# Braid groups of surfaces and one application to a Borsuk–Ulam type theorem

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Toruń 2011

## Contents



## 1 Introduction

During initial lectures we present the full and pure Artin braid groups. We give presentations of these groups and study several of their properties. We compute their centers, define a special element called Garside and study its properties. For the pure braid groups, we show how to write them as iterated product of free groups. Then we move on to the study of the full and pure braid groups of surfaces. We give special attention to the case where the surface is the 2-sphere. We show that these groups are no longer torsion free and we study some aspects of their torsion. Presentations for these groups are obtained. For the case of the pure braid groups, the Fadell–Neuwirth sequence and fibration is introduced and several properties will be discussed; in particular, the splitting of the Fadell–Neuwirth sequence of pure braid groups. In the final lecture we review the Borsuk–Ulam theorem as well as some recent results of the same type as the Borsuk-Ulam theorem. We then consider a similar question for maps between surfaces.

## 2 Classification of compact surfaces

Recall the classification of compact surfaces.[1]

**Theorem 2.1.** *Any compact surface is one of the following:*

(1) *the 2-dimensional sphere;*

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[1] Throughout these notes, the term *surface* is taken to mean a connected 2-dimensional manifold, possibly with boundary. A surface is said to be *closed* if it is compact and without boundary.

(2) *the connected sum of some number of tori;*

(3) *the connected sum of some number of projective planes;*

(4) *any of the previous ones with a finite number of open disks removed.*

A modern reference for Theorem 2.1 is [10]; for a classical treatment, see [13].

**Notation.** In what follows, $S^n$ denotes the $n$-dimensional sphere, $T_1 = S^1 \times S^1$ the torus, and $T_g$ the connected sum of $g$ tori. We also write $N_g$ for the connected sum of $g$ projective planes. In particular, $N_1 = \mathbb{R}P^2$ is the projective plane and $N_2$ is well-known to be the Klein bottle.

# 3 Braid groups of the disk

To begin with, let us consider the simplest compact surface, namely the disk $D$.

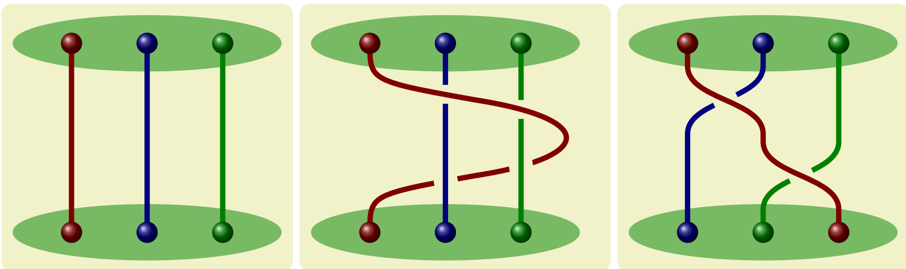
Take $n \geq 1$ and let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of any $n$ points from the interior of $D$.[2] A *geometric braid* in $D$ is a set $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of continuous paths in $D$ with the following properties:

(1) $\alpha_i(0), \alpha_i(1) \in S$ for any $i \in \{1, 2, \ldots, n\}$;

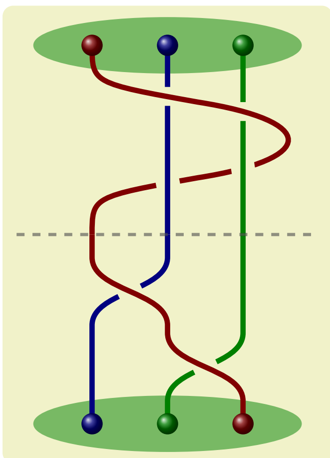
(2) given $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$, we have $\alpha_i(t) \neq \alpha_j(t)$ for any $t \in [0, 1]$.

