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On Khovanskii theory

Praca semestralna nr 3

(semestr letni 2012/13)

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ON KHOVANSKI THEORY

Abstract. We give, in an elementary way, an upper bound on number of connected components of Pfaff sets. As an application, we recall a well known result, that the structure $\mathbb{R}_{\text{Pfaff}}$ is o-minimal.

1. Introduction

In [K], Khovanski gave several formulas on upper bounds on number of connected components of Pfaff sets. The aim of this note is to give weaker, but short and elementary bound on connected components of such sets.

This note is organized as follows. In the second section we recall definition of Pfaff functions and give some easy examples. The third, mostly based on [K], is devoted to showing Bezout like theorems for Pfaff functions. In the fourth section, which is based on [C], we give a formula on an upper bound of number of connected components of Pfaff sets. This formula depends on complexity of Pfaff functions defining such sets. In the last section we recall a well known result, that the structure $\mathbb{R}_{\text{Pfaff}}$ is o-minimal.

2. Pfaff functions

Definition 2.1. Let be given a finite sequence of functions $F = (f_1, \dots, f_l)$, where $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \le i \le l$. The sequence F is called a Pfaff sequence, if there exist $P_{i,j} \in \mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_i]$, $1 \le i \le l$, $1 \le j \le n$, such that

$$\frac{\partial f_i}{\partial x_j} = P_{i,j}\left(x, f_1(x), \dots, f_i(x)\right), \quad x \in \mathbb{R}^n.$$

By a complexity of the sequence F we understand a pair (l, k), where $k = \max\{\deg P_{i,j}\}$.

Definition 2.2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a Pfaff function, if there exists a Pfaff chain $F = (f_1, \dots, f_l)$, such that $f_l = f$.

Remark 2.3. Pfaff functions are of class $\mathcal{C}^\infty$.

Example 2.4.

(i) Polynomial functions are Pfaff functions,

(ii) Let $k \in \mathbb{N}$; for $i = 1, \dots, l$ we define

$$f_1(t) = t,\ t \in \mathbb{R} \quad \text{and} \quad f_i(t) = \mathrm{e}^{f_{i-1}(t)}, \quad \text{for } 1 < i \le l.$$

This is a Pfaff sequence of length l, where $P_i = y_i \cdot y_{i-1} \cdots y_2$, for $1 < i \le l$,

(iii) Let $f_1 = (x^2+1)^{-1}$, $f_2 = \arctan x$. A sequence (f_1, f_2) is a Pfaff sequence, where $f_1' = -2xf_1^2$ and $f_2' = f_1$.

3. Bezout type theorem for Pfaff functions

The aim of this section is proving Bezout type theorem for Pfaff functions. It will be proceeded by a series of lammas.

Lemma 3.1. *Let $\Gamma \subset \mathbb{R}^n$ be a compact, connected submanifold of dimension 1, $g : \Gamma \to \mathbb{R}$ a function of class $\mathcal{C}^\infty$, and ξ a nowhere vanishing smooth vector field on Γ. Define $g_\xi(a) := g'(a)(\xi)$, $a \in \Gamma$. Then*

(i) $\Gamma = \bigcup_{i=1}^{l} \gamma_i^*$, *where $\gamma_i : [b_i, c_i] \to \Gamma$, $(b_i < c_i)$ is an arcwise parameterized, regular curve, and $\gamma_k^* \nsubseteq \gamma_j^*$, $k \neq j$, $k, j \in \{1, \ldots, l\}$.*

(ii) $\mathrm{sgn}\Big(g_\xi\big(\gamma_i(t)\big)\Big) = \mathrm{sgn}\Big(g'\big(\gamma_i(t)\big)\big(\gamma_i'(t)\big)\Big)$, $t \in [0, 1]$.

