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Recovering trivialization from triangulation in o-minimal structures

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RECOVERING TRIVIALIZATION FROM TRIANGULATION IN O-MINIMAL STRUCTURES

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Abstract. We present how trivialization of definable functions in o-minimal structures can be obtained from triangulation, by using classical model theoretical tools. The proof is based on hints given in a book [D].

1. Introduction

The main aim of this paper is to derive, by using classical model theoretical methods, the Trivialization Theorem from the Triangulation Theorem. Clues how to achieve it are given in [D] in chapters 6 and 8.

This paper is organized as follows. In the first section we present the most important informations about triangulation. Next section is devoted to some fiberwise properties of definable set. The third one collects model theoretic facts, which are used in the last section in recovering trivialization from triangulation.

Proofs of all facts from the first two sections which are not included can be found in the [D]. More detailed description of the Compactness Theorem and the Diagram Lemma is contained in a course of model theory of Bruno Poizat [P].

Notations. By R we will usually denote a real closed field. We will use the following notations:

- $\Pi_{m+n,m}$ denotes a canonical projection from R^{m+n} onto first m coordinates,
- let $A \subset R^m \times R^n$; then $A_x := \{y \in R^n : (x,y) \in A\}$,
- $e_1, \ldots, e_N$ is a canonical basis of R^N,
- if $A \subset R^n$, then $\overline{A} \subset R^n$ is a topological closure of A.

2. Triangulation

We will recall notions of simplexes and complexes. These definitions are slight different from those most often used.

Definition 2.1. Let $a_0, a_1, \ldots, a_d \in R^n$ be such points, that $a_1 - a_0, \ldots, a_d - a_0$ are linearly independent. Then the set

$$(a_0, a_1, \ldots, a_d) := \left\{ t_0 a_0 + t_1 a_1 + \ldots + t_d a_d \mid t_i > 0, \sum_{i=0}^{d} t_i = 1 \right\}$$

is called a d-simplex in R^n.
Let $a_{i_0}, a_{i_1}, \ldots a_{i_k} \in \{a_0, a_1, \ldots, a_d\}$. Then $(a_{i_0}, a_{i_1}, \ldots, a_{k_k})$ is called a k-dimensional face of a simplex $(a_0, a_1, \ldots, a_n)$.
A topological closure in R^n of $(a_0, a_1, \ldots, a_d)$ is called a closed d-simplex and is denoted by $[a_0, a_1, \ldots, a_d]$; obviously

$$[a_0, a_1, \ldots, a_d] = \left\{ t_0 a_0 + t_1 a_1 + \ldots + t_n a_d \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1 \right\}.$$

Note that $[a_0, a_1, \ldots, a_n]$ is a disjoint union of faces of a simplex $(a_0, a_1, \ldots, a_d)$.

Definition 2.2. A complex in R^n is a finite collection K of simplexes of R^n such that if $\sigma_1, \sigma_2 \in K$, then

(1) $\overline{\sigma}_1 \cap \overline{\sigma}_2 = \emptyset$, or
(2) $\overline{\sigma}_1 \cap \overline{\sigma}_2 = \overline{\sigma}_3$, where σ_3 is a common face of σ_1 and σ_2.

We put
$|K| :=$ a union of simplexes in K, (called polyhedron) and
$\mathrm{Vert}(K) :=$ a set of vertices of simplexes of a complex K.

Let $\mathcal{R} = (R, <, 0, 1, +, -, \cdot, \ldots)$ be an o-minimal structure. We have the following

Theorem 2.3 (Triangulation Theorem)**.** *Let $A \subset R^m$ be a definable set. Then there exists a complex K such that A is definably homeomorphic to a polyhedron $|K|$.*

Remark 2.4. From the proof of the Triangulation Theorem (see [D], chapter 8) we can deduce, that if $A \subset R^n$ is a definable set with parameters $B \subset R$, then a homeomorphism h from this theorem can be also definable with the same set of parameters. Similarly, if $A \subset R^n$ is a definable set with parameters $B \subset R$, then its cell decomposition $\mathcal{C}$ is also definable with parameters taken from B (see the proof in [D], chapter 3).

