EQUISINGULARITY THEORY OF ANALYTIC AND ALGEBRAIC SET GERMS. PROOF OF WHITNEY FIBERING CONJECTURE.

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Abstract. One of the main results of singularity theory states that an algebraic family of algebraic sets is generically equisingular, i.e. it is locally topologically trivial over strata of a finite stratification of the parameter space. A similar result, correctly stated, holds for analytic families of analytic set germs.

These results can be proven by means of stratification theory, in some special cases by the resolution of singularities, or by Zariski equisingularity. The main purpose of this course is to give an introduction to the latter method, that provides an algorithmic and constructive approach. First we introduce such basic tools as Puiseux Theorem, Whitney Interpolation, and arc-analytic maps. Then we show how, for Zariski equisingular families, to construct topological trivializations satisfying strong additional properties, for instance arc-analytic. Finally we show Whitney fibering conjecture in the real and complex, local analytic and global algebraic cases. (based on a recent joint paper with Laurentiu Paunescu.)

Contents

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Introduction.

Equisingularity problem. Let T be an algebraic variety (over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) and let $X\subset T\times\mathbb{P}^{n-1}_{\mathbb{K}}$ be algebraic. Denote $X_t=\pi^{-1}(t)\cap X$, where $\pi: T\times\mathbb{P}^{n-1}_{\mathbb{K}}\to T$. Is then the family $t\to X_t$ generically equisingular?

This question is an example of the equisingularity problem. To answer this question we have to first make it more precise. Usually by "generically equisingular" we mean that there exists an algebraic subset $Z\subset T$, $\dim Z<\dim T$, such that X_t and $X_{t'}$ are similar provided t,t' are in the same connected component of $T\setminus Z$. Of course then the answer depends on what we mean by "similar".

For instance, if X is nonsingular, then by Sard and Ehresmann Theorems, choosing by Z the set of critical values of the projection of X onto T (in the real case this set is only semi-algebraic), we may show that for every $t_0\in T\setminus Z$ there is a neighborhood U and a C^∞ diffeomorphism

$$\Phi(t,x)=(t,\Psi(t,x)):U\times\mathbb{P}^{n-1}_{\mathbb{K}}\to\pi^{-1}(U), \tag{0.1}$$

preserving X, i.e. $\Phi(U\times X_{t_0})=X\cap\pi^{-1}(U)$. By Nash approximation Φ can be made real analytic or even Nash, that is real analytic with semi-algebraic graph, but not complex analytic. For example the holomorphic type of elliptic curves, represented for instance by plane cubics $y^2=x^3+ax+b$ with a,b as parameters, changes continuously as a,b changes. This is the phenomenon of continuous moduli.

If X is singular then the generic equisingularity holds for topological equivalence, that is Φ can be chosen a homeomorphism, but fails for C^1 or even differentiable equivalence. For instance, in a classical Whitney example of a family of four lines $X_t=\{y(y-x)(y+x)(y-tx)=0\}\subset\mathbb{K}^2$ parametrized by t, the cross-ratio is a continuous modulus. There is no local C^1 diffeomorphism

$$\Phi(t,x,y)=(t,\Psi(t,x,y)):U\times\Omega_0\to\Omega,$$

preserving $X=\{y(y-x)(y+x)(y-tx)=0\}$. Here U denotes a neighborhood of t_0 in $\mathbb{K}$, Ω_0 of the origin in $\mathbb{K}^2$, and Ω of $t_0\times\{0\}$ in $\mathbb{K}\times\mathbb{K}^2$. In general, the best known property of Φ that does not lead to the phenomenon of continuous moduli is to be a bi-lipschitz homeomorphism [32], [35, 36].

A similar question can be asked for analytic families analytic set germs. Let T be a $\mathbb{K}$-analytic space, $U\subset\mathbb{K}^n$ an open neighborhood of the origin, and let $X\subset T\times U$ be analytic. Is it true that, at least locally on T, there is an analytic subset $Z\subset T$, $\dim Z<\dim T$ such that X can be trivialized locally over $T\setminus Z$? Again the answer is yes for the topological and bi-lipschitz equivalences and no for the differentiable one.

A related question is the existence of a locally trivial stratification of an algebraic or analytic set X. That is can X be decomposed into finite (locally finite in global analytic case) number of nonsingular subspaces, called strata, so that X is locally equisingular along each stratum?

Whitney stratification and Whitney fibering conjecture. Let V be a real or complex analytic variety. Then there exists a stratification $\mathcal{S}$ of V such that V is topologically equisingular along each stratum S. Whitney introduced in [55], [54], the regularity conditions (a) and (b) describing the way different strata glue together that guarantee, by Thom-Mather first isotopy theorem, [29], [44], [10], the topological equisingularity along each stratum. Whitney showed in [54] that any complex analytic variety admits (a) and (b) regular stratifications. The real analytic case was established in [27] and the subanalytic one in [19].

Whitney stratification can be used to show the generic topological equisingularity for two equisingularity problems considered above: algebraic families of algebraic sets and analytic families of analytic set germs. Indeed, given a Whitney stratification of X, then we define Z as the Zariski closure of the sets of critical values of π restricted to the strata.

Thus Whitney stratification approach gives a generalization of Ehresmann Theorem to the singular set-up. But the trivializations obtained by this method are not explicit and difficult to handle. These trivializations are obtained by integration of "controlled" vector fields whose existence can be theoretically established but not given by explicit formulae. Stronger regularity conditions, such as (w) of Verdier [48], or Lipschitz of Mostowski [32], [35], lead to easier constructions of such vector fields, but in general, even if these vector fields can be chosen subanalytic not much can be said about their flows.

In 1965 Whitney conjectured the existence of a better trivialization.

Conjecture. [Whitney fibering conjecture, [55] section 9, p.230] *Any analytic subvariety $V \subset U$ (U open in $\mathbb{C}^n$) has a stratification such that each point $p_0 \in V$ has a neighborhood U_0 with a semi-analytic fibration.*

By a semi-analytic fibration Whitney meant the following (that has no relation with the notion of semi-analytic set introduced about the same time by Łojasiewicz in [27]). Let p_0 belong to a stratum M and let $M_0 = M \cap U_0$. Let N be the analytic plane orthogonal to M at p_0 and let $N_0 = N \cap U_0$. Then Whitney requires that there exists a homeomorphism

$$\phi(p, q) : M_0 \times N_0 \to U_0, \tag{0.2}$$

complex analytic in p, such that $\phi(p, p_0) = p$ ($p \in M_0$) and $\phi(p_0, q) = q$ ($q \in N_0$), and preserving the strata. He also assumes that for each $q \in N_0$ fixed, $\phi(\cdot, q) : M_0 \to U_0$ is a complex analytic embedding onto an analytic submanifold $L(q)$ called the fiber (or the leaf) at q, and thus U_0 fibers continuously into submanifolds complex analytically diffeomorphic to M_0. Note that due to the existence of continuous moduli it is in general impossible to find $\phi(p, q)$ complex analytic in both variables, see [55].

Whitney stated his conjecture in the context of the conditions (a) and (b). In order to have these conditions, quoting Whitney, "one should probably require more than just the continuity of ϕ in the second variable".

Whitney fibering conjecture as stated above was proven by Hardt and Sullivan in the local analytic and global projective cases, in Theorem 6.1 of [16]. But it is not clear whether ϕ of [16] ensures the continuity of the tangent spaces to the leaves, that would imply the condition (a), or gives the condition (b). In the real algebraic case an analogue of Whitney's conjecture was proven in [15]. In this case the continuity of the tangent spaces and the condition (b) is not clear either.

Note that Whitney fibering conjecture is consistent with the following holomorphic version of the Ehresmann fibration theorem, see [50]. Let $\pi : \mathcal{X} \to B$ be a proper holomorphic submersion of complex analytic manifolds. Then, for every $b_0 \in B$ there is a neighborhood B_0 of b_0 in B and a C^∞ trivialization

$$\phi(p, q) : B_0 \times X_0 \to \mathcal{X}_{B_0},$$

holomorphic in p, where $X_0 = \pi^{-1}(b_0)$, $\mathcal{X}_0 = \pi^{-1}(B_0)$. Note that ϕ can be made real analytic but, in general, due to the presence of continuous moduli, not holomorphic. This version of Ehresmann's theorem is convenient to study the variation of Hodge structures in families of Kähler manifolds, see [50].

Zariski Equisingularity. Besides isotopy lemmas applied to stratified spaces, topological equisingularity can be obtained by means of algebro-geometric equisingularity of Zariski, as shown by Varchenko in [45, 46, 47]. Zariski's definition, see [59], is recursive and is based on the geometry of discriminants. Let $V \subset \mathbb{K}^N$ be a hypersurface. We say that V is Zariski equisingular along stratum S at $p \in S$ if, after a change of a local system of coordinates, the discriminant of a linear projection $\pi : \mathbb{K}^N \to \mathbb{K}^{N-1}$ restricted to V is Zariski equisingular along $\pi(S)$ at $\pi(p)$. The kernel of π should be transverse to S and π restricted to V should be finite at p. Stronger versions of Zariski equisingularity are obtained if one assumes that the kernel of π is not contained in the tangent cone to V at p (transverse Zariski equisingularity) or that π is generic (generic Zariski equisingularity).

The special case when S is of codimension one in V and $\mathbb{K} = \mathbb{C}$, was studied by Zariski in [57]. Note that in this case V can be considered as a family of plane curves parameterized by S. As Zariski shows, in this case Zariski equisingularity is equivalent to Whitney conditions (a) and (b) on the pair of strata $V \setminus S, S$. Such equisingular families admit uniform Puiseux representation parameterized by S, result known in literature as parametrized Puiseux or Puiseux with parameter theorem. We recall it in Section 1.

It was shown in [38] that Zariski equisingularity implies a strong version of topological triviality called arc-wise analytic triviality. We recall this result in Theorem 6.1 and present the main steps of its proof in Section 2. A trivialization $\Phi(t, x)$, as in (0.1), is called arc-wise analytic if both Φ and Φ^{-1} are analytic on real analytic arcs and moreover, for every real analytic arc $x(s)$, the map $(t, s) \to \Phi(t, x(s))$ is analytic. In particular, Φ is $\mathbb{K}$-analytic in t and fibers its range Ω into $\mathbb{K}$-analytic submanifolds, the images of $U \times \{x_0\}$.

Theorem 6.1 generalizes the results of Varchenko [45, 46, 47] but its proof is different. It is based on Whitney interpolation that gives a precise algorithmic formula for such a trivialization. The main idea is the following. Suppose V is Zariski equisingular along S and $\pi : \mathbb{K}^N \to \mathbb{K}^{N-1}$ is the projection giving this equisingularity. Thus, by inductive assumption, there is, given by a precise formula, an arc-wise analytic trivialization of $\pi(V)$ along $\pi(S)$. This trivialization is then lifted, by continuity, to a trivialization of V along S, and next extended to a trivialization of the ambient space $\mathbb{K}^n$ along S. To achieve this extension we use Whitney interpolation, c.f. Appendix I. Consequently the lift is continuous, subanalytic, and, by Puiseux with parameter theorem and the inductive assumption, arc-wise analytic.

We also note that Zariski equisingularity can be used to trivialize not only hypersurfaces but also analytic spaces of arbitrary embedding codimension. This follows from the fact that if a hypersurface V is Zariski equisingular along S and $V = \cup V_i$ is the decomposition of V into irreducible components, then an arc-wise analytic trivialization preserving V preserves also each V_i and hence any set theoretic combination of the V_i's.

Proof of Whitney fibering conjecture. Whitney fibering conjecture in the real and complex, local analytic and global algebraic cases, was shown in [38]. The main idea of this proof goes as follows. For a given germ of complex or real analytic set, it is shown the existence of a stratification such that a local trivialization $\phi(p, q)$ in (0.2) can be chosen arc-wise analytic. Since, in particular, such a $\phi(p, q)$ is $\mathbb{K}$-analytic in p it induces a local fibration by $\mathbb{K}$-analytic submanifolds (leaves or fibres), the images of $\phi(p, q_0)$ for q_0 fixed. As we show in Proposition 3.3 this guaranties the continuity of tangent spaces to the fibres and hence Whitney (a)-condition on the stratification (both in the real and complex cases). If this trivialization satisfies additionally that $\phi(p, q)$ preserves the size of the distance to the stratum, then the stratification satisfies condition (w) of Verdier [48], and hence automatically the condition (b) of Whitney.

We present the details of this proof in Part 4.

Zariski Equisingularity and regularity conditions on stratifications. In general, both equisingularity assumptions, the validity of Whitney (a) and (b) conditions and Zariski equisingularity, lead to different equisingularity conditions. We recall several classical examples in Section 10.4. By Zariski [57], they coincide for a hypersurface V along a nonsingular subvariety of codimension 1 in V.

It was shown by Speder in [41] that in the complex case Zariski equisingularity obtained by taking generic projections implies the regularity conditions (a) and (b) of Whitney. As it follows from Theorem 9.2 the assumption that the projections are transverse, in both complex and real cases, is sufficient.

Zariski equisingularity method is more explicit than the stratification one and in a way constructive. It uses the actual equations and local coordinate systems. This can be considered either as a drawback or as an advantage. Zariski equisingularity was used, for instance, by Mostowski [31], see also [3], to show that analytic set germs are always homeomorphic to algebraic ones.

In our approach we apply Zariski equisingularity to construct stratifications via corank one projections. This method was developed by Hardt and Hardt & Sullivan [13, 14, 15, 16].

Resolution of singularities and blow-analytic equivalence. Also the resolution of singularities can be used to show topological equisingularity, though the results are partial and many questions are still open. This method works for the families of isolated singularities, cf. Kuo [21], and gives local arc-analytic trivializations. But little is known if the singularities are not isolated, see e.g. [20]. Let us explain the encountered problem on a simple example. Suppose that $Y \subset V$ is nonsingular and let $\sigma : \tilde{V} \to V$ be a resolution of singularities such that $\sigma^{-1}(Y)$ is the union of some components of exceptional divisors. Fix a local projection $\pi : V \to Y$. The exceptional divisor of σ as a divisor with normal crossings is naturally stratified by the intersections of its components. Let $Z \subset Y$ be the closure of the union of all critical values of $\pi \circ \sigma$ restricted to the strata. By Sard's theorem $\dim Z < \dim Y$. We say that V is *equiresoluble along* Y if $Y \cap Z = \emptyset$. Thus V is equiresoluble along $Y' = Y \setminus Z$ and $\pi \circ \sigma$ is locally topologically (and even real analytically) trivial over Y'. If σ is an isomorphism over $V \setminus Y$ (family of isolated singularities case) then this trivialization contracts to a topological trivialization of a neighborhood of Y in V. But in the non-isolated singularity case there is no clear reason why a trivialization of $\pi \circ \sigma$ comes from a topological trivialization of σ. Thus, in general, it is not known whether equiresolubility implies topological equisingularity.

As before, one may ask how the equiresolution method is related to the other methods of establishing topological equisingularity. A non-trivial result of Villamayor [49], says that the generic Zariski equisingularity of a hypersurface implies a weak version of equiresolution, see loc. cit. for details, but the main problem remains, it does not show the existence of a topological trivialization that lifts to the resolution space.

Notation and terminology. We denote by $\mathbb{K}$ the field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers and, unless expressly stated otherwise, we work simultaneously in the real and the complex cases. Therefore we rather say complex analytic than holomorphic, so that for the complex or real case we may abbreviate it as $\mathbb{K}$-analytic.

By an analytic space we mean an analytic space in the sense of [33], though, because in the analytic case we work only locally, it is sufficient to consider only analytic set germs. For a $\mathbb{K}$-analytic space X by $Sing(X)$ we denote the set of singular points of X and by $Reg(X)$ its complement, the set of regular points of X. By a real analytic arc we mean a real analytic map $\gamma : I \to X$, where $I = (-1, 1)$ and X is a real or a complex analytic space.

For an analytic function germ F we denote by F_{red} its reduced (i.e. square free) form. Let (Y, y) be a germ of a $\mathbb{K}$-analytic space. For a monic polynomial $F \in \mathcal{O}_Y[Z]$ in Z we often consider the discriminant of F_{red}. If Y has arbitrary singularities then this discriminant should be replaced by an appropriate generalized discriminant that is a polynomial in coefficients of F, see Appendix II.

Part 1. Preliminaries.

1. Puiseux with parameter theorem.

In this section analytic means complex analytic. First we recall the classical Newton-Puiseux Theorem.

