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ON SOME APPLICATIONS OF A THEOREM OF LION

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Introduction

O-minimal geometry can be consider as response on studying geometry of semialgebraic and subanalytic sets. Let us recall, that semialgebraic sets $\mathcal{SA}_n$ of $\mathbb{R}^n$ is the smallest family containing:

(1) let $P \in \mathbb{R}[X_1, \dots X_n]$; then $\{x \in \mathbb{R}^n \mid P(x) = 0\} \in \mathcal{SA}_n$ and $\{x \in \mathbb{R}^n \mid P(x) > 0\} \in \mathcal{SA}_n$,

(2) if $A, B \in \mathcal{SA}_n$, then $A \cup B$, $A \cap B$ and $\mathbb{R}^n \setminus A \in \mathcal{SA}_n$.

It is easy to see that if $A \in \mathcal{SA}_n$, $B \in \mathcal{SA}_m$, then $A \times B \in \mathcal{SA}_{n+m}$. By famous theorem of Tarski-Seidenberg (see [**T**]) $\mathcal{SA} = \{\mathcal{SA}_n\}_{n \in \mathbb{N}}$ is stable under taking projections:

> *Let $A \in \mathcal{SA}_{n+1}$. Then $\Pi(A) \in \mathcal{SA}_n$, where Π is a natural projection on first n coordinates.*

A second source of inspiration for introducing notion of o-minimal structure were studies on *semianalytic* and *subanalytic* sets by Łojasiewicz, Gabrielov and Hironaka (see [**Ł**], [**Ga**], [**Hi**]):

> Let M be a real analytic manifold. We say $X \subset M$ is *semianalytic* if for all $a \in M$ there is an open neighbourhood U_a of a such that $X \setminus U$ is a finite union of sets
>
> $$\{x \in U \mid f_1(x) = \ldots = f_m(x) = 0,\ g_1(x) > 0 \ldots g_l(x) > 0\}$$
>
> where $f_1, \ldots, f_m, g_1, \ldots, g_l$ are analytic on U. We say that $X \in M$ is *subanalytic* if for every $a \in M$ there is an open neighborhood U_a of a and a semianalytic $Y \in M \times \mathbb{R}^m$ with compact closure, such that $U \setminus X$ is the projection of Y to M.

Semialgebraic sets as well as subanalytic with compact closure share many good "tame" properties, such as: cell decomposition, stratification, piecewise smoothness of definable maps, triangulation (see [**BR**], [**BCR**] in a semialgebraic setting and [**BM**] for properties in locally semianalytic and subanalytic context).

In 1980s van den Dries (see [**vdD**]) noticed that many properties of semialgebraic sets follow from just a few axioms. Notion of *o-minimal structure* first was used by Pillay and Steinhorn in [**PS**]. Nowadays, we can put the following definition

> We say, that $\mathcal{S} = \{\mathcal{S}_n\}_{n \in \mathbb{N}}$ is a *structure*, where each $\mathcal{S}_n$ is a set of subsets of $\mathbb{R}^n$, when it satisfies the following axioms:
>
> (1) all algebraic subsets of $\mathbb{R}^n$ are in $\mathcal{S}_n$,
> (2) for every n, $\mathcal{S}_n$ is a Boolean subalgebra of the powerset of $\mathbb{R}^n$,
> (3) if $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$, then $A \times B \in \mathcal{S}_{m+n}$,
> (4) if $\Pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ is the projection on the first n coordinates and $A \in \mathcal{S}_{n+1}$, then $\Pi(A) \in \mathcal{S}_n$.

The structure $\mathcal{S}$ is said to be *o-minimal* if, moreover, it satisfies:

the elements of $\mathcal{S}_1$ are precisely the finite unions of points and intervals.

One of the most interesting directions of studying o-minimal structures is discovering new such structures. A really inspirational result was given by Wilkie (see Section 1), which is far generalization of famous Gabrielov theorem of complement. Lion (see Section 1), using Wilkie's theorem, gave conditions on certain families of functions, under which they are definable in certain o-minimal structures. It occurs, that to establish property of belonging to o-minimal structure it is enough to check that certain systems of equations have only finitely number of regular solutions. Focusing on families of functions instead of families of sets is naturally justified by the following fact (see [**vdDM**], 4.22)

let $\mathcal{S}$ be an o-minimal structure. Let $A \in \mathcal{S}_n$ be a closed set. Then for every $p \in \mathbb{N}$ there exists $f \in \mathcal{C}^p(\mathbb{R}^n)$ such that $A = f^{-1}(\{0\})$.

O-minimal structures are also strictly connected with notion of *Hardy fields* (see [**B**], Chapter 5):

a *Hardy field* is any subfield k of the ring of all germs at $+\infty$ of differentiable functions

$$f : (A, +\infty) \longrightarrow \mathbb{R} \qquad (A \in \mathbb{R})$$

closed under differentiation.

Hardy fields were studied in 1980s in a systematic way especially by Rosenlicht (see [**Ro1**]-[**Ro4**]). They are interesting from point of view of o-minimal structures because of the observation below (see [**M**], 3.1):

If $\mathcal{S}$ is a structure, then the following conditions are equivalent

(1) *$\mathcal{S}$ is o-minimal,*
(2) *the germs of definable unary functions form a Hardy field,*
(3) *every unary definable function is either ultimately zero or ultimately nonzero.*

This thesis is devoted to find some applications of Lion's theorem. Till now it has not been used. In the first section we recall after Lion notions of geometric family and regular geometric family. We recall also property, the so called 0-regularity, under which regular geometric family lies in an o-minimal structure.

In the second section we fix a polynomially bounded o-minimal structure $\mathcal{S}$. We investigate number of regular solutions of systems of equations

$$(\star) \qquad \begin{cases} P_1\Big(\phi\big(\lambda_{11}(x)\big), \ldots, \phi\big(\lambda_{1n}(x)\big), x\Big) = 0 \\ \vdots \\ P_n\Big(\phi\big(\lambda_{n1}(x)\big), \ldots, \phi\big(\lambda_{nn}(x)\big), x\Big) = 0, \end{cases}$$

where $P = (P_1, \ldots, P_n) : U \longrightarrow \mathbb{R}^n$ is a definable map of class $\mathcal{C}^1$, $\lambda_{ij} : \mathbb{R}^n \longrightarrow \mathbb{R}$ (for $i, j = 1, \ldots n$) are affine maps, and ϕ is a function of one variable which behaves at infinity similarly to exponential function. By application of Lion-Rolin-Parusiński Theorem, we receive finiteness of number of regular solutions of systems ($\star$). For structures containing restricted exponential function we give full characterization of ϕ - it is of the form $\exp(f)$, where f is a definable function. As immediate consequence of Lion theorem we get that $\mathcal{S}$

extended by ϕ is still o-minimal. We also easily recover a famous theorem of Wilkie (see [**W1**]), which states that $\mathbb{R}(\exp)$ is o-minimal.

The third chapter contains an algorithm of microlocal transformation to normal crossings of almost $\mathcal{C}^\infty$-germs satisfying special quasianalyticity property (Definition 3.3). It is a simplified version of classical reducing of an analytic function to normal crossing given in [**BM**]. As a consequence, we give a condition (see Theorem 3.11) on families of restricted functions on lying in an o-minimal structure. We believe, that this theorem is well known among experts. We conclude chapter with application of Theorem 3.11 to classical examples: $\mathbb{R}_{an}$ and for certain Denjoy-Carleman classes.

The last chapter is devoted to investigation of almost $\mathcal{C}^\infty$-germs of the form $f = g + h$, where g is a differentially transcendental and analytic, and h is semialgebraic except zero and flat at origin. When g is strongly transcendental (see Definition 4.7), then f lies in an o-minimal structure. The differential transcendentality is not enough. We end with giving examples of Hardy fields which could not be Hardy field of any o-minimal structure. We give also examples of germs f_1, f_2, such that $\mathbb{R}(f_1)$, $\mathbb{R}(f_2)$ are o-minimal, $\mathbb{R}(f_1, f_2)$ constitutes Hardy field, but it does not belong to any o-minimal structure.

List of frequently used notations

$\mathbb{N}$	the set of natural numbers $1, 2, 3, \ldots$
$\mathbb{Q}$	the field of rational numbers
$\mathbb{R}$	the field of real numbers
$e_1, \ldots, e_n$	vectors of canonical basis of $\mathbb{R}^n$
$\lvert x\rvert$	absolute value of $x \in \mathbb{R}$
$\lvert\beta\rvert$	length of $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$; $\lvert\beta\rvert = \beta_1 + \ldots + \beta_m$
$\mathcal{P}(A)$	power set of a set A
x^β	$x_1^{\beta_1} \cdots x_n^{\beta_n}$, where $x = (x_1, \ldots, x_n)$, $\beta = (\beta_1, \ldots, \beta_n)$
Ξ_m^k	set of strictly increasing sequences from $\{1, \ldots, m\}$ of length k
$\Pi_{m,n}$	natural projection from $\mathbb{R}^m$ on n first coordinates
$\mathbb{Q}[X_1, \ldots X_k]$	the ring of polynomials of k variables with rational coefficients
$\mathbb{R}[X_1, \ldots X_k]$	the ring of polynomials of k variables with real coefficients
$\mathcal{C}^k(U)$	functions $f : U \longrightarrow \mathbb{R}$ of class $\mathcal{C}^k$, where $k \in \mathbb{N} \cup \{\infty, \omega\}$
$\mathcal{C}^k(\mathbb{R}^n, 0)$	germs of functions at 0 of class $\mathcal{C}^k$, where $k \in \mathbb{N} \cup \{\infty, \omega\}$
$\mathcal{C}^k_{0^+}$	germs of functions $f : (0, \epsilon) \longrightarrow \mathbb{R}$ ($\epsilon > 0$) of class $\mathcal{C}^k$, $k \in \mathbb{N} \cup \{\infty, \omega\}$
$f\rvert_A$	restriction of a function f to A
id_X	identity map on X
$\Gamma(f)$	graph of a function f
$\frac{\partial^{\lvert\kappa\rvert} f}{\partial x^\kappa}$	partial differential of order $\lvert\kappa\rvert$ in direction x^κ of a function f
$\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}$	jacobian of a map $f = (f_1, \ldots, f_n)$
$d_x f$	differential of f at x
rank $d_x f$	rank of $d_x f$
$j_m f(x)$	$\big(f(x), f'(x), \ldots, f^{(m)}(x)\big)$, where $f \in \mathcal{C}^m(J)$, $J \subset \mathbb{R}$
$\mathbb{S}^r$	r-dimensional sphere in $\mathbb{R}^{r+1}$
$\mathbb{P}^k(\mathbb{R})$	k-dimensional projective space
$\mathcal{S}_\mathcal{A}$	structure $\mathcal{S}$ extended by family of functions $\mathcal{A}$
$K(A)$	extension of a field K by elements of a set A

CHAPTER 1

Theorems of Lion and Wilkie

In this section we shortly describe a theorem of Lion on special algebras of functions and how it is connected with Wilkie's theorem of complement. We begin with some definitions.

DEFINITION 1.1 (see [**L**]). We say, that $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$, where $\mathfrak{G}_n \subset \mathbb{R}^{\mathbb{R}^n}$ $(n \in \mathbb{N})$ is a *geometric family*, when:
(G1) if $f, g \in \mathfrak{G}_n$, then $fg, f+g \in \mathfrak{G}_n$,
(G2) if $f \in \mathfrak{G}_n$, and for every $x \in \mathbb{R}^n$ $f(x) \neq 0$, then $1/f \in \mathfrak{G}_n$,
(G3) $\mathbb{R}[X_1, \dots, X_n] \subset \mathfrak{G}_n$,
(G4) if $L : \mathbb{R}^m \to \mathbb{R}^n$ is affine, for some $m \in \mathbb{N}$, and $f \in \mathfrak{G}_n$, then $f \circ L \in \mathfrak{G}_m$.

DEFINITION 1.2 (see [**L**]). Geometric family $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ is called *regular*, if for every $n \in \mathbb{N}$ and every $g \in \mathfrak{G}_n$, there exist affine hyperplanes $H_1, \dots, H_l$ and functions $g_1, \dots, g_n \in \mathfrak{G}_n$ such that for $U = \mathbb{R}^n \backslash (H_1 \cup \ldots \cup H_l)$, $g|_U$ is of class $\mathcal{C}^1$ and $\frac{\partial}{\partial x_i}(g|_U) = g_i|_U$.

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ and $t \in \mathbb{R}^m$. By reg $g^{-1}(t)$ we denote (after Lion [**L**]) the set of all these $x \in g^{-1}(t)$, for which there exist a neighborhood, such that $g|_U$ is a submersion of class $\mathcal{C}^1$.

DEFINITION 1.3 (see [**L**]). We say that a geometric family $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ is *0-regular*, if for every $n \in \mathbb{N}$ and every mapping $g = (g_1, \dots, g_n) : \mathbb{R}^n \to \mathbb{R}^n$, where $g_i \in \mathfrak{G}_n$ $(i = 1, \dots, n)$, and for each $t \in \mathbb{R}^n$, the set reg $g^{-1}(t)$ is finite.

DEFINITION 1.4 (see [**L**]). We say that a geometric family $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ has *uniform fibre finiteness property*, when for every $n, p \in \mathbb{N}$ and $g = (g_1, \dots, g_p)$: $\mathbb{R}^n \to \mathbb{R}^p$, where $g_i \in \mathfrak{G}_n$ $(i = 1, \dots, p)$, there exists $N \in \mathbb{N}$, such that for each $t \in \mathbb{R}^p$,

$$\sharp\left\{A \subset \mathbb{R}^n \mid A \text{ is connected component of } g^{-1}(t)\right\} < N$$

THEOREM 1.5 (**Lion** [**L**]). *Let $\mathfrak{G} = \{\mathfrak{G}_n\}_{n\in\mathbb{N}}$ be a geometric regular family. If it is 0-regular, then it has uniform fibre finiteness property.*

Now, we turn to the modification of Wilkie's theorem by Karpinski and Macintyre. Let $AG(\mathbb{R}^n)$ denote the set of all affine subspaces of $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$. Then we put

$$\gamma(A) := \min\{N \in \mathbb{N} : \text{for all } V \in AG(\mathbb{R}^n) \quad A \cap V \ \text{ has at most } N \text{ connected components}\}.$$

If such N does not exist, then we put $\gamma(A) = \infty$.

DEFINITION 1.6. A sequence $\mathcal{S} = \{\mathcal{S}_n\}_{n\geq 1}$, where $\mathcal{S}_n \subset \mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{N}$, is called a *weak o-minimal structure*, if for every $n \geq 1, m \geq 1$ the following conditions are satisfied:
(W1) if $A, B \in \mathcal{S}_n$, then $A \cap B \in \mathcal{S}_n$,
(W2) $\mathcal{S}_n$ contains all semialgebraic subsets of $\mathbb{R}^n$,

(W3) if $A \in \mathcal{S}_n$, $B \in \mathcal{S}_m$, then $A \times B \in \mathcal{S}_{n+m}$,
(W4) if $A \in \mathcal{S}_n$ and σ is a permutation of coordinates, then $\sigma(A) \in \mathcal{S}_n$,
(W5) if $A \in \mathcal{S}_n$, then $\gamma(A) < \infty$,
(W6) if $A \in \mathcal{S}_n$, then there exists $N \geq n$ and a closed set $B \in \mathcal{S}_N$ such, that $A = \Pi_{N,n}(B)$, where $\Pi_{N,n} : \mathbb{R}^N \ni (x_1, \dots, x_m) \to (x_1, \dots, x_N) \in \mathbb{R}^n$.

Definition 1.7. A weak o-minimal structure $\mathcal{S} = \{\mathcal{S}_n\}_{n \geq 1}$ satisfies DC^N *condition for all* N, if for each $A \in \mathcal{S}_n$ there exists $p \geq n$ such that for every $N \in \mathbb{N}$, $A = \Pi_{p,n}(\{f_N = 0\})$ where $f_N \in C^N(\mathbb{R}^p)$ and graph$f_N \in \mathcal{S}_{p+1}$.