Definition 2.5. Let $K \subset R^n, L \subset R^m$ be complexes. A map $V : \mathrm{Vert}(K) \longrightarrow \mathrm{Vert}(L)$ is called a vertex map, if for every simplex $(a_0, a_1, \ldots, a_d)$ of K, a set $\{V(a_0), \ldots, V(a_d)\}$ spans a simplex of L.

A vertex map can be extended to a map $|V| : |K| \longrightarrow |L|$ in the following way:
let $(a_0, a_1, \ldots, a_k) \in K$, for some $a_0, \ldots, a_k \in R^n$. Then

$$|V| \left(\sum_{i=0}^{k} t_i a_i \right) = \sum_{i=0}^{k} t_i V(a_i), \quad \text{for} \quad t_i > 0, \quad \sum_{i=0}^{k} t_i = 1.$$

Observation 2.6. *Let $K \subset R^n, L \subset R^m$ be complexes, and $V : \mathrm{Vert}(K) \longrightarrow \mathrm{Vert}(L)$ be a vertex map. Then a map $|V| : |K| \longrightarrow |L|$ is continuous.*

Proof. We can easily extend the map $|V|$ to a map $\overline{|V|} : \overline{|K|} \longrightarrow \overline{|L|}$, by defining $\overline{|V|}$ for every $[a_0, a_1, \ldots, a_k] \subset \overline{|K|}$ by

$$\overline{|V|}\left(\sum_{i=0}^{k} t_i a_i\right) = \sum_{i=0}^{k} t_i V(a_i), \quad \text{for} \quad t_i \geq 0, \quad \sum_{i=0}^{k} t_i = 1.$$

We will show that $\overline{|V|}$ is continuous. Then $|V|$, as restriction of $\overline{|V|}$ to some faces of simplexes, also will be continuous.

Let $[a_0, a_1, \ldots, a_k]$ be a closed k-simplex in R^n. Define

$$A := \left\{(t_1, \ldots, t_k) \in R^k : \ t_1 \geq 0, \ldots, t_k \geq 0, \sum_{i=1}^{k} t_i \leq 1\right\},$$

and let

$$\Phi : A \ni (t_1, \ldots, t_k) \longrightarrow a_0 + \sum_{i=1}^{k} t_i(a_i - a_0).$$

is continuous as an affine map. Since $a_1 - a_0, \ldots, a_k - a_0$ are linearly independent, there exists Φ^{-1}, and is also affine. We get, that

$$C \subset [a_0, a_1, \ldots, a_d] \text{ is closed } \Leftrightarrow \text{ Bar}(C) := \Phi^{-1}(C) \text{ is closed.}$$

Now consider $\widetilde{V} = \overline{|V|}|_{[a_0, a_1, \ldots, a_d]}$, for some $[a_0, a_1, \ldots, a_d] \subset \overline{|K|}$. Let

$$C \subset [V(a_0), V(a_1), \ldots, V(a_d)]$$

be a closed set. Then

$$\text{Bar}(C) = \text{Bar}\left(\widetilde{V}^{-1}(C)\right).$$

We have showed that $\overline{|V|}$ is continuous in restriction to each closed d-simplex $[a_0, a_1, \ldots, a_d] \subset \overline{|V|}$. Such simplexes constitute a finite, close covering of $|K|$. Now it is enough to use the following theorem

Let $f : X \longrightarrow Y$ be a map between two topological spaces, X and Y. If there exists a finte, closed covering of X, $X = X_1 \cup \ldots \cup X_l$ such that every $f|_{X_i}$ is continuous, then f is continuous. □

Combining 2.3 and 2.6, we immediately get

Corollary 2.7. *Let $A \subset R^m$ be a definable set. Then there exists $N \in \mathbb{N}$, such that A is definably homeomorphic to a union of faces of $(e_1, \ldots, e_N) \in R^N$.*

