \mathbb{Z}_n$ are conjugates of powers of R (see [16, Theorem 4.13]), it follows that N is torsionfree. The preceding discussion now implies that N is a free group, hence $\operatorname{gd} N = 1$. Applying Proposition 6.1 concludes the proof. □

We can offer the following improvement of Proposition 6.1 in the case when G is a semidirect product:

Proposition 6.4. *If a finite group H acts freely on a surface M other than the 2-sphere, the torus, the projective plane, or the Klein bottle, and N is a group with $\operatorname{cd} N < \infty$, then any semidirect product $N \rtimes H$ acts properly discontinuously and cellularly on a $(\operatorname{gd} N + 2)$-dimensional homotopy surface of type M.*

Proof. As usual, the free H-action on M gives rise to the group extension

$$1 \to \pi_1(M) \to \Gamma' \to H \to 1, \text{ where } \Gamma' \cong \operatorname{Aut}\big(\pi_1(M)\big) \times_{\operatorname{Out}(\pi_1(M))} H.$$

This in turn induces a homomorphism $\varphi\colon H \to \operatorname{Out}\big(\pi_1(M)\big)$.

Define $\psi\colon N \rtimes H \to \mathrm{Out}\big(\pi_1(M)\big)$ by $\psi(n,h) = \varphi(h)$ for any $(n,h) \in N \rtimes H$. By Proposition 2.1, ψ gives rise to the group extension

$$1 \to \pi_1(M) \to \Gamma \to N \rtimes H \to 1, \text{ with } \Gamma \cong \mathrm{Aut}\big(\pi_1(M)\big) \times_{\mathrm{Out}(\pi_1(M))} (N \rtimes H).$$

In order to conclude the proof it is enough to show that Γ is torsionfree, for in this case $\mathrm{cd}\,\Gamma < \infty$ and Proposition 3.1 is applicable. This last assertion, however, is a straightforward consequence of the observation that $\big(\sigma,(n,h)\big) \in \Gamma$ if and only if $(\sigma,h) \in \Gamma'$. □

Remark 6.5. Consult [8], [9] for many examples of finite groups which act freely on specific surfaces, particularly those of low genus.

6.2. We now turn our attention to actions on homotopy surfaces of a specified type. Let M_2 denote the orientable surface of genus 2 and N_g the nonorientable surface of genus g.

Proposition 6.6. *A group G acts properly discontinuously and cellularly on a homotopy surface X of type M_2 if and only if either* $\mathrm{cd}\,G < \infty$ *or* $G \cong G' \rtimes \mathbb{Z}_2$*, with* $\mathrm{cd}\,G' < \infty$.

Proof. The 'if' part follows from Example 3.3 and Proposition 6.4, respectively.

Conversely, if G is torsionfree, then $\mathrm{cd}\,G < \infty$ by Theorem 5.2. Assume that G has torsion and observe that if $H \subseteq G$ is a nontrivial finite subgroup, then the induced action

$$H \to \mathrm{Aut}\big(H^2(X,\mathbb{Z}_3)\big)$$

is nontrivial. Indeed, by Proposition 3.1, $H \cong \mathbb{Z}_2$ and $\chi(X/H) = -1$. On the other hand, using the transfer isomorphism $H^*(X/H;\mathbb{Z}_3) \to H^*(X;\mathbb{Z}_3)^H$, we obtain:

$$\chi(X/H) = 1 - \dim_{\mathbb{Z}_3} H^1(X;\mathbb{Z}_3)^H + \dim_{\mathbb{Z}_3} H^2(X;\mathbb{Z}_3)^H.$$

Thus if the action $H \to \mathrm{Aut}\,H^2(X,\mathbb{Z}_3)$ were trivial, it would imply the equality $\dim_{\mathbb{Z}_3} H^1(X;\mathbb{Z}_3)^H = 3$. A simple linear algebra argument shows that this is impossible.

Consequently, we have an epimorphism $G \to \mathrm{Aut}\,H^2(X,\mathbb{Z}_3) \cong \mathbb{Z}_2$ with torsionfree kernel, which – again by Theorem 5.2 – has finite cohomological dimension. An appropriate group extension splits and the conclusion follows. □

By Proposition 3.1, homotopy N_3's do not admit free and cellular actions of finite groups. Combining this with Example 3.3 and Theorem 5.2 thus yields:

Proposition 6.7. *A group acts properly discontinuously and cellularly on a homotopy surface of type N_3 if and only if it has finite cohomological dimension.*

This also gives a counterexample to the converse of Corollary 5.3: it is known that $\mathrm{Out}\big(\pi_1(N_3)\big) \cong GL(2,\mathbb{Z})$ – see, for example, [2, Theorem 3] – and this group obviously contains finite subgroups.

Remark 6.8. For the case of the 2-sphere, consult [10] and [15, Section 7]. The former paper deals with properly discontinuous and cellular actions on homotopy $2n$-spheres, while the latter treats properly discontinuous actions on $S^{2n} \times \mathbb{R}^m$ specifically.

As for the projective plane, it is clear that homotopy $\mathbb{R}P^2$'s admit properly discontinuous and cellular actions of only torsionfree groups. Furthermore, given any group G with finite cohomological dimension, there always exists a homotopy $\mathbb{R}P^2$ which admits a properly discontinuous and cellular G-action. At present, however, I do not know whether this is possible for a torsionfree group with infinite cohomological.

In general, the higher the genus of the surface in question, the more difficult the task at hand. We can, however, offer some partial results. We will be interested in virtually cyclic groups which act properly discontinuously and cellularly on homotopy surfaces of type N_{p+2}, where p is a prime number.

