Existence and positivity in CDO term structure models

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Motivation

In term structure models one often has two types of bonds p^1 and p^2, both given by

$$p^i(t, T) = e^{-\int_t^T f^i(t,u)du}.$$

If bond 1 is default-free and bond 2 is defaultable, one expects

$$f^1(t, u) \leq f^2(t, u) \tag{1}$$

for all t, u.

Our aim is twofold, however in a much more general setting:

- When does absence of arbitrage imply (1)?
- What are sufficient conditions on f^1, f^2 for (1) to hold?

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Top-Down Models for portfolio credit risk

Essentials of securitization

- Consider a pool of m defaultable entities.
- Default i occurs at τ_i with associated loss q_i
- Cumulative loss

$$L_t = \sum_{i=1}^{m} q_i 1_{\{\tau_i \leq t\}}.$$

- Normalize the total nominal to 1, set $\mathcal{I} := [0, 1]$.
- Typical products can be described via the conditional distribution of L (after discounting): A security which pays $1_{\{L_T \leq \eta\}}$ at T is called (T, η)-*bond*.
- Its price at time $t \leq T$ is denoted by $P(t, T, \eta)$.

We assume

$$P(t, T, \eta) = 1_{\{L_t \leq \eta\}} \exp\Big(- \int_t^T f(t, u, \eta) du \Big)$$

and that

$$df(t, T, \eta) = \alpha(t, T, \eta)dt + \sigma(t, T, \eta)dW_t + \int_E \gamma(t, T, \eta, x)(\mu(dt, dx) - F_t(dx)dt); \tag{2}$$

W is a possibly infinite-dimensional Brownian motion and μ is a integer-valued random measure on $\mathbb{R}^+ \times E$ with compensator $dt \otimes F_t(dx)$.

(A1) L is given by

$$L_t = \int_0^t \int_E 1_{\{L_{s-} + \ell_s(x) \leq 1\}} \ell_s(x) \mu(ds, dx),$$

where ℓ is a non-negative, predictable process such that for all $t \geq 0$ it holds that $\int_0^t 1_{\{L_{s-} + \ell_s(x) \leq 1\}} \ell_s(x) F_s(dx) ds < \infty$ (finite activity).

Under (A1), L is a non-decreasing, pure-jump process with values in $\mathcal{I}$. Furthermore, the indicator process $(1_{\{L_t \le \eta\}})_{t \ge 0}$ is cádlág and has intensity

$$\lambda(t, \eta) := F_t(\{x \in E : L_{t-} + \ell_t(x) > \eta\}); \tag{3}$$

that is,

$$M_t^\eta := 1_{\{L_t \le \eta\}} + \int_0^t 1_{\{L_s \le \eta\}} \lambda(s, \eta)\, ds \tag{4}$$

is a martingale. Moreover, $\lambda(t, \eta)$ is decreasing in η with $\lambda(t, 1) = 0$.

This framework encompasses most of the existing portfolio credit risk models.

For example, if $\tau_1, \dots, \tau_m$ are conditionally independent with intensities λ_i and the losses at t have distribution $F_{i,t}$, then

$$\lambda(t,\eta) = \sum_{\tau_i > t} \lambda_{i,t} \int_E 1_{\{L_{t-} + \ell(x) > \eta\}} F_{i,t-}(dx).$$

There are affine specifications and risk-minimizing hedging strategies have been derived. We proceed as follows:

1. Absence of arbitrage
2. Existence in an SPDE specification
3. Positivity and monotonicity

Absence of arbitrage

We call the measure $\mathbb{Q}$ a martingale measure and write $\mathbb{Q} \in \mathcal{Q}$, if

$$\left(e^{\int_0^t f(u,u,1)du} P(t,T,\eta)\right)_{t\geq 0} \quad \text{are local martingales for all } (T,\eta). \tag{5}$$

Let $\Sigma^j(t,T,\eta) := \int_t^T \sigma^j(t,s,\eta)ds$ and $\Gamma(t,T,\eta,x) := \int_t^T \gamma(t,s,\eta,x)ds$.

Theorem

Assume that (A1)–(A5) hold. Then $\mathbb{Q} \in \mathcal{Q}$, if and only if

$$\alpha(t,T,\eta) = \sum_j \sigma^j(t,T,\eta)\Sigma^j(t,T,\eta) - \int_E \gamma(t,T,\eta,x)\Big(e^{-\Gamma(t,T,\eta,x)}1_{\{L_{t-}+\ell_t(x)\leq\eta\}} - 1\Big)F_t(dx) \tag{6}$$

$$r_t(0,\eta) = r_t + \lambda(t,\eta), \tag{7}$$

where (6) and (7) hold on $\{L_t \leq \eta\}$, $\mathbb{Q}\otimes dt$-a.s.

