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# On a theorem of Lion

Praca semestralna nr 1

(semestr letni 2011/12)

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# ON A THEOREM OF LION

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Abstract. We reformulate a theorem of Lion over arbitrary o-minimal structure. Let $\mathcal{S}$ be an o-minimal structure and $\mathcal{J}$ a family of unary functions, such that $\mathcal{S}\cup\mathcal{J}$ generates a $\mathcal{S}$-geometric $\mathcal{S}$-regular family. We prove, that 0-regularity property of differential algebra, which is closed from right with respect to affine maps, generated by $\mathcal{S}\cup\mathcal{J}$, implies regularity of entire $\mathcal{S}$-geometric family.

## 0. introduction

Studying of o-minimal structures can be divided in two main branches. The first is developing a geometry of o-minimal structures from axioms. The second one concentrates on discovering new such structures. Among polynomial bounded o-minimal expansions of real field the best known are:

(1) structure of semialgebraic sets,
(2) structure $\mathbb{R}_{\text{an}}$, generated by polynomials and a family of restricted analytic functions (o-minimal by result of van den Dries [D]; by Gabrielov [G], Łojasiewicz [LOJ] and Bierstone and Milman [BM], $\mathbb{R}_{\text{an}}$-definable sets are exactly globally subanalytic sets),
(3) expansions of real field generated by certain Denjoy-Carleman classes ([RSW]).

In polynomially unbounded case the first breakthrough was made by Wilkie, who proved in [W1], that $\mathbb{R}_{exp}$ is o-minimal. It was followed by results dealing with o-minimality of structures generated by total $\mathcal{C}^\infty$-Pfaffian functions ([W2]) and Pfaffian closure ([S]).

In this paper we present in a more general situation a theorem of Lion from [L], which is a generalization of Wilkie's theorem (see [W2]). Wilkie's original result gives necessary and sufficient conditions for expansion of the field $\mathbb{R}$ by total $\mathcal{C}^\infty$-functions to be o-minimal. Theorems of this type may be helpful in answering questions concerning:

(1) o-minimal expansions of o-minimal structures by polynomially unbounded functions,
(2) generating by Hardy fields o-minimal structures.

This paper is organized as follows. In the first section we reformulate a theorem of Lion over arbitrary o-minimal structure. This part is based on Lion's paper. The second gives a connection between Wilkie's and Lion's theorems. Third deals with 0-regularity property in case, when we add to a o-minimal structure $\mathcal{S}$ a family of unary functions $\mathcal{J}$, such that $\mathcal{S} \cup \mathcal{J}$ generates a $\mathcal{S}$-geometric $\mathcal{S}$-regular family. We prove, that 0-regularity property of differential algebra, which is closed from right with respect to affine maps, generated by $\mathcal{S} \cup \mathcal{J}$, implies regularity of entire $\mathcal{S}$-geometric family.

## 1. 0-regularity implies UFF property

Throughout the paper $\mathcal{S} = \{\mathcal{S}_n\}_{n\in\mathbb{N}}$ will denote a fixed o-minimal structure.

**Definition 1.1.** We say, that $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$, where $\mathcal{G}_n \subset \mathbb{R}^{\mathbb{R}^n}$ $(n \in \mathbb{N})$ is a $\mathcal{S}$-*geometric family*, when:

(G1) if $f, g \in \mathcal{G}_n$, then $fg, f + g \in \mathcal{G}_n$,
(G2) if $f \in \mathcal{G}_n$, and for every $x \in \mathbb{R}^n$ $f(x) \neq 0$, then $1/f \in \mathcal{G}_n$,
(G3) $\mathbb{R}[X_1, \ldots, X_n] \subset \mathcal{G}_n$,
(G4) if $\psi : \mathbb{R}^m \to \mathbb{R}^n$ is definable in $\mathcal{S}$, $f \in \mathcal{G}_n$, then $f \circ \psi \in \mathcal{G}_m$.

**Definition 1.2.** $\mathcal{S}$-geometric family $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is called $\mathcal{S}$-*regular*, if for every $n \in \mathbb{N}$ and every $g \in \mathcal{G}_n$, there exists a closed, $\mathcal{S}$-definable subset $C$ of $\mathbb{R}^n$, $\dim C < n$, and functions $g_1, \ldots, g_n \in \mathcal{G}_n$ such that for $U = \mathbb{R}^n \backslash C$, $g|_U$ is of class $\mathcal{C}^1$ and $\frac{\partial}{\partial x_i}(g|_U) = g_i|_U$.

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ and $t \in \mathbb{R}^m$. By $\operatorname{reg} g^{-1}(t)$ we denote the set of all these $x \in g^{-1}(t)$, for which there exists a neighborhood, such that $g|_U$ is a submersion of class $\mathcal{C}^1$.

**Definition 1.3.** Let $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a $\mathcal{S}$-geometric family. We say that $\mathcal{G}$ is *0-regular*, if for every $n \in \mathbb{N}$ and every mapping $g = (g_1, \ldots, g_n) : \mathbb{R}^n \to \mathbb{R}^n$, where $g_i \in \mathcal{G}_n$ $(i = 1, \ldots, n)$, and for each $t \in \mathbb{R}^n$, the set $\operatorname{reg} g^{-1}(t)$ is finite.

**Definition 1.4.** We say that a $\mathcal{S}$-geometric family $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ has *uniform fibre finiteness property* (UFF), when for every $g = (g_1, \ldots, g_p)$: $\mathbb{R}^n \to \mathbb{R}^p$, where $g_i \in \mathcal{G}_n$ $(i = 1, \ldots, p)$, there exists $N \in \mathbb{N}$ such that for each $t \in \mathbb{R}^p$,

$$\sharp\left\{A \subset \mathbb{R}^n \mid A \text{ is connected component of } g^{-1}(t)\right\} < N$$

**Definition 1.5.** Let $U \subset \mathbb{R}^n$ be an open set. We say, that $U$ admits a *carpeting function of class* $\mathcal{C}^k$, when there exists $\delta : \mathbb{R}^n \to \mathbb{R}$, $\delta \in \mathcal{G}_n$, such that:

(i) for every $x \in U$, $\delta(x) > 0$,
(ii) $\delta|_U$ is of class $\mathcal{C}^k$,
(iii) for every $\epsilon > 0$ there exists a compact set $K \subset U$, such that if $x \in U \backslash K$, then $\delta(x) < \epsilon$.

**Definition 1.6.** A subset $X$ of $\mathbb{R}^n$ is called a *leaf of class* $\mathcal{C}^k$ *and codimension* $q$, when there exists a set $U \subset \mathbb{R}^n$, which admits a carpeting function of class $\mathcal{C}^k$, and $f = (f_1, \ldots, f_q) : \mathbb{R}^n \to \mathbb{R}^q$, such that

(i) $f_i \in \mathcal{G}_n$, $i = 1, \ldots, q$,

(ii) $f|_U$ is a submersion of class $\mathcal{C}^k$,
(iii) $X = f^{-1}(0) \cap U$.

In this situation we say that $X$ is a *leaf associated with a triple* $(U, \delta, f)$.

**Lemma 1.7.** *Let $k \geq 2, p + q \leq n$. Let $X$ be a leaf of codimension $q$ and class $\mathcal{C}^k$ of $\mathbb{R}^n$ associated with a triple $(U, \delta, f)$. Let $g : \mathbb{R}^n \to \mathbb{R}^p$, where $g_i \in \mathcal{G}_n, i = 1, \ldots, p$, be a map of class $\mathcal{C}^k$ on $U$. Let $\theta : U \to \mathbb{R}$ be defined as a sum of squares of $(p+q)$-minors of the Jacobian matrix of $(f, g)(x)$, that is*

$$\theta(x) = \sum_{K \in \Lambda^n_{p+q}} \left( \frac{\partial(f, g)}{\partial(x_{k_1}, \ldots, x_{k_{p+q}})} \right)^2,$$

*where $\Lambda^n_{p+q} = \{(k_1, \ldots, k_{p+q}) : \ 1 \leq k_1 < \ldots < k_{p+q} \leq n\}$. Put*

$$\delta' = \frac{\delta\theta}{1+\theta} \qquad \text{and} \qquad U' = \{x \in U : \delta' > 0\}.$$

*Then $X' = X \cap U'$ is a leaf associated with a triple $(U', \delta', f)$.*

The proof of the lemma is obvious. In fact, we just cut $X$ to these points, for which $g|_X$ is a submersion.