Proof. Existence of curves γ_i^* from the first point follows directly from definition of manifold and of possibility of arcwise parametrization of every regular curve. Set $i \in \{1, \ldots, l\}$ and γ_i. There exists a function $\alpha_i(t) : [b_i, c_i] \to \mathbb{R}$, such that $\alpha_i(t)\gamma_i'(t) = \xi(\gamma(t))$, $t \in [b_i, c_i]$. Function α_i is nowhere vanishing and continuous function . It follows that $\alpha_i > 0$ or $\alpha_i < 0$. If $\alpha_i > 0$, then

$$\mathrm{sgn}\Big(g'\big(\gamma_i(t)\big)\big(\gamma_i'(t)\big)\Big) = \mathrm{sgn}\Big(\alpha_i(t) g'\big(\gamma_i(t)\big)\big(\gamma_i'(t)\big)\Big) = \mathrm{sgn}\Big(g'\big(\gamma_i(t)\big)(\xi)\Big).$$

If $\alpha_i < 0$, then by taking an arc $\gamma_i(c_i - t)$ we can check as before in an analogous way, that

$$\mathrm{sgn}\Big(g'\big(\gamma_i(c_i - t)\big)\big(\gamma_i'(c_i - t)\big)\Big) = \mathrm{sgn}\Big(g_\xi\big(\gamma(c_i - t)\big)\Big),$$

which ends the proof of lemma. □

Lemma 3.2. *Let $\Gamma \subset \mathbb{R}^n$ be a compact, connected submanifold of dimension 1, $g : \Gamma \to \mathbb{R}$ a function of class $\mathcal{C}^\infty$, and ξ a nowhere vanishing smooth vector field on Γ. Let $g_\xi(a) = g'(a)(\xi)$, $a \in \Gamma$. If $g^{-1}(0)$ is a regular value and $x \in g^{-1}(0)$, then there exists $y \in \Gamma$, such that*

(i) $\mathrm{sgn}\big(g_\xi(x)\big) = -\mathrm{sgn}\big(g_\xi(y)\big)$,

(ii) *there exists $\gamma_{xy} : [b_x, c_x] \to \Gamma$, a curve of class $\mathcal{C}^1$ connecting y and x, such that $\gamma\big((b_x, c_x)\big) \cap g^{-1}(0) = \emptyset$.*

Proof. Let $\Gamma = \bigcup_i^l \gamma_i^*$, as in the previous lemma. We can assume, that $x \in \gamma_1^*$, $\gamma_i(c_i) \in \gamma_{i+1}^*$, $\gamma_l(c_l) \in \gamma_1^*$. We have two possibilities:

(i) $\gamma_1^* \cap (g^{-1}(0) \backslash \{x\}) \neq \emptyset$. There exist $t_1 \in [b_1, c_1]$ such that $\gamma(t_1) = x$ and the lowest $t_2 > t_1$: $\gamma_1(t_2) = y \in g^{-1}(0)$ ($g^{-1}(0)$ is a finite set). By the Rolle theorem

$$\mathrm{sgn}\Big(g_\xi(x)\Big) = \mathrm{sgn}\Big(g'\big(\gamma_1(t_1)\big)\big(\gamma_1(t_1)\big)\Big) = -\mathrm{sgn}\Big(g'\big(\gamma_1(t_2)\big)\big(\gamma_1(t_2)\big)\Big) = \mathrm{sgn}\Big(g_\xi(y)\Big).$$

(ii) $\gamma_1^* \cap (g^{-1}(0) \backslash \{x\}) = \emptyset$. We can glue together curves γ_1 and γ_2 because of their arcwise parametrization and the previous lemma. We obtain $\widetilde{\gamma}_2$, a curve of class $\mathcal{C}^1$ and we can repeat our procedure.

If for a curve $\widetilde{\gamma}_l$ created by gluing $\gamma_1, \ldots, \gamma_l$ we would have $\widetilde{\gamma}_l^* \cap \Big(g^{-1}(0)\backslash\{x\}\Big) = \emptyset$, then by taking a curve γ being a gluing of $\widetilde{\gamma}_l$ and γ_1 we get a function $G = g \circ \gamma : [b,c] \to \mathbb{R}$, $(b < c)$, and $G(t_1) = G(t_2) = G(x)$, for some $t_1 < t_2$. Then

$$\operatorname{sgn}\Big(g_\xi(x)\Big) = \operatorname{sgn}\Big(g'\big(\gamma(t_1)\big)\big(\gamma(t_1)\big)\Big) = -\operatorname{sgn}\Big(g'\big(\gamma(t_2)\big)\big(\gamma(t_2)\big)\Big).$$