Theorem 1.8 (**Wilkie, Karpinski, Macintyre**, [**W2**], [**KM**]). *Suppose $\mathcal{S} = \{\mathcal{S}_n\}_{n \geq 1}$ is an o-minimal weak structure satisfying DC^N for all N. Then there exists an o-minimal structure $\widetilde{\mathcal{S}} = \{\widetilde{\mathcal{S}}_n\}_{n \geq 1}$ which contains $\mathcal{S}$.*

It is not difficult to check that if $\mathfrak{G} = \{\mathfrak{G}_n\}_{n \in \mathbb{N}}$ is a regular geometric family with uniform fibre finiteness property, then through defining $\mathcal{S}_n$ as the family of all subsets of $\mathbb{R}^n$ of the form $f^{-1}(0)$, where $f \in \mathfrak{G}_n$, we obtain a weak o-minimal structure. Less obvious is that this structure satisfies DC^N condition for all $N \in \mathbb{N}$. This property was checked in [**A**]. Now, combining Theorems 1.5 and 1.8 we can state

Theorem 1.9 (see, [**L**], [**A**]). *Let $\mathfrak{F} = \{\mathfrak{F}_n\}_{n \in \mathbb{N}}$ be a regular, geometric and 0-regular family. Then there exists an o-minimal structure $\mathfrak{S}$ such that every $f \in \mathfrak{F}$ is definable in $\mathfrak{S}$.*

CHAPTER 2

A new proof of o-minimality of $\mathbb{R}_{\exp}$

2.1. Lion-Rolin-Parusiński Preparation Theorem

Let $f : E \longrightarrow \mathbb{R}$ be a definable function, where $E \subset \mathbb{R}^n$. We begin with the following definition:

Definition 2.1. *A reduction of f* (on the set E) ia a quadruple (θ, A, V, α), where

(1) $A, \theta : \Pi_{n-1}(E) \longrightarrow \mathbb{R}$ are definable functions,

(2) $V : E \longrightarrow \mathbb{R}$ is a definable function such that there exists $L > 1$ satisfying an inequality

$$L^{-1} \le V(x) \le L,\ x \in E,$$

called *a unit*,

(3) $\alpha \in \mathbb{Q}$, such that

$$f(x) = |x_n - \theta(x')|^\alpha A(x')V(x), \quad \text{where} \quad x' = \Pi_{n-1}(x),$$

Moreover, either $\theta \equiv 0$ or there exists $M \ge 1$ satisfying an inequality

$$(A) \qquad M^{-1}|x_n| \le |\theta(x')| \le M|x_n|, \quad x' = \Pi_n(x), \quad \text{for } x \in E.$$

By Parusiński [**P**], Lion and Rolin [**LR**] (in a subanalytic case) and Speissegger and van den Dries [**vdDS**] (in polynomially bounded o-minimal structures) we can decompose the domain E of f in such a way that we have

Theorem 2.2. *Let f be a definable function in polynomially a bounded o-minimal structure. Then there exists a cell decomposition $\mathcal{C}$ of E, such that for every $C \in \mathcal{C}$ the function $f|_C$ can be described in a reduced form, that is*

$$f|_C(x) = |x_n - \theta(x')|^\alpha A(x')V(x), \quad \text{where} \quad x' = \Pi_{n-1}(x)$$

and functions θ, A and V have properties as in Definition 2.1.

Definition 2.3. *A preparation of f* (on the set E) is a family

$$\left(\{\theta_i\}_{i=1,\dots,n}, \{A_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,i}}, \{V_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,i}}, \{\alpha_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,i}}\right)$$

such that $(\theta_1, A_{11}, V_{11}, \alpha_{11})$ is a reduction of f on E, and for $i > 1$:

(1) $(\theta_i, A_{i1}, V_{i1}, \alpha_{i1})$ is a reduction of θ_{i-1} on $\Pi_{n-i+1}(E)$,

(2) $(\theta_i, A_{ij}, V_{ij}, \alpha_{ij})$ is a reduction of $A_{(i-1)(j-1)}$ on $\Pi_{n-i+1}(E)$, when $1 < j \le i$.

Let $\mathcal{F} = \{f_\nu\}_{\nu=1,\dots,s}$ be a finite family of definable functions defined on a set E.

Definition 2.4. *A preparation* of the family $\mathcal{F}$ on the set E is a family

$$\left(\{\theta_i\}_{i=1,\dots,n}, \{A^\nu_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,i}}, \{V^\nu_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,i}}, \{\alpha^\nu_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,i}}\right)_{\nu=1,\dots,s}$$

such that for every $\nu = 1, \dots, s$ a family

$$\left(\{\theta_i\}_{i=1,\dots,n}, \{A_{ij}^\nu\}_{\substack{i=1,\dots,n,\\ j=1,\dots,i}}, \{V_{ij}^\nu\}_{\substack{i=1,\dots,n,\\ j=1,\dots,i}}, \{\alpha_{ij}^\nu\}_{\substack{i=1,\dots,n\\ j=1,\dots,i}} \right)$$

is a preparation of f_ν.

REMARK 2.5. A preparation of a function f on E can be represented in the form of triangular array

$$\begin{array}{llll} f & & & \\ \theta_1 & (A_{11}, V_{11}, \alpha_{11}) & & \\ \theta_2 & (A_{21}, V_{21}, \alpha_{21}) & (A_{22}, V_{22}, \alpha_{22}) & \\ \vdots & & & \\ \theta_n & (A_{n1}, V_{n1}, \alpha_{n1}) & \dots & (A_{nn}, V_{nn}, \alpha_{22}). \end{array}$$

then, for every $m \in \{1, \dots, n-1\}$ an array

$$\begin{array}{llll} \theta_m & & & \\ \theta_{m+1} & (A_{(m+1)1}, V_{(m+1)1}, \alpha_{(m+1)1}) & & \\ \theta_{m+2} & (A_{(m+2)1}, V_{(m+2)1}, \alpha_{(m+2)1}) & (A_{(m+2)2}, V_{(m+2)2}, \alpha_{(m+2)2}) & \\ \vdots & & & \\ \theta_n & (A_{n1}, V_{n1}, \alpha_{n1}) & \dots & (A_{n(n-m)}, V_{n(n-m)}, \alpha_{2(n-m)}). \end{array}$$

is a preparation of θ_m on $\Pi_{n-m}(E)$. Note that on E the function f has the following representation

$$f(x) = |x_n - \theta_1|^{\alpha_{11}} \cdot |x_{n-1} - \theta_2|^{\alpha_{22}} \cdot \ldots \cdot |x_1 - \theta_n|^{\alpha_{nn}} \cdot A_{nn} \cdot V_{11} \cdot V_{22} \cdots V_{nn}.$$

When $\theta_1 = \ldots = \theta_n \equiv 0$ and $f \not\equiv 0$ on E, then

$$f(x) = |x_n|^{\alpha_{11}} |x_{n-1}|^{\alpha_{22}} \cdots |x_1|^{\alpha_{nn}} \cdot A_{nn} \cdot V_{11} \cdot V_{22} \cdots V_{nn} = |x_n|^{\alpha_{11}} |x_{n-1}|^{\alpha_{22}} \cdots |x_1|^{\alpha_{nn}} \cdot \widetilde{V}$$

where

$$M^{-1} \leq |\widetilde{V}| \leq M \quad \text{for some } M > 1.$$

THEOREM 2.6. *Let $\mathcal{F} = \{f_\nu\}_{\nu=1,\dots,s}$ be a finite family of definable functions on $E \subset \mathbb{R}^n$. Then there exists a cell decomposition $\mathcal{C}$ of E such that, for every $C \in \mathcal{C}$, there exists a preparation of the family $\mathcal{F}$ on C.*

PROOF. Apply Parusiński-Lion-Rolin Preparation Theorem in the case of subanalytic structure, and Speissegger-van den Dries Theorem in full extent. □

In next chapters we will use lemma below

LEMMA 2.7. *Let $f : E \longrightarrow \mathbb{R}$ be a definable function on a definable set $E \subset \mathbb{R}^n$, on which the function f has a preparation*

$$\left(\{\theta_i\}_{i=1,\dots,n}, \{A_{ij}\}_{\substack{i=1,\dots,n,\\ j=1,\dots,i}}, \{V_{ij}\}_{\substack{i=1,\dots,n,\\ j=1,\dots,i}}, \{\alpha_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,i}} \right)$$

Then there exists $M > 1$ such that

(1) *there exist* $\nu \in \{1, \dots, n\}$ *and* $\alpha_1, \dots, \alpha_{\nu-1} \in \mathbb{Q}$ *such that*

$$M^{-1} < |x_\nu| \cdot |x_1|^{\alpha_1} \cdots |x_{\nu-1}|^{\alpha_{\nu-1}} < M, \quad x \in E,$$

or

(2) *there exists numbers* $\alpha_1, \dots, \alpha_n \in \mathbb{Q}$ *such that*

$$M^{-1} < |f(x)| \cdot |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} < M, \quad x \in E.$$

Proof. We consider two cases:

(i) $\theta_{n-\nu+1} \neq 0$ and $\theta_{n-\nu+2} = \ldots = \theta_n = 0$ on E for some $\nu \in \{1, \dots, n\}$; then (1) follows from inequality (A),

(ii) $\theta_\nu \equiv 0$ for all $\nu \in \{1, \dots, n\}$; then

$$f(x) = x_n^{\alpha_{11}} \cdots x_1^{\alpha_{nn}} V_{11}(x) \cdots V_{nn}(x),$$

and (2) follows.

□

2.2. Implicit function theorem in o-minimal structures

In this short subsection we prove a local-global of the implicit function theorem in o-minimal structures (see [**AP**]). We believe that this is widely known fact, but we have not found any source to it.

Proposition 2.8. *Let* $p \in \{1, 2, \ldots\} \cup \{\infty, \omega\}$. *Let* Ω *be a definable open subset of* $\mathbb{R}^{n+m}$ *and let* $F = (F_1, \dots, F_m) : \Omega \to \mathbb{R}^m$ *be a definable* $\mathcal{C}^p$*-mapping such that*

$$\frac{\partial(F_1, \dots, F_m)}{\partial(x_{n+1}, \dots, x_{n+m})} \neq 0 \text{ on } \Omega.$$

Then there exists a finite family $f_i : C_i \to \mathbb{R}^n$, $i \in \{1, \dots, s\}$ *of definable* $\mathcal{C}^p$*-mappings defined on definable open subsets* C_i *of* $\mathbb{R}^n$ *such that for each* i

$$F(x_1, \dots, x_n, f_i(x_1, \dots, x_n)) = 0 \text{ on } \; C_i, \text{ and } \bigcup_{i=1}^{s} \Gamma(f) = F^{-1}(0).$$

Proposition above is a consequence of more general, elementary fact.

Proposition 2.9. *Let* E *be any definable subset of* $\mathbb{R}^{n+m}$. *Let* $\pi : \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *be the natural projection. Assume that* $\pi|_E$ *is locally injective. Then there exists a finite family* $A_1, \dots, A_s$ *of open definable subsets of* E *such that* $E = A_1 \cup \ldots, \cup A_s$ *and* $\pi|_{A_i}$ *is injective for each* $i \in \{1, \dots, s\}$.

Proof. We will prove by induction on $k \in \{1, \dots, n\}$ that if C is a definable subset of E of dimension k, then there exists a finite family $A_1, \dots, A_s$ of open definable subsets of E such that $C \subset A_1 \cup \ldots \cup A_s$ and each of $\Pi|_{A_i}$ ($i \in \{1, \dots, s\}$) is injective.
By using a cell decomposition we can assume without any loss of generality that C is a k-dimensional cell. Then $\Pi|_C$ is injective and $\Pi(C)$ is a k-dimensional cell in $\mathbb{R}^n$. After perhaps a permutation of coordinates in $\mathbb{R}^n$, one can assume that

$$\Pi(C) = \big\{(x_1, \dots, x_k) \mid (x_1, \dots, x_k) \in \Omega, x_j = \; \phi_j(x_1, \dots, x_k) \; (j = k+1, \dots, n)\big\}$$

and

$$C=\big\{(x_1,\dots,x_{n+m}) \mid (x_1,\dots,x_k)\in\Omega,\ x_j=\phi_j(x_1,\dots,x_k)\ (j=k+1,\dots,n+m)\big\},$$

where Ω is definable open subset of $\mathbb{R}^k$ and $\phi_j:\Omega\longrightarrow\mathbb{R}(j=k+1,\dots,n+m)$ are definable continuous functions.
For each $u=(u_1,\dots,u_k)\in\Omega$ and $\epsilon>0$, define $\Theta(u,\epsilon)$ by

$$\big\{(x_1,\dots,x_{n+m})\in E \mid u=(x_1,\dots,x_k),\ |x_j-\phi_j(u)|<\epsilon\ (j=k+1,\dots,n+m)\big\}.$$

For each $u\in\Omega$, set

$$r(u):=\sup\{\epsilon\in(0,1] \mid \Pi|_{\Theta(u,\epsilon)} \text{ is injective}\}.$$

By the assumption of local injectivity r is well-defined and it is easy to check that r is definable. There exists a closed definable subset Z of Ω of dimension $<k$ such that r on $\Omega\backslash Z$ is continuous. It is clear that Π is injective in restriction to the set

$$\bigcup_{u\in\Omega\backslash Z}\Theta(u,r(u))=\Big\{(x_1,\dots,x_{n+m})\in E \mid u=(x_1,\dots,x_k)\in\Omega\backslash Z,$$
$$|x_j-\phi_j(u)|<r(u)\ (j=k+1,\dots,n+m)\Big\},$$

which is an open definable neighbourhood of $C\cap(\Omega\backslash Z)$ in E. Now, to finish the proof it suffices to apply the induction hypothesis to $C\cap Z$ □

2.3. Finiteness of regular solutions of systems of equations

Let $\mathcal{S}$ be a polynomially bounded o-minimal structure containing semialgebraic structure. Let us consider a function $\phi:[0,\infty)\longrightarrow\mathbb{R}$ of class $\mathcal{C}^1$ satisfying the following properties:

(1) for every $K>0$ the function $\phi|_{[0,K]}$ is definable, and increasing on $(A,\infty]$, where $A>0$,
(2) $\phi(t)/t^k\longrightarrow\infty$, for $t\longrightarrow\infty$ and for every $n\in\mathbb{N}$,
(3) let $M>0$ and $\alpha_1,\dots,\alpha_n\in\mathbb{Q}$. Let $U\subset\mathbb{R}^n$ be a set defined by an inequality

$$M^{-1}<\phi(x_1)^{\alpha_1}\cdots\phi(x_n)^{\alpha_n}<M;$$

then the function $\phi(x_1)^{\alpha_1}\cdots\phi(x_n)^{\alpha_n}:U\longrightarrow\mathbb{R}$ is definable in $\mathcal{S}$,
(4) let $\alpha_1,\dots,\alpha_n\in\mathbb{Q}$; then the function

$$\phi^{-1}\left(\phi(x_1)^{\alpha_1}\cdots\phi(x_n)^{\alpha_n}\right):\mathbb{R}^n\longrightarrow\mathbb{R}$$

is definable in $\mathcal{S}$ on

$$\mathcal{A}=\left\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid \phi(x_1)^{\alpha_1}\cdots\phi(x_n)^{\alpha_n}>\phi(A)\right\}.$$

Observation 2.10. If ϕ is monotonic and $\phi(0)=0$, then the condition (4) implies (3). Indeed, let $M>1$. Then, by (5), a map $\phi^{-1}\left(\phi(x_1)^{\alpha_1}\cdots\phi(x_n)^{\alpha_n}\right)$ is definable on the set satisfying inequalities

$$\phi^{-1}(M^{-1})<\phi^{-1}\left(\phi(x_1)^{\alpha_1}\cdots\phi(x_n)^{\alpha_n}\right)<\phi^{-1}(M).$$

By property (1), a map $\phi(x_1)^{\alpha_1}\cdots\phi(x_n)^{\alpha_n}$ is definable on

$$M^{-1}<\phi(x_1)^{\alpha_1}\cdots\phi(x_n)^{\alpha_n}<M.$$

Remark 2.11. Every function satisfying properties $(1)-(4)$ can be extended to $\widetilde{\phi}:\mathbb{R}\longrightarrow\mathbb{R}$ such that $\widetilde{\phi}$ is of class $\mathcal{C}^1$ and $\widetilde{\phi}|_{(-\infty,0]}$ is linear. From now we will denote just by ϕ this extension.

Let $U\subset\mathbb{R}^n_y\times\mathbb{R}^n_x$ be an open definable set, and $P=(P_1,\dots,P_n):U\longrightarrow\mathbb{R}^n$ a definable map of class $\mathcal{C}^1$.