Theorem 1.1. (Newton-Puiseux Theorem)
Let $F(x,z) \in \mathbb{C}\{x\}[z]$ be a monic polynomial in z

$$F(x,z) = z^N + \sum_{i=1}^{N} c_i(x) z^{N-i} \tag{1.1}$$

with coefficients $c_i(x) \in \mathbb{C}\{x\}$. Then there is a positive integer d and $\tilde a_i(y) \in \mathbb{C}\{y\}$ such that

$$F(y^d, z) = \prod_{i=1}^{N} (z - \tilde a_i(y)).$$

If $F(x,z)$ is irreducible in $\mathbb{C}\{x,z\}$ then we may take $d = N$. In the general case $d = N!$ always works. We present below an analytic proof of a more general result, Puiseux with parameter Theorem. Newton-Puiseux Theorem can be also proven algebraically. We present a classical algebraic proof, stemming from Newton, in Appendix III, though we do not use it later.

If the coefficients of F, $c_i(x)$, depend of many variables, $x = (x_1, \ldots, x_m)$, then, in general, it is impossible to express the roots of F as fractional power series in x. This is still possible if the discriminant of F is normal crossings, that is it is of the form $x^M unit(x)$. This result is known as Abhyankar-Jung Theorem, see [1], [39].

We will recall and prove below a parametrized version of Newton-Puiseux Theorem that we call Puiseux with parameter theorem, see [57] Thm. 7 and [58] Thm. 4, [40]. This theorem is a special case of Abhyankar-Jung Theorem. The proof we present is analytic.

We use the following notation: $U_{\varepsilon,r} = U_\varepsilon \times U_r$, where $U_\varepsilon = \{t \in \mathbb{C}^m; \|t\| < \varepsilon\}$, $U_r = \{x \in \mathbb{C}; |x| < r\}$. We denote $U_r \setminus \{0\}$ by U_r^*.

Theorem 1.2. (Puiseux with parameter) *Let*

$$F(t,x,z) = z^N + \sum_{i=1}^{N} c_i(t,x) z^{N-i} \tag{1.2}$$

be a monic polynomial in $z \in \mathbb{C}$ with analytic coefficients $c_i(t,x)$ defined on $U_{\varepsilon,r} = U_\varepsilon \times U_r$. Suppose that the discriminant $\Delta_F(t,x)$ of F does not vanish on $U_\varepsilon \times U_r^*$. Then there is a positive integer d and analytic functions $\tilde a_i(t,y) : U_{\varepsilon, r^{1/d}} \to \mathbb{C}$ such that

$$F(t, y^d, z) = \prod_{i=1}^{N} (z - \tilde a_i(t,y)).$$

Let θ be a d-th root of unity. Then for each i there is j such that $\tilde a_i(t,\theta y) = \tilde a_j(t,y)$.

If F has real coefficients then the family $\tilde a_i(t,y)$ is conjugation invariant.

The proof we present below is due to Łojasiewicz and Pawłucki, see [40].

Proof. Consider the polynomial in z, $P(t,w,z) := F(t, e^{2\pi i w}, z)$, whose coefficients $c_i(t, e^{2\pi i w})$ are analytic on $U_\varepsilon \times H$, where $H = \{w; 2\pi \mathrm{Im}(w) > \ln r\}$. The discriminant $\Delta_P(t,w) = \Delta_F(t, e^{2\pi i w})$ does not vanish on $U_\varepsilon \times H$ and hence P admits complex analytic roots

$$\xi_1(t,w), \cdots, \xi_N(t,w).$$

(The roots are complex analytic locally by the Implicit Function Theorem. They are well-defined globally because $U_\varepsilon \times H$ is contractible.)

The coefficients of P are periodic: $P(t,w+1,z) = P(t,w,z)$. Hence for each root $\xi_i(t,w)$ there is another root $\xi_{\varphi(i)}(t,w)$ such that $\xi_i(t,w+1) = \xi_{\varphi(i)}(t,w)$. The map $\varphi : \{1,\dots,N\} \to \{1,\dots,N\}$ is a bijection and $\varphi^{N!} = id$. Therefore, for $d = N!$,

$$\xi_i(t, w+d) = \xi_i(t,w). \tag{1.3}$$

Thus there are analytic functions $\tilde a_i(t,y) : U_\varepsilon \times U^*_{r^{1/d}} \to \mathbb{C}$ such that $\xi_i(t,w) = \tilde a_i(t, e^{2\pi i w/d})$. Since $\tilde a_i(t,y)$ are roots of $F(t, y^d, z)$ they are locally bounded on $U_{\varepsilon, r^{1/d}}$. This allows us to conclude that they are analytic on $U_{\varepsilon, r^{1/d}}$ by Riemann's theorem on removable singularities. □

The assumption of Theorem 1.2 on the discriminant of F can be express equivalently as $\Delta_F(t,x) = x^M u(t,x)$ where $u \neq 0$ on $U_{\varepsilon,r}$. If $M = 0$ then by the Implicit Function Theorem the complex roots of F, denoted by $a_1(t,x), ..., a_N(t,x)$, are distinct analytic functions of (t,x). If $M > 0$ then they become analytic in (t,y) after a ramification $x = y^d$ and, in general, they can not be chosen analytic in x. These roots are well-defined only as an unordered set. By Puiseux with parameter Theorem, for $x \in U_r \setminus \{0\}$ fixed, an ordering of the roots at $(0,x)$, $a_1(0,x), ..., a_N(0,x)$, gives a unique ordering of the roots at (t,x), $a_1(t,x), \dots, a_N(t,x)$, by continuity. By Corollary 1.3 this also holds for $x = 0$.

Corollary 1.3. *The roots of F at $(t,0)$, $a_1(t,0), \dots, a_N(t,0)$, are complex analytic functions in t. If $a_i(0,0) = a_j(0,0)$ then $a_i(t,0) \equiv a_j(t,0)$. Thus the multiplicity of $a_i(t,0)$ as a root of F is independent of t.*

Proof. The family $a_1(t,0), \dots, a_N(t,0)$ coincides with $\tilde a_1(t,0), \dots, \tilde a_N(t,0)$. If $\tilde a_i(0,0) = \tilde a_j(0,0)$ then $\tilde a_i(t,y) - \tilde a_j(t,y)$ divides y^{dM} and hence equals a power of y times a unit. □

Exercise 1.4. The Puiseux pairs of $a_i(t,x)$ and the contact exponents between different branches are independent of t. For the necessary definitions see [51], [22], [52].

The next corollary is essential for the proof of Theorem 4.1. It allows to use the bi-lipschitz property given by Proposition 10.20.

Corollary 1.5. *There is $C > 0$ such that for (t, x) from a small neighborhood of the origin*

$$\gamma(t,x) = \max_{a_i(t,x)\not\equiv a_j(t,x)} \frac{|(a_i(t,x) - a_i(0,x)) - (a_j(t,x) - a_j(0,x))|}{|a_i(0,x) - a_j(0,x)|} \le C\|t\|. \tag{1.4}$$

Let $a_i(t,x) \not\equiv a_j(t,x)$. Then

$$\frac{a_i(t,x) - a_j(t,x)}{a_i(0,x) - a_j(0,x)} \to 1 \quad as \quad t \to 0. \tag{1.5}$$

Exercise 1.6 (Abhyankar-Jung Theorem). Suppose the coefficients of (1.1), $c_i(x)$, depend of many variables, $x = (x_1, \dots, x_m)$, and are analytic on $U_{r_1} \times \cdots \times U_{r_m}$. Suppose also that the discriminant $\Delta_F(x)$ of F does not vanish on $U^*_{r_1} \times \cdots \times U^*_{r_m}$. Then there are positive integer $d_1, \dots, d_m$ and analytic functions $\tilde a_i(y) : U_{r_1^{1/d_1}} \times \cdots \times U_{r_m^{1/d_m}} \to \mathbb{C}$ such that

$$F(y_1^{d_1}, \dots, y_m^{d_m}, z) = \prod_{i=1}^N (z - \tilde a_i(y)).$$

2. Zariski Equisingularity.

In this section we recall the notion of algebro-geometric equisingularity introduced by Zariski in [59] that is also known as Zariski equisingulairty. Zariski equisingularity can be understood as a generalization of Puiseux with parameter theorem to the arbitrary number of variables. It can be defined for both complex and real analytic cases.

Definition 2.1. By a *system of pseudopolynomials in $x = (x_1, ..., x_n) \in \mathbb{K}^n$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, with parameter $t \in U \subset \mathbb{K}^m$*, we mean a family $F_i(t, x), i = 0, \dots, n$, defined on $U \times U_i$, where U is a neighborhood of the origin in $\mathbb{K}^m$, U_i is a neighbourhood of the origin in $\mathbb{K}^i$, of the form

$$F_i(t, x_1, \dots, x_i) = x_i^{d_i} + \sum_{j=1}^{d_i} A_{i,j}(t, x_1, \dots, x_{i-1}) x_i^{d_i - j}, \quad i = 1, \dots, n, \tag{2.1}$$

if $d_i > 0$, or if $d_i = 0$, then F_i is identically equal to 1, and by convention we define all $F_j, j < i$, as identically equal to 1. For each $i = 1, \dots, n$, we assume that F_{i-1} is the Weierstrass polynomial associated to the discriminant of $F_{i,red}$, and that all the coefficients $A_{i,j}$ are analytic and vanish identically on $T = U \times \{0\}$.

We consider the system $\{F_i\}$ as a family function germs at the origin in $\mathbb{K}^n$, parameterized by $t \in (\mathbb{K}^m, 0)$. If $F_0(0) \neq 0$ we call this system *Zariski equisingular.* As Varchenko showed in [45], answering a question posed by Zariski in [59], for a Zariski equisingular system, the family of analytic set germs $X_t = \{F_n(t, x) = 0\} \subset (\mathbb{K}^n, 0)$ is topologically equisingular for t close to the origin.

Theorem 2.2 (Varchenko [45])**.** *If $F_i(t, x)$, $i = 0, \dots, n$, is a Zariski equisingular system of pseudopolynomials, then there exist $\varepsilon > 0$ and a homeomorphism*

$$\Phi : B_\varepsilon \times \Omega_0 \to \Omega, \tag{2.2}$$

where $B_\varepsilon = \{t \in \mathbb{K}^m; \|t\| < \varepsilon\}$, Ω_0 *and* Ω *are neighborhoods of the origin in* $\mathbb{K}^n$ *and* $\mathbb{K}^{m+n}$ *resp., such that*

(1) $\Phi(t,0) = (t,0)$, $\Phi(0,x_1,\dots,x_n) = (0,x_1,\dots,x_n)$*;*
(2) Φ *has a triangular form*

$$\Phi(t,x_1,\dots,x_n) = (t,\Psi_1(t,x_1),\dots,\Psi_{n-1}(t,x_1,\dots,x_{n-1}),\Psi_n(t,x_1,\dots,x_n));$$

(3) Φ *preserves the zero set of* F_n *that is* $\Phi(B \times \{x \in \Omega_0; F_n(x) = 0\} = \{(t,x) \in \Omega; F_n(t,x) = 0\}$*.*

Zariski equisingularity can be used not only to trivialize topologically a hypersurface given by $F_n = 0$ but also for analytic or algebraic set germs. This follows from the fact that, by construction of topological trivialization Φ of [45], if G divides F_n, then Φ preserves the zero set of G. For more on Zariski equisingularity see [59], [45, 46, 47], [3].

3. Arc-wise analytic trivializations.

Let Z, Y be $\mathbb{K}$-analytic spaces. A map $f : Z \to Y$ is called *arc-analytic* if $f \circ \delta$ is analytic for every real analytic arc $\delta : I \to Z$, where $I = (-1,1) \subset \mathbb{R}$. The arc-analytic maps were introduced by Kurdyka in [23] and have been subsequently used intensively in real analytic and algebraic geometry, see [25]. It was shown by Bierstone and Milman in [2] (see also [37] for a different proof) that the arc-analytic maps with subanalytic graphs are continuous and that the arc-analytic maps with semi-algebraic graphs are blow-analytic, i.e. can be made real-analytic after composing with blowings-up. Therefore the arc-analytic maps are closely related to the blow-analytic trivialization in the sense of Kuo [21].

In these notes we consider arc-analytic trivializations satisfying some additional properties. We define below the notion of arc-wise analytic trivialization, that is not only arc-analytic with arc-analytic inverse, but it is also $\mathbb{K}$-analytic with respect to the parameter $t \in T$. For simplicity we assume that the parameter space T is nonsingular.

Definition 3.1. Let T, Y, Z be $\mathbb{K}$-analytic spaces, T nonsingular. We say that a map $f(t,z) : T \times Z \to Y$ is *arc-wise analytic in* t if it is $\mathbb{K}$-analytic in t and arc-analytic in z, that is if for every real analytic arc $z(s) : I \to Z$, the map $f(t, z(s))$ is real analytic, and moreover, if $\mathbb{K} = \mathbb{C}$, complex analytic with respect to t.

All arc-wise analytic maps considered in these notes are subanalytic and hence continuous.

We stress that even for complex analytic spaces we define the notion of arc-analyticity using only real analytic arcs. (A map of complex analytic spaces $f : Z \to Y$, with Z nonsingular, that is complex analytic on complex analytic arcs is, by Hartogs Theorem, complex analytic.)

The following definition has been given in [38].

Definition 3.2. Let Y, Z be $\mathbb{K}$-analytic spaces and let T be a nonsingular $\mathbb{K}$-analytic space. Let $\pi : Y \to T$ be a $\mathbb{K}$-analytic map. We say

$$\Phi(t,z) : T \times Z \to Y$$

is an *arc-wise analytic trivialization of* π if it satisfies the following properties

(1) Φ is a subanalytic homeomorphism,
(2) Φ is arc-wise analytic in t (in particular it is $\mathbb{K}$-analytic with respect to t),
(3) $\pi \circ \Phi(t,z) = t$ for every $(t,z) \in T \times Z$,
(4) the inverse of Φ is arc-analytic,
(5) there exist $\mathbb{K}$-analytic stratifications $\{Z_i\}$ of Z and $\{Y_i\}$ of Y, such that for each i, $Y_i = \Phi(T \times Z_i)$ and $\Phi_{|T\times Z_i} : T \times Z_i \to Y_i$ is a real analytic diffeomorphism.

Sometimes we say for short that such Φ is an arc-wise analytic trivialization if it is obvious from the context what the projection π is.

In the algebraic case we require Φ to be semialgebraic and that the stratification is algebraic in the sense we explain in Section 7.

If $\Phi(t,z) : T \times Z \to Y$ is an *arc-wise analytic trivialization* then, for each $z \in Z$, the map $T \ni t \to \Phi(t,z) \in Y$ is a $\mathbb{K}$-analytic embedding. We denote by L_z its image and we call it *a leaf or a fiber of* Φ. We say that Φ *preserves* $X \subset Y$ if X is a union of leaves. We denote by $T_y = T_y L_z$, $y = \Phi(t,z)$, the tangent space to the leaf through y.

Computation in local coordinates. Let $(t_0, z_0) \in T \times Z$, $y_0 = \Phi(t_0, z_0)$. Choosing local coordinates we may always assume that $(T, t_0) = (\mathbb{K}^m, 0)$, (Z, z_0) is an analytic subspace of $(\mathbb{K}^n, 0)$, and (Y, y_0) is an analytic subspace of $(T \times \mathbb{K}^n, 0)$ with $\pi(t,x) = t$. Thus we may write

$$\Phi(t,z) = (t, \Psi(t,z)). \tag{3.1}$$

We also suppose that $L_0 = \Phi(T \times \{0\}) = T \times \{0\}$ as germs at the origin..

Using local coordinates we identify T_y with an m-dimensional vector subspace of $\mathbb{K}^m \times \mathbb{K}^n$ and consider T_y as a point in the Grassmannian $G(m, m+n)$. These tangent spaces are spanned by the vector fields v_i on Y defined by

$$v_i(\Phi(t,z)) := (\partial/\partial t_i, \partial\Psi/\partial t_i) \qquad i = 1, ..., m. \tag{3.2}$$

Proposition 3.3. *Let $\Phi(t,z) : T \times Z \to Y$ be an arc-wise analytic trivialization. Then the vector fields v_i and the tangent space map $y \to T_y$ are subanalytic, arc-analytic, and continuous.*

Proof. The subanalyticity follows by the classical argument of subanalyticity of the derivative of a subanalytic map, see [24] Théorème 2.4. Let $(t(s), z(s)) : (I, 0) \to (T \times Z, (t_0, z_0))$ be a real analytic arc germ. Consider the map $\Psi : T \times I \to \mathbb{K}^n$

$$\Psi(t, z(s)) = \sum_{k \geq k_0} D_k(t) s^k. \tag{3.3}$$

The arc-analyticity of v_i on $(t(s), z(s))$ follows from the analyticity of $(t,s) \to \partial\Psi(t, z(s))/\partial t_i$. Finally, subanalytic and arc-analytic maps are continuous, cf. [2] Lemma 6.8. □

Remark 3.4. For $y = \Phi(t,z)$ fixed, $\tau \to \Phi(t + \tau e_i, z)$ is an integral curve of v_i through y. Moreover, such an integral curve is unique as follows from (5) of Definition 3.2.