It follows that $\operatorname{sgn}\Big(g_\xi(x)\Big) = 0$. We get a contradiction with regularity of zero. We can obtain a curve γ_{xy} by restriction of $\widetilde{\gamma}_i$, for suitable $i \in \{1, \ldots, l\}$. □

Definition 3.3. Let M be a smooth manifold, $\dim M = d$, and $f : M \to \mathbb{R}^n$ be a function of class $\mathcal{C}^\infty$. Let $k = \min\{d, m\}$. Define

$$\mathcal{N}(f) := \sup_{a \in M} \left\{\sharp N_a^f\right\}, \quad \text{where} \quad N_a^f := \left\{m \in f^{-1}(a) \mid \operatorname{rank} f'(m) = k\right\}.$$

Lemma 3.4. *Let $\Gamma \subset \mathbb{R}^n$ be a compact manifold of dimension 1, $g : \Gamma \to \mathbb{R}$, a function of class $\mathcal{C}^\infty$, and ξ a nowhere vanishing smooth vector field on Γ. Let $g_\xi(a) = g'(a)(\xi)$, $a \in \Gamma$. If $g^{-1}(0)$ is a regular value and $\mathcal{N}(g_\xi) \le N$, $N \in \mathbb{N}$, then $\sharp g^{-1}(0) \le N$.*

Proof. $\Gamma = \bigcup_{i=1}^{l} \Gamma_i$, where Γ_i are connected components of Γ. If $\Gamma_i \cap g^{-1}(0) \neq \emptyset$, then $\Gamma_i \cap g^{-1}(0) = \{a_{i1}, \ldots, a_{in_i}\}$. Due to the lemma 3.2 we may assume, that for every pair $a_{i,j}, a_{i,j+1}$ there exists a curve γ_{ij} connecting these two points $\gamma_{i,j}^* \cap g^{-1}(0) = \{a_{i,j}, a_{i,j+1}\}$. Take

$$A = \min_{\substack{1 \le i \le l \\ 1 \le j \le n_i}} \{|g_\xi(a_{i,j})|\}.$$

By lemma 3.2 the function g_ξ realises all values from interval $[-A, A]$ on $\gamma_{i,j}^*$. By the Sard theorem there exists $\varepsilon > 0$ such that $\varepsilon < A$, and ε is a regular value for g_ξ. It follows that $\sharp g^{-1}(0) \le \mathcal{N}(g_\xi) \le N$. □

Observation 3.5. *Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $G : \mathbb{R}^{n+1} \to \mathbb{R}$ be a function of class $\mathcal{C}^\infty$. Let $b \in \mathbb{R}$ be a regular value for F, $F^{-1}(b) = \Gamma$. Then there exists a nowhere vanishing vector field ξ na Γ, such that*

$$G'(x)\big(\xi(x)\big) = \det \frac{\partial(F_1, \ldots, F_n, G)}{\partial x_1 \ldots \partial x_{n+1}}(x), \ x \in \Gamma.$$

Proof. Using Laplace expansion we have

$$\det \frac{\partial(F_1, \ldots, F_n, G)}{\partial x_1 \ldots \partial x_{n+1}}(x) = \sum_{i=1}^{n+1} (-1)^{n+1+i} \frac{\partial G}{\partial x_1}(x) \widetilde{F}_i(x), x \in \Gamma,$$

where $\widetilde{F}_i(x)$ are determinants of matrices created by deleting i-th column and the last row . We define $\xi(x) := (\widetilde{F}_1, \ldots, \widetilde{F}_{n+1})$. We have $\xi(x) \neq 0$ for every $x \in \Gamma$, because b is a regular value for F. Let us also notice, that $\xi(x) \in \ker F'(x)$. We have showed, that ξ is a desired vector field on Γ. □

Proposition 3.6. *Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $G : \mathbb{R}^{n+1} \to \mathbb{R}$ be functions of class $\mathcal{C}^\infty$. Moreover, we assume that F is proper. Let $a \in \mathbb{R}$ be a regular value for G, $G^{-1}(a) = M$. Let*

$$H = \left(F, \det \frac{\partial(F_1, \dots, F_n, G)}{\partial x_1 \dots \partial x_{n+1}}\right).$$

If $\mathcal{N}(H) \leq N$, then $\mathcal{N}(F|_M) \leq N$.

