BROWNIAN MOTION
A tutorial

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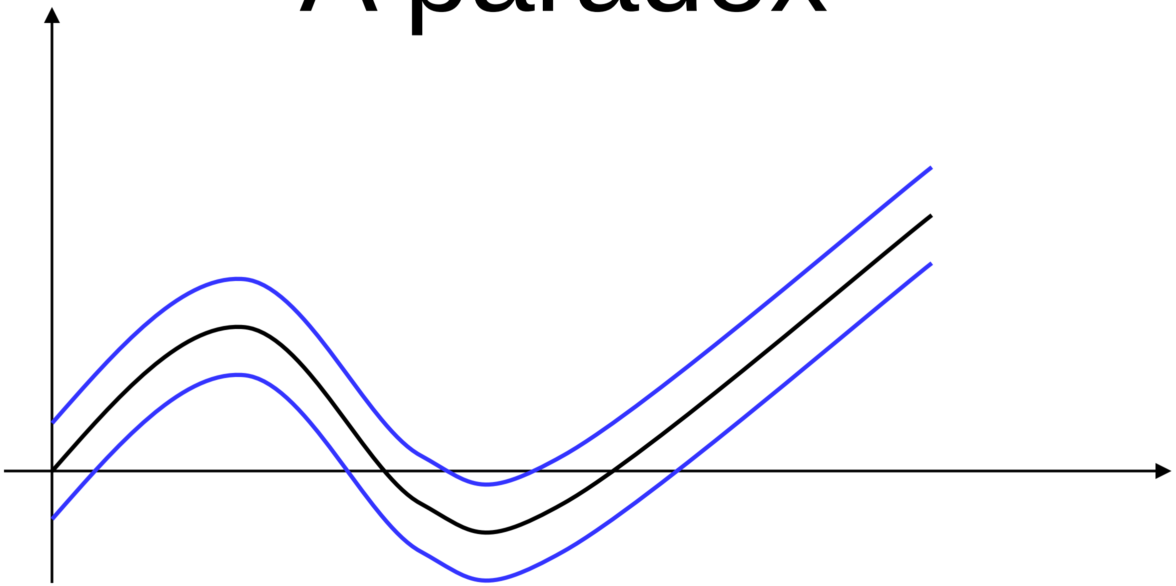
A paradox



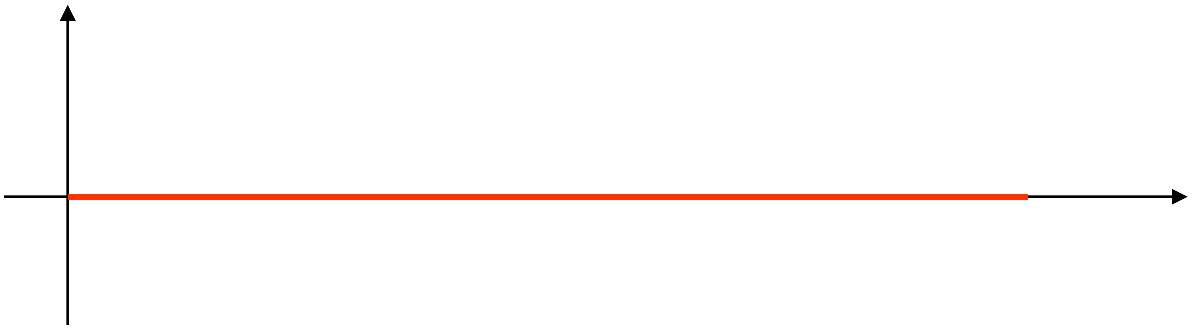
$$f:[0,1]\to R, \quad \sup_{t\in[0,1]} |f''(t)| < \infty$$