The $\alpha_i$'s are called the *strings* of $\alpha$. Note that it follows easily from the definition that

$$\{\alpha_1(0), \alpha_2(0), \ldots, \alpha_n(0)\} = \{\alpha_1(1), \alpha_2(1), \ldots, \alpha_n(1)\} = S.$$



The first question that springs to mind is whether we can define a group structure on the set of all geometric braids (the set $S$ is understood to be fixed). The natural idea is to consider the operation of concatenation of braids.



[2]This is merely a technical assumption.

Formally, given braids $\alpha$ and $\beta$, we define $\alpha * \beta$ by

$$(\alpha * \beta)_i(t) = \begin{cases} \alpha_i(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta_{\alpha_i(1)}(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases} \quad \text{for } i \in \{1, 2, \ldots, n\}.$$

Unfortunately, the collection of geometric braids is not a group under this operation: among other flaws (e.g., the natural candidate for the neutral element, the constant braid $e$, $e_i \equiv x_i$, is not an idempotent), $(\alpha * \beta) * \gamma$ is almost never the same as $\alpha * (\beta * \gamma)$.

To avoid such unpleasant problems, we switch to homotopy. Two geometric braids are said to be *homotopic* if one can be deformed to the other by simultaneous homotopies of the strings that fix endpoints, so that different strings never intersect. The reader is encouraged to write down a formal definition.

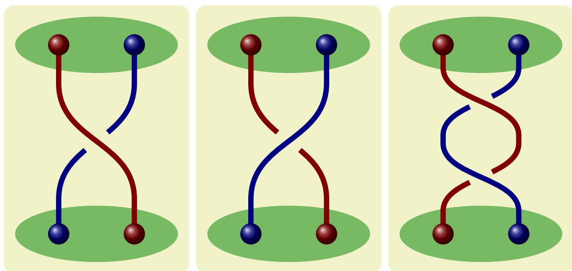
As one might expect, the relation of homotopy of braids is an equivalence relation and the operation of concatenation of geometric braids turns the quotient set into a group. This group is called the *Artin braid group of the disk* (or the *full Artin group*, or simply the *Artin group*) on $n$ strings. The usual notation is $B_n(D)$.

The proof that $B_n(D)$ is indeed a group is not difficult, although technical. We confine ourselves to stating that the neutral element of $B_n(D)$ is $[e]$, and the inverse of $[\alpha] \in B_n(D)$ is $[\alpha^{-1}]$, where $\alpha_i^{-1}(t) = \alpha_i(1-t)$ for any $i \in \{1, 2, \ldots, n\}$, $t \in [0, 1]$.

Given a group, we would like to know how it looks like. What are the generators? Perhaps a particularly nice presentation is known?

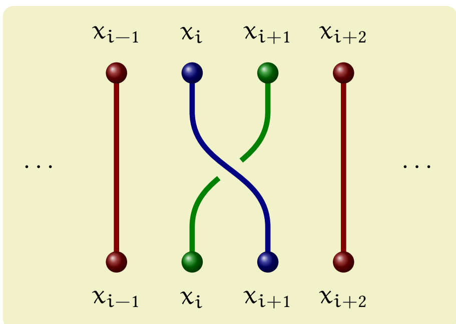
**Example 3.1.** (1) Clearly, $B_1(D)$ is the fundamental group of the disk, so $B_1(D)$ is trivial.

(2) Let us consider $B_2(D)$.



Denote the left-hand side braid by $\sigma_1$.[3] Playing around with pictures a little bit hints that $B_2(D) = \mathbb{Z}[\sigma_1]$, which is indeed the case. For example, the middle braid is $\sigma_1^{-1}$, and the right-hand side one is $\sigma_1^2$.

Fix an integer $n$. For any $i \in \{1, 2, \ldots, n-1\}$, define an element $\sigma_i \in B_n(D)$ as indicated by the picture: let all the strings apart from the $i$-th and $(i+1)$-th be constant.




[3]We usually will not distinguish between a braid and its equivalence class.