Let $a \in \mathbb{R}^n \times (0, \infty)$. Note that if $\delta$ is a carpeting function of class $\mathcal{C}^k$ for an open set $U \subset \mathbb{R}^n$, then $\delta/(||x - a||^2)$ is also carpeting function for $U$ of the same class.

**Lemma 1.8.** *Let $k > 2n + 4, p + q < n$. Let $X \subset \mathbb{R}^n$ be a leaf of codimension $q$ and of class $\mathcal{C}^k$ associated with a triple $(U, \delta, f)$. Let $g = (g_1, \ldots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$ be a map such, that $g_i \in \mathcal{G}_n$, for $i = 1, \ldots, p$. Then there exist:*

(a) *leaves $Z_1, \ldots, Z_r$ of codimension $n - p$ and of class $\mathcal{C}^{k-4}$, such that $Z_i \subset X'$ ($X'$ is a set of regular point of $(f, g)$), $i = 1, \ldots, r$,*
(b) *$R \subset \mathbb{R}^n$ - a subset of regular values of $g$, such that $\mathcal{L}^p(\mathbb{R}^p \backslash R) = 0$ (where $\mathcal{L}^p$ denotes $p$-dimensional Lebesgue measure),*

*satisfying the following conditions:*

(1) *$g|_{Z_i}$ is a submersion, for $i = 1, \ldots, r$,,*
(2) *if $t \in R$, and $V \subset g^{-1}(t) \cap X$ is a connected component of $g^{-1}(t) \cap X$, then $\left( \bigcup_{i=1}^{r} Z_i \right) \cap V \neq \emptyset$.*

*Proof.* Let $\theta, \delta', U'$ and $X'$ be as in lemma 1.7. Define

$$\delta'_a = \frac{\delta\theta}{(1+\theta)||x - a||^2}, \quad \text{for } a \in \mathbb{R}^n \times (0, \infty),$$

and a set

$$S_a =: \left\{ x \in X' : \ G_I(x) = 0, I \in \Lambda^n_{p+q+1} \right\}, \quad \text{where} \quad G_I = \frac{\partial(f, g, \delta'_a)}{\partial(x_{i_1}, \ldots, x_{i_{p+q+1}})}.$$

Let $t \in \mathbb{R}^p$ and $a \in \mathbb{R}^n \times (0, \infty)$. The set $g^{-1}(t) \cap X'$ is a leaf of codimension $p + q$ and of class $\mathcal{C}^{k-1}$ associated with a triple $((f, g - t), U', \delta'_a)$. Notice that

$$x \in g^{-1}(t) \cap X' \text{ is a singular point of } \delta'_a|_{g^{-1}(t)\cap X'}$$
$$\Leftrightarrow$$
$$x \in S_a \cap (g^{-1}(t) \cap X').$$

Indeed, the set $S_a$ is equal to the zeroes of the form

$$\gamma = df_1 \wedge \ldots \wedge df_q \wedge dg_1 \wedge \ldots \wedge dg_p \wedge d\delta'_a.$$

The form $\gamma$ is zero if and only if $df_1 \wedge \ldots \wedge df_q \wedge dg_1 \wedge \ldots \wedge dg_p$ and $d\delta'_a$ are linearly dependent. Coefficients of $\gamma$ are exactly functions $G_I, I \in \Lambda^n_{p+q+1}$.
Let $V \subset \mathbb{R}^n$ be a connected component of $g^{-1}(t) \cap X'$. Since $\delta'_a$ is bounded and tends to zero on a boundary of $U'$, $g|_V$ has local extremum. Of course it is contained in $S_a$ (because ker $d_x\delta'_a \supset$ ker $d_x(f,g) = T_x(g^{-1}(t) \cap X')$). Therefore, for every $t \in \mathbb{R}^p$, the intersection of $S_a$ with every connected component $W \subset g^{-1}(t) \cap X'$ is nonempty.
We will find $a \in \mathbb{R}^n \times (0,\infty)$, such that $S_a$ is contained in the union of some leaves $Y_1, \ldots, Y_r$ of codimension $n-p$ and of class $\mathcal{C}^{k-3}$. Define a map

$$\Omega : X' \times (\mathbb{R}^n \times (0,\infty)) \ni (x,a) \to \left(T_x g^{-1}(t) \cap X, d_x\delta'_a\right) \in G_n^{n-p-q} \times \mathbb{R}^{n*}.$$

(where $x \in g^{-1}(t)$). We may assume that $\Omega \pitchfork \mathcal{H}$ (see Appendix: in fact, for almost every $a \in S_1 \subset \mathbb{R}^n \times (0,\infty)$, $\Omega|_{X' \times \{a\}}$ is transverse to $\mathcal{H}$, where $\mathcal{L}^{n+1}(\mathbb{R}^n \times (0,\infty) \backslash S_1) = 0$), where

$$\mathcal{H} := \left\{(T,l) \in G_n^{n-p-q} \times \mathbb{R}^{n*} : \ T \subset \mathrm{ker} l\right\}.$$

As a consequence we get that (cf. [GG], theorem 4.4)

$$\Omega^{-1}(\mathcal{H}) = \{(x,a) \in X' \times (\mathbb{R}^n \times (0,\infty))\}$$

is a submanifold of dimension $n+p+1$ and of class $\mathcal{C}^{k-2}$. Indeed, codim $\mathcal{H} = n-p-q$, so dim $\Omega^{-1}(\mathcal{H}) = n-q+n+1-(n-p-q) = n+p+1$.
Take $a \in \mathbb{R}^n \times (0,\infty)$ such that $S_a$ is a manifold of dimension $p$ and $\Omega|_{X' \times \{a\}}$ is transverse to $\mathcal{H}$ (from the Sard theorem there exist $S_2 \subset \mathbb{R}^n \times (0,\infty)$ such that $\mathcal{L}^{n+1}(\mathbb{R}^n \times (0,\infty) \backslash S_2) = 0$ and for every $a \in S_1$ the set $S_a$ is a manifold; it suffices to take now $a \in S_1 \cap S_2 \neq \emptyset$). Let

$$\mathcal{K} := \{K \subset \Lambda^n_{p+q+1} : \ \sharp K = n-p-q\}.$$

Define maps

$$F_K = (G_{I_1^K}, \ldots, G_{I_{n-p-q}^K}), \text{ where } K \in \mathcal{K}, \ K = \{I_1^K, \ldots, I_{n-p-q}^K\}.$$

Consider triples $(U_K, \delta_K, f)$, obtained as in lemma 1.7 for triples $(U', \delta'_a, f)$ and functions $F_K$. Associate with the triples $(U_K, \delta_K, (f, F_K))$ leaves $Y_K$, which are of codimension $n-p$ and of class $\mathcal{C}^{k-3}$. Of course

$$(\star) \qquad S_a \subset \bigcup_{K \in \Lambda^n_{p+q+1}} Y_K.$$

Due to the Sard theorem there exists $R_1 \subset \mathbb{R}^p$, such that $\mathcal{L}^p(\mathbb{R}^p \backslash R_1) = 0$ and that every $t \in R_1$ is a regular value of $g|_{S_a}$. Use lemma 1.7 again for triples $(U_K, \delta_K, (f, F_K))$ and the map $g$. We get leaves $Z_K$ of codimension $n-p$ and of class $\mathcal{C}^{k-4}$. Of course $Z_K \subset X'$, and $g|_{Z_K}$ are submersions for every $K \in \Lambda^n_{p+q+1}$. Take $R_0$, the set of regular values of $g$ and $R = R_0 \cap R_1$. For every $t \in R$, the set $g^{-1}(t) \cap S_a \subset \bigcup Z_K$, because $t$ is a regular value of $g$ and $g|_{S_a}$. □