We will investigate the number of regular solutions of systems of equations

$$(\star)\qquad \begin{cases} P_1\Big(\phi\big(\lambda_{11}(x)\big),\dots,\phi\big(\lambda_{1n}(x)\big),x\Big)=0\\ \vdots\\ P_n\Big(\phi\big(\lambda_{n1}(x)\big),\dots,\phi\big(\lambda_{nn}(x)\big),x\Big)=0,\end{cases}$$

for $x\in U'$, where

$$U':=\big\{x\in\mathbb{R}^n \mid \big(\phi\big(\lambda_{i1}(x)\big),\dots,\phi\big(\lambda_{in}(x)\big),x\big)\in U,\ i=1,\dots,n\big\}$$

and $\lambda_{ij}:\mathbb{R}^n\longrightarrow\mathbb{R}$ (for $i,j=1,\dots n$) are affine maps.

The system $(\star)$ is equivalent to the system

$$(\star\star)\qquad \begin{cases} P_1\big(\phi(\xi_{11}),\dots,\phi(\xi_{1n}),x\big)=0\\ \vdots\\ P_1\big(\phi(\xi_{n1}),\dots,\phi(\xi_{nn}),x\big)=0\\ \xi_{11}-\lambda_{11}(x)=0,\\ \vdots\\ \xi_{nn}-\lambda_{nn}(x)=0\end{cases}$$

It follows that without loss of generality we can consider system of equations

$$(\star')\qquad \begin{cases} P_1\big(\phi(x_1),\dots,\phi(x_n),x\big)=0\\ \vdots\\ P_n\big(\phi(x_1),\dots,\phi(x_n),x\big)=0,\end{cases}$$

where $x=(x_1,\dots,x_n)$.

For our purposes we will study systems

$$(\star\star')\qquad \begin{cases} P_1\big(\phi(x_1),\dots,\phi(x_m),x\big)=0\\ \vdots\\ P_n\big(\phi(x_1),\dots,\phi(x_m),x\big)=0,\end{cases},$$

where $m\le n$. We will write further that equations of the form $(\star\star')$ are of type (n,m). We will denote by $\Phi_{n,m}$ the following mapping

$$\Phi_{n,m}:\mathbb{R}^n\ni x=(x_1,\dots,x_n)\longrightarrow(\phi(x_1),\dots,\phi(x_m),x_1,\dots,x_n)\in\mathbb{R}^m\times\mathbb{R}^n\quad(m\le n),$$

and we put $\Phi_{n,n}=\Phi_n$. We will use also the following notation

Definition 2.12. Let $V \subset \mathbb{R}^n$ be an open set and let $F : V \longrightarrow \mathbb{R}^p$ $(p \le n)$ be a map of class $\mathcal{C}^1$. We define

$$\begin{aligned} ZR(F) :=& \big\{x \in V \mid F(x) = 0,\ \text{rank } d_x F = \min\{n,p\}\big\} = \\ =& \big\{x \in V \mid F(x) = 0,\ \text{F is a submersion at } x\big\}. \end{aligned}$$

Our purpose is to prove the following

Theorem 2.13. *If P is definable and ϕ satisfies conditions* $(1)-(4)$, $\sharp\ ZR(P \circ \Phi_n) < \infty$.

Observation 2.14. Consider the system $(\star\star')$. Then

$$\Phi_{n,m}\big(ZR(P \circ \Phi_{n,m})\big) \subset ZR(P)$$

Proof. If $P \circ \Phi_{n,m}$ is a submersion at x, then P is a submersion at $\Phi_{n,m}(x)$. □

First we will now prove

Proposition 2.15. *Let $m, n \in \mathbb{N},\ m < n,\ U \subset \mathbb{R}^m_y \times \mathbb{R}^n_x$ be an open, definable set, and let $P_i : U \longrightarrow \mathbb{R}$ be definable functions of class $\mathcal{C}^1$, for $i = 1, \dots, n$. Consider a system of equations $(\star\star')$, for $x \in U' = \Phi^{-1}_{n,m}(U)$. Assume that for all systems of types $(\star\star')$ with $(n', p') < (n, p)$ (in the lexicographic order) the number of nondegenerate solutions is finite. Then*

$$\sharp\ ZR(P \circ \Phi_{n,m}) < \infty \quad \text{in} \quad U'.$$

Proof. Let $x = (t, u) \in (\mathbb{R}^m \times \mathbb{R}^{n-m}) \cap U'$ and $P = (P_1, \dots, P_n)$. The Jacobian matrix $A(x)$ of $P \circ \Phi_{n,m}$ at this point is equal to

$$\begin{bmatrix} \left(\frac{\partial P_1}{\partial x_1} + \phi'(t_1)\frac{\partial P_1}{\partial y_1}\right)(z) & \cdots & \left(\frac{\partial P_1}{\partial x_m} + \phi'(t_m)\frac{\partial P_1}{\partial y_m}\right)(z) & \frac{\partial P_1}{\partial x_{m+1}}(z) & \cdots & \frac{\partial P_1}{\partial x_n}(z) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\partial P_n}{\partial x_1} + \phi'(t_1)\frac{\partial P_n}{\partial y_1}\right)(z) & \cdots & \left(\frac{\partial P_n}{\partial x_m} + \phi'(t_m)\frac{\partial P_n}{\partial y_m}\right)(z) & \frac{\partial P_n}{\partial x_{m+1}}(\Phi(x)) & \cdots & \frac{\partial P_n}{\partial x_n}(z) \end{bmatrix}$$

where $z = \Phi_{n,m}(x)$. It follows, that

$$\Phi_{n,m}(ZR(P \circ \Phi_{n,m})) \subset \bigcup_{s=1}^{n} \left\{\frac{\partial P_s}{\partial x_n} \neq 0\right\} \cap \{P_s = 0\}.$$

From Proposition 2.8 we get, that for every $s \in \{1, \dots, n\}$

$$\left\{\frac{\partial P_s}{\partial x_n} \neq 0\right\} \cap \{P_s = 0\} = \bigcup_{j=1}^{n_s} \text{graph } f_{sj},$$

where $f_{sj} : U_{sj} \longrightarrow \mathbb{R}$ is a definable function of class $\mathcal{C}^1$, and U_{sj} is an open definable subset of $\mathbb{R}^{n-1}$. For every $s \in \{1, \dots, n\},\ j \in \{1, \dots, n_s\}$ define

$$\begin{aligned} &\widetilde{P}^{sj}_k : U_{sj} \ni (y, x') \longrightarrow P_k\big(y, x', f_{sj}(y, x')\big) \in \mathbb{R}, \quad k \neq s \\ &\widetilde{P}_{sj} := \left(\widetilde{P}^{sj}_1, \dots, \widetilde{P}^{sj}_{s-1}, \widetilde{P}^{sj}_{s+1}, \dots, \widetilde{P}^{sj}_n\right), \end{aligned}$$

where $x' = (x_1, \dots, x_{n-1})$. Assume, without loss of generality, that $s = n$, $j = 1$ and $f_{n1} = f$, $U_{n1} = V$, $\widetilde{P} = (\widetilde{P}_1, \dots, \widetilde{P}_{n-1}) = \widetilde{P}_{n1}$. We claim, that rank $d_{x'}(\widetilde{P} \circ \Phi_{n-1,m}) = n - 1$ for x' if and only if rank $d_{(x',x_n)}(P \circ \Phi_{n,m}) = n$, where $x_n = f(\Phi_{n-1,m}(x'))$. Indeed, note that for each $(\nu = 1, \dots, n-1)$:

$$\frac{\partial\left(\widetilde{P}_\nu \circ \Phi_{n-1,m}\right)}{\partial x_i} = \frac{\partial P_\nu}{\partial x_i} + \phi'(t_i)\frac{\partial P_\nu}{\partial y_i} - \left(\frac{\partial P_n}{\partial x_n}\right)^{-1}\frac{\partial P_\nu}{\partial x_n}\frac{\partial P_n}{\partial x_i} - \phi'(t_i)\left(\frac{\partial P_n}{\partial x_n}\right)^{-1}\frac{\partial P_\nu}{\partial x_n}\frac{\partial P_n}{\partial y_i},$$

for $i = 1, \dots, m$, and for $i = m+1, \dots, n-1$ we have

$$\frac{\partial\left(\widetilde{P}_\nu \circ \Phi_{n-1,m}\right)}{\partial x_i} = \frac{\partial P_\nu}{\partial x_i} - \left(\frac{\partial P_n}{\partial x_n}\right)^{-1}\frac{\partial P_\nu}{\partial x_n}\frac{\partial P_n}{\partial x_i}.$$

Let $x = (x', f(\phi(x_1), \dots, \phi(x_m), x'))$ and denote by $\widetilde{A}(x')$ the following matrix of $M_{n\times n}(\mathbb{R})$

$$\begin{bmatrix} & B(x') & & 0 \\ A_{n1}(x) & \cdots & A_{n(n-1)}(x) & A_{nn}(x) \end{bmatrix}, \quad \text{where} \quad B(x') = \left[\frac{\partial \widetilde{P}_j \circ \Phi_{n-1,m}}{\partial x_i}\right]_{\substack{i=1,\dots,n-1\\ j=1,\dots,n-1}}.$$

The matrix $\widetilde{A}(x')$ is obtained from the matrix $A(x)$ by subtracting from every verse $j \in \{1, \dots, n-1\}$ the last one multiplicated by $\left(\frac{\partial P_n}{\partial x_n}\right)^{-1}\frac{\partial P_j}{\partial x_n}(x)$. This allow us to state, that

$$\sharp\ ZR(P \circ \Phi_{n,m}) = \sharp\ \bigcup_{s=1}^{n}\bigcup_{j=1}^{n_s} ZR(\widetilde{P}_{sj} \circ \Phi_{n-1,m}).$$

It follows that $\sharp\ ZR(P \circ \Phi_{n,m}) < \infty$. □

Remark 2.16. The geometrical meaning of the Proposition 2.15 is as follows. We have

$$ZR(P) \cap \left(\bigcup_{s=1}^{n}\left\{\frac{\partial P_s}{\partial x_n} \neq 0\right\}\right) = \bigcup_{k\in K} N_k,$$

where K is a finite, and for each k the set N_k is a $\mathcal{C}^1$-submanifold of dimension m. The number of points in which the submanifold $M = \text{im}\ \Phi_{n,m}$ cuts with $ZR(P)$ transversally is in one to one correspondence with the sum of transversal cuts of $\Pi_{n-1}(M)$ with $\Pi_{n-1}(N_k)$, $k \in K$.

Let $n \in \mathbb{N}$ and assume that for every system $(\star\star')$ of type $(n', m') < (n, n)$ the number of nondegenerate solutions of the system $(\star\star')$ is finite. Then we may concentrate on solutions of $(\star\star')$, for which all coordinates are nonnegative (otherwise, by induction hypothesis, the number of nondegenerate solutions for which not all coordinates are positive, is finite).

Observation 2.17. Consider an equation $(\star')$. Let $A \subset \mathbb{R}^n_+$ be a set of regular solutions of the equation $(\star')$ of type (n, n). Let $j \in \{1, \dots, n\}$ and $k \in \mathbb{R}_+$. Put

$$U_j^k := \big\{(y, x) \in U \mid x_j \in (0, k)\big\}.$$

Then

$$\sharp A \cap U_j^k < \infty,$$

for every $N \in \mathbb{N}$.

Proof. The equation $(\star')$ on the set U_j^k is of type $(n,n-1)$, because $\phi|_{(0,k)}$ is a definable function, and

$$ZR(P\circ\Phi_n|_{U_j^k}) = ZR(P\circ\eta_j^k\circ\Phi_{n,n-1}), \quad \text{where}$$

$$\eta_j^k : \left(\mathbb{R}^{n-1}\times\mathbb{R}^n\right)\cap U_j^k \ni (y,x) \longrightarrow (y_1,\dots,y_{i-1},\phi(x_i),y_i,\dots,y_{n-1},x)\in\mathbb{R}^{2n},$$

for $k,j\in\{1,\dots,n\}$. □

Definition 2.18. Consider a system $(\star')$. We call this system of type $(n,n,n-r)$, for $r=1,\dots,n-1$, if the following condition holds:
let $x=(x_1,\dots,x_n)$ be a regular solution of (1), then

$$x_{n-r+1}=\phi(x_{n-r})V_{n-r}(x),\ \dots,\ x_n=\phi(x_{n-1})V_{n-1}(x)$$

where $V_{n-r},\dots,V_{n-1}:\Phi_n^{-1}(U)\longrightarrow\mathbb{R}$ are units.
The system $(\star')$, which does not satisfy any such condition is called of type (n,n,n).

We put the following order:

$$\ldots<(n,n-2)<\ldots<(n,n-1)<(n,n,1)<(n,n,2)<\ldots<(n,n,n)<(n+1,1)<\ldots$$

Let us now consider a system of equations of type (n,n,r), where $r\in\{1,\dots,n\}$ and a cell decomposition of $ZR(P)$ of class $\mathcal{C}^1$ such that for every cell $C\in\mathcal{C}$ functions defining them have a preparation. Note that dim $C\le n$.

Lemma 2.19. *Let $C\in\mathcal{C}$ be such, that dim $\Pi_n(C)<n$. Then there exists a definable neighbourhood $U_C\subset U$ of C such that the system $(\star')$ restricted to $\Phi_n^{-1}(U_C)$ can be reduced to the equation of type $(n,n-1)$.*

Proof. There exist $i\in\{1,\dots,n-1\}$ and a definable function $g_C:\Pi_{i-1}(C)\longrightarrow\mathbb{R}$ describing cell C such that $\Pi_{i-1}(C)$ is open and

$$ZR(P\circ\Phi_n)\cap\Phi_n^{-1}(C)\subset\left\{x\in\mathbb{R}^n \mid \phi(x_i)=g_C(\phi(x_1),\dots,\phi(x_{i-1}))\right\}.$$

By lemma 2.7, there exist $\nu\in\{2,\dots,i\}$, $M>1$ and $a_1,\dots,a_{\nu-1}\in\mathbb{Q}$ such that

$$M^{-1}<\phi(x_\nu)\cdot\phi(x_1)^{a_1}\cdots\phi(x_{\nu-1})^{a_{\nu-1}}<M,$$

if $(\phi(x_1),\dots,\phi(x_n),x)\in C$. By the property (4) of ϕ the function

$$\Phi_C:V_C\longrightarrow\phi(x_i)\cdot\phi(x_1)^{a_1}\cdots\phi(x_{\nu-1})^{a_{\nu-1}}\in\mathbb{R}$$

is definable, where

$$V_C:=\left\{x\in\mathbb{R}^n \,|M^{-1}<\phi(x_\nu)\cdot\phi(x_1)^{a_1}\cdots\phi(x_{\nu-1})^{a_{\nu-1}}<M\right\}.$$

Let $U_C:=(\mathbb{R}^n_+\times V_C)\cap U$, $\widetilde{y}=(y_1,\dots,\widehat{y}_\nu,\dots,y_n)$ and define

$$P_C:\widetilde{U}_C\ni(\widetilde{y},x)\longrightarrow P\big(y_1,\dots,y_{\nu-1},\Phi_C(x)y_1^{-a_1}\cdots y_{\nu-1}^{-a_{\nu-1}},y_{\nu+1},\dots,y_n,x\big)\in\mathbb{R}^n,$$

$$\widetilde{U}_C=\left\{(\widetilde{y},x)\in\mathbb{R}^{n-1}\times\mathbb{R}^n \mid \exists_{y_\nu\in\mathbb{R}_+}\ (y_1,\dots,y_{\nu-1},y_\nu,y_{\nu+1},\dots,y_n,x)\in U_C\right\}.$$

Then

$$ZR(P\circ\Phi_n)\cap\Phi_n^{-1}(U_C)\subset ZR(P_C\circ\Phi_{n,n-1}).$$

□

REMARK 2.20. It could seem that to reduce the system $(\star')$ to one of type $(n, n-1)$ it is enough to make a substitution $y_i = g_C(y_1, \ldots, y_{i-1})$. We will show an example that this is not true. Let $(S) = \mathbb{R}_{an}$ be an o-minimal structure generated by global subanalytic functions. Of course $\phi =$exp satisfies conditions $(1) - (4)$. Consider a map

$$P = (P_1, P_2) : \mathbb{R}^4 \ni (y_1, y_2, x_1, x_2) \longrightarrow \left(y_1^2 - y_2, x_1^2 - x_2\right) \in \mathbb{R}^2.$$

Then

$$ZR(P) = \{(y, y^2, x, x^2) \in \mathbb{R}^4 \mid (x, y) \in \mathbb{R}^2\} \subset \{y_1^2 - y_2 = 0\},$$

and $ZR(P)$ has a global parametrization. On $ZR(P)$ the first equation is a zero function therefore a system

$$\begin{cases} e^{2x_1} - e^{x_2} = 0 \\ x_1^2 - x_2 = 0 \end{cases}$$

should not have nondegenerate solutions, but $ZR(P \circ \Phi_2) = \{(0,0), (2,4)\}$.