3. Fiberwise properties of definable sets

Consider an o-minimal expansion $\mathcal{R}$ of a real closed field R;

$$\mathcal{R} = (R, <, 0, 1, +, -, \cdot, \ldots).$$

The aim of this section is to show that if a definable function $f : A \longrightarrow R^k$, with $A \subset R^{n+m}$, is a homeomorphism on each fiber A_x, then there exists a definable

partition $\mathcal{C}$ of $\Pi_{n+m,m}(A)$, such that f is an homeomorphism on any set $(C \times R^m) \cap A$, where $C \in \mathcal{C}$.
We begin with the following

Theorem 3.1. *Let $S \subset R^{n+m}$ be a definable set such that for every $x \in R^n$ the fiber S_x is an open (closed) set in R^m. Then there is a definable partition of R^n into definable cells $C_1, \ldots, C_k$ such that for each $i \in \{1, \ldots, k\}$ the set $S \cap (C_i \times R^m)$ is open (closed) in $C_i \times R^m$.*

Proofs of the theorem 3.1, as well as corollaries 3.2, 3.4 and proposition 3.5 can be found in [D], chapter 6.2.

Corollary 3.2. *Let $S' \subset S$ be definable sets in R^{n+m} and let $A \subset R^n$ be a definable set such that for every $x \in A$ the fiber S'_x is open (closed) in S_x. Then there exists a definable partition of A into definable cells $A_1, \ldots, A_k$ such that for each $i \in \{1, \ldots, k\}$ the set $S' \cap (A_i \times R^m)$ is open (closed) in $S \cap (A_i \times R^m)$.*

Definition 3.3. Let $f : A \longrightarrow R^m$, with $A \subset R^n$. We say, that f is locally bounded if for each point $a \in A$ there exists a neighbourhood $U \subset A$ such that $f(U)$ is bounded.

Remark 3.4. Let $f : A \longrightarrow R^m$, where $A \subset R^n$, be a definable map. Then the following conditions are equivalent:

(i) f is continuous,
(ii) f is locally bounded and $\Gamma(f)$ is closed in $A \times R^n$.

Then we have the following consequence of 3.2 and 3.4:

Corollary 3.5. *Let $S \subset R^{n+m}$ be a definable set, and $f : S \longrightarrow R^k$ a locally bounded definable map. Moreover, let $A \subset R^n$ be a definable set such that for every $a \in A$ a function $f_a : S_a \longrightarrow R$, where $f_a(x) = f(a, x)$, is continuous. Then there exists a definable partition $A_1, \ldots, A_l$ of A such that for every $i \in \{1, \ldots, l\}$ a function*

$$f|S \cap (A_i \times R^m) : S \cap A_i \times R^m \longrightarrow R^k$$

is continuous.

We will show that we can omit the assumptions of local boundedness of function. We have the following

Lemma 3.6. *Let $f : S \longrightarrow R^k$ be a definable map, where $S \subset R^n$. Then there exists a definable partition $S_1, \ldots S_l$ of S, such that for every $i \in \{1, \ldots, l\}$ a function*

$$f|A_i : A_i \longrightarrow R^k$$

is locally bounded.

Proof. It is enough to apply the cell decomposition theorem, [D], Chapter 3, (2.11), $(\mathrm{II})_n$, and Remark 3.4. ☐

After combining 3.5 and 3.6 we get

Proposition 3.7. *Let $S \subset R^{n+m}$ be a definable set, and $f : S \longrightarrow R^k$ a definable map. Moreover, let $A \subset R^n$ be a definable set such that for every $a \in A$ a function $f_a : S_a \longrightarrow R$, where $f_a(x) = f(a,x)$, is continuous. Then there exists a definable partition $A_1, \ldots, A_l$ of A such that for every $i \in \{1, \ldots, l\}$ a function*

$$f|S \cap (A_i \times R^m) : S \cap (A_i \times R^m) \longrightarrow R^k$$

is continuous.