**Lemma 1.9.** *Let $p+q=n$, $k>n$. Let $X$ be a leaf of codimension $q$ and of class $\mathcal{C}^k$. Suppose that $g = (\hat{g}, g_p) = (g_1, \ldots, g_{p-1}, g_p) : \mathbb{R}^n \to \mathbb{R}^p$, where $g_i \in \mathcal{G}_n, i = 1, \ldots, p$ is of class $\mathcal{C}^k$ on an open neighborhood of $X$. Moreover, let $g|_X$ be a submersion. Then for every $t = (u, t_p) = (t_1, \ldots, t_{p-1}, t_p) \in \mathbb{R}^p$, and each connected component $Z \subset \big(\hat{g}^{-1}(u)\big) \cap X)$, the following condition is satisfied*

$$Z \cap \big(g^{-1}(t) \cap X\big) \neq \emptyset \;\Rightarrow\; \sharp Z \cap \big(g^{-1}(t) \cap X\big) = 1$$

*Proof.* Let $t = (u, t_p) \in \mathbb{R}^p$. Maps $\hat{g}|_X$ and $g|_X$ are submersions, $\dim(\hat{g}^{-1}(u) \cap X) = 1$, and $\dim(g^{-1}(t) \cap X) = 0$. Let $Z \subset \hat{g}^{-1}(u) \cap X$ be a connected component of $\hat{g}^{-1}(u) \cap X$. Consider $g_p|_Z$. Suppose, that the condition does not hold, that is

$$\exists z_1 \in Z \;\; \exists z_2 \in Z : \;\; z_1 \neq z_2 \;\; \text{and} \;\; t_p = g_p(z_1) = g_p(z_2).$$

Then there exists a point $z_0 \in Z$, such that $g_p'(z_0) = 0$. This is a contradiction, because rank $g = p$. □

**Remark 1.10.** If we just take an open set $U \subset \mathbb{R}^n$ (without having a leaf $X$) lemmas 1.8 and 1.9 are also true. Proofs can be made in a similar way as above.

Notice, that any $\mathcal{S}$-definable open cell $D \subset \mathbb{R}^n$ of class $\mathcal{C}^k$ admits a carpeting function (see, for example [S]). It is enough to take a function

$$\zeta : \mathbb{R}^n \ni (x_1, \ldots, x_n) \to \frac{1}{x_1^2 + \ldots + x_n^2 + 1} \in \mathbb{R}$$

and compose it with $\mathcal{C}^k$-diffeomorphism $\phi : D \to \mathbb{R}^n$. Function

$$\delta := \begin{cases} \zeta \circ \phi, & x \in D \\ 0, & x \in \mathbb{R}^n \backslash D \end{cases}$$

has desired properties.

Let us now state

**Theorem 1.11.** *Let $g : \mathbb{R}^n \to \mathbb{R}^p$, $g_i \in \mathcal{G}_n, i = 1, \ldots, p$. Then there exist*

(1) *$R \subset \mathbb{R}^p$, such that $\mathcal{L}^p(\mathbb{R}^p \backslash R) = 0$,*

(2) *$h_i = (h_{i1}, \ldots, h_{in}) : \mathbb{R}^n \to \mathbb{R}^n, h_{ij} \in \mathcal{G}_n, i = 1, \ldots, r, j = 1, \ldots, n$,*

*such that for every $t \in R$*

$$\sharp \left\{ A \subset \mathbb{R}^n \mid A \text{ is connected component of } g^{-1}(t) \right\} \leq N$$

*where* $N := \sum_{i=1}^{r} \sharp\text{reg } h_i^{-1}(0)$.

*Proof.* If $p > n$, then for almost all $t \in \mathbb{R}^p$, the set $g^{-1}(t)$ is empty (due to the Sard theorem). We may assume, that $p \leq n$. We will proceed by induction on $(n, p)$ ($p \leq n$; we consider a lexicographic order). Let $(n, p) = (1, 1)$ and $g : \mathbb{R} \to \mathbb{R}$, $g \in \mathcal{G}_1$. There exists a set $A \subset \mathbb{R}$, which consists of finitely many points, such that on $\mathbb{R} \backslash A = I_1 \cup \ldots \cup I_r$ ($I_l$ are open intervals, $l = 1, \ldots, r$) the function $g$ is of class $\mathcal{C}^1$, and $g' \neq 0$. It is is easy to see that it is enough to take $R = \mathbb{R}^n \backslash g(A)$ and functions

$$g_l(x) := g \circ \phi_l(x) - a_l, \quad \text{where } a_l \in g(I_l),\ l = 1, \ldots, r,$$

and $\phi_l : \mathbb{R} \to I_l$ are semialgebraic diffeomorphisms.

Assume that the statement is true for every $(n, p) < (n_0, p_0)$. Take $g = (g_1, \ldots, g_{p_0}) : \mathbb{R}^{n_0} \to \mathbb{R}^{p_0}$, $g_i \in \mathcal{G}_{n_0}, i = 1, \ldots, p_0$. Let $\mathcal{D}$ be a cell decomposition of $\mathbb{R}^n$ compatible with a closed, nowhere dense, definable set $A$, outside which $g$ is of class $\mathcal{C}^{6n}$. If $n_0 = p_0$, then for every open cell $D \in \mathcal{D}$ denote by $D'$ the set of regular points of $g$. Apply 1.10, for every $D'$. We reduce the situation to the case $(n, p_0 - 1)$. Put $\mathcal{D}_n = \{D \in \mathcal{D} : \text{dim} D = n\}$. By induction hypothesis we get, for an every open cell $D$, and a map $g \circ \phi_D$, where $\phi_D : \mathbb{R} \to D$ is a $\mathcal{C}^{6n}$-diffeomorphism, sets $R_D \subset \mathbb{R}^{p_0 - 1}$, such that $\mathcal{L}^p(\mathbb{R}^{p_0-1} \backslash R_D) = 0$, functions $h_1^D, \ldots, h_{r_D}^D$, and an upper limits $N_D$ on numbers of connected components of $(g \circ \phi_D)^{-1}(t)$, for $t \in R_D$. We get an upper limit $N = \sum_{D \in \mathcal{D}_n} N_D$ for the number of connected components of almost all fibers of $g|_E$, where $E = \sum_{D \in \mathcal{D}_n} D$. Due to the Sard Theorem, there exists a $R_1 \subset \mathbb{R}^{p_0-1}$ such that $\mathcal{L}^p(\mathbb{R}^{P_0-1} \backslash R_1) = 0$ and for $t \in R_1$, the set $g|_{E'}^{-1}(t)$ is empty, where $E' = \mathbb{R}^n \backslash E$. It is enough to take $R = R_1 \cap \left(\bigcap_{D \in \mathcal{D}_n} R_D\right)$.