4. Puiseux with parameter and arc-wise analytic trivializations

In this section we use Whitney Interpolation and Puiseux with parameter theorem to construct arc-wise analytic trivializations of equisingular (in the sense of Zariski) families of plane curve singularities. In Part 2 we will extend this construction to the Zariski equisingular families of hypersurface singularities of arbitrary number of variables.

Let

$$F(t,x,z) = z^N + \sum_{i=1}^{N} c_i(t,x) z^{N-i} \tag{4.1}$$

be a unitary polynomial in $z \in \mathbb{K}$ with $\mathbb{K}$-analytic coefficients $c_i(t,x)$ defined on $U_{\varepsilon,r} = U_\varepsilon \times U_r$, where $U_\varepsilon = \{t \in \mathbb{K}^m; \|t\| < \varepsilon\}$, $U_r = \{x \in \mathbb{K}; |x| < r\}$. Suppose that the discriminant $\Delta_F(t,x)$ of F is of the form

$$\Delta_F(t,x) = x^M u(t,x), \quad u \neq 0 \text{ on } U_{\varepsilon,r}.$$

Denote by $a(t,x) = (a_1(t,x),...,a_N(t,x))$ the vector of roots and consider the self map $\Phi : U_{\varepsilon,r} \times \mathbb{C} \to U_{\varepsilon,r} \times \mathbb{C}$

$$\Phi(t,x,z) = (t,x,\psi(z,a(0,x),a(t,x))), \tag{4.2}$$

where $\psi(z,a,b)$ is the Whitney interpolation map given by (10.20).

Theorem 4.1. *For $\varepsilon > 0, r > 0$ sufficiently small the map Φ defined in (4.2) is an arc-wise analytic trivialization of the projection $U_{\varepsilon,r} \times \mathbb{C} \to U_\varepsilon$. Moreover, there is $C > 1$ such that*

$$C^{-1}|F(0,x,z)| \le |F(\Phi(t,x,z))| \le C|F(0,x,z)|. \tag{4.3}$$

In particular Φ preserves the zero level of F.

If $\mathbb{K} = \mathbb{R}$ then Φ is conjugation invariant in z.

Proof. Φ is continuous by Proposition 10.21 and Remark 10.19. By Proposition 10.20 and Corollary 1.5, if ε is sufficiently small, then for t and x fixed, $\psi_{a(0,x),a(t,x)} : \mathbb{C} \to \mathbb{C}$ is bi-Lipschitz. Therefore Φ is bijective and the continuity of Φ^{-1} follows from the invariance of domain.

Next we show (4.3). The Lipschitz constants of $\psi_{a(0,x),a(t,x)} : \mathbb{C} \to \mathbb{C}$ and of its inverse can be chosen independent of $(t,x) \in U_{\varepsilon,r'}$. Let L be a common upper bound for these constants. Then, because $\psi_{a(0,x),a(t,x)}(a_i(0,x)) = a_i(t,x)$,

$$L^{-1}|z - a_i(0,x)| \le |\psi_{a(0,x),a(t,x)}(z) - a_i(t,x)| \le L|z - a_i(0,x)| \tag{4.4}$$

Because $F(\Phi(t,x,z)) = \prod_i (\psi_{a(0,x),a(t,x)}(z) - a_i(t,x))$ taking the product over i we obtain (4.3) with $C = L^N$.

Let $(x(s), z(s))$ be a real analytic arc and we assume that $x(s)$ is not constant. Then the discriminant of $F(t, x(s), z + z(s))$ equals s^{M_1} times an analytic unit. Suppose $M_1 > 0$. By replacing F by $(t,s,z) \to F(t,x(s),z+z(s))$ we may assume that $x(s) \equiv s$ and $z(s) \equiv 0$. After (10.20) we consider

$$\varphi(t,s) = \psi(0,a(s),b(t,s)) = \frac{\sum_k \big(\sum_{j=1}^N \ Q_{k,j}(a(s),b(t,s))\big)\overline{Q}_k(a(s))}{N!(\sum_k Q_k(a(s))\overline{Q}_k(a(s)))}, \tag{4.5}$$

where $a(s) = a(0,s)$, $b(t,s) = a(t,s)$, and

$$Q_k(a) = \prod_i a_i^{N!} \ P_k(a_1^{-1},...,a_N^{-1}),$$

$$Q_{k,j}(a,b) = \prod_i a_i^{N!} \ a_j^{-1}\frac{\partial P_k}{\partial \xi_j}(a_1^{-1},...,a_N^{-1})\big(b_j - a_j\big).$$

Since $P_k(a)$ is symmetric in a, $Q_k(a)$ is a polynomial in the coefficients $c_i(0,s)$ of F. Hence $Q_k(a(s))$ and $\overline{Q}_k(a(s))$ are real analytic in $s \in \mathbb{R}$.

Next we study $\sum_{j=1}^N \ Q_{k,j}(a(s),b(t,s))$.

Lemma 4.2. *Let $Q(a,b) \in \mathbb{C}[a,b]$ be a polynomial invariant under the action of the permutation group: $Q(\sigma(a),\sigma(b)) = Q(a,b)$ for all $\sigma \in S_N$. Then $Q(a(s),b(t,s)) \in \mathbb{C}\{t,s\}$.*

Proof. By invariance under permutations $Q(a(s),b(t,s))$ is well-defined for $(t,s) \in B \times D$, where B is a neighborhood of the origin in $\mathbb{C}^m$ and D is a small disc centered at the origin in $\mathbb{C}$. Moreover it is bounded and complex analytic on $B \times (D \setminus \{0\})$. Therefore, by Riemann's theorem on removable singularities, $Q(a(s),b(t,s))$ is complex analytic on $B \times D$. □

Thus, by Lemma (4.2) the numerator of (4.5) is analytic in $t \in \mathbb{K}^m$, $s \in \mathbb{R}$. As we have shown before its denominator is analytic in (one variable) $s \in \mathbb{R}$. Moreover $\varphi(t,s)$ is bounded and therefore has to be analytic.

If $M_1 = 0$, then $a(t,x(s))$ is analytic by the IFT.

If $x(s) \equiv 0$ then consider $f(t,z) = F(t,0,z)$. By Corollary 1.3 the number of complex roots of f is independent of t and hence the discriminant of the $(f)_{red}$ does not vanish. Thus by the IFT, Φ on $x = 0$ is analytic.

The next lemma shows that the inverse of Φ is arc-analytic and completes the proof of Theorem 4.1.

Lemma 4.3. *If $(t(s),x(s),z(s))$ is a real analytic arc, then there is a real analytic $\tilde{z}(s)$ such that $(t(s),x(s),z(s)) = \Phi(t(s),x(s),\tilde{z}(s))$.*

Proof. Since Φ^{-1} is subanalytic such $\tilde{z}(s)$ exists continuous and subanalytic. Thus there is a positive integer q such that for $s \geq 0$, $\tilde{z}(s)$ is a convergent power series in $s^{1/q}$. We show that all exponents of $\tilde{z}(s)$, $s \geq 0$, are integers. Suppose that this is not the case. Then

$$\tilde{z}(s) = \sum_{i=1}^n v_i s^i + v_{p/q}s^{p/q} + \sum_{k>p} v_{k/q}s^{k/q},$$

with $p/q > n$ and $p/q \notin \mathbb{N}$. Denote $\tilde z_{an}(s) = \sum_{i=1}^n v_i s^i$. Then $\psi(\tilde z_{an}(s), a(0,x(s)), a(t(s),x(s)))$ is real analytic and by the bi-Lipschitz property, Proposition 10.20,

$$|\psi(\tilde z_{an}(s), a(0,x(s), a(t(s),x(s))) - \psi(\tilde z(s), a(0,x(s)), a(t(s),x(s)))| \sim |\tilde z_{an}(s) - \tilde z(s)| \sim s^{p/q}$$

that is impossible since $\psi(z_{an}(s), a(0,x(s)), a(t(s),x(s)))$ and $\psi(z(s), a(0,x(s)), a(t(s),x(s)))$ are real analytic in s.

This shows that $(t(s),x(s),z(s))$, $\Phi(t(s),x(s),\tilde z(s))$ are two real analytic arcs that coincide for $s \ge 0$ and therefore also for $s \le 0$. □

This ends the proof of Theorem 4.1. □

5. Further properties of arc-wise analytic trivializations

5.1. **Arc-wise analytic trivializations regular along a fiber.** We now define regular arc-wise analytic trivializations along a fiber that will be important for applications in stratification theory including our proof of Whitney fibering conjecture, c.f. section 7. Regular arc-wise analytic trivializations preserve the size of the distance to a fixed fiber.

Definition 5.1. We say that an arc-wise analytic trivialization $\Phi(t,z) : T \times Z \to Y$ is *regular at* $(t_0,z_0) \in T \times Z$ if there is a neighborhood U of (t_0,z_0) and a constant $C > 0$ such that for all $(t,z) \in U$ (in local coordinates at (t_0,z_0) and $y_0 = \Phi(t_0,z_0)$)

$$C^{-1}\|\Psi(t_0,z)\| \le \|\Psi(t,z))\| \le C\|\Psi(t_0,z)\|, \tag{5.1}$$

where as in (3.1), $\Phi(t,z) = (t, \Psi(t,z))$, $\Psi(t,z_0) \equiv 0$. We say that Φ is *regular along* L_{z_0} if it is regular at every (t,z_0), $t \in T$.

We have the following criterion of regularity which follows from the more general Proposition 5.3 that we prove in the next subsection.

Proposition 5.2. *The arc-wise analytic trivialization $\Phi(t,z)$ is regular at $(0,0)$ if and only if for every real analytic arc germ $z(s) : (I,0) \to (Z,0)$, the leading coefficient of (3.3) is nonzero: $D_{k_0}(0) \neq 0$.*

Moreover, if $\Phi(t,z)$ is regular at $(0,0)$, then in a neighborhood of $(0,0) \in T \times Z$

$$\|\frac{\partial \Psi}{\partial t}(t,z))\| \le C\|\Psi(t,z)\|. \tag{5.2}$$

5.2. **Functions and maps regular along a fiber.** In this section we generalize the notion of regularity for arc-wise analytic trivializations to $\mathbb{K}$-analytic function germs $f : (Y,y_0) \to (\mathbb{K},0)$, see Definition 5.4. First we show the following criterion that we state for f of a slightly more general form.

Proposition 5.3. *Let $\Phi(t,z) : T \times Z \to Y$ be an arc-wise analytic trivialization and let $f : (Y,y_0) \to (\mathbb{R}^k,0)$, $y_0 = \Phi(t_0,z_0)$, be a real analytic map germ. Then the following conditions are equivalent:*

(i) for a constant $C > 0$ and for all (t,z) sufficiently close to (t_0,z_0)

$$C^{-1}\|f(\Phi(t_0,z))\| \le \|f(\Phi(t,z))\| \le C\|f(\Phi(t_0,z))\|. \tag{5.3}$$

(ii) for every real analytic arc germ $z(s) : (I, 0) \to (Z, z_0)$ the leading coefficient D_{k_0} of

$$f(\Phi(t, z(s))) = \sum_{k \geq k_0} D_k(t) s^k \tag{5.4}$$

satisfies $D_{k_0}(t_0) \neq 0$.

(iii) there is $C > 0$ such that for all (t, z) sufficiently close to (t_0, z_0)

$$\|\frac{\partial (f \circ \Phi)}{\partial t}(t, z))\| \leq C \|f \circ \Phi(t, z)\|. \tag{5.5}$$

Proof. To show that (ii) implies (i) we use the curve selection lemma. If (i) fails then there is a real analytic arc germ $(t(s), z(s)) : (I, 0) \to (T \times Z, (t_0, z_0))$ along which one of the inequalities of (i) fails, that is, for instance, $\frac{\|f(\Phi(t(s), z(s)))\|}{\|f(\Phi(t_0, z(s)))\|} \to \infty$ as $s \to 0$. But this contradicts (ii). To complete this argument we note that $f(\Phi(t_0, z(s))) \equiv 0$ iff $f(\Phi(t, z(s))) \equiv 0$, that is what (ii) means in this case. Clearly (ii) follows from (i).

Similarly, it is sufficient to show (iii) on every real analytic arc and this follows immediately from (ii). Finally (i) follows from (iii). □

Definition 5.4. Let $\Phi(t, z) : T \times Z \to Y$ be an arc-wise analytic trivialization in t. We say that an analytic function germ $f : (Y, y_0) \to (\mathbb{K}, 0)$, $y_0 = \Phi(t_0, z_0)$, is *regular for* Φ if it satisfies one of the equivalent conditions of Proposition 5.3.

We say that f is *regular along* L_{z_0} if it is regular at every (t, z_0), $t \in T$.

Proposition 5.5. *Let $\Phi(t, z) : T \times Z \to Y$ be an arc-wise analytic trivialization and let $f, g : (Y, y_0) \to (\mathbb{K}, 0)$ be two analytic function germs. Then f and g are regular if and only if so is fg.*

Proof. It follows from (ii) of Proposition 5.3. □

In the complex case the regularity is a geometric notion as the following proposition shows.

Proposition 5.6. *Suppose $\mathbb{K} = \mathbb{C}$. Let $\Phi(t, z) : T \times Z \to Y$ be an arc-wise analytic trivialization and let $f : Y \to \mathbb{C}$ be a complex analytic function. Suppose that Φ preserves $V(f)$. Then f is regular for Φ at every point of $V(f)$.*

Proof. Suppose that this is not the case. Then there exists a real analytic arc $z(s) : (I, 0) \to (Z, z_0)$, such that in (5.4), $D_{k_0} \not\equiv 0$ and $D_{k_0}(t_0) = 0$. Clearly $f \circ \Phi(t_0, z(s)) \neq 0$ for $s \neq 0$. We show that for $s \neq 0$ there is $t(s)$, $t(s) \to t_0$ as $s \to 0$, such that $f \circ \Phi(t(s), z(s)) = 0$. This would contradict the assumption on Φ. For this, by restricting to a $\mathbb{K}$-analytic arc through t_0, we may suppose that t is a single variable $t \in (\mathbb{C}, 0)$. Let us then write $f \circ \Phi(t, z(s)) = s^{k_0} h(t, s)$, where

$$h(t, s) = D_{k_0}(t) + \sum_{k > k_0} D_k(t) s^{k - k_0}.$$

Since 0 is an isolated root of $h(t, 0) = 0$, Rouché's Theorem implies that $h(t, s) = 0$ has roots also for $s \neq 0$. □

5.3. **Preservation of multiplicity and of singular locus.** In this subsection we suppose that T, Z, Y are open subsets of $\mathbb{K}^m$, $\mathbb{K}^n$ and $\mathbb{K}^{n+m}$, respectively and that $\Phi : T \times Z \to Y$ is an arc-wise analytic trivialization of the standard projection $\pi : \mathbb{K}^{n+m} \to \mathbb{K}^m$.

We first show the preservation of multiplicities of functions regular for Φ. Let us denote $Y_t = Y \cap \pi^{-1}(t)$ and for a function $f : Y \to \mathbb{K}$, by f_t the restriction of f to Y_t. In the following lemma we compare the multiplicities of f at $(t,z) \in Y$ and the multiplicities of the restrictions f_t at $z \in Y_t$.