Now we can state the following proposition:

Proposition 3.8. *Let $S \subset R^{n+m}$ be a definable set and $f : S \longrightarrow R^k$ a definable map. Assume that there exists a definable set $A \subset R^n$ such that $f|S \cap (A \times R^n)$ is injective and $f_a : S_a \longrightarrow R^n$ is a homeomorphism from S_a onto $f_a(S_a)$ for every $a \in A$. Then there exists a definable partition $\mathcal{A} = \{A_1, \ldots, A_l\}$ of A such that for each $i \in \{1, \ldots, l\}$ a function*

$$f|S \cap (A_i \times R^m) : S \cap (A_i \times R^m) \longrightarrow R^k$$

is a homeomorphism from $S \cap (A_i \times R^n)$ onto $f\big(S \cap (A_i \times R^n)\big)$.

Proof. By 3.7 there exist a definable partition $\mathcal{B} = \{B_1, \ldots, B_p\}$ of A such that for every $i \in \{1, \ldots, p\}$ a function

$$f|S \cap (B_i \times R^m) : S \cap (B_i \times R^m) \longrightarrow R^k$$

is continuous. Define

$$f_i = f|S \cap (B_i \times R^m), \quad \text{for} \quad i \in \{1, \ldots, p\},$$

and sets

$$S'_i = \big\{(a,c) \in R^n \times R^k \mid a \in B_i \;\wedge\; \exists\, z \in S_a :\; f(z) = c\big\}\,.$$

Define functions

$$g_i : S'_i \ni (a,c) \longrightarrow f^{-1}(c) \in S \cap (B_i \times R^n).$$

Use again 3.7 for each of functions g_i, $i = \{1, \ldots, p\}$. We get partitions $\mathcal{A}_i = \{A_{1_i}, \ldots, A_{l_1}\}$ of B_i such that for every $S'_i \cap \big(A_{j_i} \times R^k\big)$ a function $g_i|S'_i \cap \big(A_{j_i} \times R^k\big)$ is continuous. We claim that

$$f_i|S \cap (A_{j_i} \times R^m) \longrightarrow R^k \quad \text{for} \quad j_i \in \{1_i, \ldots, p_i\}$$

is a homeomorphism onto its image. We will show that $f^{-1}|f\,(S \cap (A_{j_i} \times R^m))$ is continuous. Take $c \in f\big(S \cap (A_{j_i} \times R^m)\big)$ and unique $a \in A_{j_i}$ such that $(a,c) \in S'_i \cap \big(A_{j_i} \times R^k\big)$. Let U be a neighbourhood of (a,c). Since $g_i|S'_i \cap \big(A_{i_j} \times R^k\big)$ is continuous, there exists a neighbourhood $V = (a_1, a_2) \times W$ of $g_i^{-1}(a, c) = (a, f^{-1}(c))$, where W is a neighbourhood of c, such that

$$g_i(V) = \mathrm{im} f^{-1}(V) \subset U.$$

Then $\mathcal{A} = \bigcup_{i=1}^{p} \mathcal{A}_i$ is a partition with the desired properties. □

4. Model theoretic tools

We will need the following version of a fundamental theorem of first order model theory

Theorem 4.1. (Compactness Theorem). *Suppose that $\mathcal{T}$ is a theory in a language $\mathcal{L}$. Assume that there exists a family $\mathcal{A} = \{\phi_i\}_{i \in I}$ of $\mathcal{L}$-sentences such that for every model $\mathcal{M}$ of the theory $\mathcal{T}$ there is $i \in I$ such that $\mathcal{M} \models \phi_i$. Then we can choose a finite set $\{\phi_1, \ldots, \phi_k\} \subset \mathcal{A}$ such that for every model $\mathcal{M}$ of the theory $\mathcal{T}$ there exists $j \in \{1, \ldots, k\}$ such that $\mathcal{M} \models \phi_j$.*

Another useful tool is

Lemma 4.2. (The diagram lemma). *Let $\mathcal{A}$ and $\mathcal{B}$ be models for a language $\mathcal{L}$. Let A and B be universes of models $\mathcal{A}$ and $\mathcal{B}$, respectively. Then $\mathcal{A}$ is elementarily embedded in $\mathcal{B}$ iff $\mathcal{B}$ can be expanded to a model of* $\mathrm{Th}(\mathcal{A}_A)$.