Existence

For existence, we assume that $E = I \times G$, where $I = [0, 1]$ as previously and G is the mark space of a (homogeneous) Poisson random measure $\tilde{\mu}$.
Denote by μ^L the Poisson random measure associated to the jumps of L, such that

$$L_t = \int_0^t \int_I x \, \mu^L(ds, dx). \tag{8}$$

(A1') $L_t = \sum_{s \leq t} \Delta L_s$ is càdlàg, non-decreasing, adapted, pure jump process with values in I which admits an absolutely continuous compensator $\nu^L(t, dx)dt$ satisfying $\nu^L(t, \mathcal{I}) < \infty$. $\tilde{\mu}$ is a homogeneous Poisson random measure on $\mathbb{R}^+ \times G$ with compensator $dt \otimes \tilde{F}(dx)$ and $\int_0^t \int_G F(dx)ds < \infty$ for all $t \geq 0$. Moreover, $\tilde{\mu}$ and μ^L are independent.

We set $\mu = \mu^L \otimes \tilde{\mu}$.

We switch to the Musiela parametrization, such that

$$r(t,\xi,\eta) = f(t, t+\xi, \eta).$$

Moreover, we consider models where r is the mild solution of

$$dr_t = \left(\frac{\partial}{\partial \xi} r_t + \alpha_t\right) dt + \sigma_t dW_t + \int_G \gamma_t(x)\,\tilde{\mu}(dt,dx) + \int_I \delta_t(x) \mu^L(dt,dx). \quad (9)$$

Corollary

Under (A1')-(A5') we have that $\mathbb{Q} \in \mathcal{Q}$*, if and only if*

$$\begin{aligned} \alpha(t,T,\eta) &= \sum_j \sigma^j(t,T,\eta)\Sigma^j(t,T,\eta) \\ &\quad - \int_E \gamma(t,T,\eta,x) e^{-\Gamma(t,T,\eta,x)} \tilde{F}(dx) \\ &\quad - \int_I 1_{\{L_{t-}+x \le \eta\}} \delta(t,T,\eta,x) e^{-\Delta(t,T,\eta,x)} \nu_t^L(dx), && (10) \\ r_t(0,\eta) &= r_t + \lambda(t,\eta), && (11) \end{aligned}$$

where (6) *and* (7) *hold on* $\{L_t \le \eta\}$, $\mathbb{Q} \otimes dt$*-a.s.*

A martingale problem

(A6) $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{G} \otimes \mathcal{H}$, $\mathbb{Q}(d\omega) = \mathbb{Q}_1(d\omega_1)\mathbb{Q}_2(\omega_1, d\omega_2)$, where $\omega = (\omega_1, \omega_2) \in \Omega$, and $\mathcal{F}_t = \mathcal{G}_t \otimes \mathcal{H}_t$, where

1. $(\Omega_1, \mathcal{G}, (\mathcal{G}_t), \mathbb{Q}_1)$ is some filtered probability space carrying the market information, in particular the Brownian motions $W^j(\omega) = W^j(\omega_1)$, $j = 1, 2, \ldots$ and the Poisson random measure $\tilde{\mu}(\omega) = \tilde{\mu}(\omega_1)$,
2. $(\Omega_2, \mathcal{H})$ is the canonical space of paths for I-valued increasing marked point processes endowed with the minimal filtration $(\mathcal{H}_t)$: the generic $\omega_2 \in \Omega_2$ is a càdlàg, increasing, piecewise constant function from $\mathbb{R}_+$ to I. Let
$$L_t(\omega) = \omega_2(t)$$
be the coordinate process. The filtration $(\mathcal{H}_t)$ is therefore $\mathcal{H}_t = \sigma(L_s \mid s \leq t)$, and $\mathcal{H} = \mathcal{H}_\infty$,
3. $\mathbb{Q}_2$ is a probability kernel from $(\Omega_1, \mathcal{G})$ to $\mathcal{H}$ to be determined below.

- Under (A6), $\sigma_t(\omega) = \sigma_t(\omega_1, \omega_2)$ (and γ, δ) are functions of the loss path ω_2.
- The evolution equation (9) can thus be solved on the stochastic basis $(\Omega_1, \mathcal{G}, (\mathcal{G}_t), \mathbb{Q}_1)$ along any genuine loss path $\omega_2 \in \Omega_2$.

Regarding condition (11), note that

$$\nu^L(t, (0, \eta]) = \lambda(t, L_t) - \lambda(t, L_t + \eta), \quad \eta \in I, \tag{12}$$

where $\lambda(t, x) = 0$ for $x \geq 1$. Then (11) is equivalent to

$$\nu^L(\omega; t, dx) = -r_t(\omega; 0, \omega_2(t) + dx), \tag{13}$$

(with $r_t(0, \eta) \equiv r_t$ for $\eta \geq 1$) Hence, *unless* δ *is zero*,

$$\alpha_t(\xi, \eta) = \alpha_t(\xi, \eta, r_t)$$

becomes an explicit linear functional of the (short end of the) prevailing spread curve.