Proposition 5.7. *If $f : (Y, y_0) \to (\mathbb{K}, 0)$, $y_0 = \Phi(t_0, z_0)$, is regular for Φ then for t close to t_0 the following multiplicities are equal.*

$$\operatorname{mult}_{y_0} f = \operatorname{mult}_{\Phi(t,z_0)} f = \operatorname{mult}_{y_0} f_{t_0} = \operatorname{mult}_{\Phi(t,z_0)} f_t. \tag{5.6}$$

Proof. We use the argument of Fukui's proof of invariance of multiplicity by blow-analytic homeomorphisms, cf. [9]. It is based on the observation that, on a smooth space, $\operatorname{mult}_{y_0} f = \min_{y(s)} \operatorname{ord}_0 f(y(s))$, where the minimum is taken over all real analytic arcs $y(s) : (I, 0) \to (Y, y_0)$. Thus since Φ and Φ^{-1} are arc-analytic

$$\operatorname{mult}_{\Phi(t,z_0)} f_t = \min_{z(s) \to z_0} \operatorname{ord}_0 f(\Phi(t, z(s))).$$

For f regular such orders are preserved by Φ and this shows the last equality in (5.6). The other ones follow from (ii) of Proposition 5.3. □

Consider an ideal $\mathcal{I} = (f_1, ..., f_k)$ of $\mathcal{O}_Y$ and denote by $X = V(\mathcal{I})$ its zero set and by X_t the set $X \cap \pi^{-1}(t)$. Recall that for $y \in X \subset \mathbb{K}^{n+m}$ the Zariski tangent space $T_y X$ is the kernel of the differential $D_y(f_1, ..., f_k)$.

Proposition 5.8. *Suppose every $f_i, i = 1, ..., k$, is regular for Φ. Then, for every $y = \Phi(t,z)$, $T_y X_t = \pi^{-1}(t) \cap T_y X$ and for z fixed, $\dim_{\mathbb{K}} T_{\Phi(t,z)} X_t$ is independent of t. In particular, $Sing X_t = \pi^{-1}(t) \cap Sing X$ and Φ preserves $Sing X$.*

Proof. The equality $T_y X_t = \pi^{-1}(t) \cap T_y X$ follows from the fact that the tangent space to the leaf through y satisfies $T_y L_z \subset T_y X$ and is transverse to the fibers of π.

The differential of f at y vanishes if and only if

$$\min_{y(s)} \operatorname{ord}_0 f(y(s)) > 1$$

where the minimum is taken over all real analytic arcs $y(s) : I \to (Y, y)$. Similarly, the differentials of $f_1, ..., f_l$ at y are independent if and only if for every $i = 1, ..., l$ there is a real analytic arc $y(s) : I \to (Y, y)$ such that

$$\operatorname{ord}_0 f_i(y(s)) = 1 \text{ and } \operatorname{ord}_0 f_j(y(s)) > 1 \text{ for all } j = 1, ..., \hat{\imath}, ..., l.$$

All these conditions are preserved by Φ. □

5.4. **Preservation of multiplicities of roots.** Corollary 1.3 admits a multidimensional generalization, see Zariski [60]. We shall need in the sequel the following result that is a consequence of [60] and Proposition 5.7. We include its proof for the reader's convenience.

Lemma 5.9 (Preservation of multiplicities of roots)**.** *Let $\Phi : T \times Z \to Y$ be an arc-wise analytic trivialization, $y_0 = \Phi(t_0, z_0)$, and let A_i, $i = 1, ..., N$, be $\mathbb{K}$-analytic functions defined in a neighborhood of y_0. Let*

$$f(y, w) = w^N + \sum_i A_i(y) w^{N-i}$$

and suppose that the discriminant $\Delta(f_{red})$ is regular for Φ. Then, in a neighborhood of t_0, the roots of f at $\Phi(t, z_0)$,

$$a_1(\Phi(t, z_0)), \dots, a_N(\Phi(t, z_0)),$$

are complex analytic functions in t and if $a_i(\Phi(t_0, z_0)) = a_j(\Phi(t_0, z_0))$ then $a_i(\Phi(t, z_0)) = a_j(\Phi(t, z_0))$ for all t. In particular, the multiplicity of each $a_i(\Phi(t, z_0))$ as a root of f is independent of t.

Proof. Choose a real analytic arc germ $z(s) : I \to Z$, $z(0) = z_0$, so that $\Delta(f_{red})$ is not identically zero on $\Phi(t, z(s))$. By Corollary 1.3 it suffices to show that $F(t, s, w) = f_{red}(\Phi(t, z(s)), w)$ satisfies the assumptions of the Puiseux with parameter theorem. To show it we first note that the discriminant of F equals to $\Delta(f_{red})(\Phi(t, z(s)))$. Secondly we observe that, by regularity of $\Delta(f_{red})$ on $z(s)$ in the form (5.3), $\Delta(f_{red})(\Phi(t, z(s)))$ equals s^k times an analytic unit. □

Part 2.

6. Zariski Equisingularity implies arc-wise analytic triviality

In this section we generalize Theorem 4.1 to the arbitrary number of variables hypersurface case.

Theorem 6.1. *If $F_i(t, x)$, $i = 0, \dots, n$, is a Zariski equisingular system of pseudopolynomials, then there exist $\varepsilon > 0$ and a homeomorphism*

$$\Phi : B_\varepsilon \times \Omega_0 \to \Omega, \tag{6.1}$$

where $B_\varepsilon = \{t \in \mathbb{K}^m; \|t\| < \varepsilon\}$, Ω_0 and Ω are neighborhoods of the origin in $\mathbb{K}^n$ and $\mathbb{K}^{m+n}$ resp., such that

- (Z1) $\Phi(t, 0) = (t, 0)$, $\Phi(0, x_1, \dots, x_n) = (0, x_1, \dots, x_n)$;
- (Z2) *Φ has a triangular form*

$$\Phi(t, x_1, \dots, x_n) = (t, \Psi_1(t, x_1), \dots, \Psi_{n-1}(t, x_1, \dots, x_{n-1}), \Psi_n(t, x_1, \dots, x_n));$$

- (Z3) *For $(t, x_1, \dots, x_{i-1})$ fixed, $\Psi_i(t, x_1, \dots, x_{i-1}, \cdot) : \mathbb{K} \to \mathbb{K}$ is bi-Lipschitz and the Lipschitz constants of Ψ_i and Ψ_i^{-1} can be chosen independent of $(t, x_1, \dots, x_{i-1})$;*
- (Z4) *Φ is an arc-wise analytic trivialization of the standard projection $\Omega \to B_\varepsilon$;*

(Z5) F_n *is regular along* $B_\varepsilon \times \{0\}$.

Recall after Proposition 5.5 that (Z5) implies that for any analytic G dividing a power of F_n, there is $C > 0$ such that

$$C^{-1}|G(0,x)| \le |G(\Phi(t,x))| \le C|G(0,x)|. \tag{6.2}$$

In particular Φ preserves the zero level of G.

Proof. The proof is by induction on n. The functions Ψ_i are constructed inductively so that every

$$\Phi_i(t, x_1, \ldots, x_i) = (t, \Psi_1(t, x_1), \ldots, \Psi_i(t, x_1, \ldots, x_i)) \tag{6.3}$$

satisfies the above properties (Z1)-(Z4) and (Z5) for F_i. Thus suppose that $\Psi_1, \ldots, \Psi_{n-1}$ are already constructed and that for $i < n$ the homeomorphisms (6.3) satisfy the properties (Z1)-(Z5). To simplify the notation we write $(x_1, \ldots, x_n) = (x', x_n)$.

As we will show below the trivialization $\Phi_{n-1} : B_{\varepsilon'} \times \Omega'_0 \to \Omega'$ lifts (by continuity) to all complex roots of F_n, and then can be extended to $B_{\varepsilon'} \times \Omega'_0 \times \mathbb{C} \to \Omega' \times \mathbb{C}$ by Whitney Interpolation Formula. Then the arc-wise analyticity of Φ will be proven by reduction to the Puiseux with parameter case. Let $x(s) = (x'(s), x_n(s))$ be a real analytic arc. By the inductive assumption $\Phi_{n-1}(t, x'(s))$ is analytic in t, s. We show that $f(t, s, z) = (F_n(\Phi_{n-1}(t, x'(s)), z))_{red}$ satisfies the assumptions of Puiseux with parameter theorem, and then we conclude by Theorem 4.1. For this, we first consider the case when the discriminant of $F_{n,red}(\Phi_{n-1}(t, x'(s)), z)$ is not identically equal to zero, to show that the number and the multiplicities of the roots of F_n are constant along each leaf of Φ.

The property (5) of Definition 3.2 for Φ will be shown later in section 8 where the appropriate stratification is introduced. In the argument below we do not use this property in the inductive step.

Let us present now the details of the proof. By the inductive assumption F_{n-1}, and hence the discriminant of $F_{n,red}$, is regular for Φ_{n-1}. Therefore, by the preservation of multiplicities of roots principle, Lemma 5.9, for any $x' \in \Omega'_0$, the complex roots of F_n

$$a_1(\Phi_{n-1}(t, x')), \ldots, a_{d_n}(\Phi_{n-1}(t, x'))$$

can be chosen $\mathbb{K}$-analytic of t. Moreover, $a_i(0, x') = a_j(0, x')$ if and only if $a_i(\Phi_{n-1}(t, x')) = a_j(\Phi_{n-1}(t, x'))$ for all $t \in B_{\varepsilon'}$. Denote by $a(\Phi_{n-1}(t, x')) = (a_1(\Phi_{n-1}(t, x')), \ldots, a_{d_n}(\Phi_{n-1}(t, x')))$ the vector of such roots and set

$$\begin{aligned} \Psi_n(t, x) &:= \psi(x_n, a(0, x'), a(\Phi_{n-1}(t, x'))) \\ &= x_n + \frac{\sum_{j=1}^N \mu_j(x_n, a(0, x'))(a_j(\Phi_{n-1}(t, x')) - a_j(0, x'))}{\sum_{j=1}^N \mu_j(x_n, a(0, x'))}, \end{aligned} \tag{6.4}$$

where ψ is given by (10.20), and then define Φ by (Z2). Thus Φ satisfies (Z1). We show that Φ is a homeomorphism that satisfies (Z3)-(Z5). This will be checked on every real analytic arc applying Puiseux with parameter theorem.

Lemma 6.2. *Let $K \Subset \Omega_0'$. Then*

$$\sup_{x' \in K} \max_{a_i(0,x') \neq a_j(0,x')} \frac{|a_i(\Phi_{n-1}(t,x')) - a_j(\Phi_{n-1}(t,x'))|}{|a_i(0,x') - a_j(0,x')|} \to 1 \quad as \quad t \to 0 \tag{6.5}$$

Proof. Denote

$$\gamma(t,x') = \max_{a_i(0,x') \neq a_j(0,x')} \frac{|(a_i(\Phi_{n-1}(t,x')) - a_j(\Phi_{n-1}(t,x'))) - (a_i(0,x') - a_j(0,x'))|}{|a_i(0,x') - a_j(0,x')|}.$$

We show that γ is bounded on $B_{\varepsilon'} \times K$ and converges to 0 as t goes to 0. Let $x'(s)$ be a real analytic arc such that $(0, x'(s))$ is not entirely included in the zero set of F_{n-1}. By Corollary 1.5, γ is bounded on $(t, x'(s))$ and converges to 0 as t goes to 0. Thus, by the curve selection lemma, the claim holds on $\{(t,x'); F_{n-1}(t,x') \neq 0\}$. We extend it on the zero set of F_n by the lower semi-continuity of γ, Remark 10.22. □

Thus, taking ε' smaller if necessary, we see by Proposition 10.20 that Ψ_n of (6.4) is well-defined, continuous by Proposition 10.21, and satisfies (Z3).

Choose a neighborhood $\tilde\Omega_0'$ of the origin in $\mathbb{K}^{n-1}$, $\varepsilon \le \varepsilon'$ and $r > 0$ so that $\tilde\Omega_0' \Subset \Omega_0'$ and F_n does not vanish on $B_\varepsilon \times \tilde\Omega_0' \times \partial D$, where $\partial D = \{x_n \in \mathbb{K}; \|x_n\| = r\}$. Then we set $\Omega_0 = \tilde\Omega_0' \times D$, where $D = \{x_n \in \mathbb{K}; \|x_n\| < r\}$, and $\Omega = \Phi(B_\varepsilon \times \Omega_0)$.

Now we show (Z4) (except the property (5) of Definition 3.2 that will be shown in section 8). Let $x(s) : I \to \Omega_0$ be a real analytic arc. We show that $\Phi(t, x(s))$ is analytic in t and s. If $(0, x'(s))$ is not entirely included in the zero set of F_{n-1} then it follows from Theorem 4.1 (we argue as in the proof of Lemma 5.9). Thus, suppose $F_{n-1}(0, x'(s)) \equiv 0$. Consider

$$f(t,s,z) = (F_n(\Phi_{n-1}(t,x'(s)), z))_{red}. \tag{6.6}$$

By (6.5) the discriminant of f has to be of the form s^M times a unit, thus again the claim follows from Theorem 4.1. To show that the inverse of Φ is arc-analytic we use the inductive assumption, i.e. the assumption that the inverse of Φ_{n-1}, is arc-analytic, and then, for a real analytic arc $x'(s)$, over $(t,s) \to \Phi_{n-1}(t, x'(s))$, we use Lemma 4.3.

The proof of (Z5) is similar to the proof of (Z4). First, by Proposition 5.3, it suffices to show it over the flow of any real analytic curve $x'(s)$, that is for $(t,s) \to \Phi_{n-1}(t, x'(s))$. If $(0, x'(s))$ is not entirely included in the zero set of F_{n-1}, then it follows directly from the proof of Lemma 5.9 and Theorem 4.1. If $F_{n-1}(0, x'(s)) \equiv 0$ then we consider (6.6) and conclude again by Theorem 4.1. □

6.1. **Geometric properties.** In this subsection we summarize some geometric properties of the arc-wise analytic trivialization Φ constructed in the proof of Theorem 6.1. Firstly Φ preserves the multiplicities and the singular locus of the functions regular for Φ.

The preservation of multiplicity follows by induction from Zariski [60], or, independently from Proposition 5.7.

Proposition 6.3 (Zariski equisingularity implies equimultiplicity)**.** *Let F_i, $i = 0, \ldots, n$, be a Zariski equisingular system of analytic functions at the origin in $\mathbb{K}^m \times \mathbb{K}^n$. Then for any*

$\mathbb{K}$-analytic function G dividing F_n, the multiplicities

$$\operatorname{mult}_{(t,0)} G = \operatorname{mult}_0 G_t, \tag{6.7}$$

where $G_t(x) = G(t,x)$, are independent of t. □

Note that, by construction $\Phi(t,x) = (t, \Psi(t,x))$ is real analytic in the complement of $B_\varepsilon \times Z$, where Z is a nowhere dense $\mathbb{K}$-analytic subset of Ω_0. Let us, for t fixed, denote $x \to \Psi(t,x)$ by Ψ_t. It follows from (Z2) and (Z3) that the jacobian determinant of Ψ_t, that is well-defined in the complement of $B_\varepsilon \times Z$, is bounded from zero and infinity in a neighborhood of the origin, that is there exists $C, c > 0$ such that

$$c \le |jac \det(\Psi_t)(t,x)| \le C. \tag{6.8}$$

Consider an analytic set $X = \{f_1(t,x) = ... = f_k(t,z) = 0\} \subset \Omega$ defined by $\mathbb{K}$-analytic functions $f_1(t,x), ..., f_k(t,z)$ regular for Φ. Denote $X_t = X \cap \pi^{-1}(t)$. Then, as follows from Proposition 5.8, $Sing X_t = \pi^{-1}(t) \cap Sing X$ and Φ preserves $Sing X$ and $Reg X$.

6.2. **Generalizations.** The following proposition follows from the proof of Theorem 6.1.

Proposition 6.4. *Theorem 6.1 holds if in the definition of a system of pseudopolynomials the assumption*

(i) F_{i-1} equals to the Weierstrass polynomial associated to the discriminant of $F_{i,red}$.

is replaced by

(ii) The discriminant of $F_{i,red}$ divides F_{i-1}.

Zariski equisingularity can be also used to show the topological triviality of analytic function germs, see [3], [38]. For this application one needs the following result.

Proposition 6.5. *Theorem 6.1 holds if in the definition of a system of pseudopolynomials the assumption*

(i) The discriminant of $F_{i,red}$ divides F_{i-1}.

is replaced by

(ii) There are $q_i \in \mathbb{N}$ such that $F_i = x_1^{q_i} \tilde{F}_i$, where $\tilde{F}_i(x_1, ..., x_i)$ is a monic Weierstrass polynomial in x_i, and for $i = 1, ..., n$, the discriminant of $\tilde{F}_{i,red}$ divides F_{i-1}.

Moreover, in the conclusion we may require that $\Psi_1(t, x_1) \equiv x_1$.