We should also recall the remarkable result

Theorem 4.3. *Let $\mathcal{M}$ be an o-minimal structure in a language $\mathcal{L}$. Then every structure $\mathcal{N}$ elementarily equivalent to $\mathcal{M}$ is also o-minimal.*

Proof. Let

$$\mathcal{M} = (M, <, 0, 1, +, -, \cdot, \ldots)$$

be an o-minimal structure, and

$$\mathcal{N} = (N, <, 0, 1, +, -, \cdot, \ldots)$$

a structure elementarily equivalent to $\mathcal{M}$. Let $A \subset N$ be a definable set in $\mathcal{N}$,

$$A = \{x \in N \mid \mathcal{N} \models \phi(\underline{a}, x)\},$$

where $\underline{a} = (a_1, \ldots, a_n) \in N^n$. Consider a definable set

$$B_M := \left\{(\underline{t}, x) \in M^{n+1} : \ \mathcal{M} \models \phi(\underline{t}, x)\right\}$$

By 2.4, B_M has a 0-definable cell decomposition $\mathcal{C}$. Since $\mathcal{M} \equiv \mathcal{N}$, the set

$$B_N := \left\{(\underline{t}, x) \in N^{n+1} : \ \mathcal{N} \models \phi(\underline{t}, x)\right\}$$

has the same cell decomposition $\mathcal{C}$. Therefore the set $A = B_{N,\underline{a}}$ is a finite unions of intervales. □

Remark 4.4. This theorem was first originally proved in [KPS].

5. Triangulation and trivialization

Let $\mathcal{R} = (R, <, 0, 1, +, -, \cdot, \ldots)$ be a fixed model of an o-minimal theory $T = \mathrm{Th}(\mathcal{R})$. From now we will mean by a definable set A a set definable with parameters in R, if it will not be otherwise stated. We will recall the definition of a trivialization of a definable function $f : A \longrightarrow B$:

Definition 5.1. Let $f : A \longrightarrow B$ be a definable function, with $A \subset R^m$ and $B \subset R^n$. We say that f is definably trivial over a definable set $C \subset B$, if there exist a definable set $F \subset R^N$, for some N, and a definable homeomorphism $h : f^{-1}(C) \longrightarrow C \times F$, such that the following diagram commutes

$$\begin{array}{ccc} A \supset f^{-1}(C) & \xrightarrow{\quad h \quad} & C \times F\ , \\ & {\scriptstyle f} \searrow \qquad \swarrow {\scriptstyle p} & \\ & C & \end{array}$$

where p is the canonical projection onto the first n coordinates.

The following proposition is the most important step in recovering trivialization from triangulation:

Proposition 5.2. *Let $A \subset R^m \times R^n$ be definable. Then there exists a definable map $f : A \longrightarrow R^N$, such that for each $b \in B = \Pi_{m+n,n}(A)$ the map $f_b : A_b \longrightarrow R^N$ is a homeomorphism onto a union of faces of the simplex $(e_1, \ldots, e_N)$.*

Proof. Let $\mathcal{L}$ be the language of the structure $\mathcal{R}$. Augment the language $\mathcal{L}$ to

$$\mathcal{L}_R = \mathcal{L} \cup \{c_a : \ a \in \mathcal{R}\}$$

by adding new constant symbols for every $a \in \mathcal{R}$. We will identify a with c_a. Add new constants $\underline{d} = (d_1, \ldots, d_m)$ and define a language

$$\mathcal{L}^* := \mathcal{L}_R \cup \{d = (d_1, \ldots, d_m)\}.$$

Consider any model $\mathcal{S} = (S, <, 0, 1, +, -, \cdot, \ldots)$ of theory T in the language $\mathcal{L}^*$. We interpret constants $d_1, \ldots, d_m$ as some elements $b_1, \ldots, b_m$. By the diagram lemma 4.2, $\mathcal{S}$ is an elementary extension of $\mathcal{R}$. Moreover, by 4.3, the structure $\mathcal{S}$ is o-minimal. Consider a formula ϕ, which describes a definable set

$$A = \{(\underline{s}, \underline{t}) \in R^m \times R^n \mid \ \mathcal{R} \models \phi(\underline{s}, \underline{t})\}.$$