Theorem

Assume (A6) holds. Let r_0, $\sigma_t, \gamma_t(x)$ *and* $\delta_t(x)$ *satisfy (A2), (A4) and (A5'), respectively. Define* $\nu^L(t, dx)$ *by* (13) *and* α_t *by* (10) *for all* (t, T, x)*. Suppose, for any loss path* $\omega_2 \in \Omega_2$*, there exists a solution* $r_t(\xi, \eta)$ *of* (9) *such that* $r_t(0, \eta)$ *is progressive, decreasing and càdlàg in* $\eta \in I$*. Then*

1. *(A3) is satisfied.*
2. *There exists a unique probability kernel* $\mathbb{Q}_2$ *from* $(\Omega_1, \mathcal{G})$ *to* $\mathcal{H}$*, such that the loss process* $L_t(\omega) = \omega_2(t)$ *satisfies (A1') and the no-arbitrage condition* (5) *holds.*

A SPDE formulation

We consider the SPDE

$$dr_t = \left(\frac{d}{d\xi}r_t + \alpha(r_t)\right)dt + \sigma(r_t)dW_t + \int_G \gamma(r_{t-},x)\tilde{\mu}(dt,dx) + \int_I \delta(r_{t-},x)\mu^L(dt,dx). \quad (14)$$

with vector fields $\sigma : H \to H$, $\gamma : H \times G \to H$ and $\delta : H \times I \to H$ and

$$\begin{aligned}\alpha(\omega_2,t,h)(\xi,\eta) &= \sum_j \sigma^j(h)(\xi,\eta)\Sigma^j(\xi,\eta) \\ &\quad - \int_E \gamma(h,x)(\xi,\eta)e^{-\Gamma(h,x)(\xi,\eta)}\tilde{F}(dx) \\ &\quad - \int_I 1_{\{\omega_2(t-)+x\le\eta\}}\delta(h,x)(\xi,\eta)e^{-\Delta(h,x)(\xi,\eta)}h(0,\omega_2(t)+dx).\end{aligned}$$

Theorem

Suppose that (A6) holds and that

$$\|\sigma(h_1) - \sigma(h_2)\|_{L_2^0(H)} \leq K\|h_1 - h_2\|,$$

$$\left(\int_E \|\gamma(h_1, x) - \gamma(h_2, x)\|^2 \tilde{F}(dx)\right)^{1/2} \leq K\|h_1 - h_2\|,$$

$$\int_0^t \int_I \|\delta(h_1, x) - \delta(h_2, x)\| \mu^{\omega_2}(ds, dx) \leq K(\omega_2, t)\|h_1 - h_2\|,$$

$$\|\alpha(\omega_2, t, h_1) - \alpha(\omega_2, t, h_2)\| \leq K(\omega_2, t)\|h_1 - h_2\|.$$

Then, for each $\omega_2 \in \Omega_2$ there is a unique solution $r(\cdot, \omega_2) : \Omega_1 \times \mathbb{R}_+ \to H$ for

$$dr_t(\cdot, \omega_2) = \left(\frac{d}{d\xi} r_t(\cdot, \omega_2) + \alpha(r_t(\cdot, \omega_2))\right)dt + \sigma(r_t(\cdot, \omega_2))dW_t + \int_G \gamma(r_{t-}(\cdot, \omega_2), x)\tilde{\mu}(dt, dx) + \int_I \delta(r_{t-}(\cdot, \omega_2), x)\mu^{\omega_2}(dt, dx)$$

on the probability space $(\Omega_1, \mathcal{G}, \mathbb{Q}_1)$.

Positivity and Monotonicity

Consider the closed, convex cone

$$C = \bigcap_{\xi \in \mathbb{R}_+} \bigcap_{\substack{\eta_1, \eta_2 \in \mathcal{I} \\ \eta_1 \leq \eta_2}} \{h \in H : h(\xi, \eta_1) \geq h(\xi, \eta_2)\} \cap \bigcap_{\xi \in \mathbb{R}_+} \{h \in H : h(\xi, 1) \geq 0\}.$$

Definition

The CDO model (14) is called *positive and monotone* if for all $h_0 \in C$ we have

$$\mathbb{P}(r_t \in C) = 1, \quad t \geq 0$$

where $(r_t)_{t \geq 0}$ denotes the mild solution for (14) with $r_0 = h_0$.

Theorem

Assume that

$$h + \gamma(h,x) + \delta(h,y) \in C$$

for all $h \in C$*, all* $t \geq 0$ *and* $F \otimes \nu^L(t,\cdot)$*-almost all* $(x,y) \in G \times I$*. Moreover,*

$$\sigma^j(h)(\xi,1) = 0,$$
$$\alpha(h)(\xi,1) \geq 0$$

for all $\xi \in (0,\infty)$ *and all* $h \in H$ *with* $h(\xi) = 0$*, as well as*

$$\sigma^j(h)(\xi,\eta_1) = \sigma^j(h)(\xi,\eta_2)$$
$$\alpha(h)(\xi,\eta_1) \geq \alpha(h)(\xi,\eta_2)$$

for all $\xi \in (0,\infty)$*, all* $\eta_1 \leq \eta_2$ *and all* $h \in H$ *with* $h(\xi,\eta_1) = h(\xi,\eta_2)$*. Then the CDO model is positive and monotone.*

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Thank you for your attention!

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