Proof. We can always require $\Psi_1(t,x_1) \equiv x_1$ as it follows from the construction. Then x_1 is constant on the fibers of Φ and hence regular along $B_\varepsilon \times \{0\}$. Therefore, under the above assumptions, if we choose $\Psi_1(t,x_1) \equiv x_1$, then in the inductive step if Φ_i satisfies the properties (Z1)-(Z4) and (Z5) for $\tilde{F}_i$, then it satisfies them for $F_i = x_1^{q_i} \tilde{F}_i$. □

Part 3. Stratifications.

7. Arc-wise analytic stratifications.

Let X be a $\mathbb{K}$-analytic space of dimension n. By *an analytic stratification of* X we mean a filtration of X by analytic subspaces

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0$$

such that each X_j is of dimension j or X_j is empty, and $Sing(X_j) \subset X_{j-1}$. (Then, by definition of $Sing(X_j)$, $X_j \setminus X_{j-1}$ is a locally closed $\mathbb{K}$-analytic submanifold of X of pure dimension j.) This filtration induces a decomposition $X = \sqcup S_i$, where the S_i are connected components of all $X_j \setminus X_{j-1}$. The analytic locally closed submanifolds S_i of X are called *strata* and their collection $\mathcal{S} = \{S_i\}$ is usually called a stratification of X. In what follows we simply say that $\mathcal{S} = \{S_i\}$ is an analytic stratification of X, meaning that it comes from an analytic filtration. Similarly we define an algebraic stratification of an algebraic variety X by additionally requiring that each X_j is an algebraic subvariety of X.

Stratifications are often considered with extra regularity conditions such as Whitney (a) and (b) conditions and (w) condition of Verdier. These are conditions imposed on pairs of strata. The definition of Lipschitz stratification of Mostowski [32], see also [35], involves conditions on many strata.

For more details on stratifications we refer the reader to [54], [55], [52], [10], [48], [12], [11] and the references therein. Recall that for real analytic stratification the (w) condition implies the conditions (a) and (b), see [48]. For complex analytic stratification the conditions (w) and (b) are equivalent, c.f. [43].

We say that a stratification $\mathcal{S} = \{S_i\}$ is compatible with $Y \subset X$ if Y is a union of strata.

7.1. **(Arc-a) stratifications.**

Let X be a $\mathbb{K}$-analytic space and let $\mathcal{S}$ be an analytic stratification of X. Let p be a point of a stratum $S \in \mathcal{S}$. We say that $\mathcal{S}$ is *arc-wise analytically trivial at* p or, *satisfies condition (arc-a) at* p, if there are a neighborhood Ω of p, $\mathbb{K}$-analytic coordinates on Ω such that $B = S \cap \Omega$ is a neighborhood of the origin in $\mathbb{K}^m \times \{0\}$, and an arc-wise analytic trivialization of the projection π on the first m coordinates

$$\Phi(t, x) : B \times \Omega_0 \to \Omega, \tag{7.1}$$

where $\Omega_0 = \Omega \cap \pi^{-1}(0)$, such that $\Phi(B \times \{0\}) = B$ and Φ preserves the stratification. By the last condition we mean that each stratum of $\mathcal{S}$ is the union of leaves of Φ, see Section 3.

We say that $\mathcal{S}$ is *arc-wise analytically trivial*, or *satisfies condition (arc-a)*, if it does it at every point of X.

We say that *the condition (arc-a) is satisfied along a stratum* S if it is satisfied at every $p \in S$.

Proposition 7.1. *If a stratification $\mathcal{S}$ satisfies (arc-a) at $p_0 \in S_0$ then there is a neighborhood U of p_0 such that for every $S_1 \in \mathcal{S}$, the pair S_1, S_0 satisfies Whitney condition (a) at every $p \in U \cap S_0$.*

Proof. Recall that a pair S_1, S_0 satisfy Whitney condition (a) at $p_0 \in S_0$ if (in local coordinates at p_0) for every sequence $S_1 \ni p_n \to p_0$ such that the tangent spaces $T_{p_n}S_1 \to T$ then $T \supset T_{p_0}S_0$.

Proposition 7.1 is an immediate consequence of Proposition 3.3, more precisely of the continuity of the tangent spaces of the leaves of Φ. Indeed, if $S_1 \ni p_n \to p_0 \in S_0$ then the tangent spaces T_{p_n} to the leaves of Φ at p_n converge to $T_{p_0} = T_{p_0}S_0$. Since $T_{p_n} \subset T_{p_n}S_1$ the limit T of $T_{p_n}S_1$ contains the limit of T_{p_n}, that is $T_{p_0}S_0$. □

8. Canonical stratification associated to a system of pseudopolynomials.

Consider a *system of pseudopolynomials* in $x = (x_1, ..., x_n) \in \mathbb{K}^n$

$$F_i(x_1, \ldots, x_i) = x_i^{d_i} + \sum_{j=1}^{d_i} A_{i,j}(x_1, \ldots, x_{i-1}) x_i^{d_i - j}, \quad i = 1, \ldots, n, \tag{8.1}$$

with $\mathbb{K}$-analytic coefficients $A_{i,j}$, satisfying

(1) there are $\varepsilon_j > 0$, $j = 1, \ldots, n$, such that F_i are defined on $U_i = \prod_{j=1}^{i} D_j$, where $D_j = \{|x_j| < \varepsilon_j\}$.
(2) F_i does not vanish on $U_{i-1} \times \partial D_i$, where $\partial D_i = \{|x_i| = \varepsilon_i\}$.
(3) for every i, the discriminant of $F_{i,red}$ divides F_{i-1}.

It may happen that $d_i = 0$. Then $F_i \equiv 1$ and we set by convention $F_j \equiv 1$ for $j < i$. For $i < k$ we denote by $\pi_{k,i} : U_k \to U_i$ the standard projection.

For each i we define a filtration

$$U_i = X_i^i \supset X_{i-1}^i \supset \cdots \supset X_0^i, \tag{8.2}$$

where

(1) $X_0^1 = V(F_1)$. It may be empty.
(2) $X_j^i = (\pi_{i,i-1}^{-1}(X_j^{i-1}) \cap V(F_i)) \cup \pi_{i,i-1}^{-1}(X_{j-1}^{i-1})$ for $1 \le j < i$.

As we show below every connected component S of $X_j^i \setminus X_{j-1}^i$ is a locally closed j-dimensional $\mathbb{K}$-analytic submanifold of U_i and hence (8.2) defines a stratification $\mathcal{S}_i$ of U_i. We call $\mathcal{S} = \mathcal{S}_n$ *the canonical stratification associated to a system of pseudopolynomials.*

Proposition 8.1. *For all $j \le i \le n$ every connected component S of $X_j^i \setminus X_{j-1}^i$ is a locally closed j-dimensional $\mathbb{K}$-analytic submanifold of U_i of one of the following two types:*

(I) *$S \subset V(F_i)$ and there is a connected component S' of $X_j^{i-1} \setminus X_{j-1}^{i-1}$ such that $\pi_{i,i-1}$ induces a finite $\mathbb{K}$-analytic covering $S \to S'$.*
(II) *There is a connected component S'' of $X_{j-1}^{i-1} \setminus X_{j-2}^{i-1}$ such that S is a connected component of $\pi_{i,i-1}^{-1}(S'') \setminus V(F_i)$.*

Moreover, the canonical stratification $\mathcal{S}_i$ satisfies condition (arc-a) at every $p \in U_i$.

Proof. Induction on n. Let S' be a stratum of $\mathcal{S}_{n-1}$ of dimension j and let $p' \in S'$. By the inductive assumption there are a local system of coordinates $y_1, ..., y_{n-1}$ at p' such that

$(S',p')=(\mathbb{K}^j,0)$, neighborhoods B', Ω'_0 and Ω' of p' in $\mathbb{K}^j$, $\mathbb{K}^{n-1-j}$, and $\mathbb{K}^{n-1}$ resp., and an arc-wise analytic trivialization

$$\Phi' : B'\times\Omega'_0\to\Omega'$$

preserving $\mathcal{S}_{n-1}$ and such that F_{n-1} is regular for Φ'. Since the discriminant of $F_{n,red}$ divides F_{n-1} it is also regular for Φ'. Therefore, by Lemma 5.9, the restriction of projection $\pi_{n,n-1}$

$$\pi^{-1}_{n,n-1}(B')\cap V(F_n)\to B'$$

is a finite analytic covering. This shows that the connected components of $\pi^{-1}_{n,n-1}(S')\cap V(F_n)$ and of $\pi^{-1}_{n,n-1}(S')\setminus V(F_n)$ are locally closed submanifolds of Ω_n of type (I) or (II).

Let S be a connected component of $\pi^{-1}_{n,n-1}(S')\setminus V(F_n)$ and let $p\in S'$ be such that $p'=\pi_{n,n-1}(p)$. Then $\mathcal{S}_n$ near p is the product of $\mathcal{S}_{n-1}\times\mathbb{K}$. Therefore the conclusion follows from the inductive assumption and the fact that $F_n(p)\neq 0$.

If S is a connected component of $\pi^{-1}_{n,n-1}(S')\cap V(F_n)$ we show that Φ' can be lifted to an arc-wise analytic trivialization

$$\Phi : B\times\Omega'_0\times\mathbb{K}\to\Omega'\times\mathbb{K},$$

so that Φ preserves $\mathcal{S}_n$ and F_n is regular for Φ. This can be done exactly as in the proof of Theorem 6.1 as follows. Denote by $a(y)=(a_1(y),...,a_{d_n}(y))$ the vector of complex roots of F_n and set

$$\Phi(y,x_n)=(\Phi'(y),\psi(x_n,a(0,y_{j+1},...,y_{n-1}),a(\Phi'(y))),$$

where ψ is given by Whitney interpolation formula (10.20). This ends the proof. □

8.1. **Canonical stratification associated to a system of polynomials.** Given a polynomial $F\in\mathbb{K}[x_1,...,x_n]$ we construct an associated system of polynomials $F_i\in\mathbb{K}[x_1,\dots,x_i]$, $i=1,...,n$, as follows. First we set $F_n=F$ that after a linear change of coordinates we may assume monic in x_n. Then let F_{n-1} be the discriminant of $F_{n,red}$. We again make a linear change of coordinates $x_1,...,x_{n-1}$ so that we may assume F_{n-1} monic in x_{n-1} and we continue until we get F_j a non-zero constant. This construction is algorithmic except taking generic system of coordinates. As follows from Proposition 8.1 the canonical stratification associated to a system of polynomials satisfies condition (arc-a).

8.2. **Φ of the proof of Theorem 6.1 satisfies condition (5) of Definition 3.2.** Let $\mathcal{S},\mathcal{S}_0$ be the canonical stratifications associated to the families $\{F_i(t,x)\}$ and $\{F_i(0,x)\}$ respectively. We show that Φ induces a real analytic diffeomorphism between the strata of $B_\varepsilon\times\mathcal{S}_0$ and $\mathcal{S}$. By induction on n we may suppose that the corresponding property holds for Φ_{n-1}. Let S be a stratum of $\mathcal{S}$ of type (I), that is a covering space over a stratum S'. Denote $S_0=S\cap\{t=0\}$, $S'_0=S'\cap\{t=0\}$. By construction, Φ restricted to $B_\varepsilon\times V(F_n(0,x))$ is a lift of Φ_{n-1}. Therefore, if $\Phi_{n-1}:B_\varepsilon\times S'_0\to S'$ is an analytic diffeomorphism, consequently so is its lift $\Phi:B_\varepsilon\times S_0\to S$.

Now suppose that S is of type (II). By assumption, $a_i(t,x')$ of (6.4) are real analytic on $B_\varepsilon\times S''_0$ and hence, by Whitney interpolation formula (10.20), so is Ψ_n on $B_\varepsilon\times S'_0$. This shows the claim. □

Remark 8.2. In general for an (arc-a) stratification, we have to substratify to obtain the condition (5) of Definition 3.2. For the canonical stratification associated to a system of pseudopolynomials the arc-wise analytic trivializations constructed in the proof of Proposition 8.1 are real analytic on its strata.

Part 4. Proof of Whitney Fibering Conjecture

9. Zariski Equisingularity with transverse projections.

Definition 9.1. We say that a system of pseudopolynomials $F_i(t,x)$, $i=1,\dots,n$, is *transverse* at the origin in $\mathbb{K}^m \times \mathbb{K}^n$, if for every $i=2,\dots,n$, the multiplicity $\operatorname{mult}_0 F_i(0,x)$ of $F_i(0,x)$ at $0 \in \mathbb{K}^i$ is equal to d_i.

We always have the upper semi-continuity condition. If we denote $F_t(x) = F(t,x)$, then $\operatorname{mult}_0 F_t \le \operatorname{mult}_0 F_0$ for t close to 0. Therefore the transversality is a closed condition (in euclidean or analytic Zariski topology) in parameter t.

If the system $\{F_i\}$ is Zariski equisingular then, by Proposition 6.3, the transversality is also an open condition. Thus in this case the system is transverse at any $(t,0) \in U$, keeping the notation from Definition 2.1. Therefore, writing F instead of F_n, we have $d_n = \operatorname{mult}_0 F_t = \operatorname{mult}_0 F_0$ and also $d_n = \operatorname{mult}_{(0,0)} F = \operatorname{mult}_{(t,0)} F$.

Denote $X = F^{-1}(0)$, $X_t = X \cap \{t\} \times \mathbb{K}^n$. Geometrically the assumption $\operatorname{mult}_0 F(0,x) = d_n$ means that the kernel of the standard projection $\mathbb{K}^n \to \mathbb{K}^{n-1}$ is transverse to the tangent cone of X_0 at the origin, i.e. the vertical line $\{0\} \times \mathbb{K} \subset \mathbb{K}^{n-1} \times \mathbb{K}$ is not entirely included in this tangent cone. Moreover, in the Zariski equisingular case, by Proposition 6.3, the kernel of the standard projection $\pi : \mathbb{K}^m \times \mathbb{K}^n \to \mathbb{K}^m \times \mathbb{K}^{n-1}$ is transverse to the tangent cone of X at the origin.

Theorem 9.2. *Let $F_i(t,x)$, $i=0,\dots,n$, be a Zariski equisingular system of pseudopolynomials transverse at the origin in $\mathbb{K}^m \times \mathbb{K}^n$. Let $\Phi(t,x) = (t, \Psi(t,x)) : B_\varepsilon \times \Omega_0 \to \Omega$ be the homeomorphism constructed in the proof of Theorem 6.1. Then*

(Z6) *Φ is an arc-wise analytic trivialization regular along $B_\varepsilon \times \{0\}$.*

Proof. We have to show that, after shrinking the neighborhood Ω if necessary, there is a constant $C > 1$ such that for all $(t,x) \in B_\varepsilon \times \Omega_0$,

$$C^{-1}\|x\| \le \|\Psi(t,x))\| \le C\|x\|. \tag{9.1}$$

This will be shown by induction on n. Let us write for short $x = (x', x_n)$ and $\Phi(t,x) = (t, \Psi(t,x)) = (t, \Psi'(t,x'), \Psi_n(t,x))$. By the inductive assumption

$$C_1^{-1}\|x'\| \le \|\Psi'(t,x'))\| \le C_1\|x'\|. \tag{9.2}$$

We keep the notation of the proof of Theorem 6.1. Thus let $a_1(t,x'), \dots, a_{d_n}(t,x')$ denote the complex roots of F_n. By the equimultiplicity assumption, for all $j = 0, ..., d_n - 1$,

$|A_{n,j}(t,x')| \le C_2\|x'\|^j$, and hence the roots satisfy $|a_i(t,x')| \le C_3\|x'\|$, that, by the inductive assumption, is equivalent to

$$|a_i(t,\Psi'(t,x'))| \le C_4\|x'\|. \tag{9.3}$$

By the formula (6.4) and the Lipschitz property for Whitney Interpolation, Proposition 10.20,

$$|\Psi_n(t,x) - a_i(t,\Psi'(t,x'))| \le C_5|x_n - a_i(0,x')|. \tag{9.4}$$

By (9.3) and (9.4)

$$|\Psi_n(t,x)| \le C_6(|x_n - a_i(0,x')| + |a_i(t,\Psi'(t,x'))|) \le C_7\|(x',x_n)\|$$

that shows the second inequality in (9.1). The proof of the first one is similar.

This ends the proof of Theorem9.2. □

10. Generic Arc-wise Analytic Equisingularity

We use Zariski equisingularity to show that an analytic family of analytic set germs $\mathcal{X} = \{X_t\}$, $t \in T$, is "generically" equisingular. That is, locally on the parameter space T, this family is equisingular in the complement of an analytic subset $Z \subset T$, $\dim Z < \dim T$. In this section the parameter space T may be singular.