Let

$$A^{\mathcal{S}} := \{(\underline{s}, \underline{t}) \in S^m \times S^n \mid \ \mathcal{S} \models \phi(\underline{s}, \underline{t})\},$$

be the counterpart of the definable set A in the structure $\mathcal{S}$. $A^{\mathcal{S}}$ does not depend on ϕ since $\mathcal{R} \prec \mathcal{S}$. Define

$$A^{\mathcal{S}}_{\underline{b}} = \Big\{(\underline{b}, \underline{t}) \in S^m \times S^n \mid \ \mathcal{S} \models \phi(\underline{b}, \underline{t})\Big\}.$$

By the theorem on definable triangulation and corollary 2.7, there exists a definable homeomorphism $f_{\mathcal{S},\underline{b}} : A_{\underline{b}}^{\mathcal{S}} \longrightarrow S^N$, for some $N = N(\mathcal{S},\underline{b})$, onto a union of faces $\triangle = \triangle(\mathcal{S},\underline{b})$ of the simplex $(e_1,\ldots,e_N)$. The fact that $f_{\mathcal{S},\underline{b}}$ is a homeomorphism is described by an $\mathcal{L}^*$-sentence $\chi(\underline{d},\underline{t},\underline{u}) = \chi_{\mathcal{S},\underline{b}}(\underline{d},\underline{t},\underline{u})$, where $\underline{u} = (u_1,\ldots,u_N)$ and $\underline{t} = (t_1,\ldots,t_n)$. Observe that the sentence $\chi(\underline{d},\underline{t},\underline{u})$ containes (possibly many times) as subformulas $\phi(\underline{d},\underline{t})$. which defines fiber of the set A over $\underline{b}$, and $\psi(\underline{d},\underline{t},\underline{u}) = \psi_{\mathcal{S},\underline{b}}(\underline{d},\underline{t},\underline{u})$, describing $\Gamma(f_{\mathcal{S},\underline{b}})$. By the compactness theorem we can choose a finite number of sentences

$$\chi_1(\underline{d},\underline{t},\underline{u}),\ldots,\chi_k(\underline{d},\underline{t},\underline{u})$$

from sentences $\chi_{\mathcal{S},\underline{b}}(\underline{d},\underline{t},\underline{u})$, such that for each model $\mathcal{S}$ of T and each choice of elements $b_1,\ldots,b_m$ there exists $i \in \{1,\ldots,k\}$, such that a homeomorphism $f_{\mathcal{S},\underline{b}}$ is described by the sentence χ_i, and its graph by the formula ϕ_i.
Consider definable sets

$$A_i = \{\underline{b} \in R^m : \quad \mathcal{R} \models \chi_i(\underline{b},\underline{t},\underline{u})\}\,.$$

Clearly, $\bigcup A_i = R^m$. Consider a cell decomposition $\mathcal{C}$ of R^m compatible with $A_1,\ldots,A_k$. Let $C \in \mathcal{C}$ be such that $C \subset A_i$, for some $i \in \{1,\ldots,k\}$. Define $f|_C$ as a function whose graph is described by a formula $\phi_i(\underline{d},\underline{t},\underline{u})$. Now it suffices to take $N = \max\{N_1,\ldots N_k\}$ and glue together functions $f|_C$. □

We can now formulate the trivialization theorem for definable sets:

Theorem 5.3. *Let $S \subset R^{m+n}$ be a definable set. There exists a definable partition $\mathcal{A} = \{A_1,\ldots,A_k\}$ of R^m, such that for each $i \in \{1,\ldots,k\}$ there is a definable set $F_i \subset R^n$ and a definable homeomorphism*

$$h_i : S \cap (A_i \times R^n) \longrightarrow A_i \times F_i$$

such that the diagram

$$\begin{array}{ccc} S \cap (A_i \times R^n) & \xrightarrow{\ h_i\ } & A_i \times F_i\ , \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & A_i & \end{array}$$

where p is the canonical projection onto the first m coordinates, is commutative.