Recall the following classification theorem due to Wall.

Theorem 6.9 ([19, Theorem 5.12]). *Virtually cyclic groups come in three types:*

(1) *finite groups,*

(2) *semidirect products $H \rtimes \mathbb{Z}$, with H finite,*

(3) *$G_1 *_H G_2$, with $[G_i : H] = 2$ for $i = 1, 2$ and H finite.*

Following [11], we will say that a group is of 'type I', if it satisfies (2), and of 'type II', if it satisfies (3).

Remark 6.10. It will be useful to know in the next section that a finitely generated group has two ends if and only if it is either of 'type I' or of 'type II'. In fact, this assertion is also a part of [19, Theorem 5.12].

By Proposition 3.1, the only (nontrivial) finite group which can act freely and cellularly on a homotopy surface of type N_{p+2} is the cyclic group of order p. That such an action actually exists has been proven by Fujii [9, Proposition 1.4]. Corollary 3.4 shows that this action can be taken to be cellular.

Proposition 6.11. (1) *A nontrivial virtually cyclic group G acts properly discontinously and cellularly on a homotopy surface of type N_4 if and only if G is either cyclic of order 2, infinite cyclic, the direct product $\mathbb{Z}_2 \oplus \mathbb{Z}$, or the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$.*

(2) *If a nontrivial virtually cyclic group G acts properly discontinuously and cellularly on a homotopy surface of type N_{p+2}, $p \geq 3$ a prime, then G is either cyclic of order p, infinite cyclic, or a semidirect product $\mathbb{Z}_p \rtimes \mathbb{Z}$.*

Proof. The preceding discussion disposes of the case when G is finite.

If G is of 'type I', then $G \cong H \rtimes \mathbb{Z}$, where H is a finite group. By the previous case, H is either trivial or $H \cong \mathbb{Z}_p$. That both $\mathbb{Z}$ and $\mathbb{Z}_p \oplus \mathbb{Z}$ act properly discontinuously and cellularly on $N_{p+2} \times \mathbb{R}$ is clear.

We move on to groups of 'type II'. Recall that both G_1 and G_2 can be regarded as subgroups of $G_1 *_H G_2$; see, for example, [3, Chapter II, Lemma 7.4]. Consequently, in the first case, $G \cong \mathbb{Z}_2 * \mathbb{Z}_2$. It is, however, well-known that $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$. This group acts properly discontinuously and cellularly on a homotopy surface of type N_4 by Proposition 6.4.

Finally, in the second case, G cannot be of 'type II', since $\mathbb{Z}_p$, $p \geq 3$, does not contain an index 2 subgroup. $\square$

At this point I do not know which of the semidirect products $\mathbb{Z}_p \rtimes \mathbb{Z}$ actually act properly discontinuously and cellularly on homotopy N_{p+2}'s.

7 Groups that act on $M_2 \times \mathbb{R}$

We hinted in the introduction that investigating transformation groups of homotopy surfaces is not only of intrinsic interest, but should also yield information regarding transformation groups of manifolds of the form $M \times \mathbb{R}^m$. To back that sentiment up, we will show that the elaborated methods allow to determine all groups which act properly discontinuously on $M_2 \times \mathbb{R}$. Even though this is only one example, it feels like it deserves its own section: virtually every result presented during the course of this paper is used in its proof.

Proposition 7.1. *A nontrivial group G acts properly discontinuously on $M_2 \times \mathbb{R}$ if and only if it either cyclic of order 2, infinite cyclic, the direct product $\mathbb{Z} \oplus \mathbb{Z}_2$, or the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$.*

Proof. Suppose a nontrivial group G acts properly discontinuously on $M_2 \times \mathbb{R}$. By Proposition 6.6, $\operatorname{vcd} G < \infty$, hence Theorem 4.2 is applicable and implies that $\operatorname{vcd} G \leq 1$. If $\operatorname{vcd} G = 0$, then G is finite and $G \cong \mathbb{Z}_2$ by Proposition 3.1. Since this group acts freely on M_2, obviously it acts freely also on $M_2 \times \mathbb{R}$.

Suppose that $\operatorname{vcd} G = 1$. Again by Theorem 4.2, the orbit space $M_2 \times \mathbb{R}/G$ is a closed manifold. Therefore, by Corollary 3.5, G is finitely generated. The theory of ends (see, for example, [19, Section 5]) now gives $e(G) = e(M_2 \times \mathbb{R}) = 2$. Using Wall's criterion (Remark 6.10), we see that G is either infinite cyclic, the direct product $\mathbb{Z} \oplus \mathbb{Z}_2$, or the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$. That the first two groups act properly discontinuously on $M_2 \times \mathbb{R}$ is clear. It remains to show that this is also the case for $\mathbb{Z} \rtimes \mathbb{Z}_2$.

To see this, write $F_2 = \langle a, b \rangle$ for the free group on two generators. Define $\varphi \colon F_2 \to \operatorname{Homeo}(M_2 \times \mathbb{R})$ by setting

$$\begin{cases} a \mapsto \big((x,t) \mapsto (x, t+1)\big) \\ b \mapsto \big((x,t) \mapsto (bx, -t)\big) \end{cases} \quad \text{for any } (x,t) \in M_2 \times \mathbb{R},$$

where bx comes from a free $\mathbb{Z}_2$-action on M_2. One easily sees that φ is trivial on the normal closure of $\{b^2, baba\}$, hence it induces a $\mathbb{Z} \rtimes \mathbb{Z}_2$-action on $M_2 \times \mathbb{R}$. It is routine to verify that this action is in fact properly discontinuous. $\square$

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