**Theorem 3.2.** *For any $n \geq 1$, the Artin group on $n$ strings has the following presentation:*

$$\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1, 2, \dots, n-2 \rangle .$$

Verifying that these relations actually hold in any $B_n(D)$ is not difficult. The hard part is to show that they form a complete set of relations. This is not obvious at all: it took Artin 22 years to provide a proof.[4]

**Remark 3.3.** (1) Recall the usual presentation of the symmetric group on $n$ letters $\mathcal{S}_n$:

$$\left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \sigma_i^2 = 1, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \right\rangle .$$

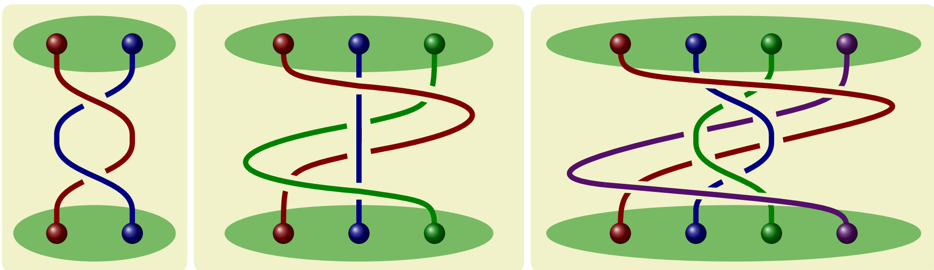
It is hard to overlook the similarities between the two presentations. From this point of view, $B_n(D)$ can be thought of as a generalization of $\mathcal{S}_n$.

(2) It is well-known that $\mathcal{S}_n$ is generated by two elements. It turns out that this is also the case for $B_n(D)$: the reader will verify that $\sigma_1\sigma_2 \cdots \sigma_{n-1}$ and $\sigma_{n-1}$ generate $B_n(D)$.

## 4 The full twist and the center of $B_n(D)$

We will now briefly investigate the center of $B_n(D)$. To begin with, we need some intuition about what kind of elements belong in there.

Let us introduce a special element of $B_n(D)$, the so called *full twist*. We will define it geometrically first; the pictures below represent full twists in $B_2$, $B_3$ and $B_4$, respectively.



(Think of pieces of string attached to the tips of your fingers and rotate one hand by 180°; this is the *half-twist*, also also called the *Garside element*. Rotating the hand once more gives the full twist.)

Algebraically, the full twist in $B_n(D)$ is given by $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$. It turns out that the center of $B_n(D)$ is generated by the full twist. We will only prove:

**Proposition 4.1.** *The full twist belongs to the center of $B_n(D)$.*

*Proof.* We confine ourselves to an algebraic proof. The reader is recommended to carry out a geometric proof.

Note that if $j < n-1$, then

$$\begin{aligned}(\sigma_1\sigma_2 \cdots \sigma_{n-1})\sigma_j &= \sigma_1\sigma_2 \cdots \sigma_{j-1}\underline{\sigma_j\sigma_{j+1}\sigma_j}\sigma_{j+2}\sigma_{j+3} \cdots \sigma_{n-1} \\ &= \sigma_1\sigma_2 \cdots \sigma_{j-1}\sigma_{j+1}\sigma_j\sigma_{j+1}\sigma_{j+2} \cdots \sigma_{n-1} \\ &= \sigma_{j+1}(\sigma_1\sigma_2 \cdots \sigma_{n-1}).\end{aligned}$$

Similarly, $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^2\sigma_{n-1} = \sigma_1(\sigma_1\sigma_2 \cdots \sigma_{n-1})^2$. Combining these two observations together, we see that

$$\begin{aligned}(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n\sigma_j &= (\sigma_1\sigma_2 \cdots \sigma_{n-1})^{j+1}\sigma_{n-1}(\sigma_1\sigma_2 \cdots \sigma_{n-1})^{n-j-1} \\ &= (\sigma_1\sigma_2 \cdots \sigma_{n-1})^{j-1}\sigma_1(\sigma_1\sigma_2 \cdots \sigma_{n-1})^{n-j+1} \\ &= \sigma_j(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n.\end{aligned}$$

□

[4] Artin claimed in a 1925 paper that this is indeed the case, but the proof was published only in 1947.