Proof. Let us assume at the beginning, that $b \in \mathbb{R}^n$ is a regular value for F and $\widetilde{F} = F|_M$. Let $F^{-1}(b) = \Gamma$. Observe that (b, a) is a regular value for (F, G). Indeed, $\ker G'(x) = T_x M$, $\operatorname{im} G'(x) = \mathbb{R}$ and $F'|_M(x)(TM_x) = \mathbb{R}^n$, when $x \in (F, G)^{-1}(b, a)$. It follows that $\operatorname{im}(F', G')(b, a) = \mathbb{R}^{n+1}$. Let us take $g := G|_\gamma$ and nowhere vanishing vector field ξ on Γ, such that

$$g'(x)(\xi(x)) = \det \frac{\partial(F_1, \dots, F_n, G)}{\partial x_1 \dots \partial x_{n+1}}(x), x \in \Gamma.$$

Since b is a regular value for g, and if $x \in N_c^{g_\xi}$, then also $x \in N_c^H$ (for every $c \in \mathbb{R}$), we can use lemma 3.4. We get, that $\sharp \widetilde{F}^{-1}(b) \leq \mathcal{N}(H) \leq N$. Let now $b \in \mathbb{R}^n$ be a critical value. Assume, for contrary, that $\sharp N_b^{\widetilde{F}} > N$. Then $\{a_1, \dots, a_{N+1}\} \subset N_b^{\widetilde{F}}$. By the inverse function theorem, for every $i \in \{1, \dots, N+1\}$ there exist U_i, such that $a_i \in U_i$ and $\widetilde{F}|_{U_i}$ are homeomorphisms on their images. Take an open set $V \in \mathbb{R}^n$, such that $b \in V \subset \bigcap_{i=1}^{N+1} \widetilde{F}(U_i)$. By the Sard theorem there exists $c \in V$ such that c is the regular value for F. As a result we get, that $\sharp \widetilde{F}^{-1}(c) \geq N + 1$; a contradiction. □

Theorem 3.7. *Let $F = (f_1, \dots, f_l)$ be a Pfaff chain, $f_i : \mathbb{R}^n \to \mathbb{R}$, $P_1, \dots P_n \in \mathbb{R}[x, z, y_1, \dots y_k]$, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$. Consider a system of equations*

$$(\star) \qquad \begin{cases} P_1\Big(x, z, f_1(x), \dots, f_k(x)\Big) = 0 \\ \vdots \\ P_{m+n}\Big(x, z, f_1(x), \dots, f_k(x)\Big) = 0 \end{cases}$$

Number of nondegenerate solutions of the system (power of $N_0^{(P_1, \dots, P_{n+m})}$) is finite, depends only on complexity (l, k) of the Pfaff chain, numbers $q_1 = \deg P_1, \dots, q_{n+m} = \deg P_{n+m}$, m and n. Moreover, it can be estimated from above by

$$\varphi(q_1, \dots, g_{n+m}, l, k, m, n) = 2^{\frac{(l-1)l}{2}} q_1 \dots q_{n+m} \left(\sum_{j=1}^{n+m} q_j + nk - n - m + 1\right)^l.$$

Proof. We may assume, that all polynomials are nonconstant. Otherwise, the system of equations would not have nondegenerate solutions.