Definition 10.1. Let T be a $\mathbb{K}$-analytic space, $U \subset \mathbb{K}^n$ an open neighborhood of the origin, $\pi : T \times U \to T$ the standard projection, and let $\mathcal{X} = \{X_k\}$ be a finite family of analytic subsets of $T \times U$. We say that $\mathcal{X}$ is *arc-wise analytically equisingular along* $T \times \{0\}$ *at* $t \in Reg(T)$, if there are neighborhoods B of t in $Reg(T)$ and Ω of $(t,0)$ in $T \times \mathbb{K}^n$, and an arc-wise analytic trivialization $\Phi : B \times \Omega_t \to \Omega$, where $\Omega_t = \Omega \cap \pi^{-1}(t)$, such that $\Phi(B \times \{0\}) = (B \times \{0\})$ and for every k, $\Phi(T \times X_{k,t}) = X_k$, where $X_{k,t} = X_k \cap \pi^{-1}(t)$.

We say that $\mathcal{X}$ is *regularly arc-wise analytically equisingular along* $T \times \{0\}$ *at* $t \in T$ if moreover Φ is regular at $(t,0)$.

Theorem 10.2. *Let $\mathcal{X} = \{X_k\}$ be a finite family of analytic subsets of a neighborhood of $T \times U$ and let $t_0 \in T$. Then there exist an open neighborhood T' of t_0 in T and a proper $\mathbb{K}$-analytic subset $Z \subset T'$, containing $Sing(T')$, such that for every $t \in T' \setminus Z$, $\mathcal{X}$ is regularly arc-wise analytically equisingular along $T \times \{0\}$ at t.*

Moreover, there is an analytic stratification of an open neighborhood of t_0 in T such that for every stratum S and every $t \in S$, $\mathcal{X}$ is regularly arc-wise analytic equisingular along $S \times \{0\}$ at t.

Proof. For each X_k fix a finite system of generators $F_{k,i} \in \mathcal{O}_{T,t_0}$ of the ideal defining it. The first claim follows from Lemma 10.3 applied to the product of all $F_{k,i}$. The second claim follows by induction on $\dim T$.

Lemma 10.3. *Let T be a $\mathbb{K}$-analytic space, $t_0 \in T$. Let F be a $\mathbb{K}$-analytic function defined in a neighborhood of $(t_0,0) \in T \times \mathbb{K}^n$. Then there exist a neighborhood T' of t_0 in T and a proper $\mathbb{K}$-analytic subset $Z \subset T'$, $\dim Z < \dim T$, $Sing(T) \subset Z$, such that, after a linear* *change of coordinates in* $\mathbb{K}^n$*, for every* $t \in T' \setminus Z$ *there is a Zariski equisingular transverse system of functions* F_i*,* $i = 0, ..., n$*, at* $(t, 0)$*, with* F_n *the Weierstrass polynomial associated to* F *at* $(t, 0)$*.*

Proof. We may suppose that T is a subspace of $\mathbb{K}^m$, $t_0 = 0$, and $(T, 0)$ is irreducible.

We construct a new system of coordinates $x_1, ..., x_n$ on $\mathbb{K}^n$, analytic subspaces $(Z_i, 0) \subset (T, 0)$ and analytic function germs $G_i(t, x_1, ..., x_i)$, $i = n, n-1, ..., 0$, such that for every $t \in T \setminus Z$, $Z = Sing(T) \cup \bigcup Z_i$, the following condition is satisfied. Let F_i be the Weierstrass polynomial in x_i associated to the germ of G_i at $(t, 0)$. Then the discriminant of $F_{i,red}$ divides F_{i-1}.

The G_i are constructed by descending induction. First we set $G_n = F$. Then we construct G_{n-1} in three steps.

Step 1. Write

$$G_n(t, x) = \sum_{|\alpha| \geq m_0} A_\alpha(t) x^\alpha,$$

where m_0 is the minimal integer $|\alpha|$ for which $A_\alpha \not\equiv 0$. We may assume $m_0 > 0$ otherwise we simply take $Z = Sing(T)$. After a linear change of x-coordinates we may assume $A(t) = A_{(0,...,0,m_0)}(t) \not\equiv 0$.

Step 2. We define $A(t) * x := (A(t)^2 x_1, ..., A(t)^2 x_{n-1}, A(t) x_n)$ and set

$$\tilde{G}_n(t, x) = (A(t))^{-(m_0+1)} G_n(t, A(t) * x) = \sum_{|\alpha| \geq m_0} \tilde{A}_\alpha(t) x^\alpha.$$

Then $\tilde{A}_{(0,...,0,m_0)} \equiv 1$ and $\tilde{G}_n$ is regular in x_n.

Step 3. Denote by H_n the Weierstrass polynomial in x_n associated to $\tilde{G}_n$. It is of degree m_0 in x_n. Let $\mathcal{K}$ be the field of fractions of $\mathcal{O}_{T \times \mathbb{K}^{n-1}, 0}$ and consider H_n as a polynomial of $\mathcal{K}[x_n]$. Let d be the degree of $H_{n,red}$. We define G_{n-1} as the dth generalized discriminant of H_n, see Appendix II, and set $Z_n = A^{-1}(0)$.

Then we repeat these steps for G_{n-1} and so on.

To see that the sequence G_i satisfies the required properties we note that if F_n denotes the Weierstrass polynomial at $(t, 0) \in T \setminus (Z_n \cup Sing(T))$ associated to G_n, then, as a germ at $(t, 0)$, the discriminant of $F_{n,red}$ divides G_{n-1}.

This ends the proof of Lemma 10.3 and Theorem 10.2. □

10.1. **(Arc-w) stratifications.** Let X be a $\mathbb{K}$-analytic space and let $\mathcal{S}$ be an analytic stratification of X. Let p be a point of a stratum $S \in \mathcal{S}$. We say that $\mathcal{S}$ is regularly *arc-wise analytically trivial at* p or *satisfies condition (arc-w) at* p, if it satisfies condition (arc-a) at p with a trivialization Φ of (7.1) that is regular along $B \times \{0\}$.

We say that $\mathcal{S}$ is *regularly arc-wise analytically trivial*, or *satisfies condition (arc-w)*, if it does it at every point of X. We say that *the condition (arc-w) is satisfied along a stratum* S if it is satisfied at every $p \in S$.

Theorem 10.4. *If a stratification $\mathcal{S}$ satisfies (arc-w) condition along a stratum S then it satisfies the condition (w) of Verdier along S.*

Proof. Recall that a pair of strata S_1, S_0 satisfies Verdier condition (w) at $p_0 \in S_0$ if there are a neighborhood U of p_0 in X and a constant $C > 0$ such that for every $p \in S_1 \cap U$, $p' \in S_0 \cap U$ the angle between T_pS_1 and $T_{p'}S_0$ satisfies

$$|\measuredangle(T_pS_1, T_{p'}S_0)| \le C \operatorname{dist}(p, p'). \tag{10.1}$$

The condition (w) can be checked in any analytic system of coordinates and hence we may use the coordinates in which the trivialization Φ of (7.1) is defined. Then $T_{p'}S_0$ is independent of p' and equals $\mathbb{K}^m \times \{0\}$. By Proposition 5.2 the tangent space T_p to the leaves of Φ at $p = (t, \Psi(t, x))$ satisfies

$$|\measuredangle(T_p, \mathbb{K}^m \times \{0\})| \le C\|\Psi(t, x)\|. \tag{10.2}$$

Since $T_p \subset T_pS_1$, (10.1) follows. □

By [43] for complex analytic space conditions (b) of Whitney and (w) of Verdier are equivalent. Therefore Theorem 10.4 gives the following result due to Speder [41].

Corollary 10.5. *If a complex analytic hypersurface X is Zariski equisingular along a non-singular subspace $Y \subset Sing(X)$, and the equisingularity is defined by "generic" projections, then the pair $Reg(X), Y$ satisfies Whitney's conditions (a) and (b).*

10.2. **Local Isotopy Lemma.** Let X be a Whitney stratified space, $p \in X$, and let S be the stratum containing p. Then, as follows from Thom's first isotopy lemma, [29], [44], [10], any local submersion onto S, restricted to X, can be trivialized over a neighborhood of p in S. As it follows from Proposition 10.6 below an analogous property holds for (arc-a) and (arc-w) stratifications.

Suppose that X is a $\mathbb{K}$-analytic subspace of a neighborhood of the origin in $\mathbb{K}^N$, Ω a neighborhood of the origin in X, and let $B = \Omega \cap (\mathbb{K}^m \times \{0\})$. Let f be a $\mathbb{K}$-analytic function on X and let $\mathcal{X} = \{X_k\}$ be a finite family of analytic subsets of X. Let $\pi : \Omega \to B$ denote the standard projection onto the first m coordinates.

Proposition 10.6. *Suppose that there exist an arc-wise analytic trivialization Φ of π preserving B and a family of analytic subsets $\mathcal{X} = \{X_k\}$ of X and such that f is regular for Φ. Let $\tilde\pi$ be another analytic submersion $\Omega \to B$. Then, after restricting to a smaller neighborhood of the origin, there is an arc-wise analytic trivialization $\tilde\Phi$ of $\tilde\pi$, preserving B and the family $\mathcal{X}$ and such that f is regular for $\tilde\Phi$. Moreover, if Φ is regular along B then $\tilde\Phi$ can be chosen regular along B.*

Proof. Let $H : B \times \Omega_0 \to B \times \Omega_0$ be given by

$$H(t, x) = (h(t, x), x) = (\tilde\pi(\Phi(t, x)), x).$$

We show that H is a local homeomorphism, arc-wise analytic in t, such that H^{-1} is also arc-wise analytic in t. Then $\tilde\Phi = \Phi \circ H^{-1}$ satisfies the claim.

Firstly H is a local homomorphism by the implicit function theorem, Theorem 2.5 of [17]. Let $\gamma := x(s)$ be a real analytic arc. Consider

$$H_\gamma(t,s) = (h(t,x(s)),s) : (B \times I, 0) \to (B \times I, 0).$$

Clearly H_γ is $\mathbb{K}$-analytic in t and real analytic in s. Since $h(t,x(s)) = t + \varphi(t,s)$ with $\varphi(t,s) \in \mathfrak{m}_s$, H_γ is an analytic diffeomorphism and its inverse is $\mathbb{K}$-analytic in t and real analytic in s. □

Let $\mathcal{S}$ be an analytic stratification of X satisfying Whitney condition (a). One says after Définition 4.1.1 of [5], that $\mathcal{S}$ satisfies *stratified local triviality condition*, (TLS) condition for short, if any local submersion onto a stratum is locally topologically trivial by a preserving strata trivialization. Thus Lemma 10.6 gives the following result. (If we assume that $\tilde\pi$ is only a C^1 submersion then, by Theorem 2.5 of [17], H constructed in the above proof is a homeomorphism and so is $\tilde{\Phi}$.)

Corollary 10.7. *A stratification satisfying the (arc-a) condition also satisfies the condition (TLS) of* [5].

10.3. **Proof of Whitney fibering conjecture.** We show below that every $\mathbb{K}$-analytic space admits locally an (arc-w) stratification. In the algebraic case such a stratification exists globally. Since an (arc-w) stratification satisfies all the properties required by Whitney it shows Whitney fibering conjecture in the algebraic and local analytic cases.

Theorem 10.8. *Let $\mathcal{V} = \{V_i\}$ be a finite family of analytic subsets of an open $U \subset \mathbb{K}^N$. Let $p_0 \in U$. Then there exists an open neighborhood U' of p_0 and an analytic stratification of of U' compatible with each $U' \cap V_i$ and satisfying the condition (arc-w).*

Theorem 10.9. *Let $\mathcal{V} = \{V_i\}$ be a finite family of algebraic subsets of $\mathbb{P}^n_{\mathbb{K}}$. Then there exists an algebraic stratification of $\mathbb{P}^n_{\mathbb{K}}$ compatible with each V_i and satisfying the condition (arc-w). Moreover, the local arc-wise analytic trivializations can be chosen semi-algebraic.*

Theorem 10.9 will be proven in Section 5. It follows from Lemma 10.17, the same way as Theorem 10.8 follows from the following version of Lemma 10.3.

Lemma 10.10. *Let F be a $\mathbb{K}$-analytic function defined in a neighborhood of $0 \in \mathbb{K}^N$ and let Y be a $\mathbb{K}$-analytic subset of a neighborhood of $0 \in \mathbb{K}^N$, $\dim Y = m$. Then there exist a neighborhood U of $0 \in \mathbb{K}^N$ and a $\mathbb{K}$-analytic subset $Z \subset Y \cap U$, $\dim Z < m$, $Sing(Y) \subset Z$, such that for every $p \in Y \cap U \setminus Z$, there are a local system of coordinates at p in which $(Y,p) = (\mathbb{K}^m \times \{0\}, 0)$ and a Zariski equisingular transverse system pseudopolynomials F_i, $i = 0, ..., n = N - m$, at p, such that F_n is the Weierstrass polynomial associated to F.*

Proof. Choose a local system of coordinates at p such that the projection on the first m coordinates restricted to Y is finite. Let

$$\varphi : Y \times \mathbb{K}^n \to \mathbb{K}^N, \qquad \varphi(y,x) = y + (0,x).$$

Then apply Lemma 10.3 to $T = Y$ and $F(\varphi(t,x))$. □

Proof of Theorem 10.8. We construct a sequence of analytic set germs at p_0

$$U = X_n \supset X_{n-1} \supset \cdots \supset X_0$$

whose representatives in a sufficiently small neighborhood U' of p_0 define a stratification satisfying the statement. For simplicity of notation we assume p_0 to be the origin.

First for each analytic space V_i choose a finite system of generators of its ideal $I(V_i) = (g_{i,j})_{j=1,\ldots n_i}$ in the local ring $\mathcal{O}_0$, and let f_{n-1} be the product of all of them: $f_{n-1} = \prod_{i,j} g_{i,j}$. In the first step we apply Lemma 10.10 to $F = f_{n-1}$ and $Y = U$ and we set $X_{n-1} = Z$. If $\dim(X_{n-1}, 0) < n-1$ then we set $X_{n-2} = X_{n-1}$. Otherwise apply Lemma 10.10 to $F = f_{n-1}$ and $Y = X_{n-1}$ and we set X_{n-2} equal to the obtained Z.

If $\dim(X_{n-2}, 0) < n-2$ then we set $X_{n-3} = X_{n-2}$. Otherwise choose a finite system of generators $I(X_{n-1}) = (h_{n-1,j})$ and let $f_{n-2} = f_{n-1} \prod_j h_{n-1,j}$. Next apply Lemma 10.10 to $F = f_{n-2}$ and $Y = X_{n-2}$ and we set X_{n-3} equal to the obtained Z.

The inductive step is then the following. Given $U = X_n \supset X_{n-1} \supset \cdots \supset X_i$ and a function f_i that is the product of f_{i+1} and a finite set of generators of $I(X_{i+1})$ in $\mathcal{O}_0$. If $\dim(X_i, 0) < i$ then we set $X_{i-1} = X_i$. Otherwise we apply Lemma 10.10 to $F = f_{n-i}$ and $Y = X_i$ and we set X_{i-1} equal to the obtained Z.

Let $p \in X_k \setminus X_{k-1}$. Then by construction there is a local system of coordinates at p in which $(X_k, p) = (\mathbb{K}^k, 0)$ and an arc-wise analytic trivialization Φ of the coordinate projection on $\mathbb{K}^k$, preserving X_k and such that f_k is regular for Φ. Therefore, Φ preserves the zero set of every factor of f_i and hence every V_i and every X_i for $i > k$. This ends the proof. □

10.4. **Examples.** There are several classical examples describing the relation between Zariski equisingularity and Whitney's conditions that we recall below. The general set-up for these examples is the following. Consider a complex algebraic hypersurface $X \subset \mathbb{C}^4$ defined by a polynomial $F(x, y, z, t) = 0$ such that $Sing X = T$, where T is the t-axis. Let $\pi : \mathbb{C}^4 \to T$ be the standard projection. In all these examples $X_t = \pi^{-1}(t)$, $t \in T$, is a family of isolated singularities, topologically trivial along T. These examples relate the following conditions :

(1) X is Zariski equisingular along T, i.e. there is a local system of coordinates in which F can be completed to a Zariski equisingular system of polynomials, see Definition 2.1.
(2) X is Zariski equisingular along T for a transverse coordinate system, i.e. there is a local system of coordinates in which F can be completed to a Zariski equisingular transverse system of polynomials, Section 9.1.
(3) X is Zariski equisingular along T for a generic system of coordinates, i.e. for generic system of local coordinates, F can be completed to a Zariski equisingular system of polynomials
(4) The pair $(X \setminus T, T)$ satisfies Whitney conditions (a) and (b).