$$P(f(t)-\varepsilon < B_t < f(t)+\varepsilon, 0<t<1)$$

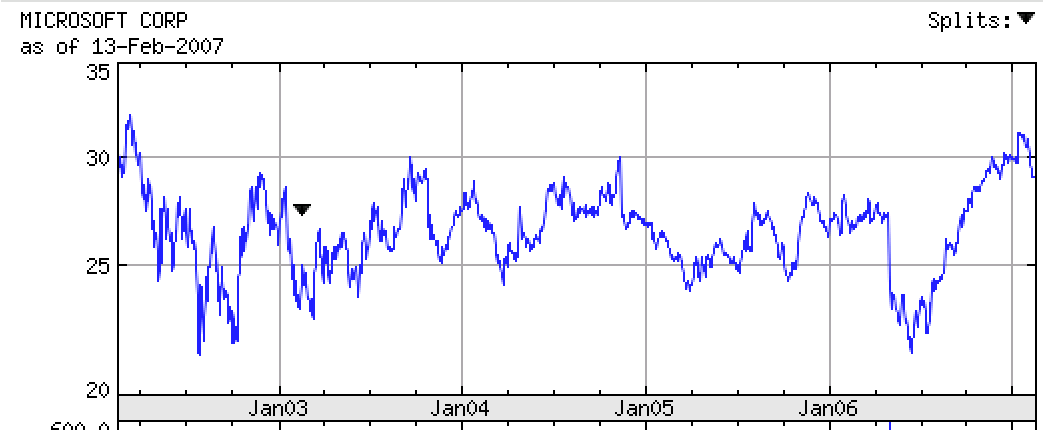
$$\approx c(\varepsilon)\exp\left(-\frac{1}{2}\int_0^1 (f'(t))^2\,dt\right) \qquad (*)$$

(*) is maximized by f(t) = 0, t>0

The most likely (?!?) shape of a Brownian path:



Microsoft stock



Definition of Brownian motion

Brownian motion is the unique process with the following properties:

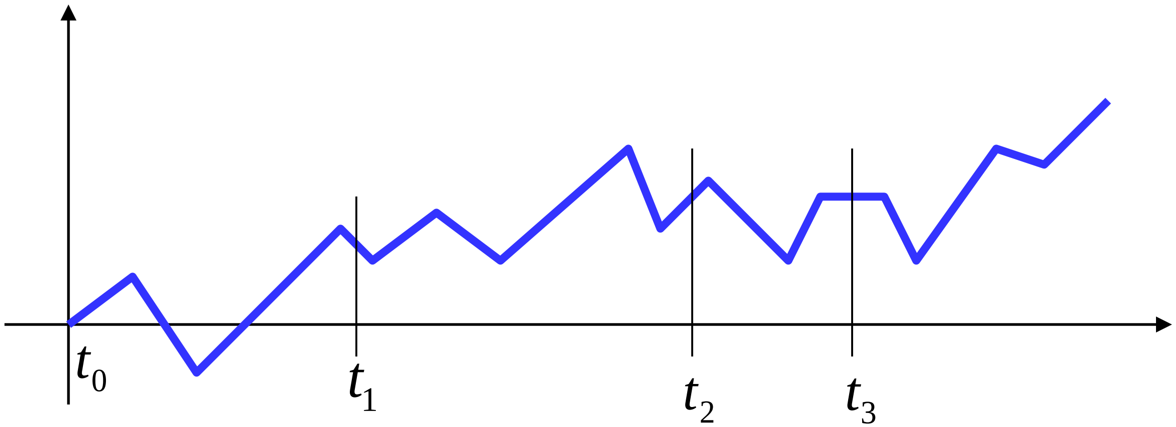
(i) No memory

(ii) Invariance

(iii) Continuity

(iv) $B_0 = 0, \quad E(B_t) = 0, \quad Var(B_t) = t$

Memoryless process



$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots$$

are independent

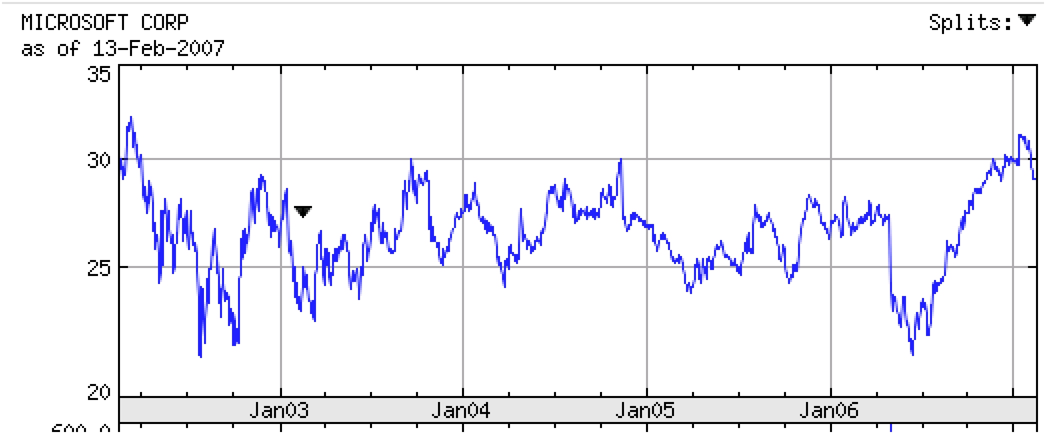
Invariance

The distribution of $B_{t+s} - B_s$ depends only on t.

Path regularity

(i) $t \to B_t$ is continuous a.s.

(ii) $t \to B_t$ is nowhere differentiable a.s.



Why Brownian motion?

Brownian motion belongs to several families of well understood stochastic processes:

(i) Markov processes

(ii) Martingales

(iii) Gaussian processes

(iv) Levy processes

Markov processes

$$\mathcal{L}\{B_t, t \geq s \mid B_s\} \quad = \quad \mathcal{L}\{B_t, t \geq s \mid B_u, 0 \leq u \leq s\}$$

The theory of Markov processes uses
tools from several branches of analysis:
(i) Functional analysis (transition semigroups)
(ii) Potential theory (harmonic, Green functions)
(iii) Spectral theory (eigenfunction expansion)
(iv) PDE’s (heat equation)

Martingales

$$s < t \Rightarrow E(B_t \mid B_s) = B_s$$

Martingales are the only family of processes for which the theory of stochastic integrals is fully developed, successful and satisfactory.

$$\int_0^t X_s \, dB_s$$

Gaussian processes

$B_{t_1}, B_{t_2}, \ldots, B_{t_n}$ is multidimensional normal (Gaussian)

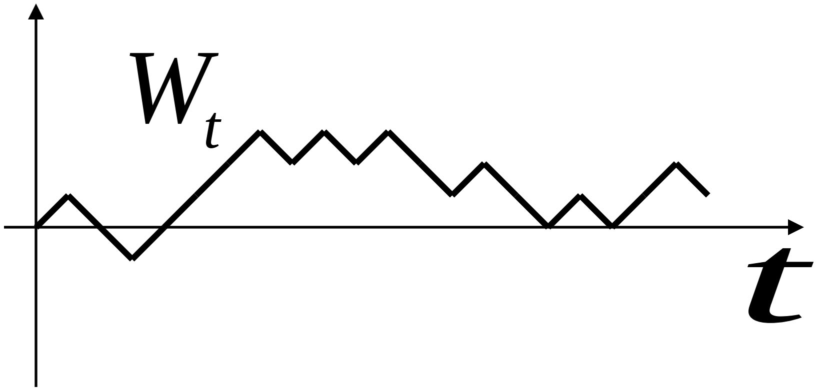
(i) Excellent bounds for tails
(ii) Second moment calculations
(iii) Extensions to unordered parameter(s)

The Ito formula

$$\int_0^t X_s \, dB_s = \lim_{n\to\infty} \sum_{k=0}^{nt} X_{k/n} (B_{(k+1)/n} - B_{k/n})$$