We will prove the theorem by induction on length of Pfaff chain F. For $l = 0$, according to Bezout theorem the number of nondegenerate solutions of the system is not bigger than $q_1 \dots q_{m+n}$. Assume now, that the theorem is true for every $s < l$ and let F be a Pfaff chain of length l of n variables. Consider the system $(\star)$. Let us add to it a new function $S:\mathbb{R}^{n+m+1}\to\mathbb{R}$ such that $S(x_0,x,z)=x_0^2+\|(x,z)\|^2-R^2$, where $R\in\mathbb{R}$. Now $(P_1,\dots,P_{n+m},S)$ is a proper map and the number of solutions of the system $(\star)$ in $\overline{K}(0,R)$ is half of the number of the new one. Let us take a new variable v and a function $\widetilde{P}(x,z,v)=f_k(x)-v$. Zero is a regular value for $\widetilde{P}$ and

$$P_1=\ldots=P_{n+m}=0\ \Leftrightarrow\ P_1=\ldots=P_{n+m}=\widetilde{P}=0.$$

Due to the proposition 3.6 the number of nondegenerate solution of the system $(\star)$ in $\overline{K}(0,R)$ is not greater than

$$(\star')\qquad P_1=\ldots=P_n=\det(\widehat{P}_1,\dots,\widehat{P}_{n+m},\widehat{\widetilde{P}},\widehat{S})=S=0,$$

where $\widehat{f}:=(\frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_{n+m}},\ \frac{\partial f}{\partial v})$, for a smooth function $f:\mathbb{R}^{m+n+2}\to\mathbb{R}$. According to the inductive assumption number of nondegenerate solutions of the system $(\star)$' is finite and not greater than

$$\varphi(q_1,\dots,g_{n+m},\widetilde{g},2,l{-}1,k,m{+}2,n),\ \text{ where } \widetilde{g}=\deg P,\ P=\det(\widehat{P}_1,\dots,\widehat{P}_{n+m},\widehat{\widetilde{P}},\widehat{S}).$$

Since a radius R may be chosen arbitrarily big, we get the finiteness of $N_0^{(P_1,\dots,P_n)}$. We will now estimate the number of solutions of the system $(\star)$. Note that the degree of P is not greater than $\sum_{i=1}^{m+n}q_i+nk-n-m+1$. Indeed, we have

- $\deg\frac{\partial P_i}{\partial x_j}\le q_i+k-1$, $i=1,\dots,m+n$, $j=1,\dots,n$,
- $\deg\frac{\partial P_i}{\partial z_j}\le q_i-1$, $i=1,\dots,m+n$, $j=1,\dots,m$,
- $\frac{\partial P_i}{\partial v}=\frac{\partial P_i}{\partial x_0}=\frac{\partial S}{\partial v}=\frac{\partial \widetilde{P}}{\partial x_0}=\frac{\partial \widetilde{P}}{\partial z_i}=0$,
- $\frac{\partial \widetilde{P}}{\partial v}=-1$, $\frac{\partial S}{\partial v}=0$,
- $\deg\frac{\partial S}{\partial x_1}=1$, $i=0,\dots,n$, $\deg\frac{\partial P_i}{\partial z_i}=1$, $i=1,\dots,m$.

We get, that

- $S(x_0,x,z,v)=-2x_0\det(\widehat{P}_1,\dots,\widehat{P}_{m+n})$,
- $\deg\det(\widehat{P}_1,\dots,\widehat{P}_{m+n})\le\sum_i^{m+n}q_1+nk-m$.

Finally, $\widetilde{q}=\deg P\le\sum_{i=1}^{m+n}q_i+nk-n-m+1$. Due to the inductive assumption

$$\begin{aligned}
&N_0^{(P_1,\dots,P_{n+m})}\le\\
&\le 1/2\ N_0^{(P_1,\dots,P_{n+m},P,S)}\le\\
&\le 1/2\ \varphi(q_1,\dots,g_{n+m},\widetilde{q},2,l-1,k,m+2,n)=\\
&=2^{\frac{(l-2)(l-1)}{2}}q_1\dots q_{n+m}\widetilde{q}\left(\sum_{j=1}^{n+m}q_j+\widetilde{q}+2+nk-n-(m+2)+1\right)^{l-1}\\
&\le 2^{\frac{(l-2)(l-1)}{2}}q_1\dots q_{n+m}\left(\sum_{i=1}^{m+n}q_i+nk-n-m+1\right)\cdot\\
&\quad\cdot\left(\sum_{j=1}^{n+m}q_j+\left(\sum_{i=1}^{m+n}q_i+nk-n-m+1\right)+2+nk-n-(m+2)+1\right)^{l-1}=
\end{aligned}$$

$$= 2^{\frac{(l-2)(l-1)}{2}} q_1 \dots q_{n+m} 2^{l-1} \left(\sum_{i=1}^{m+n} q_i + nk - n - m + 1 \right)^l =$$
$$= \varphi(q_1, \dots, g_{n+m}, l, k, m, n).$$