## 5 Pure braid groups

Given $\alpha \in B_n(D)$, we have $\alpha_i(1) = x_{\sigma(i)}$ for some permutation $\sigma \in \mathcal{S}_n$ and any $i \in \{1, 2, \ldots, n\}$. This allows us to define a homomorphism[5] $\pi\colon B_n(D) \to \mathcal{S}_n$ by setting $\pi\big([\alpha]\big) = \sigma$.

**Proposition 5.1.** *$\pi$ is surjective.*

*Proof.* Observe that $\pi(\sigma_i) = (i, i{+}1)$ for any $i \in \{1, 2, \ldots, n{-}1\}$, and the collection of consecutive transpositions is well-known to generate the symmetric group $\mathcal{S}_n$ (see Remark 3.3). □

The kernel of $\pi$ is called the *pure braid group* on $n$ strings, and is usually denoted by $P_n(D)$. Note that this is a very natural object: it corresponds to braids $\alpha \in B_n(D)$ which satisfy $\alpha_i(0) = \alpha_i(1)$ for any $i \in \{1,\ 2,\ \ldots,\ n\}$.

We have thus arrived at a short exact sequence

$$1 \to P_n(D) \to B_n(D) \to \mathcal{S}_n \to 1.$$

The bottom line is that in order to understand $B_n(D)$, we need to gain some knowledge of $P_n(D)$. In general, the pure braid groups have much nicer properties than the full braid groups. A particularly nice thing is that there exists an epimorphism $P_n(D) \to P_{n-1}(D)$, defined by forgetting the last string.

**Remark 5.2.** Note that all these results do not depend on the choice of $D$ as a surface.

## 6 Configuration spaces and the Fadell–Neuwirth fibration

Our goal now is to develop some techniques to study the short exact sequence which arises from the epimorphism $P_n(M) \to P_{n-1}(M)$. It turns out that the best way to understand it is not exactly using algebra, but rather some algebraic topology, in particular homotopy theory.

Given a space $M$ and an integer $n \geq 1$, the *n-th configuration space* of $M$ is

$$F_n(M) = \big\{(x_1, \ldots, x_n) \in \underbrace{M \times M \times \cdots \times M}_{n \text{ times}} \mid x_i \neq x_j \text{ for any } i,\, j \in \{1, 2, \ldots, n\},\, i \neq j\big\}.$$

In particular, if $n = 2$, then $F_2(M) = M \times M \setminus \Delta$, where $\Delta = \big\{(x, x) \in M \times M \mid x \in M\big\}$ is the diagonal of $M \times M$.

We will now see that braids are intimately connected with configuration spaces.

Let $M$ be a connected manifold without boundary. Since a path in $F_n(M)$ gives rise to $n$ paths in $M$, loops in $F_n(M)$ with a fixed basepoint are in one-to-one correspondence with pure geometric braids in $M$. With a little technical work, this correspondence can be seen to define an isomorphism

$$P_n(M) \cong \pi_1\big(F_n(M)\big).$$

How about the full braid groups? Again algebraic topology comes in handy. Observe that the symmetric group $\mathcal{S}_n$ acts naturally on $F_n(M)$; moreover, this action is free. Consider the *unordered configuration space* $D_n(M) = F_n(M)/\mathcal{S}_n$. As usual, $F_n(M) \to D_n(M)$ is a covering. From the theory of covering spaces, we have the short exact sequence

$$1 \to \pi_1\big(F_n(M)\big) \to \pi_1\big(D_n(M)\big) \to \mathcal{S}_n \to 1.$$

This turns out to be the same (up to equivalence) as the short exact sequence

$$1 \to P_n(M) \to B_n(M) \to \mathcal{S}_n \to 1$$

derived in Section 5. Our immediate goal is to show that this new approach is in fact useful.