Let now $C \in \mathcal{C}$ and $\dim \Pi_n(C) = n$. Then there exist definable functions of class $\mathcal{C}^1$

$$f_1, \ldots, f_n : \Pi_n(C) \longrightarrow \mathbb{R}$$

such that

$$C = \{(y, x) \in \mathbb{R}^n \times \mathbb{R}^n \mid x_1 = f_1(y), \ldots, x_n = f_n(y),\ y \in \Pi_n(C)\}$$

The set $ZR(P \circ \Phi_n) \cap \Phi_n^{-1}(C)$ is equal to nondegenerate solutions of the system

$$(\odot) \qquad \begin{cases} x_1 = f_1(\phi(x_1), \ldots, \phi(x_n)) \\ \vdots \\ x_n = f_n(\phi(x_1), \ldots, \phi(x_n)) \end{cases} \qquad x \in \Phi_n^{-1}(C).$$

LEMMA 2.21. *Let $C \in \mathcal{C}$ be such, that* $\dim \Pi_n(C) = n$. *Consider a system $(\odot)$. Let*

$$\left(\{\theta_i\}_{i=1,\ldots,n}, \{A_{ij}^\nu\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,i}}, \{V_{ij}^\nu\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,i}}, \{\alpha_{ij}^\nu\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,i}}\right)_{\nu=1,\ldots,n}$$

be a preparation of $f_1, \ldots, f_n$ and assume that there exists $i \in \{1, \ldots, n\}$ such that $\theta_i \neq 0$. Then there exists a neighbourhood $U_C \subset U$ of C such that the system $(\star\star')$ restricted to $\Phi_n^{-1}(U_C)$ can be reduced to a system $(\star\star')$ of type $(n, n-1)$.

PROOF. Let i be the greatest number from $\{1, \ldots, n\}$ for which $\theta_i \neq 0$. Then

$$\theta_i(y) = y_{n-i}^{\alpha_{(i+1)1}} \cdots y_1^{\alpha_{n(n-i)}} V(y), \quad \text{where} \quad V = V_{(i+1)1} \cdots V_{n(n-i)} \cdot A_{n(n-i)}.$$

Moreover, there exists $M > 1$ such that

$$M^{-1}\theta_i < y_{n-i+1} < M\theta_i, \quad M^{-1} < |V(y)| < M,$$

and consequently

$$M^{-2} < \phi(x_{n-i+1}) \cdot \phi(x_{n-1})^{-\alpha_{i1}} \cdots \phi(y_1)^{-\alpha_{n1}} < M^2 \quad \text{on} \quad \Phi_n^{-1}(C).$$

We can continue as in proof in lemma 2.19. ☐

Lemma 2.22. *Let $f : C \longrightarrow \mathbb{R}$ be a function defined on an open set $C \subset \mathbb{R}^n$. Let $j \in \{1, \dots, n-1\}$. Suppose that $f(x) = \phi(x_n)^{\alpha_n} \cdots \phi(x_{n-j})^{\alpha_{n-j}} V(x)$, where V is a unit, and $\alpha_1, \dots, \alpha_{j+1}$ are rational numbers. Consider an equation*

$$x_{n-j} = \phi(x_n)^{\alpha_n} \cdots \phi(x_{n-j})^{\alpha_{n-j}} V(x), \quad x \in C.$$

Suppose moreover, that there exists $M > 1$ and a sequence $\{x^\nu\}_{\nu \in \mathbb{N}}$ of solutions of above equation with

$$x_i^\nu \to \infty, \quad i = n-j, \dots, n \quad \text{and} \quad x_k^\nu = \phi(x_{k-1}^\nu) \cdot V_k^\nu, \; k = n-j+1, \dots, n,$$

with $M^{-1} < V_k^\nu < M$, for $\nu \in \mathbb{N}$ and $k = n-j+1, \dots, n$. Then $\alpha_n = \ldots = \alpha_{n-j} = 0$.

Proof. Note that for every $k = n-j+1, \dots, n$ and $l \in \mathbb{Z}$

$$\lim_{\nu \longrightarrow \infty} \frac{\left(x_{k-1}^\nu\right)^l}{x_k^\nu} = \lim_{\nu \longrightarrow \infty} \frac{\left(x_{k-1}^\nu\right)^l}{\phi(x_{k-1}^\nu) \cdot V_k^\nu} = 0 \quad \text{and}$$

$$\lim_{\nu \longrightarrow \infty} \frac{[\phi(x_{k-1}^\nu)]^l}{\phi(x_k^\nu)} = \lim_{\nu \longrightarrow \infty} \frac{(x_k^\nu)^l}{\phi(x_{k-1}^\nu) \cdot (V_k^\nu)^l} = 0.$$

Consequently, for every $q \in \mathbb{Q}_+, r \in \mathbb{Q}$ and $k = n-j, \dots, n-1$

$$\lim_{\nu \longrightarrow \infty} \frac{(x_{n-j}^\nu)^r}{[\phi(x_n^\nu)]^q} = 0 \quad \text{and} \quad \lim_{\nu \longrightarrow \infty} \frac{[(\phi(x_k^\nu)]^r}{[\phi(x_n^\nu)]^q} = 0$$

Ideed, for $r = \frac{r_1}{s}, q = \frac{q_1}{s}$

$$\frac{(x_{n-j}^\nu)^r}{[\phi(x_n^\nu)]^q} = \frac{(x_{n-j}^\nu)^r}{(x_{n-j+1}^\nu)^r} \frac{(x_{n-j+1}^\nu)^r}{(x_{n-j+2}^\nu)^r} \cdots \left(\frac{(x_n^\nu)^{r_1}}{[\phi(x_n^\nu)]^{q_1}} \right)^{1/s} \to 0.$$

Similarly we show the rest of limits.
If $\alpha_n \neq 0$, then $x_{n-j}^\nu \cdot \phi(x_n^\nu)^{-\alpha_n} \cdots \phi(x_{n-j}^\nu)^{-\alpha_{n-j}}$ would be bounded from up and down, which is impossible. Inductively we get that $\alpha_n = \ldots = \alpha_{n-j} = 0$. $\square$

Lemma 2.23. *Let $C \in \mathcal{C}$ be such, that $\dim \Pi_n(C) = n$. Let $r \in \{2, \dots, n\}$ and consider a system $(\odot)$ of type (n, n, r). Let*

$$\left(\{\theta_i\}_{i=1,\dots,n}, \{A_{ij}^\nu\}_{\substack{i=1,\dots,n \\ j=1,\dots,i}}, \{V_{ij}^\nu\}_{\substack{i=1,\dots,n \\ j=1,\dots,i}}, \{\alpha_{ij}^\nu\}_{\substack{i=1,\dots,n \\ j=1,\dots,i}} \right)_{\nu=1,\dots,n}$$

be a preparation of $f_1, \dots, f_n$ and assume that $\theta_i \equiv 0$, for $i \in \{1, \dots, n\}$. Then the system of equations $(\star\star')$ can be reduced to the equation of type $(n, n-1)$ or of type $(n, n, r-1)$.

Proof. Note that equation $(\odot)$ is of the form

$$(\otimes) \qquad \begin{cases} x_1 = \phi(x_n)^{a_{11}} \cdots \phi(x_1)^{a_{1n}} V_1\left(\phi(x_1), \dots, \phi(x_n)\right) \\ \vdots \\ x_n = \phi(x_n)^{a_{n1}} \cdots \phi(x_1)^{a_{nn}} V_n\left(\phi(x_1), \dots, \phi(x_n)\right) \end{cases} \qquad x \in \Phi_n^{-1}(C),$$

where $a_{ij} = \alpha^i_{(n-j+1)(n-j+1)}$ and $V_i(y) = V_{11}^i \cdots V_{nn}^i$, for $i = 1, \dots, n$. We have two cases:
(I). $a_{(n-r)(n-r)} = \ldots = a_{(n-r)n} = 0$. Then, by lemma 2.22, if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is a solution of the above equation, then there exists $R > 0$ such that $x_r < R$. Define

$$\widetilde{U}_C = \left\{ (y', x) \in \mathbb{R}^{n-1} \times \mathbb{R}^n \mid \exists_{z \in \mathbb{R}} \; (y'_1, \dots, y'_{k-1}, z, y'_k, \dots, y'_{n-1}, x) \in U, \; x_r < R \right\},$$

$$P_C : \widetilde{U}_C \ni (y', x) \longrightarrow P\big(y'_1, \ldots, y'_{k-1}, \phi(x_r), \ldots, y'_{n-1}, x\big) \in \mathbb{R}^n.$$

It follows that

$$ZR(P \circ \Phi_n) \cap \Phi_n^{-1}(C) \ \subset \ ZR(P \circ \Phi_n) \cap \Phi_n^{-1}\left(U \cap \{x_r < R\}\right) \ \subset \ ZR(P_C \circ \Phi_{n,n-1}).$$

In this case we reduce the system to one of type $(n, n-1)$.

(II). There exists $s < r$ such that $a_{(n-r)s} \neq 0$. Without loss of generality we may assume, that $s = r - 1$. Take a definable change of coordinates

$$\widetilde{x}_{n-r-1} = \phi^{-1}\left(\phi(x_n)^{a_{(n-r)n}} \cdots \phi(x_1)^{a_{(n-r)1}}\right) \quad \text{and} \quad \widetilde{x}_i = x_i, \ \ i \neq n-r-1$$

If x is a regular solution of $(\otimes)$, then

$$\begin{aligned}\widetilde{x}_{n-r} = x_{n-r} &= \phi(x_n)^{a_{(n-r)1}} \cdots \phi(x_1)^{a_{(n-r)n}} V_1\left(\phi(x_1), \ldots, \phi(x_n)\right) = \\ &= \phi(\widetilde{x}_{n-r-1}) V_{n-r}\left(\phi(x_1), \ldots, \phi(x_n)\right).\end{aligned}$$

Take a definable change of coordinates $\Lambda_C = (\Lambda_C^1, \Lambda_C^2)$, where

$$\begin{aligned}\Lambda_C^1 : \mathbb{R}^n \ni (x_1, \ldots, x_n) \longrightarrow \big(&x_1, \ldots, x_{n-r-2}, \\ &\phi^{-1}\left(\phi(x_n^{\nu})^{a_{(n-r)n}} \cdots \phi(x_1^{\nu})^{a_{(n-r)1}}\right), x_{n-r}, \ldots, x_n\big) \in \mathbb{R}^n, \\ \Lambda_C^2 : \mathbb{R}^n \ni (y_1, \ldots, y_n) \longrightarrow \big(&y_1, \ldots, y_{n-r-2}, y_n^{a_{(n-r)n}} \cdots y_1^{a_{(n-r)n}}, y_{n-r}, \ldots, y_n\big) \in \mathbb{R}^n.\end{aligned}$$

Define sets $\widetilde{U}_C := \Pi_n(C) \times \mathbb{R}^n_+$, $U_C := \Lambda_C^{-1}(\widetilde{U}_C)$ and maps

$$\begin{aligned}\widetilde{P}_C &: \widetilde{U}_C \ni (x, y) \longrightarrow \big(x_1 - f_1(y), \ldots, x_n - f_n(y)\big) \in \mathbb{R}^n, \\ P_C &: U_C \ni (x, y) \longrightarrow \widetilde{P}_C \circ \Lambda_C^{-1}(x, y) \in \mathbb{R}^n.\end{aligned}$$

Then

$$\Lambda\big(ZR(P \circ \Phi_n) \cap \Phi_n^{-1}(C)\big) \ \subset \ ZR(P_C \circ \Phi_n).$$

In this case we reduce equation to one of type $(n, n, r-1)$. □

PROPOSITION 2.24. *Consider a system of equations* $(\star')$ *of type* (n, n, r)*, where* $r \in \{1, \ldots, n\}$*. There exist:*

(1) *a finite family* $\{P_k\}_{k \in K_1}$ *of definable maps*

$$P_k : \mathbb{R}^{n-1} \times \mathbb{R}^n \supset U_k \longrightarrow \mathbb{R}^n \quad (U_k - \textit{open})$$

such the equation $(\star\star')$ *is of type* $(n, n-1)$*,*

(2) *a finite family* $\{P_k\}_{k \in K_2}$ *of definable maps*

$$P_k : \mathbb{R}^n \times \mathbb{R}^n \supset U_k \longrightarrow \mathbb{R}^n \quad (U_k - \textit{open})$$

such the equation $(\star')$ *is of type* $(n, n, r-1)$ *(in the case* $r > 1$*),*

such that

$$\sharp ZR(P \circ \Phi_n) < \sum_{k \in K_1} \sharp ZR(P_k \circ \Phi_{n,n-1}) + \sum_{k \in K_2} \sharp ZR(P_k \circ \Phi_n).$$

PROOF. In the case of $r > 1$ the thesis is a consequence of lammas 2.19, 2.21 and 2.23. It is enough now to consider the case $(n, n, 1)$ and situation, when solutions of the system $(\star\star')$ correspond the solutions of the system $(\otimes)$. Then, by lemma 2.22, x_1 is bounded, and we may follow as in the proof of lemma 2.23 in the case (I). □

Proof of the Theorem 2.13. We will proceed on induction on type of the system. For $n = 1$ the theorem is true by the property (3) of ϕ. In higher dimension we use Propositions 2.15 and 2.24. $\square$

Now we give a full characterization of functions satisfying conditions $(1) - (4)$ when, $\mathcal{S}$ contains a family $\mathcal{E} = \{\exp|_{[0,k]}\}_{k\in\mathbb{N}}$.

Proposition 2.25. *Let $\mathcal{S}$ be an o-minimal polynomially bounded structure containing $\mathcal{E}$. Let ϕ be a germ at infinity of function satisfying conditions* $(1) - (4)$. *Then it is of the form $e^{P(t)}$, where P is a definable germ and $\lim_{t\to\infty} P(t) = \infty$.*

Proof. Of course, if ϕ is of the form $e^{P(t)}$, P is a definable and $\lim_{t\to\infty} P(t) = \infty$, then it satisfies conditions $(1) - (4)$. We will show the opposite.
Consider a function

$$Q : U \ni (x_1, x_2) \longrightarrow \frac{\phi(x_1)}{\phi(x_2)} \in \mathbb{R}, \quad \text{on} \quad U = \left\{(x_1, x_2 \mid \frac{1}{2}\phi(x_2) < \phi(x_1)) < 2\phi(x_2)\right\}.$$

Q is definable by the condition (3) and $\Delta_2 \subset U$. Consequently

$$\frac{\partial Q}{\partial x_1}(x_1, x_2) = \frac{\phi'(x_1)}{\phi(x_2)} \quad \text{and} \quad f(x_1) := \frac{\partial Q}{\partial x_1}(x_1, x_1) = \frac{\phi'(x_1)}{\phi(x_1)}$$

are also definable. We get, that $\phi(t) = e^{P(t)}$, where $P'(t) = f(t)$. Also, P has to be growing to infinity (by definability of f and condition (2)).
We will now show that P has to be definable. Since $Q(x_1, x_2) = e^{P(x_1)-P(x_2)}$ is bounded on U, and $\mathcal{S}$ contains $\mathcal{E}$, we get that

$$P(x_1) - P(x_2) = \ln\big(Q(x_1, x_2)\big)$$

is definable on U. Also

$$\frac{\partial \ln(Q)}{\partial x_2}(x_1, x_2) = P(x_1) - f(x_2), \quad h(t) = \frac{\partial \ln Q}{\partial x_2}(t, t)$$

are definable. Consequently

$$P(t) = f(t) + h(t) \in \mathcal{S}.$$

$\square$

As corollaries of 1.9 and 2.13 we get following well known and important facts

Corollary 2.26. *Let $\mathcal{S}$ be a polynomially bounded o-minimal structure containing $\mathcal{E}$. Then the structure $\mathcal{S}_{\exp}$ generated by $\mathcal{S}$ and exponential function is o-minimal.*

Corollary 2.27. *Let $\mathcal{S}$ be a semialgebraic structure. Then $\mathcal{S}_{\exp}$ is o-minimal.*

Proof. The structure $\mathcal{S}'$ generated by $\mathcal{S}$ and $\mathcal{E}$ is o-minimal, since it is contained in $\mathbb{R}_{an}$. It follows that $\mathcal{S}'_{\exp}$ is o-minimal. But $\mathcal{S}_{\exp}$ has to contain $\mathcal{E}$, and we get, that $\mathcal{S}'_{\exp} = \mathcal{S}_{\exp}$. $\square$

CHAPTER 3

Lion theorem and normal crossings

3.1. Transforming to normal crossings in algebras of almost $\mathcal{C}^\infty$-germs

In the beginning we will recall a definition of almost $\mathcal{C}^\infty$-germs:

Definition 3.1. A germ $f \in \mathcal{C}(\mathbb{R}^m, 0)$ of function is called almost $\mathcal{C}^\infty$, if for every $n \in \mathbb{N}$ there exist $\epsilon_n \in \mathbb{R}_+$ and a representation of f

$$\widetilde{f} : K(0, \epsilon_n) \longrightarrow \mathbb{R},$$

such that

$$\widetilde{f} \in \mathcal{C}^n\big(K(0, \epsilon_n)\big).$$

The class of almost $\mathcal{C}^\infty$-differential germs will bo denoted by $\mathcal{C}^{(\infty)}(\mathbb{R}^m, 0)$.