Proof. Take, as in proposition 5.2, a map $f : S \longrightarrow R^N$ and define

$$g : S \ni (a,b) \longrightarrow (a, f(a,b)) \in R^m \times R^N.$$

Consider a family $\mathcal{B}$ of all possible choices of faces of the simplex $[e_1,\ldots,e_N]$. Let $B \in \mathcal{B}$ and take

$$\widetilde{A}_B := \Pi_{m+n,n} g^{-1}\left(R^m \times B\right).$$

Then, for each $a \in \widetilde{A}_B$, the function $g_a : S_a \longrightarrow B$ is a homeomorphism and g restricted to the set $S \cap \left(\widetilde{A}_B \times R^n\right)$ is injective. By 3.8 there exists a definable partition $\mathcal{A}_B = \{A_{1_B}, \ldots, A_{k_B}\}$ of $\widetilde{A}_B$, such that $g|S \cap \left(\widetilde{A}_{i_B} \times R^n\right)$ is a homeomorphism, for every $i \in \{1_B, \ldots, k_B\}$. Let $a_{i_B} \in A_{i_B}$ and define

$$p_{i_B} : B \ni b \longrightarrow g^{-1}(a_{i_B}, b) \in F_i \quad \text{where} \quad F_i := g^{-1}(a_{i_B}, B),$$

and a map

$$h_{i_B} : S \cap (A_{i_B} \times R^n) \ni (a, b) \longrightarrow \left(a, p_{i_B}^{-1}\big(f(a,b)\big)\right) \in A_i \cap F_i.$$

h_{i_B} is a trivialization of $S \cap (A_{i_B} \times R^n)$. Now it is enough to take

$$\mathcal{A} = \bigcup_{B \in \mathcal{B}} \mathcal{A}_B.$$

□

As an easy consequence we get

Corollary 5.4 (Trivialization Theorem)**.** *Let $f : A \longrightarrow R^m$, where $A \subset R^n$, be a definable map. Then there exists a definable decomposition $\mathcal{C}$ of $B := \operatorname{im} f$, such that for every $C \in \mathcal{C}$ the map f is trivial over C.*

Proof. Let $\Gamma(f) \subset R^n \times R^m$ denotes the graph of the map f. By theorem 5.3, there exists a definable partition $\mathcal{A} = \{A_1, \ldots, A_k\}$ of R^m, such that for each $A_i \in \mathcal{A}$ there are a definable set $F_i \subset R^n$ and a definable homeomorphism g_i, such that the diagram

$$\begin{array}{ccc} \Gamma(f) \cap (R^n \times A_i) & \xrightarrow{\quad g_i \quad} & A_i \times F_i \\ & \tilde{p} \searrow \qquad \swarrow p & \\ & A_i & \end{array}$$

is commutative, where $\tilde{p}$ and p are projections, onto the last and the first m coordinates, respectively. Let $F : A \longrightarrow R^{n+m}$, $F := (\mathrm{id}_{R^n}, f)$. It is now enough to take a homeomorphism $g_i := h_i \circ F|_{A_i}$ to get a diagram

$$\begin{array}{ccccc} f^{-1}(A_i) & \xrightarrow{F} & \Gamma(f) \cap (R^n \times A_i) & \xrightarrow{g_i} & A_i \times F_i \\ & f \searrow & \downarrow \tilde{p} & \swarrow p & \\ & & A_i & & \end{array}$$

with the desired properties. □

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