When $p_0 < n_0$, then, by induction hypothesis, (in a similar way as above) for every cell $D$ of dimension smaller than $n_0$ we obtain $R_D, N_D$ and functions $h_1^D, \ldots, h_{r_D}^D$. Consider, for $D \in \mathcal{D}_n$, the function $g|_D$. Apply remark 1.10. We get leaves $Z_1^D, \ldots, Z_{r_D}^D$, $Z_i \subset D'$, $i = 1, \ldots, r_D$ (where $D'$ is a set of regular points of $g|_D$), set $R_D^1 \subset \mathbb{R}^{p_0}$, such that $\mathcal{L}^{p_0}(\mathbb{R}^{p_0} \backslash R_D^1) = 0$ and the following condition are satisfied: if $t \in R_D^1$, and $V \subset g|_D^{-1}(t)$ is a connected component of $g^{-1}(t) \cap D$, then $\left(\bigcup_{i=1}^{r_D} Z_i\right) \cap V \neq \emptyset$. Now we can use lemma 1.9, for every $Z_i^D$, $D'$ and $g$. We reduce the situation to the case, when we are having leaves $Z_1^D, \ldots, Z_{k_D}^D$ of class $\mathcal{C}^{6n-4}$, an open cell $D$, and a map $(g_1, \ldots, g_{p_0-1})$. Proceeding as before, using lemmas 1.8 and 1.9, after $p_0 - 1$ steps, we will get leaves $W_1^D = h_{1_D}^{-1}(0), \ldots, W_{r_D}^D = h_{r_D}^{-1}(0)$ of dimension 0 and class $\mathcal{C}^{2n}$. Put $W_D = \bigcup_{i=1}^{r_D} W_i^D$. Then $\sharp W_D = N_D < \infty$, since $\mathcal{G}$ is 0-regular. Let $R_D^i \subset \mathbb{R}^{p_0-i}$, for $i = 1, \ldots, p_0 - 1$, are sets obtain by using lemma 1.8 in an $i$-th step. Now, for $t \in R_D$, where

$$R_D = S_D \cap (R_D^1 \times \mathbb{R}) \cap (R_D^2 \times \mathbb{R}^2) \cap \ldots \cap (R_D^{p_0-1} \times \mathbb{R}^{p_0-1}),$$

where $S_D$ is a set of regular values of $g|_D$, the number of connected components of $g|_D^{-1}$ is not greater than $N_D$. Finally,

- $N_D = \sum_{D\in\mathcal{D}} N_D$,
- $R = \bigcap_{D\in\mathcal{D}} R_D$
- $h_i^D, D \in \mathcal{D}, i = 1,\ldots,r_D$

satisfy desired conditions. □

Before we will state the final theorem, we will need one more lemma:

**Lemma 1.12.** *Let $g = (g_1,\ldots,g_p) : \mathbb{R}^n \to \mathbb{R}^p$, $g_i \in \mathcal{G}_n, i = 1,\ldots,p$. Let $U \subset \mathbb{R}^n$ be an open set, such that it admits carpeting function $\delta$ of class $\mathcal{C}^k$ and $g|_U$ is od class $\mathcal{C}^k$. Let $R \subset \mathbb{R}^{2+p}$ be such, that $\mathcal{L}^{2+p}(\mathbb{R}^{2+p}\setminus R) = 0$. Define*

$$G : U \times \mathbb{R}^2 \times \mathbb{R}^p \ni \big(x,u,v,t\big) \to \big(\delta(x) - u^2, ||g(x) - t||^2 + v^2, t\big) \in \mathbb{R}^2 \times \mathbb{R}^p.$$

*Let $(\eta,\varepsilon,T) \in (0,\infty) \times \mathbb{R}^p$. Then*

(1) *$G^{-1}(\eta,\varepsilon,T)$ is compact,*
(2) *$\Pi_{n+2+p,n}\big(G^{-1}(\eta,\varepsilon,T)\big) = \{x \in U : \ ||g(x) - t||^2 \le \varepsilon, \delta(x) \ge \eta\}$,*
(3) *for every $t \in \mathbb{R}^p$ and a sequence $\{(\eta_j,\varepsilon_{i,j},T_{i,j})\}_{(i,j)\in\mathbb{N}^2}$ such that*
   - $\eta_j \searrow 0$,
   - $\forall j \in \mathbb{N}\ \varepsilon_{i,j} \searrow 0$,
   - $||t - T_{i,j}|| < \sqrt{\varepsilon_{i,j}}$, $||T_{i+1,j} - T_{i,j}|| + \sqrt{\varepsilon_{i+1,j}} < \sqrt{\varepsilon_{i,j}}$

   *we have*
$$g^{-1}(t) = \bigcap_{j\in\mathbb{N}}\left(\bigcup_{i\in\mathbb{N}} \Pi_{n+2+p,n}\big(G^{-1}(\eta_j,\varepsilon_{i,j},T_{i,j})\big)\right)$$

**Theorem 1.13.** *Let $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be $\mathcal{S}$-regular, 0-regular $\mathcal{S}$-geometric family. Then it has UFF property.*

*Proof.* Let $g = (g_1,\ldots,g_p) : \mathbb{R}^n \to \mathbb{R}^p$, where $g_i \in \mathcal{G}_n, i = 1,\ldots,p$. Apply 1.11 for a map $G$, defined as in lemma 1.12. There exists $N \in \mathbb{N}$, such that for all $w \in R \subset \mathbb{R}^{2+p}$, where $\mathcal{L}^{2+p}(\mathbb{R}^{2+p}\setminus R) = 0$, the number of connected components of $G^{-1}(w)$ is not greater than $N$. Take $t \in \mathbb{R}^p$. There exists $w_t = (\eta_t,\varepsilon_t,T_t) \in R$ such that $V \cap \Pi_{n+2+p,n}\big(G^{-1}(w_t)\big) \neq \emptyset$, for every connected component $V$ of $g^{-1}(t)$. It follows that $\mathcal{G}$ has UFF property. □

## 2. Wilkie's and lion's theorems

Now, we recall the modification of Wilkie's theorem by Karpinski and Macintyre. Let $AG(\mathbb{R}^n)$ denote the set of all affine subspaces of $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$. Then we put

$$\gamma(A) := \min\{N \in \mathbb{N} : \text{for all } V \in AG(\mathbb{R}^n)$$
$$A \cap V \ \text{ has at most } N \text{ connected components}\}.$$

If such $N$ does not exist, then we put $\gamma(A) = \infty$.

**Definition 2.1.** A sequence $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$, where $\mathcal{F}_n \subset \mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{N}$, is called a *weak o-minimal structure*, if for every $n \in \mathbb{N}, m \in \mathbb{N}$ the following conditions are satisfied:

(W1) if $A, B \in \mathcal{F}_n$, then $A \cap B \in \mathcal{F}_n$,
(W2) $\mathcal{F}_n$ contains all semialgebraic subsets of $\mathbb{R}^n$,
(W3) if $A \in \mathcal{F}_n$, $B \in \mathcal{F}_m$, then $A \times B \in \mathcal{F}_{n+m}$,
(W4) if $A \in \mathcal{F}_n$ and $\sigma$ is a permutation of coordinates, then $\sigma(A) \in \mathcal{F}_n$,
(W5) if $A \in \mathcal{F}_n$, then $\gamma(A) < \infty$,
(W6) if $A \in \mathcal{F}_n$, then there exist $m \geq n$ and a closed set $B \in \mathcal{F}_m$, such that $A = \Pi_{m,n}(B)$, where $\Pi_{m,n} : \mathbb{R}^m \ni (x_1, \dots, x_m) \to (x_1, \dots, x_n) \in \mathbb{R}^n$.

**Definition 2.2.** A weak o-minimal structure $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$ satisfies $DC^N$ *condition for all* $N$, if for each $A \in \mathcal{F}_n$ there exists $p \geq n$, such that for every $N \in \mathbb{N}$, $A$ is equal to $\Pi_{p,n}(\{f_N = 0\})$, where $f_N \in C^N(\mathbb{R}^p)$ and $\mathrm{graph} f_N \in \mathcal{F}_{p+1}$.