Clearly (3)⇒(2)⇒(1). Speder showed (3)⇒(4) in [41] and (2)⇒(4) for families of complex analytic hypersurfaces with isolated singularities in $\mathbb{C}^3$ in his thesis [42] (not published). Theorem 7.1 gives (2)⇒(4) in the general case. As the examples below show, all the other implications are false.

Example 10.11 ([6]).

$$F(x,y,z,t) = z^5 + ty^6z + y^7x + x^{15} \tag{10.3}$$

This example satisfies (1) for the projections $(x,y,z) \to (y,z) \to x$ but (4) fails. As follows from Theorem 7.1, (2) fails as well.

Example 10.12 ([7]).

$$F(x,y,z,t) = z^3 + tx^4z + y^6 + x^6 \tag{10.4}$$

In this example (4) is satisfied and (3) fails. This example satisfies (1) for the projections $(x,y,z) \to (x,z) \to x$.

Example 10.13 ([28]).

$$F(x,y,z,t) = z^{16} + tyz^3x^7 + y^6z^4 + y^{10} + x^{10} \tag{10.5}$$

In this example (2) is satisfied and (3) fails.

Example 10.14 ([34]).

$$F(x,y,z,t) = x^9 + y^{12} + z^{15} + tx^3y^4z^5 \tag{10.6}$$

In this example (4) is satisfied and (1) fails. This shows also that (4) does not imply (2).

Part 5. Algebraic Case

In the algebraic case we have a global version of Theorem 10.2. Here by a *real algebraic variety* we mean an affine real algebraic variety in the sense of Bochnak-Coste-Roy [4]: a topological space with a sheaf of real-valued functions isomorphic to a real algebraic set $X \subset \mathbb{R}^N$ with the Zariski topology and the structure sheaf of regular rational functions. For instance, the set of real points of a reduced projective scheme over $\mathbb{R}$, with the sheaf of regular functions, is a real algebraic variety in this sense.

Theorem 10.15. *Let T be an algebraic variety (over $\mathbb{K}$) and let $\mathcal{X} = \{X_k\}$ be a finite family of algebraic subsets $T \times \mathbb{P}^{n-1}_{\mathbb{K}}$. Then there exists an algebraic stratification $\mathcal{S}$ of T such that for every stratum S and for every $t_0 \in S$ there is a neighborhood U of t_0 in S and a semialgebraic arc-wise analytic trivialization of π, preserving the family $\mathcal{X}$,*

$$\Phi : U \times \mathbb{P}^{n-1}_{\mathbb{K}} \to \pi^{-1}(U), \tag{10.7}$$

$\Phi(t_0,x) = (t_0,x)$, where $\pi : T \times \mathbb{P}^{n-1}_{\mathbb{K}} \to T$ denotes the projection.

Proof. We may assume that T is affine irreducible. Let $G_i(t,x)$, $t \in T$, $x = (x_1,\dots,x_n)$, be a finite family of polynomials, homogeneous in x, defining the sets X_k and let $F_n(t,x)$ be the product of all G_i. We consider F_n as a homogeneous polynomial over $\mathcal{K} = \mathbb{K}(T)$ and let

$$F_n(x) = \sum_{|\alpha|=d_n} A_\alpha x^\alpha, \quad A_\alpha \in \mathcal{K}.$$

After a linear change of coordinates x, we may suppose $A_n = A_{(0,...,0,d_n)} \neq 0$. Then we define $F_{n-1}(x_1, ..., x_{n-1})$ as the discriminant of $F_{n,red}$, and proceed inductively by constructing the system of homogeneous polynomials $F_j \in \mathcal{K}[x_1, ..., x_i]$ until $F_i \in \mathcal{K}$ is a non-zero constant. Then we take as $Z \subset T$ the union of zero sets of the denominators of the coefficients of all F_j and the numerators of the leading coefficients of all F_j. We show below that the statement of theorem holds for $T \setminus Z$ as an open stratum. Then the stratification $\mathcal{S}$ can be constructed by induction on $\dim T$.

By Theorem 6.1, $V(F_n)$ is arc-wise analytically equisingular along $(T \setminus Z) \times \{0\}$. By construction (6.4), the trivialisation $\Phi(t, x) = (t, \Psi(t, x))$ is semi-algebraic and $\mathbb{K}^*$-equivariant in the variable x, as follows from the interpolation formula, see Remark 10.19. Moreover, by construction, it is regular along $U \times \{0\}$, U being a neighborhood of t_0 in $T \setminus Z$. Then the trivialization $U \times \mathbb{P}^{n-1}_{\mathbb{K}} \to \pi^{-1}(U)$ induced by Φ, is arc-wise analytic. □

We have the following versions of Lemmas 10.3 and 10.10.

Lemma 10.16. *Let T be a $\mathbb{K}$ algebraic variety and let $F \in \mathbb{K}[T \times \mathbb{K}^n]$, $F \not\equiv 0$. Then there exists a subvariety $Z \subset T$, $\dim Z < \dim T$, such that, after a linear change of coordinates in $\mathbb{K}^n$, F can be completed to a system of polynomials $\{F_i\}$, $F_n = F$, such that for every $t \in T \setminus Z$ the system $\{F_i\}$ is transverse and Zariski equisingular at $(t, 0)$.* □

Lemma 10.17. *Let $F \in \mathbb{K}[X_1, ..., X_N]$, $F \not\equiv 0$, and let $Y \subset \mathbb{K}^N$ be an algebraic subset. Then there exist an algebraic $Z \subset Y$, $\dim Z < \dim Y$, and polynomials $\{F_i\}$, $F_n = F$, such that the following holds. For every $p \in Y \setminus Z$ there is a local system of coordinates at p in which $(Y, p) = (\mathbb{K}^m \times \{0\}, 0)$, such that the germs of $\{F_i\}$ at p form a transverse and Zariski equisingular system of polynomials.* □

Lemma 10.17 implies the algebraic Whitney fibering conjecture: Theorem 10.9.

Appendix I. Whitney Interpolation.

We generalize the classical Whitney Interpolation formula [55], [16].

Fix positive integers d, N and consider a family of functions $f_i : \mathbb{C}^N \to \mathbb{C}$, $i = 1, 2, ..., N$. We assume that, for a constant $C > 1$, this family satisfies the following properties

(1) f_i are continuous, differentiable on $(\mathbb{C}^*)^N$, and $\mathbb{R}$-homogeneous of degree d.
(2) for every permutation $\sigma \in S_N$: $f_i(\xi_{\sigma(1)}, ..., \xi_{\sigma(N)}) = f_{\sigma(i)}(\xi_1, ..., \xi_N)$.
(3) $|f_j(\xi_1, \dots, \xi_N)| \le C|\xi_j|(max_i|\xi_i|)^{d-1}$.
(4) for all k, j, $|\xi_k^2| \|(\partial f_j/\partial \xi_k, \partial f_j/\partial \bar{\xi}_k)\| \le C|\xi_j|(max_i|\xi_i|)^d$
(5) $f = \sum_i f_i$ is real valued and satisfies $C^{-1}(max_i|\xi_i|)^d \le f(\xi_1, \dots, \xi_N) \le C(max_i|\xi_i|)^d$.

For examples see Examples 10.23 and 10.24.

Given two subsets $\{a_1, \dots, a_N\} \subset \mathbb{C}$, $\{b_1, \dots, b_N\} \subset \mathbb{C}$, of cardinality N such that if $a_i = a_j$ then $b_i = b_j$. Define $D_i = b_i - a_i$ and set

$$\gamma = \max_{a_i \neq a_j} \frac{|D_i - D_j|}{|a_i - a_j|}. \tag{10.8}$$

Then

$$|D_i - D_j| \le \gamma |a_i - a_j|. \tag{10.9}$$

Let

$$\mu_i(z) := f_i((z - a_1)^{-1}, \dots, (z - a_N)^{-1}), \quad \mu(z) := f((z - a_1)^{-1}, \dots, (z - a_N)^{-1}).$$

Define the interpolation map $\psi : \mathbb{C} \to \mathbb{C}$ by

$$\psi(z) = z + \frac{\sum_{i=1}^N \mu_i(z) D_i}{\mu(z)}, \tag{10.10}$$

if $z \notin \{a_1, \dots, a_N\}$ and $\psi(a_i) = b_i$. Then ψ is continuous as follows from the following lemma.

Lemma 10.18.

$$\lim_{z \to a_j} \psi(z) = b_j.$$

Proof. Let $I_j = \{i; a_i = a_j\}$. We rewrite the interpolation formula (10.10) as

$$\psi(z) = z + D_j + \frac{\sum_{i \notin I_j} \mu_i(z)(D_i - D_j)}{\mu(z)}. \tag{10.11}$$

By the properties (3) and (5), for $i \notin I_j$, $\frac{\mu_i(z)}{\mu(z)} \to 0$ as $z \to a_j$. □

Remark 10.19. **Symmetries.**
The map ψ is also invariant under permutations $\sigma \in S_N$, $\sigma(a) = (a_{\sigma(1)}, \dots, a_{\sigma(N)})$

$$\psi(z, \sigma(a), \sigma(b)) = \psi(z, a, b).$$

Let $\tau : \mathbb{C} \to \mathbb{C}$ be complex affine. Then

$$\psi(\tau(z), \tau(a), \tau(b)) = \tau(\psi(z, a, b)).$$

Proposition 10.20. *The map $\psi : \mathbb{C} \to \mathbb{C}$ is Lipschitz with Lipschitz constant $4N^2C^4\gamma + 1$. If $\gamma < (4N^2C^4)^{-1}$ then ψ is a bi-Lipschitz homeomorphism, with $(1 - 4N^2C^4\gamma)^{-1}$ a Lipschitz constant of ψ^{-1}.*

Proof. It suffices to show that for $z \notin \{a_1, \dots, a_N\}$ and for every unit vector $v \in \mathbb{C}$

$$|(\psi(z) - z)'| \le 4N^2C^4\gamma, \tag{10.12}$$

where by "prime" we denote any directional derivative $\frac{\partial}{\partial v}$, $v \in \mathbb{C}$, $|v| = 1$. Indeed, if (10.12) holds then clearly ψ is Lipschitz. Moreover, if $\gamma < (4N^2C^4)^{-1}$ then for any $p \in \mathbb{C}$, $z \to p + z - \psi(z)$ is a contraction and hence admits a unique fixed point z_p, that is a unique z_p such that $\psi(z_p) = p$. Hence ψ is a homeomorphism by the invariance of domain. By (10.12) for any $p, q \in \mathbb{C}$, $|(\psi(p) - p) - (\psi(q) - q)| \le 4N^2C^4\gamma|p - q|$, that gives

$$|p - q| \le (1 - 4N^2C^4\gamma)^{-1}|\psi(p) - (\psi(q)| \tag{10.13}$$

if $\gamma < (4N^2C^4)^{-1}$.

To show (10.12) we use the following bounds that follow from the conditions (3)-(5).

$$\begin{aligned} |\mu_i(z)| &\le C^2|z - a_i|^{-1}\mu(z)^{\frac{d-1}{d}}, \\ |\mu_i'(z)| &\le NC^2|z - a_i|^{-1}\mu(z), \\ |\mu'(z)| &\le N^2C^2\mu(z)^{\frac{d+1}{d}}, \end{aligned} \tag{10.14}$$

Given $z \in \mathbb{C}$, choose j such that $|z - a_j| = \min_i |z - a_i|$. Then, for all i,

$$|a_i - a_j| \le 2|z - a_i|. \tag{10.15}$$

By differentiating (10.11)

$$|(\psi(z) - z)'| \le \frac{\sum_{i \notin I_j} |\mu_i'(z)(D_i - D_j)|}{\mu(z)} + \frac{(\sum_{i \notin I_j} |\mu_i(z)(D_i - D_j)|)|\mu'(z)|}{(\mu(z))^2}. \tag{10.16}$$

By (10.9) and (10.14)

$$|\mu_i'(z)(D_i - D_j)| \le 2NC^2\gamma\mu(z)$$

and

$$|\mu_i(z)(D_i - D_j)||\mu'(z)| \le 2N^2C^4\gamma(\mu(z))^2.$$

This shows (10.12) and hence ends the proof of Proposition 10.20. □

Consider ψ as a function of three arguments $z \in \mathbb{C}$, $a \in \mathbb{C}^N$, $b \in \mathbb{C}^N$,

$$\psi(z, a, b) = \psi_{a,b}(z) = z + \frac{\sum_{j=1}^N \mu_j(z, a)(b_j - a_j)}{\mu(z, a)}, \tag{10.17}$$

where $\mu_i(z,a) = f_i((z-a_1)^{-1},\dots,(z-a_N)^{-1})$, $\mu(z,a) = \sum_i \mu_i(z,a)$, and $\psi_{a,b}(a_i) = b_i$. Then $\psi(z,a,b)$ can also be considered as a family of functions $\psi_{a,b} : \mathbb{C} \to \mathbb{C}$, parameterized by a, b.

Proposition 10.21. *Let $a(x) : X \to \mathbb{C}^N, b(x) : X \to \mathbb{C}^N$ be continuous functions defined on a topological space X such that for every $x \in X$ and i, j, if $a_i(x) = a_j(x)$ then $b_i(x) = b_j(x)$. Then $\psi(z, a(x), b(x))$ is continuous as a function of (x, z).*

Proof. Let $(z,a,b) \to (z_0,a_0,b_0)$. Clearly $\psi(z,a,b) \to \psi(z_0,a_0,b_0)$ if $z_0 \notin \{a_{01},\dots,a_{0N}\}$. Thus suppose $z_0 = a_{01} = a_{0j}$ for $j \in J$. Then $\psi(z_0,a_0,b_0) = b_{01}$. We have

$$\psi(z,a,b) - \psi(z_0,a_0,b_0) = (z - z_0) + \frac{\sum_{i\in J} \mu_i(z,a)((b_i - b_{01}) - (a_i - a_{01}))}{\mu(z,a)} + \frac{\sum_{i\notin J} \mu_i(z,a)((b_i - b_{01}) - (a_i - a_{01}))}{\mu(z,a)}.$$

Clearly the first two summands converge to 0 as $(z,a,b) \to (z_0,a_0,b_0)$. So does the third one because

$$\frac{\mu_i(z,a)}{\mu(z,a)} \to 0$$

if $i \notin J$. To show this last property we note that $\mu(z,a) \to \infty$, the limit of $z - a_i$ is nonzero if $i \notin J$, and use the first inequality of (10.14). □

Remark 10.22. If $(a,b) \to (a_0,b_0)$ then $\gamma(a_0,b_0) \le \liminf \gamma(a,b)$, thus γ is lower semi-continuous.

Example 10.23. In the original Whitney interpolation $f_i(\xi) = |\xi_i|$, cf. [55], see also [16].

Example 10.24. In this paper we use the following family. For $\xi_1,\dots,\xi_N \in \mathbb{C}$ we denote by $\sigma_i = \sigma_i(\xi_1,\dots,\xi_N)$ the elementary symmetric functions of $\xi_1,\dots,\xi_N$. Let $P_k = \sigma_k^{\alpha_k}$, where $\alpha_k = (N!)/k$. Define

$$f(\xi) = \sum_k P_k(\xi)\bar{P}_k(\xi). \tag{10.18}$$

and

$$f_j(\xi) = \frac{1}{N!}\sum_k \xi_j \frac{\partial P_k}{\partial \xi_j}\bar{P}_k. \tag{10.19}$$

Then ψ equals

$$\psi(z,a,b) = z + \frac{\sum_k \left(\sum_{j=1}^N \ \xi_j \frac{\partial P_k}{\partial \xi_j}(b_j - a_j)\right)\bar{P}_k(\xi)}{N!(\sum_k P_k\bar{P}_k(\xi))}, \tag{10.20}$$

where $\xi = ((z-a_1)^{-1},\dots,(z-a_N)^{-1})$.

Appendix II. Generalized discriminants.

We recall below the classical generalized discriminants, see e.g. [56] Appendix IV. Let $\mathcal{K}$ be a field of characteristic zero and let

$$F(Z) = Z^p + \sum_{j=1}^{p} a_i Z^{p-i} = \prod_{j=1}^{p} (Z - \xi_i) \in \mathcal{K}[Z],$$

with the roots $\xi_i \in \overline{\mathcal{K}}$. Then the expressions

$$D_j = \sum_{r_1 < \cdots < r_j} \prod_{k<l;\, k,l \in \{r_1,\ldots,r_j\}} (\xi_k - \xi_l)^2$$

are symmetric in $\xi_1, \ldots, \xi_p$ and hence polynomials in $a_1, \ldots, a_p$. Thus D_n is the standard discriminant and F has exactly d distinct roots if and only if $D_{d+1} = \cdots = D_n = 0$ and $D_d \neq 0$. The following lemma is obvious.