This ends the proof of the theorem. □

4. Number of connected components of Pfaff sets

We turn now to theorems which allows us to estimate the number of connected components of sets defined by zeroes of Pfaff functions. Moreover, we will show that the number of connected components of sets created from cutting such sets with an arbitrary affine set is finite and bounded from above.

Lemma 4.1. *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^\infty$ and $U = P^{-1}(0)$ be a compact manifold of dimension $n-1$. Consider a system of equations*

$$(*) \begin{cases} P(x) = 0 \\ \frac{\partial P}{\partial x_1}(x) = 0 \\ \vdots \\ \frac{\partial P}{\partial x_{n-1}}(x) = 0 \end{cases}$$

Then there exists a rotation ψ, such that all solutions of this system are nondegenerate.

Proof. Consider a map
$$\phi := \frac{\mathrm{grad} P^2}{\|\mathrm{grad} P^2\|} : U \to \mathbb{S}^{n-1}.$$
By the Sard theorem there exists $b \in \mathbb{S}^{n-1}$, such that b and $-b$ are regular values. There exists a rotation $\psi : \mathbb{R}^n \to \mathbb{R}^n$, such that $\psi(b) = (0, \dots, 0, 1)$. It follows, that we may assume that $(0, \dots, 0, \pm 1)$ are regular values of ϕ. Notice, that

$$a \text{ is a solution of the system } (*) \ \Leftrightarrow \ \phi(a) = (0, \dots, \pm 1).$$

Since $T_a U = T_{\phi(a)} \mathbb{S}^{n-1} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$, we can calculate a derivative of ϕ in coordinates $(x_1, \dots, x_{n-1})$:

$$\det \left(\frac{\partial \phi}{\partial x_1 \dots \partial x_{n-1}} \right) = \frac{1}{\|\mathrm{grad} P(a)\|^{n-1}} \det \left(\frac{\partial P}{\partial x_i \partial x_j} \right)_{\substack{i=1,\dots,n-1 \\ j=1,\dots,n-1}} \neq 0$$

If a is a solution of the system $(*)$, then jacobian of $\left(P, \frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_{n-1}}\right)$ is equal to

$$\frac{\partial P}{\partial x_n} \det \left(\frac{\partial^2 P}{\partial x_i \partial x_j} \right)_{\substack{i=1,\dots,n-1 \\ j=1,\dots,n-1}} \neq 0$$

□

Proposition 4.2. *Let $P : \mathbb{R}^n \to \mathbb{R}$, $P \in \mathbb{R}[x_1, \dots, x_n, f(x)]$, where f is a Pfaff function of complexity (l, k) and $\deg P = q$. Moreover, let $U = P^{-1}(0)$ be a compact* *manifold of dimension $n-1$. Then the number of connected components of the set U is not greater than*

$$\Phi(q,l,k,n) = 2^{\frac{(l-1)l}{2}} q(q+k-1)^{n-1}(qn+2nk-2n-k+2)^l.$$

Proof. $U = \bigcup_{i=1}^s U_i$, where U_i are connected components of U. Let

$$\pi_i : U_i \to \mathbb{R}, \quad \pi_i(x_1,\dots,x_n) = x_n.$$

The function π_i approaches its maximum and minimum U_i. It means, that $\frac{\partial P}{\partial x_1} = \dots = \frac{\partial P}{\partial x_{n-1}} = 0$. It follows that the number of connected components of U is not greater than a half of the number of solutions of a system of equations

$$\begin{cases} P(x) = 0 \\ \frac{\partial P}{\partial x_1}(x) = 0 \\ \vdots \\ \frac{\partial P}{\partial x_{n-1}}(x) = 0 \end{cases}$$