**Theorem 2.3 (Wilkie, Karpinski, Macintyre).** *Suppose $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is an o-minimal weak structure satisfying $DC^N$ for all $N$. Then there exists an o-minimal structure $\widetilde{\mathcal{F}} = \{\widetilde{\mathcal{F}}_n\}_{n\in\mathbb{N}}$ which contains $\mathcal{F}$.*

It is not difficult to check that if $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is a $\mathcal{S}$-regular geometric family with uniform fibre finiteness property, then through defining $\mathcal{F}_n$ as the family of all subsets of $\mathbb{R}^n$ of the form $f^{-1}(0)$, where $f \in \mathcal{G}_n$, we obtain a weak o-minimal structure. Less obvious is that this structure satisfies $DC^N$ condition for all $N \in \mathbb{N}$. We will check this in detail.

**Proposition 2.4.** *Let $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a $\mathcal{S}$-regular geometric family with the uniform fibre finiteness property, closed with respect to compositions from the right with $\mathcal{S}$-maps. Then, for each $n \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that if $N \in \mathbb{N}$ and $F = (F_1, \dots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$, where $F_i \in \mathcal{G}_n, i = 1, \dots, k$, $A := F^{-1}(0)$, then there exists $\widetilde{F} : \mathbb{R}^{n+l} \to \mathbb{R}$ of class $\mathcal{C}^N$ such that $\widetilde{F}_i \in \mathcal{G}_{n+l}$ for every $i = 1, \dots, n+k$ and $A = \Pi_{n+l,n}(\widetilde{F}^{-1}(0))$.*

*Proof.* We will prove the proposition by induction on $n$. For $n = 1$ it is obvious, because sets on the real line are finite sums of points and intervals. Now assume a proposition to be true for every $k < n+1$. Take $F : \mathbb{R}^{n+1} \to \mathbb{R}^k$, where $F_i \in \mathcal{G}_{n+1}$ and let $A = F^{-1}(0)$. Let $V \subset \mathbb{R}^{n+1}$ be a closed, nowhere dense definable set such that $F|_{\mathbb{R}^{n+1}\setminus V}$ is of class $\mathcal{C}^N$. Take a cell decomposition $\mathcal{B}$ of $\mathbb{R}^{n+1}$ of class $\mathcal{C}^N$ compatible with $V$. $\mathcal{B} = \mathcal{B}_0 \cup \dots \cup \mathcal{B}_{n+1}$, where

$$\mathcal{B}_i = \{B \in \mathcal{B}_i \mid \dim B = i\}, \qquad i = 0, 1, \dots, n+1.$$

Let $B \in \mathcal{B}$. Consider two cases:

(1) $B \in \mathcal{B}_{n+1}$. There exists $\varphi_B = (\varphi_B^1, \dots, \varphi_B^{n+1}) : \mathbb{R}^{n+1} \to B$, a definable diffeomorphism of class $\mathcal{C}^N$. Then

$$A \cap B = \left\{x \in \mathbb{R}^{n+1} \mid \exists\, z \in \mathbb{R}^{n+1}\ \psi_B(x,z) = 0\right\},$$

where $\psi_B(x,z) = \big(F \circ \varphi_B\big)^2(z) + \sum_{i=1}^{n+1} \big(\varphi_B^i(z) - x_i\big)^2$ is a function of class $\mathcal{C}^N$ and $\psi_B \in \mathcal{G}_{2n+2}$.

(2) $B \in \mathcal{B}_j$, for some $j = 1, \dots, n$ . There exists $\varphi_B = (\varphi_B^1, \dots, \varphi_B^{n+1}) : \mathbb{R}^j \to B$, a definable diffeomorphism of class $\mathcal{C}^\infty$. By induction hypothesis there exist $l_j \in \mathbb{N}$ and $\mathcal{C}^N$-map $\widehat{F}_B : \mathbb{R}^{j+l_j} \to \mathbb{R}$ such that $\Pi_{j+l_j}(\widehat{F}^{-1}(0)) = (F \circ \varphi_B)^{-1}(0)$. Now

$$A \cap B = \left\{x \in \mathbb{R}^{n+1} \mid \exists\, t^j \in \mathbb{R}^j\ \exists\, u^j \in \mathbb{R}^{l_j}\ \psi_B(x, t^j, u^j) = 0\right\},$$

where $\psi_B(x, t^j, u^j) = \widehat{F}^2(t^j, u^j) + \sum_{i=1}^{n+1} \big(\varphi_B^i(t^j) - x_i\big)^2$ is a function of class $\mathcal{C}^N$ and $\psi_B \in \mathcal{G}_{n+j+l_j+1}$.

Define $B_0 = \bigcup_{B \in \mathcal{B}_0} B$ and $A_0 = B_0 \cap A$. Consider $\mathcal{C}^N$-function

$$\Psi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^n \times \mathbb{R}^{l_n} \times \ldots \times \mathbb{R}^1 \times \mathbb{R}^{l_1} \longrightarrow \mathbb{R}, \quad \text{such that}$$

$$\Psi(x, z, t^n, u^{l_n}, \dots, t^1, u^{l_1}) = \prod_{B \in \mathcal{B}_{n+1}} \psi_B(x,z) \prod_{j=1}^{n} \prod_{B \in \mathcal{B}_j} \psi_B(x, t^j, u^j) \prod_{a \in A_0} \left( \sum_{i=1}^{n+1} (a_i - x_i)^2 \right).$$

Let $l = 2n + 2 + n(n+1)/2 + \sum_{j=1}^n l_j$. It is easy to see that $A = \Pi(\Psi^{-1}(0))$, where $\Pi : \mathbb{R}^l \to \mathbb{R}^{n+1}$ is a projection onto $n+1$ first coordinates. □

## 3. 0-regularity

**Lemma 3.1.** *Let $n, k \in \mathbb{N}$, and $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ be a definable immersion. Then there exists a cell decomposition $\mathcal{B}$ of $\mathbb{R}^n$, such that for every $B \in \mathcal{B}$ a map $f|_B$ is an injection.*

*Proof.* Since $f$ is an immersion, there exists $N \in \mathbb{N}$ such that, for every $t \in \mathbb{R}^{n+k}$, the power of fibers $f^{-1}(t)$ is not greater than $N$. Define

$$\widetilde{graph} f := (f(x), x) \subset \mathbb{R}^{n+k} \times \mathbb{R}^n.$$

Let $\mathcal{B}_1$ be a cell decomposition of $\mathbb{R}^{n+k} \underset{\sim}{\times \mathbb{R}^n}$ compatible with $\widetilde{graph} f$. Notice that every cell $A \in \mathcal{B}_1$, such that $A \subset \underset{\sim}{graph} f$ is of type 'graph' above $\Pi_1(A)$, where $\Pi_1$ is a projection on first $n+k$ coordinates. Suppose, for contrary, that $\dim \Pi_1(A) < \dim A$. Notice that $\dim A \le n$. Let $A_i := \Pi_{2n+k, n+k+i}(A)$, where $\Pi_{2n+k, n+k+i}$ is a projection from $\mathbb{R}^{2n+k}$ on first $n+k+i$ coordinates and $i \in \{0, \dots, n\}$. There exists $i \ge 1$, such that

$$A_i = \left\{(x,y) \in \mathbb{R}^{n+k+i-1} \times \mathbb{R} :\ g_1(x) < y < g_2(x), x \in A_{i-1}\right\},$$

where $g, f : A_{i-1} \to \mathbb{R}$ and $g_1(x) < g_2(x)$, for every $x \in A_{i-1}$. Then $\sharp f^{-1}(z) = \infty$, $z \in A_0$. We get a contradiction with the uniform finiteness of power of fibers. Define

$$\mathcal{A} := \{\Pi_2(A) \mid A \in \mathcal{B}_1,\ A \subset \widetilde{graph f}\},$$

where $\Pi_2$ is a projection on last $n$ coordinates. Then, for every $A \in \mathcal{A}$, a map $f|_A$ is an injection. Now it suffices to take a cell decomposition $\mathcal{B}$ of $\mathbb{R}^n$ compatible with the family $\mathcal{A}$. □