[5] Veryfing that $\pi$ is indeed a well-defined homomorphism is not difficult, albeit somewhat technical

Recall that a continuous map $p\colon E \to B$ is called a *locally trivial fibration* with fiber $F$ if for every point $b \in B$ there exists and open neighbourhood $U$ of $b$ such that $p^{-1}(U)$ is homeomorphic to $F \times U$. One prominent feature of a fibration (not necessarily a locally trivial one) is that there exists the associated long exact sequence of homotopy groups

$$\cdots \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \to \cdots \to \pi_0(B).$$

**Theorem 6.1.** *Let $M$ be a manifold without boundary. For any $1 \leq r \leq n$, the projection $p\colon F_n(M) \to F_r(M)$ defined by forgetting the last $n-r$ coordinates is a locally trivial fibration.*

The proof is not simple at all: it has a substantial idea, and is very technical. An interested reader should consult [9].

**Theorem 6.2.** *If $S$ is a surface other the 2-dimensional sphere or the projective plane, then $F_n(S)$ and $D_n(S)$ are $K(\pi, 1)$'s.*

*Proof.* Since $F_n(S)$ is a covering space of $D_n(S)$, it suffices to carry out the proof for $F_n(S)$.

Proceed inductively. For $n = 1$, $F_1(S) = S$ and the conlusion follows from our knowledge of surfaces. Suppose the conclusion is true for $n > 1$. To see that it is also true for $n+1$, use the Fadell–Neuwirth fibration $S \setminus \{x_1, x_2, \ldots, x_n\} \to F_{n+1}(S) \to F_n(S)$ to obtain the long exact sequence of homotopy groups

$$\cdots \to \pi_k\big(S \setminus \{x_1, \ldots, x_n\}\big) \to \pi_k\big(F_{n+1}(S)\big) \to \pi_k\big(F_n(S)\big) \to \pi_{k-1}\big(S \setminus \{x_1, \ldots, x_n\}\big) \to \cdots$$

It is now obvious that the higher homotopy groups of $F_{n+1}(S)$ vanish. □

We now want to profit from the fact that $F_n(M)$ and $D_n(M)$ are aspherical. Recall that the cohomology of a group $G$ is the same as the cohomology of $K(G, 1)$.

**Corollary 6.3.** $H^i\big(P_n(S), A\big) = H^i\big(B_n(S), A\big) = 0$ *for $i > 2n$ and any coefficients $A$.*

*Proof.* This follows immediately from the fact that $\dim F_n(S) = 2n$. □

How to use this to obtain information about properties of the group? A well-known theorem from the theory of group cohomology says that if a group $G$ has torsion, then it has infinite cohomological dimension.[6]

**Corollary 6.4.** *If $S$ is a surface other than the 2-dimensional sphere or the projective plane, then $P_n(S)$ and $B_n(S)$ are torsion free.*

**Remark 6.5.** It is not difficult to show that $P_n(S)$ is torsion free using the Fadell–Neuwirth fibration.

Proceed inductively. For $n = 1$, $P_1(S) = \pi_1(S)$ is well-known to be torsion free. Assume the result for $n > 1$. To see that it also holds for $n+1$, consider the short exact sequence

$$1 \to P_1\big(S \setminus \{x_1, \ldots, x_{n-1}\}\big) \to P_{n+1}(S) \to P_n(S) \to 1$$

which arises from the appropriate Fadell-Neuwirth fibration. Since $P_1\big(S \setminus \{x_1, \ldots, x_{n-1}\}\big)$ is torsion free (as the fundamental group of a punctured surface) and $P_n(S)$ is torsion free by induction, it follows that this is also the case for $P_n(S)$.