Lemma 10.25. *Let $F \in \mathbb{K}[Z]$ be a monic polynomial of degree p that has exactly d distinct roots in $\xi_i \in \overline{\mathcal{K}}$ of multiplicities $\mathbf{m} = m_1, ..., m_d$. Then there is a positive constant $C = C_{p,\mathbf{m}}$ such that the generalized discriminant $D_{d,F}$ of F and the standard discriminant $\Delta_{F_{red}}$ of F_{red} are related by*

$$D_{d,F} = C \Delta_{F_{red}}.$$

We often use the following consequence of the Implicit Function Theorem.

Lemma 10.26. *Let $F \in \mathbb{K}\{x_1, ..., x_n\}[Z]$ be a monic polynomial in Z such that the discriminant $\Delta_{F_{red}}$ does not vanish at the origin. Then, on a neighborhood U of $0 \in \mathbb{K}^n$, the complex roots $\xi_i(x_1, ..., x_n)$ of F are $\mathbb{K}$-analytic, distinct, and of constant multiplicities.*

Appendix III. Newton's algorithm.

We present below the Newton algorithm for finding Newton-Puiseux roots. For the details see [51], [22].

Let $f(x, z) \in \mathbb{C}[x, z]$ denote a two variable polynomial with complex coefficients

$$f(x, z) = \sum a_{ij} z^i x^j. \tag{10.21}$$

We suppose f not identically equal to 0.

For each $a_{ij} \neq 0$, let us plot in $\mathbb{R}^2$ a dot at (i, j), called a *Newton dot*. The set of Newton dots is called *the Newton diagram* and will be denoted by $\Delta(f, 0)$. The convex hull of translated quadrants $a_{i,j} + (\mathbb{R}^+)^2$ will be denoted by $\tilde{\Delta}(f, 0)$. The boundary of this convex hull consists of two half-lines parallel to the axes and a polygonal line (maybe reduced to a point). This polygonal line will be called *the Newton polygon of f at the origin* and denoted by $\mathbb{P}(f, 0)$.

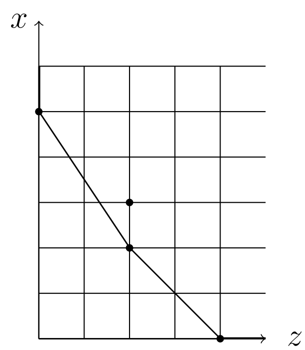

FIGURE 1.

Example: Newton Polygon of $f(x, z) = z^4 - z^2x^2 + 5z^2x^3 - x^5$ on Figure 1.

Exercise 10.27. Draw the Newton polygons of $z^5 - z^2x^5 - z^3x^3 + x^8$, $z(z^2 - x^3)$, $x(z^2 - x^3)$, $zx(z^2 - x^3)$.

Exercise 10.28. Similarly we may define the Newton polygon for infinite sums $\sum a_{ij} x^i z^j$ provided $(i, j) \in \mathbb{N}^2$ or even allowing denominators $(i, j) \in \frac{1}{d}\mathbb{N} \times \frac{1}{d}\mathbb{N}$, d being a positive integer. Show that in this more general case the Newton polygon consists of finitely many edges.

Newton polygon $\mathbb{P}(f, \lambda)$ relative to an arc $z = \lambda(x)$. By a fractional power series we mean a series of the form

$$z = \lambda(x) = c_1 x^{n_1/d} + c_2 x^{n_2/d} + \cdots, \quad c_i \in \mathbb{C},$$

where $d, n_1 < n_2 < \cdots$ are positive integers, such that $\lambda(y^d)$ has positve radius of convergence $\rho > 0$. We can identify λ with the analytic arc $\lambda : z = c_1 y^{n_1} + c_2 y^{n_2} + \cdots, z = x^d$, defined for $|y| < \rho$.

Let us apply the change of variables

$$X = x, \quad Z = z - \lambda(x),$$

to $f(x,z)$, yielding

$$F(X,Z) := f(X, Z + \lambda(X)) = \sum c_{ij} Z^i X^{j/d}. \tag{10.22}$$

Then we define *the Newton polygon of f relative to λ*, denoted by $\mathbb{P}(f,\lambda)$, as the Newton polygon of F at the origin in the usual sense.

The Newton edges E_s and their associated Newton angles (co-slopes) θ_s are defined in an obvious way as illustrated in the following example. Take $f(x,z) = z^2 - x^3 + x^4$, $\lambda(x) = x^{3/2}$. Then $\mathbb{P}(f,\lambda)$ has compact edges $\mathrm{E}_1, \mathrm{E}_2$ with $\tan\theta_1 = 3/2$, $\tan\theta_2 = 5/2$.

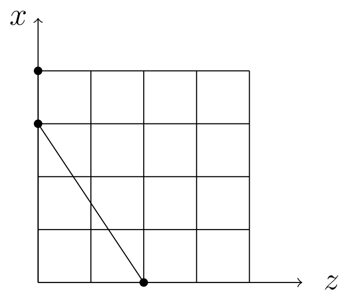

FIGURE 2. $\mathbb{P}(f,0)$

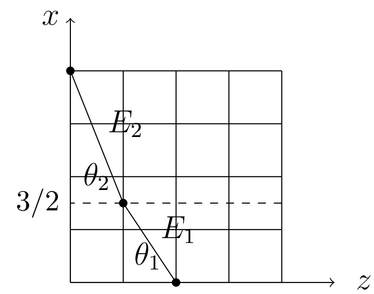

FIGURE 3. $\mathbb{P}(f,\lambda)$

Take any edge E_s. *The associated weighted homogeneous form $\tilde{\mathcal{E}}_s(z)$ and the associated polynomial $\mathcal{E}_s(z)$* are defined as follows

$$\tilde{\mathcal{E}}_s(X,Z) := \sum c_{ij} Z^i X^{j/d}, \quad (i, j/d) \in \mathrm{E}_s\,.$$
$$\mathcal{E}_s(\xi) := \tilde{\mathcal{E}}_s(\xi, 1).$$

In the above example $\mathcal{E}_1(\xi) = \xi^2 + 2\xi$, $\mathcal{E}_2(\xi) = 2\xi + 1$. If $i < s$ then always $\mathcal{E}_i(0) = 0$.

Exercise 10.29. Draw the Newton polygon of

(1) $f(x,z) = z^5 - z^2x^5 - z^3x^3 + x^8$ relative to $\lambda(x) = x^{8/5}$,
(2) $f(x,z) = (z^2 - x^3)^2 - zx^5$, relative to $\lambda(x) = x^{3/2}$.

The highest Newton edge E_s will be sometimes denoted by E_H and the associatted polynomial by $\mathcal{E}_H$. Assume that $\mathcal{E}_H(0) \neq 0$, that is there are dots on the (vertical) x-axis. Expanding f along the arc $z = \lambda(x)$ as a fractional power series is equivalent to setting $Z = 0$ in (10.22)

$$f(x, \lambda(x)) = \sum c_{0.j} x^{j/d} \tag{10.23}$$

We call the lowest exponent in (10.23) with non-zero coefficients *the order* f *on* λ, and denote it by $\operatorname{ord}_x f(x, \lambda(x))$. It coincides with the x-coordinate of the intersection of the Newton polygon with the x-axis.

10.5. **Newton-Puiseux roots of** $f(x,z)$**.** If $f(x, \lambda(x)) \equiv 0$ then we say that the arc $z = \lambda(x)$ is *a Newton-Puiseux root of* f *(at the origin).*

Theorem 10.30. *$z = \lambda(x)$ is* a Newton-Puiseux root of f *if and only if* $\mathbb{P}(f, \lambda)$ *contains no Newton dots on the x-axis.*

If the highest edge of $\mathbb{P}(f, \lambda)$ ends on the line $z = m$, that means in particular that there are no Newton dots left to this line, then we say that $z = \lambda(x)$ is *a Newton-Puiseux root of* f *of multiplicity* m.

Exercise 10.31. Find a Newton-Puiseux root of $z^5 - z^2x^5 - z^3x^3 + x^8$.

10.6. **Sliding.** Fix $\eta \in \mathbb{Q}$ positive, and $c \in \mathbb{C}, c \neq 0$. Then $\mathbb{P}_1 = \mathbb{P}(f, \lambda + cx^\eta)$ is obtained from $\mathbb{P} = \mathbb{P}(f, \lambda)$ as follows. We denote by $\theta_1 < \ldots < \theta_s$ the co-slopes of $\mathbb{P}(f, \lambda)$.

Case $\eta < \theta_1$.
Then $\mathbb{P}_1$ contains a single edge of co-slope η.

Case $\theta_j < \eta < \theta_{j+1}$. Then the edges $\mathrm{E}_1, \ldots, \mathrm{E}_j$ and their associated forms of both Newton polygons are identical and $\mathbb{P}_1$ contains exactly one more edge. This extra edge is of co-slope η.

Case $\eta = \theta_j$. Then the edges $\mathrm{E}_1, \ldots, \mathrm{E}_{j-1}$ and their associated forms of both Newton polygons are identical.

If c is not a root of $\mathcal{E}_j(\xi)$ then $\mathbb{P}_1$ contains exactly one more edge, and this edge is of co-slope η. The associated polynomial of this edge is obtained from $\mathcal{E}_j(\xi)$ by a shift

$$\mathcal{E}_{\mathbb{P}_1, j}(\xi) = \mathcal{E}_j(\xi + c) \tag{10.24}$$

If c is a root of $\mathcal{E}_j(\xi)$ and $j < s$ then $\mathbb{P}_1$ contains an edge of co-slope η and (10.24) holds. Moreover, then $\mathbb{P}_1$ contains more edges, of co-slopes higher than η, provided $z = \lambda(x) + cx^\eta$ is not a multiple root of f, of multiplicity equal to the multiplicity of c as a root of $\mathcal{E}_j(\xi)$. A similar situation happens if $\eta = \theta_s$, with an extra possibility that c is the only root of $\mathcal{E}_s(\xi)$. Then $\mathbb{P}_1$ does not contain an edge of co-slope η and may or may not contain more edges of co-slopes higher than η.

Case $\theta_s < \eta$. Then $\mathbb{P}(f, \lambda + cx^\eta) = \mathbb{P}(f, \lambda)$ with the same associated forms.

Exercise 10.32. Perform the sliding $\mathbb{P}(f, \lambda) \to \mathbb{P}(f, \lambda + cx^\eta)$ in the following cases.

(1) $f(x,z) = z^2 - x^3 + x^4$, $\lambda = x^{3/2}$, $\eta = 5/2$, $c = 1$ and $c = -1/2$.
(2) $f(x,z) = (z^2 - x^3)^2 - zx^5$, $\lambda(x) = x^{3/2}$, $\eta = 7/4$, try $c = 1, 1/2$.

Example 10.33. Consider $f(x,z) = z+z^2-x^2$. Draw the Newton polygon of f and try to find the Newton Puiseux root of f using sliding. This leads to an infinite procedure producing $z = \lambda(x) = x^2 - x^4 + \sum_{i\geq 3} a_{2i}x^{2i}$. But, $\partial f/\partial z(0,0) \neq 0$, and therefore, by the Implicit Function Theorem (IFT), z can be solved as a function of x, i.e. there is a convergent power series (analytic function germ) $\lambda : (\mathbb{C},0) \to (\mathbb{C},0)$ such that $f(x,\lambda(x)) \equiv 0$, and there are no other solutions of $f(x,z) = 0$ near the origin.

In general, suppose that the highest edge E_H of $\mathbb{P}(f,\lambda)$ has vertices $(0,k_0),(1,k)$, $k<k_0$ and let $\theta = \theta_{H-1}$ denote the co-slope of the preceding edge E_{H-1}. Let us apply to $F(X,Y)$ the change of variables

$$X = \tilde{X}^M, Z = \tilde{Z}\tilde{X}^{M\theta},$$

where M will be specified later, yielding

$$\tilde{F}(\tilde{X},\tilde{Z}) := F(\tilde{X}^M, \tilde{Z}\tilde{X}^{M\theta},) := \tilde{X}^{M(k+\theta)}(c_{1,k}Z + \sum_{i+j\geq 2} \tilde{c}_{ij}\tilde{Z}^i\tilde{X}^j).$$

Define $G(\tilde{X},\tilde{Z}) = c_{1,k}\tilde{Z} + \sum_{i+j\geq 2} \tilde{c}_{ij}\tilde{Z}^i\tilde{X}^j)$ and choose M so that all exponents of the series are integers. Then, by the IFT, $G(\tilde{X},\tilde{Z}) = 0$ admits near the origin a unique analytic solution $\tilde{Z} = \beta(\tilde{X})$. Thus

$$z = \lambda(x) + x^\theta\beta(x^{1/M}))$$

is a Newton-Puiseux root of $f(x,z)$.

10.7. **Finding Newton-Puiseux roots of $f(x,z)$.** We suppose that $f(x,0) \not\equiv 0$. If this not the case we divide f by an appropriate power of x. Denote by m the lowest exponent of $f(0,z)$ with nonzero coefficient. We may suppose $m>0$.

Consider the Newton polygon $\mathbb{P}(f,0)$. The polynomial associated to the first edge, $\mathcal{E}_1(\xi)$, is of degree m. Take any root c_1 of $\mathcal{E}_1(\xi)$ and consider $\mathbb{P}(f,\lambda_1)$, where $\lambda_1(x) = c_1x^{\theta_1}$. If $c_1 = 0$ then $\mathbb{P}(f,\lambda_1) = \mathbb{P}(f,0)$ otherwise it is obtained from $\mathbb{P}(f,0)$ by sliding.

Then consider the polynomial associated to the second edge, $\mathcal{E}_2(\xi)$, of $\mathbb{P}(f,\lambda_1)$. Its degree equals to the multiplicity of c_1 as a root of $\mathcal{E}_1(\xi)$, say m_1. Take any root c_2 of $\mathcal{E}_2(\xi)$ and consider $\mathbb{P}(f,\lambda_2)$, where $\lambda_2(x) = \lambda_1(x) + c_2x^{\theta_2}$.

This way we construct recursively an arc $\lambda_i(x) = c_1x^{\theta_1}+c_2x^{\theta_2}+\cdots+c_ix^{\theta_i}$ and the associated Newton polygon $\mathbb{P}(f,\lambda_i)$. The algorithm depends on the choices of roots $c_1,c_2,\ldots$. Note that at the i-th stage a different choice of c_i leads to a different Newton polygon $\mathbb{P}(f,\lambda_i)$, and hence the sequence of exponents $\theta_1,\theta_2,\ldots$ may be depend on the choices as well.

Suppose that after k steps the associated polynomial of highest edge is of degree 1. Then, instead of continuing the algorithm (possibly infinitely), we may terminate it by the IFT. This does not lead to the complete formula for a Newton-Puiseux root but means that there exists a unique Newton-Puiseux root of the form

$$\lambda(x) = \sum_{i=1}^{k} c_kx^{\theta_k} + H.O.T.,$$

where $H.O.T.$ stands for the higher order terms, that is a convergent power series in $y^{1/M}$, where M is the smallest common multiple of the denominators in θ_i's.

Theorem 10.34. *Suppose all Newton-Puiseux roots of $f(x,z)$ are simple (i.e. of multiplicity 1). Then after a finite number of steps, regardless the choices of the roots, the above algorithm terminates.*

Exercise 10.35. Search for the Newton Puiseux roots

(1) $f(x,z) = z^2 + x^3 - x^4$
(2) $f(x,z) = (z^2 + x^2)^2 - (z^2 - x^2)$
(3) $f(x,z) = z^2(z+1) - x^2$
(4) $f(x,z) = (z^2 - x^3)^2 - zx^5$

Exercise 10.36.

(1) Consider $f(x,z) = (z^2 - x^3)^2$. Compare the Newton polygons $\mathbb{P}(f,0)$ and $\mathbb{P}(\partial f/\partial z, 0)$. What are the Newton-Puiseux roots of $\partial f/\partial z$? What are that of f, and what are their multiplicities?
(2) For a polynomial $f(x,z)$ and arc $z = \lambda(x)$ compare the Newton polygons $\mathbb{P}(f,\lambda)$ and $\mathbb{P}(\partial f/\partial z, \lambda)$.

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