**Proposition 3.2.** *Let $n, k \in \mathbb{N}$, and $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ be a definable immersion. Then $f(\mathbb{R}^n) = \bigcup_{i=1}^{l} M_i$, where $l \in \mathbb{N}$ and $M_i$ are connected, definable manifolds of* $\dim M_i = n$*, for every $i = 1, \dots, l$.*

*Proof.* Define a set

$$C := \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid f(u) \neq f(v)\}.$$

For every $u \in \mathbb{R}^n$, the sets

$$C_u := \{v \in \mathbb{R}^n \mid (u, v) \in C\} \ \cup\ \{u\}$$

are open and definable. By lemma 3.1, there exists a cell decomposition of $\mathbb{R}^n$, such that for every $B \in \mathcal{B}$, a map $f|_B$ is an injection. Take $B \in \mathcal{B}$. If $\dim B = n$, then $f(B)$ is a connected, definable manifold. Consider the case, when $\dim B < n$. Take $u \in B$. Define $C_u := C_B$. Define a map

$$F_B := C_B \times C_B \ni (x, y) \to (f(x), f(y)) \in \mathbb{R}^{n+k} \times \mathbb{R}^{n+k}.$$

Let $\Delta_{A \times A} := \{(x, x),\ x \in A\}$, for $A \in \mathbb{R}^l$. Then a set

$$D_B := \big((C_B \times C_B)\big) \cap F_B^{-1}\big((\mathbb{R}^{n+k} \times \mathbb{R}^{n+k}) \setminus (\Delta_{\mathbb{R}^{n+k} \times \mathbb{R}^{n+k}})\big)$$

is open in $\mathbb{R}^n \times \mathbb{R}^n$. Put $\widetilde{E}_B = \widetilde{\Pi}(D_B)$, where $\widetilde{\Pi}$ is a projection from $\mathbb{R}^n \times \mathbb{R}^n$ onto the first $n$ coordinates, and let $E_B$ a connected component of $\widetilde{E}_B$, which contains $u \in B$. Observe that $E_B$ is an open set (since projections are open maps). Notice that $f|_{E_B}$ is injective. Suppose, for contrary, that there exist $a, b \in E_B$, $a \neq b$, such that $f(a) = f(b)$. Then $(a, b), (b, a) \in C_B$ and $F(a, b) = F(b, a)$. It follows that $a, b \notin E_B$, a contradiction. It is easy to see, that $B \subset E_B$, and

$$f(\mathbb{R}^n) = \bigcup_{B \in \mathcal{B}} f(E_B).$$

□

**Remark 3.3.** We can always take immersions defined on an open definable set $U \subset \mathbb{R}^n$. Each $M_i$ has its global parametrization, that is, there exist an open set $U_i \subset U$, such that $f|_{U_i} : U_i \to M_i$ is a homeomorphism onto its image of maximal rank.

**Remark 3.4.** In fact, we can always cover an image of a definable immersion $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ of class $\mathcal{C}^k$ by finite number of images of injective definable $\mathcal{C}^k$-immersions $\psi_i : \mathbb{R}^n \to \mathbb{R}^{n+k}$, $i \in I$, $\sharp I < \infty$. Indeed, take one of $E_B$ from the previous proposition. Then we can take a $\mathcal{C}^{k+1}$-cell decomposition $\mathcal{D}$ of $E_B$. Let $D \in \mathcal{D}$. If $\dim D = n$, then there exists a $\mathcal{C}^{k+1}$-definable diffeomorphism $\phi_D : \mathbb{R}^n \to D$ and $\psi_D = f \circ \phi_D$. When $\dim D < n$, we can proceed in the following way. A normal vector bundle $ND$ of $D$ is a manifold of class $\mathcal{C}^k$. From the definable tubular neighborhood theorem (cf. [C], Theorem 6.11) there exist:

- definable function $\tilde{\epsilon} : D \to \mathbb{R}_+$ of class $\mathcal{C}^k$,
- an open, definable set $U = \{(x, v) \in ND : ||v|| < \tilde{\epsilon}(x)\}$,
- an open, definable set $\Omega_D \subset \mathbb{R}^n$, such that $D \subset \Omega_D$,
- a definable $\mathcal{C}^k$-diffeomorphism $\gamma_D : U \to \Omega_D$, such that $f((x, 0)) = x$, $x \in D$.

Then a function

$$\xi_D(x) := \inf_{v \in A_x} \{\mathrm{dist}(x, \phi_D(x, v))\}, \quad \text{where} \quad A_x = \left\{(x, v) \in ND : ||v|| = \frac{\epsilon}{2}\right\}$$

is continuous and definable. Now put

$$\zeta_D(x) := \mathrm{dist}(x, \partial E_B), \quad \eta_D(x) := \min(\xi_D(x), \zeta_D(x)), \quad x \in D.$$

Of course $\eta_D$ is a definable, positive, continuous function. We get an open set

$$E := \{(x, v) \in ND : ||(x, 0) - \gamma_D(x, v)|| < \eta_D(x)\}.$$

Define a function $\mu : D \to \mathbb{R}_+$

$$\mu(x) := \sup\left\{r \in \mathbb{R}_+ : \{x\} \times \overline{K}(0, r) \subset E_x\right\},$$

which is positive, definable and locally bounded from below by positive constants (since $E$ is open). By lemma 6.12 from [C] there exists $\epsilon : D \to \mathbb{R}_+$, a definable, positive function of class $\mathcal{C}^k$ such that $\epsilon < \mu$. Of course there exist following diffeomorphisms of class $\mathcal{C}^k$:

- $\alpha_D : \widetilde{U} \to ND$, where $\widetilde{U} = \{(x, v) \in NM : ||v|| < \epsilon(x)\}$,
- $\beta_D : ND \to D \times \mathbb{R}^{n-k}$ (D is a cell, so we can find global trivialisation of normal bundle),
- $\phi_D : \mathbb{R}^k \to D$.

We can now define

$$\psi_D := f \circ \gamma_D \circ \alpha_D^{-1} \circ \beta_D^{-1} \circ (\phi_D, \mathrm{id}_{\mathbb{R}^{n-k}}).$$

Now we can get

**Proposition 3.5.** *Let $U \subset \mathbb{R}^n$ be an open definable set. Let $f : U \to \mathbb{R}^{n+k}$ be a definable, injective immersion. Let $M := f(U)$. Then there exist definable functions $h_1, \ldots, h_l$, where $h_i : \mathbb{R}^{n+k} \to \mathbb{R}^k$, such that $M = \bigcup_{i=0}^{l} h_i^{-1}(0)$. Moreover, zero is a regular value of each $h_i$ (restricted to the set, where $h_i$ is of class $\mathcal{C}^1$).*

*Proof.* We have

$$U = \bigcup_{I \in \Lambda_n^{n+k}} U_I, \text{ where}$$

$$U_i = \left\{ x \in U : \frac{\partial f^I}{\partial(x_1, \ldots, x_n)} \neq 0 \right\}, \ f^I = (f_{i_1}, \ldots, f_{i_n}), \ I = (i_1, \ldots, i_n).$$

Each $U_I$ is open. Let $I_0 = (1, \ldots, n)$ and assume, that $U_{I_0} \neq \emptyset$. We will prove the statement in the case of $f : U_{I_0} \to \mathbb{R}^{n+k}$. Others are similar. Consider a map

$$g : U_{I_0} \times \mathbb{R}^k \to \mathbb{R}^{n+k}, \quad g(x, y) = f(x) + (0, y).$$