Let $\mathcal{R} \subset \mathcal{C}^{(\infty)}(\mathbb{R}^m, 0)$ be an algebra. We define the smallest algebra $\mathcal{S} \subset \mathcal{C}^{(\infty)}(\mathbb{R}^m, 0)$ containing $\mathcal{R}$ and closed by taking the following operations:

(1) $x_i \in \mathcal{S}$, for $i = 1, \ldots, m$,
(2) if $P \in \mathcal{S}$, $\phi = (\phi_1, \ldots, \phi_m) \in \mathcal{S}^m$ and $\phi_i(0) = 0$ for every $i = 1, \ldots, m$, then $P \circ \phi \in \mathcal{S}$,
(3) if $P \in \mathcal{S}$, then $\frac{\partial P}{\partial x_i} \in \mathcal{S}$ for every $i = 1, \ldots, m$,
(4) let $P \in \mathcal{S}$; if $\frac{\partial P}{\partial x_i}(0) \neq 0$ for some $i = 1, \ldots, m$, then $\alpha \in \mathcal{S}$, where α is a germ of an implicit function for $P = 0$ at zero with respect to the variable x_i,
(5) let $P \in \mathcal{S}$; if $\frac{\partial P}{\partial x_i}(0) \neq 0$ for some $i = 1, \ldots, m$, then $\alpha \in \mathcal{S}$, where α is a germ of an implicit function for $P - y = 0$ at zero with respect to the variable x_i,
(6) let $i \in \{1, \ldots, n\}$, $P \in \mathcal{S}$ and $X_i := \{x \in \mathbb{R}^n \mid x_i = 0\}$; if $P|_{X_i} \equiv 0$ then $\frac{P}{x_i}$ is well defined and belongs to $\mathcal{S}$.

We call operations $(1) - (6)$ *admissible operations.*

Remark 3.2. If $f \in \mathcal{S}$ then, by operations (1) and (2), also

$$f(x_1, \ldots, x_k, \underbrace{0, \ldots, 0}_{n-k})$$

belongs to $\mathcal{S}$.

Note that every $f \in \mathcal{S}$ has a Taylor series at zero. Denote it by T_f. In order to get transformation to normal crossings of germs from $\mathcal{R}$ we will demand that every nonzero germ f have a nonzero expansion T_f at zero. This is so-called *quasianalytic* property.

Definition 3.3. We say that algebra $\mathcal{S}$ is *quasianalytic*, when for every $f \in \mathcal{S}$

$$f \neq 0 \Longleftrightarrow T_f \neq 0.$$

We will proof a microlocal version of reducing $R \in \mathcal{S}$ to normal crossings in case when $\mathcal{S}$ is quasianalytic, that is, when all elements of $\mathcal{S}$ are quasianalytic. It is an adaptation to our purposes of classical reducing of an analytic function to normal crossings given in [**BM**] and for quasianalytic functions from [**RSW**]. Firstly we define maps, which will play a crucial role in our considerations:
let $k, m \in \mathbb{N}$ and $k < m$. Take $w \in \mathbb{P}^k(\mathbb{R})$. We may assume, that there exist $\alpha_1, \ldots, \alpha_{k+1}$ and $k_0 \in \{1, \ldots, k+1\}$ such that

$$w = (\alpha_1 : \ldots : \alpha_{k+1}) \ \text{ and } \ \alpha_{k_0} = 1$$

Define a map $L^{w,k_0} = (\widehat{L}^{w,k_0}, L^{w,k_0}_m) = (L^{w,k_0}_1, \ldots, L^{w,k_0}_{m-1}, L^{w,k_0}_m) : \mathbb{R}^m \longrightarrow \mathbb{R}^m$:

I. In case $k_0 < k+1$,

$$L^{w,k_0}_j(\xi_1, \ldots, \xi_m) = \begin{cases} \xi_j, & j \in \{k+1, \ldots, m-1\} \cup \{k_0\} \\ \xi_{k_0}(\xi_j + \alpha_j), & j \in \{1, \ldots, k\} \backslash \{k_0\} \\ \xi_{k_0}(\xi_m + \alpha_{k+1}), & j = m \end{cases}$$

II. In case $k_0 = k+1$,

$$L^{w,k_0}_j(\xi_1, \ldots, \xi_m) = \begin{cases} \xi_j, & j \in \{k+1, \ldots, m\}, \\ \xi_m(\xi_j + \alpha_j), & j \in \{1, \ldots, k\} \end{cases}$$

Define

$$\widetilde{\Pi}_{m,k} : \mathbb{R}^m \ni (x_1, \ldots, x_m) \longrightarrow (x_1, \ldots, x_k, x_m) \in \mathbb{R}^{k+1}$$

Put $W'_w := \{z \in \mathbb{R}^{k+1} \mid z \in w\}$ and take $W_w := \widetilde{\Pi}^{-1}_{m,k}(W'_w)$. Then L^{w,k_0} is a diffeomorphism on some cone neighbourhood of W_w, that is, on $\widetilde{\Pi}^{-1}_{m,k}(U_w) \backslash \{0\}$, where U_w contains every line from some open neighbourhood of w in $\mathbb{P}^k(\mathbb{R})$.
We need also the following definition

Definition 3.4. Let $f \in \mathcal{C}^{(\infty)}(\mathbb{R}^m, 0)$. We say that f *admits a decomposition of type* (d, k, σ), where $d, k \in \mathbb{N}$ and $\sigma \in \mathbb{N}^{m-1}$ if it can be written in the form

$$f(x) = (x')^\beta l_1^{p_1}(x) \cdots l_k^{p_k}(x) W(x) U(x), \quad x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R},$$

where

- $p_1, \ldots, p_k \in \mathbb{N}, \beta \in \mathbb{N}^{m-1}$,
- $l_1, \ldots, l_k, W, U \in \mathcal{C}^{(\infty)}(\mathbb{R}^m, 0)$,
- U is a unit,
- $l_i(x', x_m) = U_i(x) \cdot x_m + B_i(x')$, where U_i is a unit, $\mathrm{ord} B_i(x') > 0$, $i = 1, \ldots, k$,
- $W = A_0(x') + \ldots + A_{d-2}(x') x_m^{d-2} + V(x) x_m^d$, where $\mathrm{ord} A_j \geq d - j$ for $j = 1, \ldots, d-2$ and V is a unit,
- $A_j^{d!/(d-j)}(x') = (x')^{\gamma^j} Z_j(x')$, $\gamma^j \in \mathbb{N}^{m-1}$, Z_j is a unit, for $j = 1, \ldots, d-2$
- $B_i^{d!}(x') = (x')^{\delta^i} Y_i(x')$, $\delta^i \in \mathbb{N}^{m-1}$, $Y_i(0)$ is a unit, for $i = 1, \ldots, k$,
- exponents δ^i and γ^j are totally ordered with respect to partial ordering from $\mathbb{N}^{m-1}$, that is, $\alpha \leq \beta$ when $\alpha_i \leq \beta_i$, for $i = 1, \ldots, m-1$,
- $\sigma = \min\{\gamma^j, \delta^i, \quad j = 1, \ldots, d-2, \ i = 1, \ldots, k\}$.

We say that f is in *normal crossings form*, when it is of the form

$$f(x) = x^\beta U(x), \quad U \text{ is a unit}, \quad \beta \in \mathbb{N}^m.$$

Let $\mathcal{A} \subset \mathcal{C}^{(\infty)}(\mathbb{R}^m, 0)$. We say that f *admits a decomposition of type* $(d, k, |\sigma|)$ *in a family* $\mathcal{A}$, when $l_i, \dots, l_k, W, U \in \mathcal{A}$.

REMARK 3.5. Note that if a germ f is of type (d, k, σ), then always $|\sigma| > d!$.

From now we fix a quasianalytic algebra $\mathcal{S}$ of almost $\mathcal{C}^\infty$-germs closed under operations (1) – (6).

PROPOSITION 3.6. *Let $P \in \mathcal{S}$. Assume that P admits a decomposition of type (d, k, σ) in $\mathcal{S}$. Let $\{x_\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^m$ be a sequence such that $x_\nu \longrightarrow 0$. Then there exist*
(1) $\{t_\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^m$, $t_\nu \longrightarrow 0$,
(2) $\phi = (\phi_1, \dots, \phi_m) \in \mathcal{S}^m$ such that $\phi(0) = 0$,
such that $P \circ \phi$ is of type (d', k', σ') with $(d', k', |\sigma'|) < (d, k, |\sigma|)$ (in lexicographic order) and $\phi(t_\nu) = x_\nu$, after possibly choice of a subsequence of x_ν.

THEOREM 3.7. *Let $P \in \mathcal{S}$ and $\{x_\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^m$ be a sequence such that $x_\nu \longrightarrow 0$. Then there exist*
(1) $\{t_\nu\}_{k \in \mathbb{N}} \subset \mathbb{R}^m$, $t_\nu \longrightarrow 0$,
(2) $\phi = (\phi_1, \dots, \phi_m) \in \mathcal{S}^m$ such that $\phi(0) = 0$,
such that $P \circ \phi$ is in normal crossings form and $\phi(t_\nu) = x_\nu$, after possibly choice of a subsequence of x_ν.

Before we prove Proposition 3.6 and Theorem 3.7 we need the following lemma

LEMMA 3.8. *Let $p \in \mathbb{N}$ and $h_1, \dots h_p \in \mathcal{S}$ such that $h_j - h_k \neq 0, j \neq k$. Consider H, a product of h_i, $i = 1, \dots, p$ and $h_k - h_l$, $l, k = 1, \dots, p$, $k \neq l$. Assume that H is in normal crossings form. Then each $h_i, i = 1, \dots, p$ is in normal crossings form,*

$$h_i(x) = x^{\beta_j} U_j(x), \quad \text{where } \beta_j \in \mathbb{N}^m,\ U_j \in \mathcal{S},\ U_j(0) \neq 0.$$

Moreover, exponents $\beta_1, \dots \beta_p$ are totally ordered with respect to partial ordering from $\mathbb{N}^m$, that is, $\gamma \leq \delta$ when $\gamma_i \leq \delta_i$, for $i = 1, \dots, m$.

PROOF. We will show that every f, where f is from the family

$$A := \{h_i, i = 1, \dots, p\} \cup \{h_i - h_j, i, j \in \{1, \dots, p\},\ i \neq j\}$$

is in normal crossings form. Indeed,

$$H = \prod_{f \in A} f = x_1^{\kappa_1} \cdots x_m^{\kappa_m} U, \quad U(0) \neq 0.$$

If $(\kappa_1, \dots, \kappa_m) = 0$, then every $f \in A$ is in normal crossings form, since f is nonzero at the origin. Assume that $\kappa_i > 0$, and $\kappa_j = 0$, for $j < i$. Suppose, for the contrary, that every $f_i := f|_{\{x_i = 0\}} \neq 0$. But then $0 = S_{H_i} = \prod_{f \in A} S_{f_i} \neq 0$, which is impossible. There exists f_0' such that $f_0 = x_i f_0'$. Proceeding this way with $\frac{H}{x_i}$, and $(A \backslash \{f_0\}) \cup \{f_0'\}$ we conclude, that every element of A is in normal crossings form.
The rest of the proof follows as in Lemma 4.7 from [BM]. □

We will prove Proposition 3.6 and Theorem 3.7 simultaneously by induction on m. Note that for $m = 1$ every function $P \in \mathcal{S}$ is already in normal crossings form. Assume, that for

$m' < m$ Theorem 3.7 is true. Let $\{x_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^m$ be a sequence converging to zero and $P \in \mathcal{S}$ be in the form

$$P(x) = (x')^\beta l_1^{p_1}(x)\cdots l_k^{p_k}(x)W(x)U(x),$$

where $\beta \in \mathbb{N}^{m-1}$, $k, p_1, \ldots, p_k \in \mathbb{N}$, ord $l_i = 1$, for $i = 1, \ldots, k$, $l_h \neq l_j$ for $j, h = 1, \ldots, k$, $j \neq h$, ord $W = d$, and $U(0) \neq 0$. We claim, that we can find $\widetilde{\phi} \in \mathcal{S}$ such that $P \circ \widetilde{\phi}$ is of type $(d, k, |\sigma|)$, for some $\sigma \in \mathbb{N}^{m-1}$ and moreover, after possibly choice of a subsequence of $\{x_\nu\}_{\nu\in\mathbb{N}}$ we may assume, that there exists a sequence $\{\widetilde{x}_\nu\}_{\nu\in\mathbb{N}}$ such that $\widetilde{\phi}(\widetilde{x}_\nu) = x_\nu$ and $\widetilde{x}_\nu \longrightarrow 0$. Indeed, after appropriate linear change of coordinates L

$$\begin{aligned} W(x', x_m) =& W(x', 0) + \frac{\partial W}{\partial x_m}(x', 0)\cdot x_m + \ldots + \frac{1}{(d-1)!}\frac{\partial^{d-1} W}{\partial x_m^{d-1}}(x', 0)\cdot x_m^{d-1} + \\ &+ \left(\frac{1}{d!}\frac{\partial^d W}{\partial x_m^d}(x', 0) + W_1(x)\cdot x_m\right)\cdot x_m^d, \\ l_i(x', x_m) =& U_i(x)\cdot x_m + B_i(x'), \end{aligned}$$

where

(1) ord $A_s = \frac{1}{s!}\frac{\partial^s W}{\partial x_m^s} \geq d - s$, for $s = 1, \ldots, m-1$, and $A_d = \frac{1}{d!}\frac{\partial^d W}{\partial x_m^d}$ is a unit,
(2) $V(x) = \frac{1}{d!}\frac{\partial^d W}{\partial x_m^d}(x', 0) + W_1(x)\cdot x_m$ is a unit, since $A_d(0) \neq 0$,
(3) ord $B_i \geq 1$, $i = 1, \ldots, k$.