**Proposition 6.6.** *If $S$ is a surface other than the 2-dimensional sphere, the projective plane, or the disk, then the centers of $P_n(S)$ and $B_n(S)$ are trivial.*

*Proof.* In the case of $P_n(S)$, proceed similarly as above.

How to proceed in the case of $B_n(S)$? Let $x \in \mathcal{Z}\big(B_n(S)\big)$. Since $P_n(S)$ has finite index in $B_n(S)$, $x^k \in P_n(S)$ for a suitable $k$; in fact, $x^k \in \mathcal{Z}\big(P_n(S)\big)$, hence $x^k = 1$. But we know that $B_n(S)$ is torsion free, hence $x = 1$. □

[6] See [2, Chapter VIII].

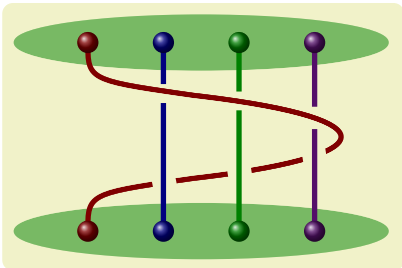
## 7 Braid groups of the 2-dimensional sphere

We will now consider the braid groups of the 2-dimensional sphere. The first thing that we would like to know is the presentation of $B_n(S^2)$. Furthermore, we want study torsion of the group $B_n(S^2)$. We will profit from our knowledge of $B_n(D)$.

Let us look at some specific braids on the sphere. Consider the inclusion $D \hookrightarrow S^2$. Let $S = \{x_1, \ldots, x_n\} \subseteq D \subseteq S^2$; for geometrically convenient reasons, assume that all the $x_i$'s lie in the equator. This inclusion provides braids on $S^2$. In particular, we have $\sigma_1, \ldots, \sigma_{n-1} \in B_n(S^2)$ by abuse of notation.

The inclusion $i\colon D \hookrightarrow S^2$ induces a homomorphism $i_*\colon B_n(D) \to B_n(S^2)$. It turns out to be surjective: given a geometric braid $\alpha$ there is another braid $\alpha'$ isotopic to the first one such that the north pole does not belong to the image of $\alpha'_i$ for any $i$. Consequently, $[\alpha'] \in B_n(D)$. This shows that $B_n(S^2)$ is a quotient of $B_n(D)$.

It is not clear, however, whether $i_*$ is not is an isomorphism. Consider the following braid (here pictured for $n = 4$, but it should be obvious how to proceed for higher values of $n$).



The reader will convice himself that this braid in the disk is not trivial, unlike in $S^2$. Consequently, $i_*$ cannot possibly be an injection. In other words, this element should be a part of relations of a potential presentation of $B_n(S^2)$. Algebraically, this particular braid is encoded as $\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1$. And in fact, we have

**Theorem 7.1** ([4, Section 4])**.** *For any $n \geq 1$, a presentation of $B_n(S^2)$ is given by:*

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1 \rangle .$$

**Theorem 7.2** ([4, Theorem 2])**.** *$P_3(S^2) = \mathbb{Z}_2$. Consequently, $B_3(S^2)$ is finite. Furthermore, $B_n(S^2)$ has torsion for any $n$.*

Of course, $B_1(S^2) = P_1(S^2) = \pi_1(S^2)$ is trivial. To compute $P_2(S^2)$, consider $\pi_1\big(F_2(S^2)\big) = \pi_1(S^2 \times S^2 \setminus \Delta)$. We have the projection $S^2 \setminus \{x_1\} \to S^2 \times S^2 \setminus \Delta \to S^2$. Applying the long exact sequence of homotopy groups gives $P_2(S^2) = 1$. Using the short exact sequence with $\mathcal{S}_2$ as the quotient and $P_2(S^2)$ as the kernal, we see that $B_2(S^2) = \mathbb{Z}_2$.