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds$$

Random walk



Independent steps, P(up)=P(down)

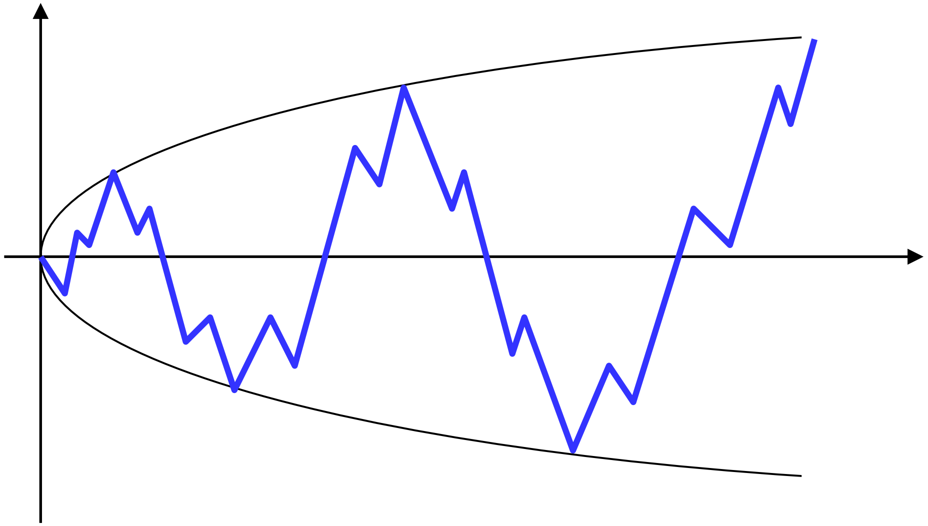
$$\left\{\sqrt{a}W_{t/a}, t \geq 0\right\} \xrightarrow{a\to\infty} \left\{B_t, t \geq 0\right\}$$

(in distribution)

Scaling

Central Limit Theorem (CLT), parabolic PDE's

$$\{B_t, 0 \le t \le 1\} \overset{D}{=} \{\sqrt{a} B_{t/a}, 0 \le t \le 1\}$$



Cameron-Martin-Girsanov formula

Multiply the probability of each Brownian path $\{B_t, 0 \le t \le 1\}$ by

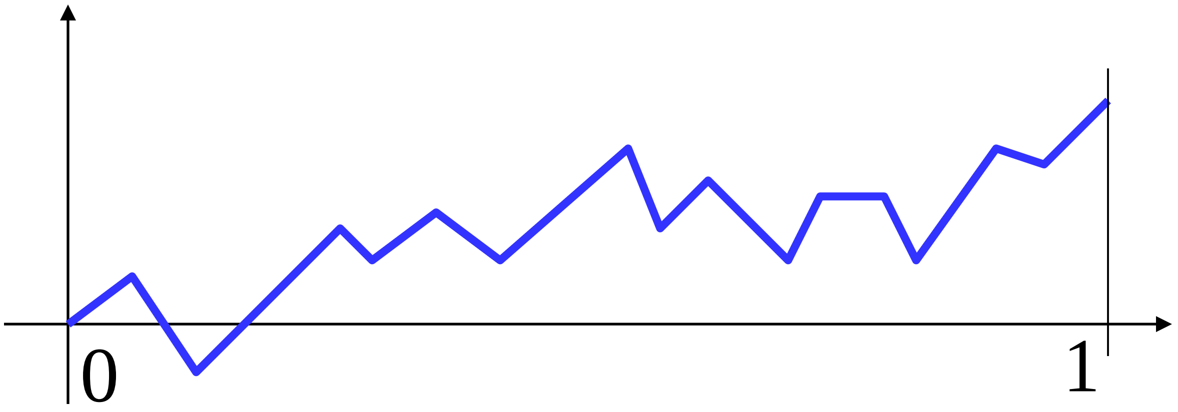
$$\exp\left(\int_0^1 f'(s)\,dB_s - \frac{1}{2}\int_0^1 (f'(s))^2\,ds \right)$$