Note that after this operation β and $k, p_1, \ldots, p_k$ may change. Take $W(x', x_m + \alpha(x'))$, where α is a germ of an implicit function for $\frac{\partial^{d-1} W}{\partial x_m^{d-1}} = 0$ at zero with respect to the variable x_m; therefore we may assume, that $A_{d-1}(x') = \frac{1}{(d-1)!}\frac{\partial^{d-1} W}{\partial x_m^{d-1}}(x', 0) = 0$. Note that $\widetilde{L} = (L \circ L_1)$, where $L_1(x) = (x', x_m + \alpha(x'))$ is a diffeomorphism. Define $z_\nu := \widetilde{L}^{-1}(x_\nu)$ and $H(x')$ as a product of nonzero $A_s^{d!/(d-s)}, B_i^{d!}$ and their nonzero differences. Let $z'_\nu := (z_\nu^1, \ldots, z_\nu^{m-1})$, for $\nu \in \mathbb{N}$. By induction, after possibly choice of subsequence of z'_ν, there exist $\phi_1 \in \mathcal{S}$ and $\{\widetilde{z}_\nu\}_{\nu\in\mathbb{N}}$ such that $H \circ \phi_1$ is normal crossings form, $\widetilde{z}_\nu \longrightarrow 0$ and $\phi_1(\widetilde{z}_\nu) = z'_\nu$. Then, by Lemma 3.8, we may assume, that A_s, B_i are in normal crossings form,

$$\begin{aligned} &A_s^{d!/(d-s)}(x') = (x')^{\gamma^s} Z_s(x') && \gamma^s \in \mathbb{N}^{m-1},\ Z_s(0) \neq 0 && s = 1, \ldots, d-2 \\ &B_i^{d!}(x') = (x')^{\delta^i} Y_i(x') && \delta^i \in \mathbb{N}^{m-1},\ Y_i(0) \neq 0 && s = 1, \ldots, k, \end{aligned}$$

and exponents γ^s, δ^i are totally ordered. We define

$$\widetilde{\phi} := \widetilde{L} \circ (\phi_1, \mathrm{id}_{x_m}) \quad \text{and} \quad \widetilde{x}_\nu := (\widetilde{z}_\nu, z_\nu).$$

Proof of the Proposition 3.6 and Theorem 3.7. We have to consider two possibilities:

Case 1. $d > 1$. Let σ be the smallest of exponents γ^s, δ^i, for $s = 1, \ldots, d-2$, $i = 1, \ldots, k$. There exist $r \in \{1, \ldots, m-1\}$ and $\iota \in \Xi_{m-1}^r$ such that

$$S = \sigma_{\iota(1)} + \ldots + \sigma_{\iota(r)} \geq d! \ \text{ and } S - \sigma_q < d!,\ q = \iota(1), \ldots, \iota(r).$$

After permutation of coordinates we may assume, that $\iota(1) = 1, \ldots, \iota(r) = r$. After possibly choice of a subsequence x_ν, there exists $a \in \mathbb{S}^r$ such that

$$\lim_{\nu\longrightarrow+\infty}\frac{(x_{\nu 1}, \ldots, x_{\nu r}, x_{\nu m})}{\|(x_{\nu 1}, \ldots, x_{\nu r}, x_{\nu m})\|} = a,$$

otherwise $(x_{\nu 1},\dots,x_{\nu r},x_{\nu m})=0$ for every $\nu\in\mathbb{N}$. In the first case we choose $w\in\mathbb{P}^r(\mathbb{R})$ in such a way that $a\in W'_w$. There are three possibly situations:

I. $\underline{w=(0:\ldots:1)}$ or $\underline{(x_{\nu 1},\dots,x_{\nu r},x_{\nu m})=0}$, for every $\nu\in\mathbb{N}$. Put $r_0=r+1$. Then $W\circ L^{w,r_0}$ is in normal crossings form, because

$$l_i\circ L^{w,r_0}(\xi)=\xi_m\left(U'_i(\xi)+B'_i(\xi)\right),$$
$$W\circ L^{w,r_0}(\xi)=\xi_m^d\left(A'_0(\xi)+\ldots+A'_{d-2}(\xi)+V\circ L^{w,r+1}(\xi)\right),$$

where

$$B'_i(\xi)=\frac{1}{\xi_m}B_i\circ\widehat{L}^{w,r_0}(\xi),\ U'_i(\xi)=U_i\circ L^{w,r_0}(\xi),\qquad i=1,\dots,k,$$
$$A'_s(\xi)=\frac{1}{\xi_m^{d-s}}A_s\circ\widehat{L}^{w,r_0}(\xi)\qquad s=1,\dots,d-2,$$

and

$$B'_i(0)=0,\ A'_s(0)=0,\quad\text{for}\quad i=1,\dots,k,\ s=1,\dots,d-2.$$

II. $\underline{w=(\alpha_1:\ldots:\alpha_{r+1}),\ \alpha_{r+1}\neq 0\text{ and }w\neq(0:\ldots:1)}$. We may assume, that there exists $r_0<r+1$ such that $\alpha_{r_0}=1$. Then

$$l_i\circ L^{w,r_0}(\xi)=\xi_{r_0}\Big(U'_i(\xi)\cdot\xi_m+B'_i(\xi)\Big)=\xi_{r_0}l'_i(\xi),\quad B'_i(\xi)=U'_i(\xi)\cdot\alpha_{r+1}+\frac{1}{\xi_{r_0}}B_i\circ\widehat{L}^{w,r_0}(\xi)$$
$$W\circ L^{w,r_0}(\xi)=\xi_{r_0}^d\Big(A'_0(\xi)+\ldots+A'_{d-2}(\xi)\cdot(\xi_m+\alpha_{r+1})^{d-2}+V'(\xi)\cdot(\xi_m+\alpha_{r+1})^d\Big)=$$
$$=\xi_{r_0}^d\Big(A''_0(\xi)+\ldots+A''_{d-2}(\xi)\cdot\xi_{d-2}+V'(\xi)\cdot(d\alpha_{r+1}\xi_m^{d-1}+\xi_m^d)\Big),$$

where $A'_s(\xi')=\frac{1}{\xi_{r_0}^{d-s}}A_s\circ\widehat{L}^{w,r_0}(\xi)$, $U'_i(\xi)=U_i\circ L^{w,r_0}(\xi)$ and $V'(\xi)=V\circ L^{w,r_0}(\xi)$. Note that $\operatorname{ord} l_i=1$ or $\operatorname{ord} l_i=0$, for every $i=1,\dots,k$. Indeed, we have two possibilities:

(1) $B'_i(0)\neq 0$, then $\operatorname{ord} l'_i=0$,
(2) $B'_i(0)=0$; then

$$\frac{\partial l'_i}{\partial\xi_m}(0)=\frac{\partial\left(U'_i\cdot\xi_m\right)}{\partial\xi_m}(0)+\frac{\partial B'_i}{\partial\xi_m}(0)=$$
$$=U'_i(0)+\alpha_{r+1}\cdot\frac{\partial\left(U_i\circ L^{w,r_0}\right)}{\partial\xi_m}(0)+\frac{\partial}{\partial\xi_m}\left(\frac{1}{\xi_{r_0}}B_i\circ\widehat{L}^{w,r_0}\right)(0)=$$
$$=U'_i(0)+\alpha_{r+1}\cdot\frac{\partial U_i}{\partial x_m}(0)\cdot\frac{\partial L_m^{w,r_0}}{\partial\xi_m}(0)=U'_i(0)\neq 0$$

Put $B=\{i\mid l'_i(0)=0\}$. Then $P\circ L^{w,r_0}(\xi)$ is of the form

$$(x')^{\beta+d'e_{r_0}}\prod_{i\in B}l'_i(x)\cdot W'(x)U'(x),\quad d'=d+p_1+\ldots+p_k,$$

where $\operatorname{ord} W'<d$, $\operatorname{ord} l'_i=1$, $i\in B$ and U' is a unit.

III. $\underline{w=(\alpha_1:\ldots:\alpha_r),\alpha_r=0}$. Again we may assume, that there exists $r_0<r+1$ such that $\underline{\alpha_{r_0}=1}$. Put $I=\{q\mid\alpha_q\neq 0,q=1,\dots,r\}$. Then

$$l'_i(\xi)=l_i\circ L^{w,r_0}(\xi)=\xi_{r_0}\Big(U'(\xi)\cdot\xi_m+B'_i(\xi')\Big)$$

$$W'(\xi) = W \circ L^{w,r_0}(\xi) = \xi_{r_0}^d \Big(A_0'(\xi) + \ldots + A_{d-2}'(\xi') \cdot \xi_m^{d-2} + V'(\xi) \cdot \xi_m^d \Big),$$

where

$$B_i'(\xi') = \frac{1}{\xi_{r_0}} B_i \circ \widehat{L}^{w,r_0}(\xi), \qquad U_i'(\xi') = \frac{1}{\xi_{r_0}} U_i \circ L^{w,r_0}(\xi),$$

$$A_s'(\xi') = \frac{1}{\xi_{r_0}^{d-s}} A_s \circ \widehat{L}^{w,r_0}(\xi), \qquad V'(\xi) = V \circ L^{w,r_0}(\xi).$$

Notice that exponents $\widetilde{\delta}^i, \widetilde{\gamma}^s$ of $(B_i')^{d!}$ and $(A_s')^{d!/d-s}$ are still linearly ordered and are equal to

$$\begin{aligned}
&\widetilde{\delta}_q^i = \delta_q^i,\ \widetilde{\gamma}_q^s = \gamma_q^s, && q \notin I, \quad i = 1, \ldots, k, \quad s = 1, \ldots d-2,\\
&\widetilde{\delta}_q^i = 0,\ \widetilde{\gamma}_q^s = 0, && q \in I \backslash \{r_0\}, \quad i = 1, \ldots, k, \quad s = 1, \ldots d-2,\\
&\widetilde{\delta}_{r_0}^i = \Sigma_{j=1}^r \delta_j^i - d!, && i = 1, \ldots, k,\\
&\widetilde{\gamma}_{r_0}^s = \Sigma_{j=1}^r \gamma_j^s - d!, && s = 1, \ldots, d-2
\end{aligned}$$

with the smallest exponent $\widetilde{\sigma}$ such that

$$\begin{aligned}
&\widetilde{\sigma}_q = \sigma_q, && q \notin I,\\
&\widetilde{\sigma}_q = 0, && q \in I \backslash \{r_0\},\\
&\widetilde{\sigma}_{r_0} = \Sigma_{j=1}^r \sigma_j - d!.
\end{aligned}$$

If $|\widetilde{\sigma}| < d!$ then ord $W' < d$, or there exists k_0 such that ord $l_{k_0}' = 0$ and $P' = P \circ L^{w,r_0}$ is of type $\big(d, \widehat{k}, \widehat{\sigma}\big)$ with $\widehat{k} < k$. Otherwise, P' is of type $(d, k, \widetilde{\sigma})$ with $|\widetilde{\sigma}| < |\sigma|$.
In all cases we define $t_\nu := \left(L^{w,r_0}\right)^{-1}(x_\nu)$. It is enough to show that for every $i = 1, \ldots, m$ a sequence $t_{\nu i}$ converges to zero. It is obvious, when $(x_{\nu 1}, \ldots, x_{\nu r}, x_{\nu m}) = 0$ for every $\nu \in \mathbb{N}$. Otherwise, we have to show this property for $i \in \{1, \ldots, r, m\} \backslash \{r_0\}$. Note that then $x_{\nu i}/x_{\nu r_0}$ is well defined ($x_{\nu r_0} \neq 0$ for sufficiently big ν) and

$$\frac{x_{\nu i}}{x_{\nu r_0}} \longrightarrow \frac{a_i}{a_{r_0}} = \alpha_i.$$

After easy computation, we get that

$$t_{\nu i} = \frac{x_{\nu i}}{x_{\nu r_0}} - \alpha_i \longrightarrow 0.$$

Case 2. $d = 0$. Let us assume in the beginning, that P is of the form

$$P(x) = l_1^{p_1}(x) \cdots l_k^{p_k}(x) U(x), \quad l_i(x) = U_i(x) \cdot x_m + B_i(x'),$$

where ord$B_i > 0$ and U_i are units, for $i = 1, \ldots, k$. Note that we may assume, that $B_1 \equiv 0$. Indeed, after change of coordinates $(x', \widetilde{x}_m) = \Phi\big(x', l_1(x)\big)$ we have $l_1 = \widetilde{x}_m$. Moreover, $\Phi^{-1}(x', \widetilde{x}_m) = \big(x', \phi(x', \widetilde{x}_m)\big)$, where ϕ is a germ of an implicit function for

$$\widetilde{x}_m - l_1(x) = \widetilde{x}_m - U_i(x) \cdot x_m - B_i(x') = 0.$$

at zero with respect to the variable x_m. Then, for $i > 1$

$$\frac{\partial \left(l_i \circ \Phi^{-1}\right)}{\partial \widetilde{x}_m}(0) = \frac{\partial l_i}{\partial x_m}(0) \cdot \frac{\partial \phi}{\partial \widetilde{x}_m}(0) = U_i(0) \cdot \frac{1}{U_1(0)} \neq 0$$

We can now proceed as before the main proof of the Proposition 3.6 and Theorem 3.7 and we may assume, that

$$B_i(x') = (x')^{\delta^i} Y_i(x'), \quad \delta^i \in \mathbb{N}^{m-1}, \; Y_i(0) \neq 0 \quad i = 2, \ldots, k,$$

Moreover, (by Lemma 3.8) exponents δ^i, $i = 2, \ldots, k$ are totally ordered. Let σ be the smallest among these exponents. Take $q_0 \in \{1, \ldots, m-1\}$ such that $\sigma_{q_0} > 0$. After appropriate permutation of coordinates we may assume, that $q_0 = 1$. After possibly choice of a subsequence of x_ν there exists $a \in \mathbb{S}^1$ such that

$$\lim_{\nu \longrightarrow +\infty} \frac{(x_{\nu 1}, x_{\nu m})}{\|(x_{\nu 1}, x_{\nu m})\|} = a,$$

otherwise $(x_{\nu 1}, x_{\nu m}) = 0$ for every $\nu \in \mathbb{N}$. In the first case we can find $w \in \mathbb{P}(\mathbb{R})$ such that $a \in W'_w$. Again we consider three situations:

I. $\underline{w = (0:1)}$ or $\underline{(x_{\nu 1}, x_{\nu m}) = 0}$ for every $\nu \in \mathbb{N}$. Put $r_0 = 2$. Then $P \circ L^{w,2}$ is in normal crossings form, because

$$l_i \circ L^{w,2}(\xi) = \xi_m \left(U_i(\xi) + B'_i(\xi) \right), \quad i = 1, \ldots, k,$$

where

$$B'_i(\xi) = \frac{1}{\xi_m} B_i \circ \widehat{L}^{w,2}(\xi), \; B'_i(0) = 0 \;\text{ and }\; U'_i(\xi) = U_i \circ L^{w,2}(\xi).$$

II. $\underline{w = (\alpha_1 : \alpha_2), \; \alpha_2 \neq 0, \text{ and } w \neq (0:1)}$. Put $r_0 = 1$. Then

$$\xi_1 l'_i(\xi) = l_i \circ L^{w,1}(\xi) = \xi_1 \Big(U'_i(\xi) \cdot \xi_m + B'_i(\xi') \Big), \quad B'_i(\xi') = U'_i(\xi) \cdot \alpha_2 + \frac{1}{\xi_1} B_i \circ \widehat{L}^{w,1}(\xi).$$

Put $B = \{ i \mid l'_i(0) = 0 \}$. Of course 1 does not belongs to B. Then $P \circ L^{w,1}(\xi)$ is of the form

$$(x')^{\beta + d' e_1} \prod_{i \in B} l'_i(x) \cdot U'(x), \quad d' = p_1 + \ldots + p_k,$$

where $\operatorname{ord} l'_i = 1$, $i \in B$, l'_i are regular with respect to ξ_m (we use an argument as in case 1.II) and U' is a unit. We can now modify $P \circ L^{w,1}(\xi)$ to the type $(0, \widetilde{k}, \widetilde{\sigma})$, for some $\widetilde{\sigma}$, where $\widetilde{k}$ is equal to the number of elements of the set B.

III. $\underline{w = (1:0)}$. Put $r_0 = 1$. Then

$$l_i \circ L^{w,1}(\xi) = \xi_1 \Big(U'_i(\xi) \cdot \xi_m + B'_i(\xi') \Big) = \xi_1 l'_i(\xi), \quad B'_i(\xi') = \frac{1}{\xi_1} B_i \circ \widehat{L}^{w,1}(\xi).$$

Exponents $\widetilde{\delta}^i$ of B'_i are still linearly ordered, since they are equal to

$$\begin{aligned} \widetilde{\delta}^i_q &= \delta^i_q, & q \neq 1, \; i = 1, \ldots, k, \\ \widetilde{\delta}^i_1 &= \delta^i_1 - 1, & i = 1, \ldots, k. \end{aligned}$$

with the smallest exponent $\widetilde{\sigma}$, where

$$\begin{aligned} \widetilde{\sigma}_q &= \sigma_q, & q \neq 1, \\ \widetilde{\sigma}_1 &= \sigma_1 - 1, & q = 1. \end{aligned}$$

If $|\widetilde{\sigma}| = 0$, then $P \circ L^{w,1}(\xi)$ is of type $(0, \widetilde{k}, \sigma_1)$ with $(0, \widetilde{k}, |\sigma_1|) < (0, k, |\sigma|)$ and $\widetilde{k} < k$, because all l'_i such that $\operatorname{ord} l'_i = 1$ are regular with respect to x_m and their exponents $\widetilde{\delta}_q$ are

totally ordered. Otherwise, $P \circ L^{w,1}(\xi)$ is of type $(0,k,\widetilde{\sigma})$ and we proceed with one of three situations described above.
In all cases we define $t_\nu := \left(L^{w,r_0}\right)^{-1}(x_\nu)$. We check, that $t_\nu \longrightarrow 0$ in the same way as before. □

3.2. Conditions on o-minimality

Let $\mathcal{R}$ be an algebra of almost $\mathcal{C}^\infty$-germs. Assume, that its extension $\mathcal{S}$ by operations $(1)-(6)$ is quasianalytic. Then we have the following corollary of Theorem 3.7:

PROPOSITION 3.9. *Number of isolated roots of $f \in \mathcal{S}$ is finite.*

PROOF. Suppose that $\{x_\nu\}_{\nu\in\mathbb{N}}$ is an infinite sequence of points such that
(1) $\lim_{\nu\to\infty} x_\nu = 0$
(2) for every $\nu \in \mathbb{N}$ point x_ν is an isolated zero of f.