Due to the lemma 4.1 we can choose such a coordinate system, that all solutions of this system are nondegenerate. By the theorem 3.7 the number of nondegenerate solutions is not greater than

$$\frac{1}{2}\,\varphi(q,\underbrace{q+k-1,\dots,q+k-1}_{n-1},l,k,0,n) \le 2^{\frac{(l-1)l}{2}} q(q+k-1)^{n-1}(nq+2nk-2n-k+2)^l.$$

□

Proposition 4.3. *Let $P \in \mathbb{R}[x_1,\dots,x_n,f(x)]$, where f is a Pfaff function. Suppose that $U = P^{-1}(0)$ consists of at least two connected components. Then for sufficiently big $R > 0$ there exists P_R, such that $P_R \in \mathbb{R}[x_1,\dots,x_n,P(x)]$, $\deg P_R = 2$ and $M_R = P_R^{-1}(0)$ is a compact hypersurface and number of its connected components is not lower than the number of connected components of the set $U \cap \overline{K}(0,R)$.*

Proof. Let us take sufficiently big $R > 0$, such that $U \cap \overline{K}(0,R)$ has at least two connected components. Let

$$\mathcal{F} = \{V_i \subset \mathbb{R}^n \mid V_i - \text{a connected scomponent of } U \cap \overline{K}(0,R)\}.$$

By the compactness $\sharp\mathcal{F} < \infty$, $\mathcal{F} = \{V_1,\dots,V_s\}$. Let us take open sets W_i, such that $W_i \supset V_i$, $i = 1,\dots,s$ and $W_i \cap W_j = \emptyset$, $j \neq i$. Let us take $W = \bigcup_{i=1}^s W_i$ and $Z = \overline{K}(0,R)\backslash W$. Then every connected component of W (which are W_i) contains exactly one of connected components of the family $\mathcal{F}$. Let us define a function

$$Q_\varepsilon(0) = P^2 + \varepsilon(\|x\|^2 - R^2).$$

Then $W_R^\varepsilon = Q_\varepsilon^{-1}(0) \subset \overline{K}(0,R)$. Let us take $\varepsilon_0 = \min_{z\in Z} P^2(z)/R^2 > 0$. Notice, that $W_R^\varepsilon \cap Z = \emptyset$, gdy $\varepsilon < \varepsilon_0$. Indeed,

$$P^2(z) + \varepsilon(\|z\|^2 - R^2) \ge R^2\varepsilon_0 + \varepsilon(\|z\|^2 - R^2) \ge R^2(\varepsilon_0 - \varepsilon) > 0, \; z \in Z.$$

Let us take one of W_i. $W_R^\varepsilon \cap W_i \neq \emptyset$, since $Q_\varepsilon|_{V_i} < 0$ and $Q_\varepsilon|_Z > 0$. It follows that number of connected components of W_R^ε is not smaller then $\sharp\mathcal{F}$. Note that a set

$$X = \{(x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}_+ \mid Q_\varepsilon(x) = 0\}$$

is a smooth hypersurface. Since $\frac{\partial Q_\varepsilon}{\partial \varepsilon}(x) = \|x\|^2 - R^2$, gradient of the function Q_ε may vanish only when $\|x\| = R$. In this case $\frac{\partial Q_\varepsilon}{\partial x_i}(x) = 2\varepsilon x_i$. We have showed, that the set X is a hypersurface. Consider a function

$$f : X \ni (x, \varepsilon) \longrightarrow \varepsilon \in \mathbb{R}.$$

Due to the Sard theorem there exists ε , such that it is a regular value for f. We have proved that $M := W_R^\varepsilon$ is a manifold of dimension $n - 1$. □

Definition 4.4. Let $AG(\mathbb{R}^n)$ denote the set of all affine subspaces of $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$. Then we put

$$\gamma(A) := \min\{N \in \mathbb{N} : \text{for all } V \in AG(\mathbb{R}^n) \quad A \cap V \ \text{ has at most } N \text{ connected components}\}.$$

If such N does not exist, then we put $\gamma(A) = \infty$.