The effect is the same as replacing $\{B_t, 0 \le t \le 1\}$ with $\{B_t + f(t), 0 \le t \le 1\}$

Invariance (2)

Time reversal

$$\{B_t, 0 \le t \le 1\} \overset{D}{=} \{B_{1-t} - B_1, 0 \le t \le 1\}$$



Brownian motion and the heat equation

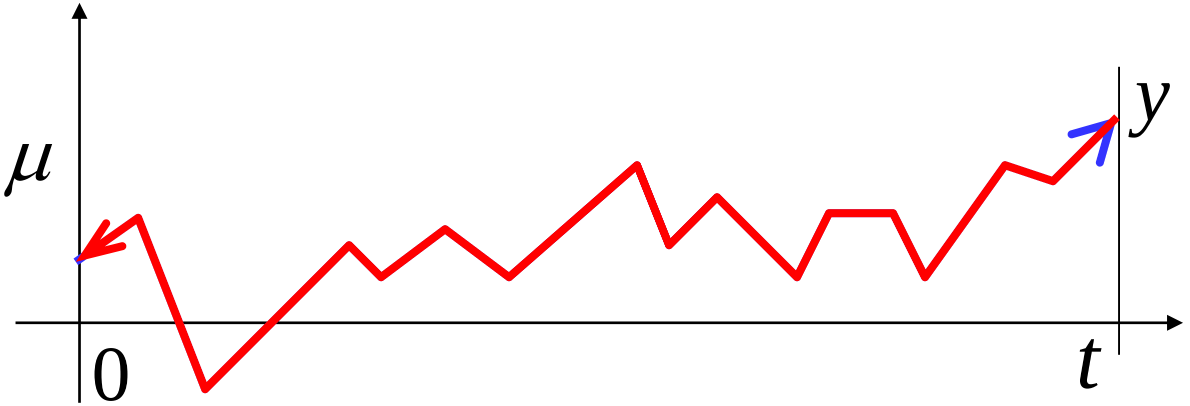
$u(x,t)$ – temperature at location x at time t