**Remark 7.3.** We will now briefly describe what is known about torsion in $B_n(S^2)$.

- Murasugi [12] classified all finite cyclic subgroups of $B_n(S^2)$.
- Thompson [14] proved that the quaternion group $Q_8$ lives inside $B_4(S^2)$. This result was later extended by Gonçalves–Guashi [7], who proved that $Q_8 \subseteq B_n(S^2)$ if and only if $n \geq 4$ is even.
- Finally, Gonçalves–Guashi provided a full classification of all possible finite subgroups of $B_n(S^2)$.

## 8 Borsuk–Ulam type theorems

Recall the celebrated Borsuk–Ulam[7] theorem:

**Theorem 8.1** ([1, Sätz II]). *If $f\colon S^n \to \mathbb{R}^n$ is a continuous map, then there exists $x \in S^n$ such that $f(x) = f(-x)$.*

*Proof.* Let $f\colon S^n \to \mathbb{R}^n$ be a continuous map. Suppose that $f(x) \neq f(-x)$ for any $x \in S^n$. Define a map $g\colon S^n \to S^{n-1}$ by setting

$$g(x) = \frac{f(x) - f(-x)}{||f(x) - f(-x)||},\ x \in S^n.$$

Observe that $g(-x) = -g(x)$ for any $x \in S^n$, i.e., $g$ is $\mathbb{Z}_2$-equivariant with respect to antipodal actions on both spheres. But such a map cannot possibly exist, for it would give rise to a map $\tilde{g}\colon \mathbb{R}P^n \to \mathbb{R}P^{n-1}$, which in turn induces a homomorphism $\tilde{g}^*\colon H^*(\mathbb{R}P^{n-1};\mathbb{Z}_2) \to H^*(\mathbb{R}P^n;\mathbb{Z}_2)$. Recall that $H^*(\mathbb{R}P^{n-1};\mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^n)$ and $H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[y]/(y^{n+1})$. Computation shows that $\tilde{g}^*(x) = y$. By naturality of the cup product, $x^n \mapsto y^n \neq 0$. A contradiction. □

The Borsuk–Ulam theorem can be stated in terms of involutions. Recall that a continuous map $\tau\colon X \to X$ is called a *free involution* if $\tau(x) \neq x$ for any $x \in X$ and $\tau \circ \tau = \mathrm{id}$. In the case of the classical Borsuk–Ulam theorem, $\tau$ is taken to be the antipodal map, but the proof presented above makes it clear that the Borsuk–Ulam theorem holds for any free involution of $S^n$.

Let $M$ be a manifold, $\tau$ a free involution of $M$. We say that $(M, \tau, \mathbb{R}^n)$ satisfies the *Borsuk–Ulam property* if for any continuous map $f\colon M \to \mathbb{R}^n$ there exists $x \in M$ such that $f(x) = f\big(\tau(x)\big)$.

Two general remarks: if $n > \dim M$, the Borsuk–Ulam property never holds. On the other hand, if $n = 1$, it always holds. Thus the problem is reduced to the interval $2 \leq n \leq \dim M$.

In the case of surfaces, this boils down to only one question: does the triple $(S, \tau, \mathbb{R}^2)$ satisfy the Borsuk–Ulam property? This problem was adressed in [6] for closed surfaces. For example, if $S$ is orientable and its Euler characteristic is congruent to $2 \bmod 4$, then $(S, \tau, \mathbb{R}^2)$ satisfies the Borsuk–Ulam property for any free involution $\tau$. On the other hand, there exists a pair of free involutions $(\tau_1, \tau_2)$ of $N_3$ such that the Borsuk–Ulam property holds for $\tau_1$ and does not hold for $\tau_2$.

What can be said in a more general situation when the target space is not necessarily $\mathbb{R}^n$, but another manifold, or even a CW complex $Y$? The obvious new layer of complication is that we do not know how to subtract in $Y$. To remedy this, define a map $\varphi\colon M \to Y \times Y$ by $\varphi(x) = \big(f(x), f\big(\tau(x)\big)\big)$ for any $x \in M$. Now things start to fit together really, really well.