It follows almost immediately

Theorem 4.5. *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a Pfaff function of complexity (l, k). Let $U = P^{-1}(0)$. Then $\gamma(U) < \infty$.*

Proof. Let V be an affine subspace of $\mathbb{R}^n$. We can write V as

$$V = \{x \in \mathbb{R}^n \mid L_1(x) + a_1 = \ldots = L_{k_V}(x) + a_{k_V} = 0\},$$

where L_i are linear operators and $a_i \in \mathbb{R}$, $i = 1, \ldots, k_V$. Observe that

$$x \in V \cap U \ \Leftrightarrow \ (L_1(x) + a_1)^2 + \ldots + (L_{k_V}(x) + a_{k_V})^2 + P^2(x) = 0.$$

By using 4.2 and 4.3 we get, that for $R > 0$ the set $\overline{K}(0, R) \cap V \cap U$ has finitely many connected components and its number is bounded from above by $\Phi(2, l, k, n)$. Since this number does not depend on R and on the choice of the subspace V, the proof of the theorem is ended. □

5. Structure $\mathbb{R}_{\text{Pfaff}}$

Definition 5.1. A sequence $\mathcal{S} = \{\mathcal{S}_n\}_{n\in\mathbb{N}}$, where $\mathcal{S}_n \subset \mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{N}$, is called a weak o-minimal structure, if for every $n, m \in \mathbb{N}$, the following conditions are satisfied:

(W1) if $A, B \in \mathcal{S}_n$, then $A \cap B \in \mathcal{S}_n$,
(W2) $\mathcal{S}_n$ contains all semialgebraic subsets of $\mathbb{R}^n$,
(W3) if $A \in \mathcal{S}_n$, $B \in \mathcal{S}_m$, then $A \times B \in \mathcal{S}_{n+m}$,
(W4) if $A \in \mathcal{S}_n$ and σ is a permutation of coordinates, then $\sigma(A) \in \mathcal{S}_n$,

(W5) if $A \in \mathcal{S}_n$, then $\gamma(A) < \infty$,

(W6) if $A \in \mathcal{S}_n$, then there exist $m \geq n$ and a closed set $B \in \mathcal{S}_m$, such that $A = \Pi_{m,n}(B)$, where $\Pi_{m,n} : \mathbb{R}^m \ni (x_1, \dots, x_m) \to (x_1, \dots, x_n) \in \mathbb{R}^n$.

Definition 5.2. A set $A \subset \mathbb{R}^n$ is called a Pfaff set, if there exists $P : \mathbb{R}^{n+k} \to \mathbb{R}$, such that $A = P^{-1}(0)$, $P \in \mathbb{R}[x, f_1(x), \dots, f_k(x)]$, and $f_1, \dots, f_k$ are Pfaff functions. The family of these sets will be denoted by $\mathbb{R}^w_{\text{Pfaff}}$.

Observation 5.3. *The family* $\mathbb{R}^w_{\text{Pfaff}}$ *is a weak o-minimal structure.*

Definition 5.4. Let be given a weak o-minimal structure $\mathcal{S} = \{\mathcal{S}_n\}_{n\in\mathbb{N}}$. We say, that $\mathcal{S}$ is of DSF type, if for every $A \in \mathcal{S}_n$ there exists $m \geq n$ and $f \in \mathcal{S}^\infty(\mathbb{R}^m)$, such that graph$f \in \mathcal{S}$ and

$$A = \Pi_{m,n}\big(Z(f)\big), \quad \text{where} \quad \Pi_{m,n} : \mathbb{R}^m \ni (x_1, \dots, x_n) \longrightarrow (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Theorem 5.5 (**Wilkie**, [W])**.** *Suppose* $\mathcal{S} = \{\mathcal{S}_n\}_{n\in\mathbb{N}}$ *is a weak o-minimal structure satisfying* DSF *condition. Then there exists an o-minimal structure* $\widetilde{\mathcal{S}} = \{\widetilde{\mathcal{S}}_n\}_{n\in\mathbb{N}}$ *which contains* $\mathcal{S}$*.*

An immediate consequence of 4.5 and 5.5 is the following

Theorem 5.6. *A structure* $\mathbb{R}_{\text{Pfaff}} = \widetilde{\mathbb{R}}^w_{\text{Pfaff}}$ *is o-minimal.*

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