By Theorem 3.7 there exist (after possibly choice of a subsequence of x_ν)
(1) $\{t_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^m$, $t_\nu \longrightarrow 0$,
(2) $\phi = (\phi_1,\ldots,\phi_m) \in \mathcal{S}^m$, $\phi_i(0) = 0$ for every $i = 1,\ldots,m$,

such that $P\circ\phi$ is in normal crossings form and $\phi(t_\nu) = x_\nu$. But then $Z(P\circ\phi)$ has infinitely many connected components. We get a contradiction, because $P\circ\phi$ is in normal crossings form. □

Let $\mathcal{A} = \{\mathcal{A}_n\}_{n\in\mathbb{N}}$ be a family of restricted $\mathcal{C}^\infty$-functions on $I^n = [-1,1]^n$, that is if $f \in \mathcal{A}_n$, then exist a neighbourhood U of I^n and $\mathcal{C}^\infty$-function $\widetilde{f} : U \longrightarrow \mathbb{R}$ such that $\widetilde{f}|_{I^n} = f$. Assume moreover, that $\mathcal{A}$ is closed under operation of taking partial differentials. Let us generate geometric family $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$ from $\mathcal{A}$. Then, for every $F \in \mathcal{G}_n$ we can find polynomials $P, Q \in \mathbb{R}[T_1,\ldots,T_s,X_1,\ldots,X_n]$, and for $i = 1,\ldots,s$ functions $f_i \in \mathcal{A}_{n(i)}$ and affine maps $\lambda_i : \mathbb{R}^n \longrightarrow \mathbb{R}^{n(i)}$ such that

$$F(x) = \frac{P\Big(f_1(\lambda_1(x)),\ldots,f_s(\lambda_s(x)),x\Big)}{Q\Big(f_1(\lambda_1(x)),\ldots,f_s(\lambda_s(x)),x\Big)},$$

where $Q\Big(f_1(\lambda_1(x)),\ldots,f_s(\lambda_s(x)),x\Big) \neq 0$ for every $x \in \mathbb{R}^n$. For every $a \in I^n$ we can consider a germ $F(x+a)$ at zero,

$$F_a(x) = \frac{P\Big(\widetilde{f}_1(\lambda_1(x+a)),\ldots,\widetilde{f}_s(\lambda_s(x+a)),x+a\Big)}{Q\Big(\widetilde{f}_1(\lambda_1(x+a)),\ldots,\widetilde{f}_s(\lambda_s(x+a)),x+a\Big)}.$$

Note that the number of isolated roots of F is not bigger that the number of isolated roots of

$$\widehat{F} : \mathbb{R}^{n(1)} \times \ldots \times \mathbb{R}^{n(s)} \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\widehat{F}(\xi^1,\ldots,\xi^s,x) = P\Big(f_1(\xi^1),\ldots,f_s(\xi^s),x\Big) + \sum_{j=1}^{s}(\lambda_j(x)-\xi^j)^2.$$

We have the following

THEOREM 3.10. *For every $n \in \mathbb{N}$, $H \in \mathcal{G}_n$ and $a \in I^n$ consider the algebra $\mathcal{R}(H_a)$ generated by H_a. Assume that its extension $\mathcal{S}(H_a)$ by admissible operations is quasianalytic. Then for every $F \in \mathcal{G}$ the number of isolated roots is finite.*

PROOF. Let $F \in \mathcal{G}_n$ and take $\widehat{F}$. Consider

$$\mathcal{W} = \mathcal{P}\big(\{1,\dots,n(1)\}\big) \times \dots \times \mathcal{P}\big(\{1,\dots,n(s)\}\big) \times \mathcal{P}\big(\{1,\dots,n\}\big)$$

and decomposition

$$\mathbb{R}^{n(1)} \times \dots \times \mathbb{R}^{n(s)} \times \mathbb{R}^n = \bigcup_{w\in\mathcal{W}} U_w,$$

where $U_w = U_{w_1} \times \dots \times U_{w_s} \times U_{w_{s+1}}$ and

$$U_{w_k} = \Big\{\xi^k \in \mathbb{R}^{n(k)} \mid |\xi_i^k| \le 1,\ i \notin w_k,\ |\xi_i^k| \ge 1,\ i \in w_k\Big\},\ k = 1,\dots,s$$
$$U_{w_{s+1}} = \Big\{x \in \mathbb{R}^n \mid |x_i| \le 1,\ i \notin w_{s+1},\ |x_i| \ge 1,\ i \in w_{s+1}\Big\}.$$

For $w \in \mathcal{W}$ we define

$$\widehat{F}_w(\xi^1,\dots,\xi^s,x) = P\Big(\phi_1^w(\xi^1),\dots,\phi_s^w(\xi^s),\psi^w(x)\Big) + \sum_{j=1}^{s}(\lambda_j(\psi_j^w(x)) - \psi_{s+1}^w(\xi^j))^2,$$

where

$$\begin{aligned}
&\phi_k^w = 0, && \text{if} \quad w_k \neq \emptyset,\\
&\phi_k^w = \widetilde{f}_k, && \text{if} \quad w_k = \emptyset,\\
&(\psi_k^w)_i(\xi^k) = \xi_i^k, \quad i \notin w_k, && (\psi_k^w)_i(\xi^k) = \frac{1}{\xi_i^k}, \quad i \in w_k, \quad \text{for } k = 1,\dots,s,\\
&(\psi_{s+1}^w)_i(x) = x_i, \quad i \notin w_{s+1}, && (\psi_{s+1}^w)_i(x) = \frac{1}{x_i}, \quad i \in w_{s+1}.
\end{aligned}$$

For every $\widehat{F}_w$ there exists a polynomial $\widehat{H}_w$, such that

$$\widehat{F}_w = \frac{\widehat{H}_w\Big(\phi_1^w(\xi^1),\dots,\phi_s^w(\xi^s),\xi^1,\dots\xi^s,x\Big)}{x^\kappa \Pi_{k=1}^s \Pi_{i\in w_k}(\xi_i^k)^{\alpha_{ki}}}$$

for suitable $\kappa \in \mathbb{N}^n$, $\alpha_{ki} \in \mathbb{N}$. On U_w number of isolated roots of $\widehat{F}_w$ does not exceed the number of isolated roots of $\widehat{H}_w\Big(\phi_1^w(\xi^1),\dots,\phi_s^w(\xi^s),\xi^1,\dots\xi^s,x\Big)$ on $I = I^{n(1)} \times \dots \times I^{n(s)} \times I^n$. For every $a \in I$ we can use Proposition 3.9 - the number of isolated roots in a neighbourhood of a is finite. By compactness of I and finiteness of decomposition on sets U_w the result follows. □

As an immediate corollary of Theorem 3.10 and theorem of Lion 1.5, we get

THEOREM 3.11. *Let $\mathcal{A}$ be an algebra of restricted smooth functions. Let $\mathcal{G}(\mathcal{A})$ be an geometric family generated by $\mathcal{A}$. Consider for every $n \in \mathbb{N}$, $H \in \mathcal{G}(\mathcal{A})_n$ and $a \in I^n$ an algebra $\mathcal{R}(H_a)$ generated by H_a. Assume, that its extension $\mathcal{S}(H_a)$ by admissible operations is quasianalytic. Then $\mathcal{A}$ is contained in a certain o-minimal structure.*

In the end of this paragraph, we give two well known examples:

Example 3.12. Let $\mathcal{A}_{\mathrm{an}}$ be a family of restricted analytic functions. Then of course it satisfies assumptions of Theorem 3.11. It follows, that $\mathbb{R}_{\mathrm{an}}$ is o-minimal.

Example 3.13. Let $M = (M_0, M_1, \ldots)$ be an infinite sequence of real numbers satisfying inequalities

$$\left(\frac{M_i}{i!}\right)^2 \leq \frac{M_{i-1}}{(i-1)!}\frac{M_{i+1}}{(i+1)!}, \quad i > 0 \quad (\text{strongly log} - \text{convex condition})$$

$$\sum_{i=0}^{\infty} \frac{M_i}{M_{i+1}} = +\infty.$$

Let $\{\mathcal{A}_{M,n}\}_{n\in\mathbb{N}}$ be a family of smooth restricted functions such that for every $f \in \mathcal{A}_{M,n}$ and its extension $\widetilde{f} : U \longrightarrow \mathbb{R}$ there exists $K > 0$ such that

$$|\widetilde{f}^{\alpha}(x)| \leq K^{|\alpha|+1} M_{|\alpha|}, \quad \text{for } x \in U, \alpha \in \mathbb{N}^n.$$

In [**RSW**] (section 6) it was shown, that $\mathcal{A}_M$ satisfies conditions of Theorem 3.11. It follows, that $\mathcal{A}_M$ is contained in a certain o-minimal structure.

CHAPTER 4

Almost $\mathcal{C}^\infty$-germs

4.1. Almost $\mathcal{C}^\infty$-differential germ of function

In this section, following Grelowski [**Gr**], we recall an example of almost $\mathcal{C}^\infty$-germ of function at zero, which does not admit any differential polynomial relation. We begin with the following

EXAMPLE 4.1 (see [**Gr**], section 4). Define a function $\widetilde{h} : (0,1) \longrightarrow \mathbb{R}$ by formula

$$\widetilde{h}(x) := \left(x - \frac{1}{n}\right)^{n+1} \cdot \left(\frac{1}{n-1} - x\right)^{n+1}, \quad x \in \left[\frac{1}{n}, \frac{1}{n-1}\right), \ n = 2, 3, \ldots$$

and $h : (-1,1) \longrightarrow \mathbb{R}$ by

$$h(x) := \begin{cases} \widetilde{h}(x), & x \in (0,1) \\ 0, & x = 0 \\ \widetilde{h}(-x), & x \in (-1,0) \end{cases}$$

Then h has properties

(h1) $h \in \mathcal{C}_0^{(\infty)}$, that is, $h \in \mathcal{C}^n\left(-\frac{1}{n-1}, \frac{1}{n-1}\right) \backslash \mathcal{C}^{n+1}\left(-\frac{1}{n-1}, \frac{1}{n-1}\right)$,

(h2) $\lim_{x \to 0} \frac{h^{(k)}(x)}{x^s}, \quad k, s \in \mathbb{N}$

(h3) h is semialgebraic outside every neighbourhood of zero.

We will need some elements of differential field theory :

DEFINITION 4.2. Let $K \subset F$ be fields such that F is a differential field. An element $f \in F$ is said to be *differentially algebraic* over K if it satisfies a nontrivial polynomial differential equation with coefficients from K: there exist $n \in \mathbb{N}$ and $P \in K[T_0, \ldots, T_n] \backslash \{0\}$ such that $P(f, \ldots, f^{(n)}) = 0$. We say that F is differentially algebraic over K, if each element of F is differentially algebraic over K. An element $f \in F$ is said to be *differentially transcendental* over K if it is not differentially algebraic over K.

PROPOSITION 4.3 (see [**Gr**], Proposition 2). *Let $F_1 \subset F_2 \subset F_3$ be fields and assume that F_2 and F_3 are differential fields that F_2 is differentially algebraic over F_1. Then $f \in F_3$ is differentially algebraic over F_2 if and only if it is differentially algebraic over F_1*

EXAMPLE 4.4. Let $\{a_n\}_{n=0}^\infty \subset \mathbb{R}$ be an algebraically independent sequence over $\mathbb{Q}$, such that

$$g(x) := \sum_{n=0}^\infty a_n x^n \in \mathbb{R}[[x]]$$

is an analytic function in a neighbourhood of $\left(-\frac{1}{2}, \frac{1}{2}\right)$. It is easy to see that g is differentially independent over $\mathbb{Q}$. Indeed, let $P \in \mathbb{Q}[T_0, \ldots, T_k] \backslash \{0\}$. Then $P(g(0), \ldots, g^{(k)}(0)) \neq 0$, and

g can not satisfy any differential polynomial relation with coefficients in $\mathbb{Q}$. Moreover, by Proposition 4.3 the function g is differentially transcendental over $\mathbb{Q}(x)$.

PROPOSITION 4.5 (see [**Gr**], section 4). *Consider the germ $f = g + h \in \mathcal{C}^{(\infty)}_{0^+}$. Then f is differentially transcendental over $\mathbb{R}(x)$ and the field $\mathbb{R}(x)(f)$ is a Hardy field.*

The following natural question arose:

Is $f|_{[-\frac{1}{2},\frac{1}{2}]}$ definable in any o-minimal strucutre?

An example below shows that in general we have to require more of the analytic function g than in Example 4.4.

EXAMPLE 4.6. Let us consider gamma Euler function

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}.$$

Γ has following properties:

(1) Γ is a meromorphic function, and analytic on $\mathbb{C}\backslash\{0, -1, -2, \ldots\}$,
(2) $\Gamma(\mathbb{R}_+) \subset \mathbb{R}$,
(3) for every $a > 0$ a radius of convergence of its expansion at a is equal to a,
(4) satisfies a functional equation $\Gamma(x + 1) = x\Gamma(x)$,
(5) Γ is differentially transcendental; this fact was proved by Hölder in 1887 (see [**Ru**], [**Ho**]).

Claim. Let ϕ be a differentially transcendental analytic function defined on an open set $U \subset \mathbb{R}$. Let $K \subset U$ be a compact interval. Then there exists a countable subset Q such that for every $a \in K\backslash Q$ a sequence $\{a_i\}_{i=0}^{\infty}$ given by

$$\phi(t) := \sum_{i=0}^{\infty} a_i (t - a)^i,$$

is algebraically independent.
Indeed, let us denote by

$$\mathcal{R}_{n,p} := \big\{R \in \mathbb{Q}[T_0, T_1, \ldots, T_n] : \deg R = p\big\}.$$

Then, for $R \in \mathcal{R}_{n,p}$ we have

$$\phi_R(t) := R\big(\phi(t), \phi'(t), \ldots, \phi^{(n)}(t)\big) \neq 0,$$

by transcendentality of ϕ. Moreover, a set $Z(\phi_R) \cap K$ is finite, and it follows that

$$Q := \bigcup_{(n,p)\in\mathbb{N}^2} \bigcup_{R\in\mathcal{R}_{n,p}} Z(\phi_R) \cap K$$

is at most countable, and is equal exactly to these points of $a \in K$, for which sequences $\{a_i\}_{i=0}^{\infty}$, being coefficients of expansion of ϕ at a, are not algebraically independent.
Let $K = [5, 6]$ and $b_0 \in K\backslash Q$. Put $\widetilde{\Gamma}(t) := \Gamma(b_0 + 4t)$. Then $\widetilde{\Gamma}$ is analytic in a neighbourhood of $(-1, 1)$ and determined by its expansion at zero. Let us consider a polynomial

$$R(s, s_0, t, t_0) = s_0 - \left(s - \frac{1}{4}\right)^5 \left(\frac{1}{3} - s\right)^5 - (b_0 + 4t)t_0$$

. Put $s = t + \frac{1}{4}$ and $f = \widetilde{\Gamma} + h$. Then

$$R(t) = R\left(t + \frac{1}{4}, f\left(t + \frac{1}{4}\right), t, f(t)\right) = (b_0 + 4t)h(t)$$

for $t \in \left[0, \frac{1}{12}\right)$. Since $R \neq 0$ and R has infinitely many zeros, then f does not belong to any o-minimal structure.

4.2. O-minimality

We will give now a definition of strongly transcendental functions. Under this condition on g, a function $f = g + h$ defined as in the previous paragraph with g - strongly transcendental analytic function generates an o-minimal structure.