Then

$$\forall (x, y) \in U_{I_0} \times \mathbb{R}^k \ \det d_{(x,y)} g = \frac{\partial f^{I_0}}{\partial(x_1, \ldots x_n)} \neq 0.$$

We can find (thank to 3.3) a finite number of open sets $V_i \subset U_{I_0} \times \mathbb{R}^k$, $i = 1, \ldots p$, such that, for every $i = 1 \ldots, p$, $g|_{V_i}$ is an injection and $g(V_1 \cup \ldots \cup V_p) \supset f(U_{I_0})$. Take $j^{V_i} = (j_1^{V_i}, \ldots j_{n+k}^{V_i}) = (g|_{V_i})^{-1}$, which is definable of class $\mathcal{C}^1$, because of definable inverse function theorem. It is easy to see, that $f(U_{I_0}) \cap g(V_i) = (j_{n+1}^{V_i}, \ldots, j_{n+k}^{V_i})^{-1}(0)$, and zero is a regular value of $\widetilde{h}^{V_i} = (j_{n+1}^{V_i}, \ldots, j_{n+k}^{V_i})^{-1}(0)$. We can extend $\widetilde{h}^{V_i}$ to $h^{V_i} \mathbb{R}^{n+k} \to \mathbb{R}^k$, by defining it outside $g(V_i)$ as 1. Taking all such functions, over all $U_I$, $I \in \Lambda_n^{n+k}$ we get a finite family of defining functions of $M$. □

**Definition 3.6.** Let $\mathcal{S}$ be an o-minimal structure and $\mathcal{J} = \{\mathcal{J}_n\}_{n \in \mathbb{N}}$ be a family of functions, where $\mathcal{J}_n \subset \mathbb{R}^{\mathbb{R}^n}$. We say that the family $\mathcal{J} = \{\mathcal{J}_n\}_{n \in \mathbb{N}}$ is *almost 0-regular with respect to the structure* $\mathcal{S}$, when $\mathcal{J}$ and $\mathcal{S}$ generate an $\mathcal{S}$-geometric $\mathcal{S}$-regular family, and a family $\widetilde{\mathcal{G}} = \{\widetilde{\mathcal{G}}_n\}_{n \in \mathbb{N}}$ generated by $\mathcal{S} \cup \mathcal{J} = \{\mathcal{S}_n \cup \mathcal{J}_n\}_{n \in \mathbb{N}}$ satisfying following properties

($\widetilde{G}1$) if $f, g \in \widetilde{\mathcal{G}}_n$, then $fg, f + g \in \widetilde{\mathcal{G}}_n$,
($\widetilde{G}2$) if $f \in \widetilde{\mathcal{G}}_n$, and for every $x \in \mathbb{R}^n$ $f(x) \neq 0$, then $1/f \in \widetilde{\mathcal{G}}_n$,
($\widetilde{G}3$) $\mathbb{R}[X_1, \ldots, X_n] \subset \widetilde{\mathcal{G}}_n$,
($\widetilde{G}4$) if $L : \mathbb{R}^m \to \mathbb{R}^n$ is an affine map, $f \in \widetilde{\mathcal{G}}_n$, then $f \circ L \in \widetilde{\mathcal{G}}_n$.

is 0-regular.

Fix now a family of functions $\mathcal{J} = \{\mathcal{J}_n\}_{n \in \mathbb{N}}$, such that $\mathcal{J}_n = \emptyset$, for $n > 1$. Before our last statement, we will need the following

**Lemma 3.7.** *Let the family $\mathcal{J} = \{\mathcal{J}_n\}_{n \in \mathbb{N}}$ be almost 0-regular with respect to the structure $\mathcal{S}$. Let $\widetilde{\mathcal{G}} = \{\widetilde{\mathcal{G}}_n\}_{n \in \mathbb{N}}$ be a $\mathcal{S}$-regular family of functions defined as in 3.6, generated by $\mathcal{S} \cup \mathcal{J} = \{\mathcal{S}_n \cup \mathcal{J}_n\}_{n \in \mathbb{N}}$. Let $f \in \mathcal{G}_n$. Then there exist $k, m \in \mathbb{N}$, a $\mathcal{S}$ - definable map $\phi : \mathbb{R}^n \to \mathbb{R}^{n+k}$, affine functions $L_1, \ldots, L_m : \mathbb{R}^{n+k} \to \mathbb{R}$, $f_1, \ldots f_m \in (\mathcal{S} \cup \mathcal{J})_1$ and $Q \in \mathbb{R}(X_1, \ldots, X_m)$ such that*

(1) *$f = Q \circ (f_1 \circ L_1, \ldots, f_m \circ L_m) \circ \phi$,*
(2) *$g = Q \circ (f_1 \circ L_1, \ldots, f_m \circ L_m) \in \widetilde{\mathcal{G}}_{n+k}$, and $g$ is of class $\mathcal{C}^1$ outside $H = H_1 \cup \ldots \cup H_l$, where each $H_i \subset \mathbb{R}^{n+k}$ is an affine hyperplane, $i = 1, \ldots, l$.*

*Proof.* (1) easily follows from properties (G1) and (G4). (2) is a consequence of the fact, that unary functions of $\mathcal{G}$ are differentiable outside a finite number of points. ☐

**Theorem 3.8.** *Let $\mathcal{J} = \{\mathcal{J}_n\}_{n\in\mathbb{N}}$ be almost 0-regular with respect to the structure $\mathcal{S}$. Then an $\mathcal{S}$-geometric $\mathcal{S}$-regular family generated by $\mathcal{S} \cup \mathcal{J} = \{\mathcal{S}_n \cup \mathcal{J}_n\}_{n\in\mathbb{N}}$ is 0-regular.*

*Proof.* We will proceed by induction on $n$. Let $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{G}_1$. Then, by 3.7, there exist $\phi : \mathbb{R} \to \mathbb{R}^k$, $\phi \in \mathcal{S}$, $g \in \widetilde{\mathcal{G}}_k$, such that $f = g \circ \phi$. The function $f$ is differentiable outside $A \subset \mathbb{R}$, $\sharp A < \infty$, $\phi$ - outside $B \subset \mathbb{R}$, $\sharp B < \infty$, and $g$ - outside a finite number of affine hyperplanes $H_i \subset \mathbb{R}^k$, $i = 1, \ldots, l$. Consider a cell decomposition $\mathcal{D}$ of $\mathbb{R}$ of class $\mathcal{C}^1$ compatible with $A, B$ and such that, if $D \in \mathcal{D}$, then $\phi(D) \subset \mathbb{R}^k \setminus \bigcup H_i$, or $\phi(D) \subset H_D = H_{i_1} \cap \ldots \cap H_{i_s}$, for some $\{i_1, \ldots, i_s\} \subset \{1, \ldots, l\}$, and $f|_{H_D}$ is of class $\mathcal{C}^1$. Take a cell $D \in \mathcal{D}$. Then $g \circ \phi|_D$ is differentiable. Let $t \in \mathbb{R}$. Consider following cases:

(a) $D$ is an open cell and $\phi(D) \subset \mathbb{R}^k \setminus \bigcup H_i$; let $x \in D$. Then $f'(x) = d_{\phi(x)} g \circ d_x \phi$. If $x$ is a regular point of $f$, then rank $d_x \phi = 1$ and dim $d_{\phi(x)} g(\text{im } d_x \phi) = 1$. Let $D' \subset D$ be a set of regular points of $g$. Use 3.3 and 3.5 to obtain a family of defining functions $h_1^D, \ldots, h_{p_D}^D : \mathbb{R}^k \to \mathbb{R}^{k-1}$ for $M_D = g(D')$. Now

$$N_t^D = \text{reg } g|_D^{-1}(t) = \sum_{i=1}^{p_D} \text{reg } (f, h_i^D)^{-1}(t, 0)$$