The Borsuk–Ulam property does not hold for $f$ if and only if $\varphi$ factors through $F_2(Y)$, the second configuration space of $Y$. Furthermore, $\varphi$ is $\mathbb{Z}_2$-equivariant with respect to $\tau$ on $M$ and an involution on $F_2(Y)$ given by $(x, y) \mapsto (y, x)$. Passing to the map of orbit spaces yields $\tilde{\varphi}\colon M/\tau \to D_2(Y)$. This in turn gives rise to the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & F_2(Y) \\ \big\downarrow & & \big\downarrow \\ M/\tau & \longrightarrow & D_2(Y) \end{array}$$

[7]An inquisitive reader will undoubtedly find that there is no joint Borsuk–Ulam paper in the literature. Ulam, however, proposed this problem.

in which the vertical maps are double coverings. Applying the fundamental group functor[8] gives

$$\begin{array}{ccc} \pi_1(M) & \longrightarrow & P_2(Y) \\ \downarrow & & \downarrow \\ \pi_1(M/\tau) & \longrightarrow & B_2(Y) \\ & \searrow \quad \swarrow & \\ & \mathbb{Z}_2 & \end{array}$$

The new triagle is also commutative.

Let us see how much of this procedure can be reversed. First of all, observe that given an equivariant map $\varphi\colon M \to F_2(Y)$, we obtain $f$ as the first coordinate of $\varphi$. With some minor algebraic assumptions on $\pi_1$'s (we need to be able to lift maps) we can also produce $\varphi$ given $\tilde{\varphi}$. To construct a geometric diagram out of the algebraic one, we need a stronger assumption: namely, the spaces in question need to be $K(\pi, 1)$'s.

**Proposition 8.2** ([8, Proposition 13])**.** *Let $M$, $Y$ be closed surfaces, $\tau$ a free involution of $M$. The triple $(M, \tau, Y)$ satisfies the Borsuk–Ulam property if and only there exists a homomorphism $\pi_1(M/\tau) \to B_2(Y)$ which makes the following diagram commute*

$$\begin{array}{ccc} \pi_1(M/\tau) & & B_2(Y) \\ & \searrow \quad \swarrow & \\ & \mathbb{Z}_2 & \end{array}$$

Let's look at a very simple example of this situation.

**Example 8.3.** Let $M = S^2$, $Y = \mathbb{R}^2$. Then $B_2(Y) = \mathbb{Z}$ and $\pi_1(S^2/\mathbb{Z}_2) = \mathbb{Z}_2$. And it is impossible to complete the algebraic triangle.

**Theorem 8.4** ([8, Corollary 1, Theorem 2])**.** *Let $S$ be a closed surface.*

(1) *The triple $(S, \tau, S^2)$ does not satisfy the Borsuk–Ulam property for any free involution $\tau$.*

(2) *The triple $(S, \tau, \mathbb{R}P^2)$ satisfies the Borsuk–Ulam property if and only if $S$ is the 2-sphere.*

The remaining cases are when the target space is different from the 2-dimensional sphere or the projective plane. This problem can be divided into four subcases, according to whether the surfaces in question are orientable or nonorientable. The most difficult one is when $S_1/\tau$ is nonorientable and $S_2$ is orientable. In particular, the case when $S_1/\tau$ is the Klein bottle is especially dauting. It is equivalent to considering the following equation in $B_2(S_2)$: $v'[a', c'] = 1$.

**Remark 8.5.** One afterthought, in view of all these results: perhaps the 'correct' formulation of the Borsuk–Ulam question is the following. Given a triple $(M, \tau, Y)$, which homotopy classes of maps $M \to Y$ satisfy the Borsuk–Ulam property? Observe that if $Y$ is conctractible, this is exactly the same as the original Borsuk–Ulam question.

[8]Of course, the reader is welcome to experiment with other functors!

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