Definition 4.7 (see [**LeG**], section 1). Let $f \in \mathcal{C}^\infty(I)$, where I is an open interval in $\mathbb{R}$. Put

$$\Delta_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \exists_{i \neq j}\ x_i = x_j\}.$$

We say, that f is *strongly transcendental* if

$$\forall_{n \in \mathbb{N}}\ \forall_{(x_1,\ldots,x_n) \in I^n \setminus \Delta_n} \exists_{B \in \mathbb{N}}\ \forall_{m \in \mathbb{N}}$$
$$\text{tr.deg}\left(x_1, \ldots, x_n, j_m f(x_1), \ldots, j_m f(x_n)\right) \geq n(m+2) - B.$$

We say that $f : J \longrightarrow \mathbb{R}$, where J is closed, bounded subset of $\mathbb{R}$, is restricted (strongly transcendental) $\mathcal{C}^\infty$-function, when there exists an extension $\widetilde{f} \in \mathcal{C}^\infty(I)$ of f, where I is an open set containing J, such that $\widetilde{f}$ is (strongly transcendental) $\mathcal{C}^\infty$-function.
for an open bounded subset I of $\mathbb{R}$ and $R \in \mathbb{R}$ we define

$$\mathcal{A}_R := \left\{f \in \mathcal{C}^\infty(I) \mid \exists_{C \in \mathbb{N}}\ \forall_{k \geq 0}\ \forall_{x \in I} |f^{(k)}(x)| \leq CR^k k!\right\}.$$

Let $g \in \mathcal{A}_R$. We define a norm

$$\|g\|_R = \sup_{k \in \mathbb{N}, x \in I} \left|\frac{g^{(k)}(x)}{R^k k!}\right|.$$

Theorem 4.8 (see [**LeG**], Proposition 2.2). *The set of strongly transcendental analytic functions is residual in $\mathcal{A}_R$ with topology given by the norm $\|\cdot\|_R$.*

Fix now strongly transcendental analytic function $g \in \mathcal{C}^\infty(0,1)$. Let $m \in \mathbb{N}$, $P \in \mathbb{R}[\underline{T}^1, \ldots, \underline{T}^m]$, where $\underline{T}^j = \left(T_0^j, \ldots, T_{l_j+1}^j\right)$, $l_j \in \mathbb{N}$ for $j = 1, \ldots, m$ and $a \in [-\frac{1}{2}, \frac{1}{2}]^m$. We have a decomposition

$$\{1, \ldots, m\} = I_1^a \cup I_2^a \cup I_3^a$$

such that

$$I_1^a = \left\{i \mid \exists_{\ n\ \in \mathbb{Z} \setminus \{0\}} :\ a_i = \frac{1}{n}\right\}, \quad I_2^a = \{i \mid a_i = 0\}, \quad I_3^a = \{1, \ldots, m\} \setminus (I_1^a \cup I_2^a).$$

Without loss of generality we may assume that $I_1^a = \{1, \ldots, k_a\}$. We define functions

$$f_{n,+}(x) = g(x) + \left(x - \frac{1}{n}\right)^{n+1} \cdot \left(\frac{1}{n-1} - x\right)^{n+1}, \quad n = 2, 3, \ldots$$
$$f_{n,-}(x) = g(x) + \left(x - \frac{1}{n+1}\right)^{n+2} \cdot \left(\frac{1}{n} - x\right)^{n+2}, \quad n = 2, 3, \ldots$$

$$f_{n,+}(x) = f_{-n,-}(x), \quad n = -2, -3, \ldots,$$
$$f_{n,-}(x) = f_{-n,+}(x), \quad n = -2, -3, \ldots.$$

Let $K := \{+,-\}^{k_a}$. For $k = k_a$ and $\kappa = (\kappa_1, \ldots, \kappa_k) \in K$ we put

$$P_a^\kappa(x_1, \ldots, x_m) := P\big((x_1 + a_1), j_{l_1} f_{a_1,\kappa_1}(x_1 + a_1), \ldots, (x_k + a_k), j_{l_k} f_{a_k,\kappa_k}(x_k + a_k),$$
$$(x_{k+1} + a_{k+1}), j_{l_{k+1}} f(x_{k+1} + a_{k+1}), \ldots, (x_m + a_m), j_{l_m} f(x_m + a_m)\big).$$

Note that for every a a function P_a^κ is of class $\mathcal{C}^{(\infty)}$ at a. Our purpose is to show, that for every a and κ number of isolated roots of P_a^κ in a certain neighbourhood of zero is finite. If $a_1 \cdots a_n \neq 0$ it follows from o-minimality of the germ P_a^κ.
Define algebras

$$\mathcal{R}_a^\kappa := \Big\{ P_a^\kappa \mid P \in \mathbb{R}[\underline{T}^1, \ldots, \underline{T}^m], \underline{T}^j = (T_1^j, \ldots, T_{l_j}^j),\ l_1, \ldots, l_m \in \mathbb{N}) \Big\}$$

and their extensions $\mathcal{S}_a^\kappa$ by admissible operations from chapter 3.
In [**LeG**] author showed quasianalycity property for strongly transcendental smooth germs. In our case (germs of $\mathcal{S}_a^\kappa$) proof can be repeated in an axactly same way.

Lemma 4.9 (see [**LeG**], Lemma 3.4). *Let $f \in \mathcal{S}_a^\kappa$. Put*

$$C = \{|a_i|, i = 1, \ldots, m\} = \{c_1, \ldots, c_n\}.$$

There exist $k \in \mathbb{N}$, $b = (b_1, \ldots, b_k) \in \mathbb{R}^k$ such that for every $\alpha \in \mathbb{N}^m$ there exist

$$m(\alpha, 1), \ldots, m(\alpha, n) \in \mathbb{N},\ P_\alpha \in \mathbb{Q}[\underline{T}^1, \underline{T}^2, \underline{T}^3, \ldots, \underline{T}^{n+2}]$$

where $\underline{T}^j = (t_o^j, \ldots, T_{m(\alpha,j)+1}^j)$ such that

$$\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(0) = P_\alpha\big(b, c_1, \ldots, c_n, j_{m(\alpha,1)} g(c_1), \ldots, j_{m(\alpha,n)} g(c_n)\big).$$

Proposition 4.10 (see [**LeG**], Lemma 3.6 and 3.5). *Let $f \in \mathcal{S}_a^\kappa$. Then $f \equiv 0$ if and only if $S_f = 0$.*

Let

$$C_j := \{x \in [-1/2, 1/2] \mid f \text{ is of class } \mathcal{C}^j\}$$

We define for $j \in \mathbb{N}$ functions $F^j : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$F^j(x) = \begin{cases} f^{(j)}(x), & x \in C_j \\ 0, & x \in (\mathbb{R} \setminus [-1/2, 1/2]) \cup C_j \end{cases}$$

Let $P \in \mathbb{R}[\underline{T}^1, \ldots, \underline{T}^n]$, where $\underline{T}^j = (T_0^j, \ldots, T_{l_j+2}^j)$ and $l_j \in \mathbb{N}$, for $j = 1, \ldots, n$. We define $\widehat{P} : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\widehat{P}(x_1, \ldots, x_n) = P(x_1, F^0(x_1), \ldots, F^{l_1}(x_1), \ldots x_n, F^0(x_n), \ldots, F^{l_n}(x_n)).$$

An easy consequence of Proposition 3.9 is

Theorem 4.11. *Number of isolated roots of $\widehat{P}$ is finite.*

PROOF. We will proceed by induction on n. For n the statement is clear. Let $\mathcal{N} = \mathcal{P}^{\{1,\dots,n\}}$. Then $\mathbb{R}^n = \bigcup_{N\in\mathcal{N}} U_N$, where

$$U_N = \left\{(x_1,\dots,x_n) \in \mathbb{R}^n \mid |x_i| > \frac{1}{2},\ i \in N, |x_i| \leq \frac{1}{2},\ i \in \{1,\dots,n\}\backslash N\right\}.$$

Consider firstly $\widehat{P}$ on $U_\emptyset$. To see, that number of isolated zeros is finite it is enough to show, that for every $a \in U_\emptyset$ there exists a neighbourhood V_a with finite number of isolated roots. Let $a \in U_\emptyset$. If $I_a^1 \neq \emptyset$, then for every $I \subset I_a^1$ define

$$\begin{aligned}
\phi^a_{Ii}(x_1,\dots,x_n) &= (a_i, F^0(a_i),\dots,F^{l_i}(a_i)), \quad i \in I\\
\phi^a_{Ii}(x_1,\dots,x_n) &= (x_i, j_{l_i}(f(x_i))), \quad i \in \{1,\dots,n\}\backslash I\\
\widehat{P}_{a,I}(x_1,\dots,x_n) &= P(\phi_{I1}(x),\dots,\phi_{In}(x)).
\end{aligned}$$

Number of isolated roots $\widehat{P}_{a,I}$ is finite, by induction hypothesis. Consequently, the number of isolated roots of $\widehat{P}$ in a neighbourhood of a is finite, because it is bounded from above by sum of number of isolated roots of $\widehat{P}^\kappa_a$, $\kappa \in \{+,-\}^{k_a}$ and $\widehat{P}_{a,I}$, $I \subset I_a^1$.
For a nonempty $N \in \mathcal{N}$ put

$$\begin{aligned}
\xi_i(x) &= \left(\frac{1}{4x_i}, 0,\dots,0\right), \quad i \in N\\
\xi_i(x) &= (x_i, j_{l_i}(f(x_i))), \quad i \in \{1,\dots,n\}\backslash N.
\end{aligned}$$

Then

$$\widehat{P}(\xi_1(x_1),\dots,\xi_n x_n) = \frac{\widehat{P}_N(x)}{\prod_{i\in N} x_i^{p_i}}$$

and

$$\widehat{P}_N(x) = P_N(x_1, F^0(x_1),\dots,F^{l_1}(x_1),\dots x_n, F^0(x_n),\dots,F^{l_n}(x_n))$$

for some $P_N \in \mathbb{R}[\underline{T}^1,\dots,\underline{T}^n]$. Now we can proceed as in case $N = \emptyset$. $\square$

As corollary of Theorems 4.11, 1.9 we get

THEOREM 4.12. *F is contained in a certain o-minimal structure.*

4.3. Examples of a Hardy field which does not generate o-minimal structure

We will give now a rich family of examples of Hardy fields which can not lie in any o-minimal structure. Let $g \in \mathcal{C}^{(\infty)}_{0^+}$ be a germ from Example 4.4. We begin with the following

OBSERVATION 4.13. Let $P \in \mathbb{R}[\underline{T}^1,\dots,\underline{T}^s]$, where $\underline{T}^j = (T_0^j,\dots,T^j_{l_j+1})$ and $l_j \in \mathbb{N}$, for $j = 1,\dots,s$. Define a germ of function $\widehat{P} \in \mathcal{C}^{(\infty)}_{\mathbb{R}^s,0}$ by

$$\widehat{P}(t_1,\dots,t_s) = P\left(t_1, j_{l_1} g(t_1),\dots,t_s, j_{l_s} g(t_s)\right).$$

Then $\widehat{P} \neq 0$.

PROOF. We will proceed by induction on s. For $s = 1$ thesis is a consequence of Proposition 4.5 and Example 4.4. Let us take $s > 1$ and assume, that the statement is true for every $k < s$. For a polynomial $P \in \mathbb{R}[\underline{T}^1, \dots, \underline{T}^s]$ we have the following representation

$$P\left(\underline{T}^1, \dots, \underline{T}^s\right) = \sum_{\alpha \in \mathbb{N}^{l_s+2}} P_\alpha\left(\underline{T}^1, \dots, \underline{T}^{s-1}\right)\left(\underline{T}^{s-1}\right)^\alpha.$$

Assume that $\widehat{P} \equiv 0$. Take a sufficiently small box $B = B'' \times B' = [-\epsilon, \epsilon]^k \times [\epsilon, \epsilon]^{l_s+2}$, where $k = l_1 + \dots l_{s-1} + 2(s-1)$ and fix $a \in B'$. Then $\widehat{P}|_{B_a} \equiv 0$, where $B_a = \{a\} \times B'$. By Proposition 4.5 $P_\alpha(a) = 0$, for every $\alpha \in \mathbb{N}^{l_s+2}$ and $a \in B'$. By the induction hypothesis $P_\alpha \equiv 0$ for every $\alpha \in \mathbb{N}^{l_s+2}$, and consequently, $P \equiv 0$. $\square$

LEMMA 4.14. *There exists $\epsilon \in \mathbb{R}_+$ such that for every $P \in \mathbb{Q}[\underline{T}^1, \underline{T}^2]\backslash\{0\}$ (where $\underline{T}^j = (T^j_0, \dots, T^j_{l_j})$ for $j = 1, 2$) a germ $P(j_{l_1}g(t), j_{l_2}g(\epsilon t))$ is nonzero.*

PROOF. Let $P \in \mathbb{Q}[\underline{T}^1, \underline{T}^2]$ and consider

$$Q(s,t) := P\left(j_{l_1}g(t), g(s), \frac{s}{t}g'(s), \dots, \frac{s^{l_2}}{t^{l_2}}g^{(l_2)}(s)\right).$$

Then there exist $R \in \mathbb{Q}[T_1, \underline{T}^1, T_2, \underline{T}^2]\backslash\{0\}$ and $d \in \mathbb{N}$ such that

$$Q(s,t) = \frac{1}{s^d}R(t, j_{l_1}g(t), s, j_{l_2}g(s)).$$

By Observation 4.13, $Q(s,t) \neq 0$. From o-minimality, there exist $\epsilon_{1_P}, \dots, \epsilon_{n_P} \in \mathbb{R}$ such that for every $\epsilon \in \mathbb{R}_+\backslash\{\epsilon_{1_P}, \dots, \epsilon_{n_P}\}$ a function $Q(t, \epsilon t)$ is nonzero. By the countability of polynomials with rational coefficients, we can find ϵ with desired property. $\square$

PROPOSITION 4.15. *Let f and g be germs of functions of $\mathcal{C}^{(\infty)}_{0^+}$ as in Example 4.4, $\epsilon \in \mathbb{R}_+$ as in Lemma 4.14. Then f is differentially transcendental over the field $\mathbb{Q}(g_\epsilon)$, where $g_\epsilon(t) = g(\epsilon t)$ and $\mathbb{Q}(g_\epsilon, f)$ is a Hardy field.*

PROOF. Let $P \in \mathbb{Q}[\underline{T}^1, \underline{T}^2]\backslash\{0\}$, where $\underline{T}^j = (T^j_0, \dots, T^j_{l_j})$ for $j = 1, 2$. It is enough to show, that

$$P(j_{l_1}f(t), j_{l_2}g_\epsilon(t)) \neq 0,$$

for sufficiently small $t > 0$. Indeed,

$$\begin{aligned} P\big(j_{l_1}f(t), j_{l_2}g_\epsilon(t)\big) &= P\big((g+h)(t), \dots, (g+h)^{l_1}(t), g(\epsilon t), \dots, g^{l_2}(\epsilon t)\big) = \\ =&P\big(j_{l_1}g(t), j_{l_2}g_\epsilon(t)\big) + \sum_{\substack{\kappa=(\kappa_0,\dots\kappa_{l_1})\in\mathbb{N}^{l_1+1}\\ |\kappa|>0}} P_\kappa\big(j_{l_1}g(t), j_{l_2}g_\epsilon(t)\big)\big[h(t)\big]^{\kappa_0}\cdots\big[h(t)^{l_1}\big]^{\kappa_{l_1}} = \\ =&P\big(j_{l_1}g(t), j_{l_2}g_\epsilon(t)\big) + A(t) \end{aligned}$$

By Lemma 4.14 and analyticity of g, there exists $m \in \mathbb{N}$ such that $\left|P_\kappa\big(j_{l_1}g(t), j_{l_2}g_\epsilon(t)\big)\right| \geq t^m$ (for sufficiently small t). On the other hand, by property (h2) of h we have $A(t) < t^n$, for every $n \in \mathbb{N}$ and sufficiently small t. We conclude, that

$$\left|P\big(j_{l_1}f(t), j_{l_2}g_\epsilon(t)\big)\right| \geq t^m - t^{m+1} > 0.$$

$\square$

Corollary 4.16. *A differential field* $\mathbb{R}(x)(g_\epsilon, f)$ *is a Hardy field.*

Proof. Follows from Proposition 4.15 and Proposition 4.3 for fields $F_1 = \mathbb{Q}(g_\epsilon)$, $F_2 = \mathbb{R}(x)(g_\epsilon)$ and $F_3 = \mathbb{R}(x)(g_\epsilon, f)$. □

Corollary 4.17. *Hardy field* $\mathbb{R}(x)(g_\epsilon, f)$ *can not be a Hardy field of any o-minimal structure*

Proof. Consider a function $P(s,t) = f(s) - g_\epsilon(t)$ and $P\left(s, \frac{1}{\epsilon}s\right)$. □

Corollary 4.18. *There exist germs of functions* f_1, f_2*, such that* $\mathbb{R}_{f_1}$ *and* $\mathbb{R}_{f_2}$ *are o-minimal,* $\mathbb{R}(x)(f_1, f_2)$ *is Hardy field, but* $\mathbb{R}_{f_1,f_2}$ *is not an o-minimal structure.*

Proof. Take an analytic function g for which $g + h$ lies in an o-minimal structure. Put $f_1 = g + h$ and $f_2 = g_\epsilon$ with ϵ as in Lemma ϵ and use Corollary 4.3. □

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