(because ker $d_x h_i = T_x M_D$ and dim im $d_x f(T_x M_D) = 1$). Since $\widetilde{\mathcal{G}}$ is 0-regular, and $(f, h_i^D) \in \widetilde{\mathcal{G}}$, then $\text{reg} g|_D^{-1}(t) < \infty$,

(b) $D$ is an open cell and $\phi(D) \subset H_D = H_1 \cap \ldots \cap H_s$; $g = Q \circ (f_1 \circ L_1, \ldots, f_m \circ L_m)$, for some affine functions $L_1, \ldots, L_m : \mathbb{R}^k \to \mathbb{R}$, $f_1, \ldots f_m \in \mathcal{J}_1$ and $Q \in \mathbb{R}(X_1, \ldots, X_m)$. Then, there exists $C \subset \{1, \ldots, m\}$, such that $f_i \circ L_i|_{H_D} = a_i$, where $a_i \in \mathbb{R}$. We can assume that $C = \{1, \ldots, q\}$, $q \leq m$. Consider $\widetilde{g} = (a_1, \ldots, a_q, f_{q+1} \circ L_{q+1}, \ldots, f_m \circ L_m)$. We can proceed as in the case (a), and get that $N_t^D = \text{reg} g|_D^{-1}(t) < \infty$.

Now

$$\text{reg } g^{-1}(t) \leq \sum_{D \in \mathcal{D}_1} N_t^D + \sharp \mathcal{D}_0,$$

where $\mathcal{D}_i = \{D \in \mathcal{D} : \ \text{dim} D = i\}$, $i = 0, 1$.

Now assume, that the theorem is true for every $m < n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathcal{D}$ be a cell decomposition of $\mathbb{R}^n$ of class $\mathcal{C}^1$, which satisfies same conditions as in the case of $n = 1$. Let $\mathcal{D}_i = \{D \in \mathcal{D} : \ \text{dim} D = i\}$, for $i = 0, \ldots, n$. When $D \in \mathcal{D}_n$, we can proceed in the same way as in the case of $n = 1$. When dim $\mathcal{D} < n$, take $\psi_D : \mathbb{R}^{\text{dim} D} \to D$, a definable $\mathcal{C}^1$-diffeomorphism, consider functions $f^I \circ \psi_D$, where $I \in \Lambda^n_{\text{dim} D}$, and use an induction hypothesis. ☐

## 4. Appendix

We will show, that the map

$$\Omega : X' \times (\mathbb{R}^n \times (0,\infty)) \ni (x,a) \to \left(T_x g^{-1}(t) \cap X, d_x \delta'_a\right) \in G_n^{n-p-q} \times \mathbb{R}^{n*},$$

defined as in 1.8 is transverse to

$$\mathcal{H} := \left\{(T,l) \in G_n^{n-p-q} \times \mathbb{R}^{n*} : \ T \subset \mathrm{ker} l\right\}.$$

Define

$$U_I := \left\{(x_1,\dots,x_n) \in X' : \ \frac{\partial(f,g)}{\partial(x_{i_1},\dots x_{i_{p+q}})} \neq 0\right\}, \text{ where } I = (i_1,\dots i_{p+q}) \in \Lambda^n_{p+q}$$

Take $I \in \Lambda^n_{p+q}$. Put $l = n-p-q$. After changing of coordinates we may assume that $\frac{\partial(f,g)}{\partial(x_{l+1},\dots x_n)} \neq 0$. For $x \in U_I$, the tangent space of $T_x g^{-1}(t) \cap X$ is generated by vectors

$$v_1 = (\underbrace{A, 0, \dots, 0}_{n-p-q}, A_{11}, \dots, A_{1(p+q)})$$

$$\vdots$$

$$v_l = (\underbrace{0, 0, \dots, A}_{n-p-q}, A_{l1}, \dots, A_{l(p+q)}), \quad \text{where}$$

$$A = \frac{\partial(f,g)}{\partial(x_{l+1},\dots x_n)} \neq 0, \quad A_{ji} = \frac{\partial(f,g)}{\partial(x_{l+1},\dots,x_{l+i},\underset{\check{i}}{x_j},x_{l+2},\dots,x_n)},$$

for $j = 1,\dots,l, \ i = 1,\dots,p+q$. It is sufficient to consider a map

$$h = (h_1,\dots,h_l) : X' \times (\mathbb{R}^n \times (0,\infty)) \to \mathbb{R}^l,$$

where

$$h_1 = A \cdot \frac{\partial \delta'_a}{\partial e_1} + \sum_{i=1}^{l} \frac{\partial \delta'_a}{\partial e_{i+l}} \cdot A_{1i}$$

$$\vdots$$

$$h_{n-p-q} = A \cdot \frac{\partial \delta'_a}{\partial e_{n-p-q}} + \sum_{i=1}^{l} \frac{\partial \delta'_a}{\partial e_{i+l}} \cdot A_{li}$$

and show, that zero is a regular value of $h$. We will need the following lemma (its idea comes from [C]):

**Lemma 4.1.** *We can choose coordinates in $\mathbb{R}^{l+1}$ such that all solutions of the system $h(x) = 0$ are nondegenerate.*

*Proof.* Consider a map

$$(*) \qquad \phi := \frac{(\delta'_a, h_1, \dots, h_l)}{||(\delta'_a, h_1, \dots, h_l)||} : X' \times (\mathbb{R}^n \times (0,\infty)) \to \mathbb{S}^l.$$

Due to the Sard theorem, there exists a regular value $b \in \mathbb{S}^l$ of $\phi$. After using of appropriate linear isometry of $\mathbb{R}^l$, we may assume, that $b = (1, 0, \dots, 0)$. Solutions of system of equations $h = 0$ are exactly these, for which $\phi(x) = (1, 0, \dots, 0)$. Notice, that

$$\frac{\partial \phi_i}{\partial v_j} = \frac{1}{||(\delta'_a, h_1, \dots, h_l)||} \frac{\partial h_i}{\partial v_j}, \quad \text{for } i \geq 2,\ j = 1, \dots, 2n - q + 1.$$

Since the matrix

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_{2n-q+1}} \\ \frac{\partial \phi_2}{\partial x_1} & \cdots & \frac{\partial \phi_2}{\partial x_{2n-q+1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_{l+1}}{\partial x_1} & \cdots & \frac{\partial \phi_{l+1}}{\partial x_{2n-q+1}} \end{bmatrix}$$

is of maximal rank, then, in particular, the matrix created from last $l$ lines of this matrix is of rank $l$. □

Now, we can find common $b \in \mathbb{S}^l$ as in the last lemma, such that for every $I \in \Lambda^n_{p+q}$, $b$ is a regular value of the map $(*)$ (we want to find a common isometry of $\mathbb{R}^l$). In this way we obtain transversality of the map $\Omega$. To get transversality of $\Omega|_{X' \times \{a\}}$ for almost all $a \in \mathbb{R}^n \times (0, \infty)$ it suffices to use the following *parametric transversality theorem*:

**Theorem 4.2** (cf. [H], Chapter 3, Theorem 2.7)**.** *Let $V, M, N$ be a $\mathcal{C}^r$-manifolds without boundary and $A \subset N$ a $\mathcal{C}^r$-submanifold. Let $\Omega : V \to \mathcal{C}^r(M, N)$ satisfy the following condition:*

(i) *the evaluation map $\Omega^{ev} : V \times M, (v, x) \to \Omega(v)(x)$ is of class $\mathcal{C}^r$,*
(ii) *$\Omega^{ev}$ is transverse to $A$,*
(iii) *$r > \max\{0, \dim M + \dim A - \dim N\}$.*

*Then the set*

$$\pitchfork(\Omega; A) = \{v \in V : \Omega(v) \pitchfork A\}$$

*is residual.*

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