Heat equation: $$\frac{\partial}{\partial t}u(x,t)=\frac{1}{2}\Delta_x u(x,t)$$

$$\mu(dx)=u(x,0)dx$$

Forward representation $$u(y,t)dy=P^{\mu}(B_t\in dy)$$

Backward representation (Feynman-Kac formula) $$u(y,t)=Eu(B_t+y,0)$$

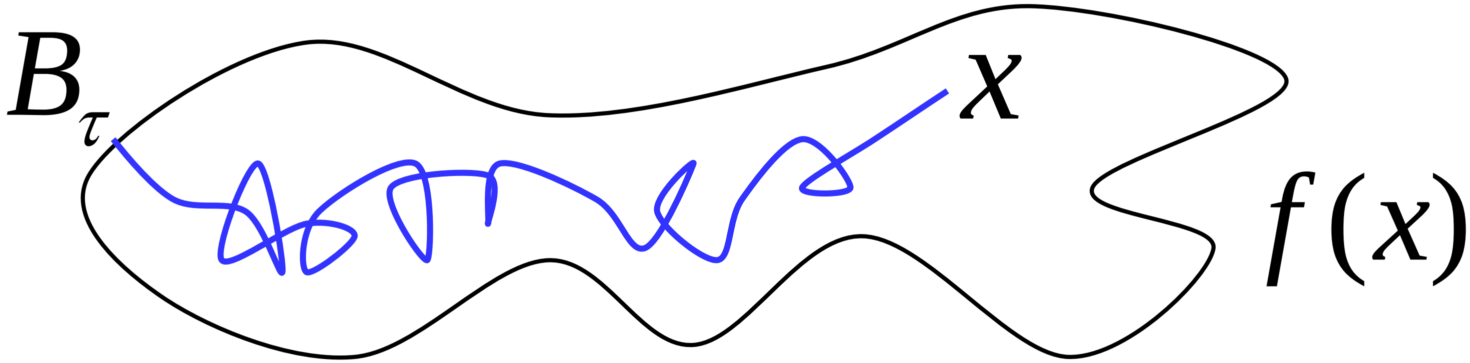


Multidimensional Brownian motion

$B_t^1, B_t^2, B_t^3, \ldots$ - independent 1-dimensional Brownian motions

$(B_t^1, B_t^2, \ldots, B_t^d)$ - d-dimensional Brownian motion

Feynman-Kac formula (2)



$$\frac{1}{2}\Delta u(x) - V(x)u(x) = 0$$

$$u(x) = E^x\left(f(B_\tau)\exp\left[-\int_0^\tau V(B_s)\,ds \right] \right)$$

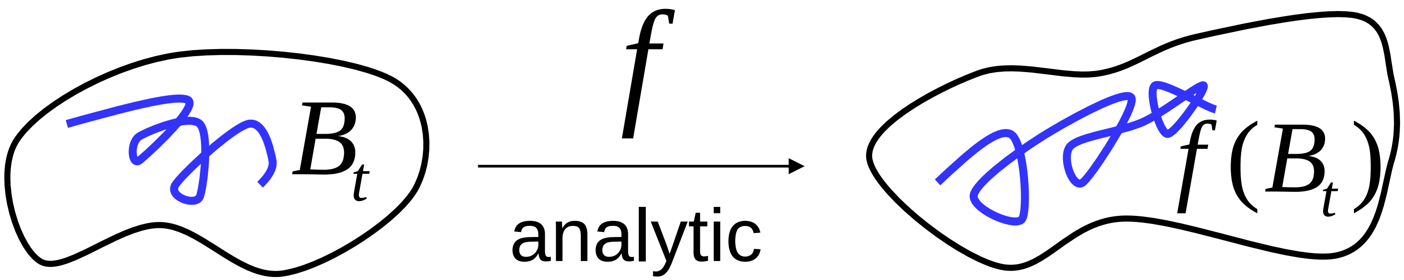
Invariance (3)

The d-dimensional Brownian motion is invariant under isometries of the d-dimensional space. It also inherits invariance properties of the 1-dimensional Brownian motion.

$$\frac{1}{\sqrt{2\pi}}\exp(-x_1^2/2)\frac{1}{\sqrt{2\pi}}\exp(-x_2^2/2)$$

$$=\frac{1}{2\pi}\exp(-(x_1^2+x_2^2)/2)$$

Conformal invariance



$$\{ f(B_t) - f(B_0), t \geq 0\}$$

has the same distribution as

$$\{B_{c^{-1}(t)}, t \geq 0\}, \quad c(t) = \int_0^t | f'(B_s)|^2 \, ds$$

The Ito formula
Disappearing terms (1)

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s)\,dB_s + \frac{1}{2}\int_0^t \Delta f(B_s)\,ds$$

If $\Delta f \equiv 0$ then

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_s)\,dB_s$$

Brownian martingales

Theorem (Martingale representation theorem).
{Brownian martingales} = {stochastic integrals}

$$M_t = \int_0^t X_s dB_s$$

$$E(M_t \mid F_s^B) = M_s, \quad M_t \in F_t^B = \sigma\{B_s, s \leq t\}$$

The Ito formula
Disappearing terms (2)

$$f(t,B_t)-f(t,B_0)=\int_0^t \frac{\partial}{\partial x} f(s,B_s)dB_s$$

$$+\int_0^t \frac{\partial}{\partial s} f(s,B_s)ds+\frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f(s,B_s)ds$$

$$Ef(t,B_t)-Ef(t,B_0)$$

$$=E\int_0^t \frac{\partial}{\partial s} f(s,B_s)ds+\frac{1}{2}E\int_0^t \frac{\partial^2}{\partial x^2} f(s,B_s)ds$$

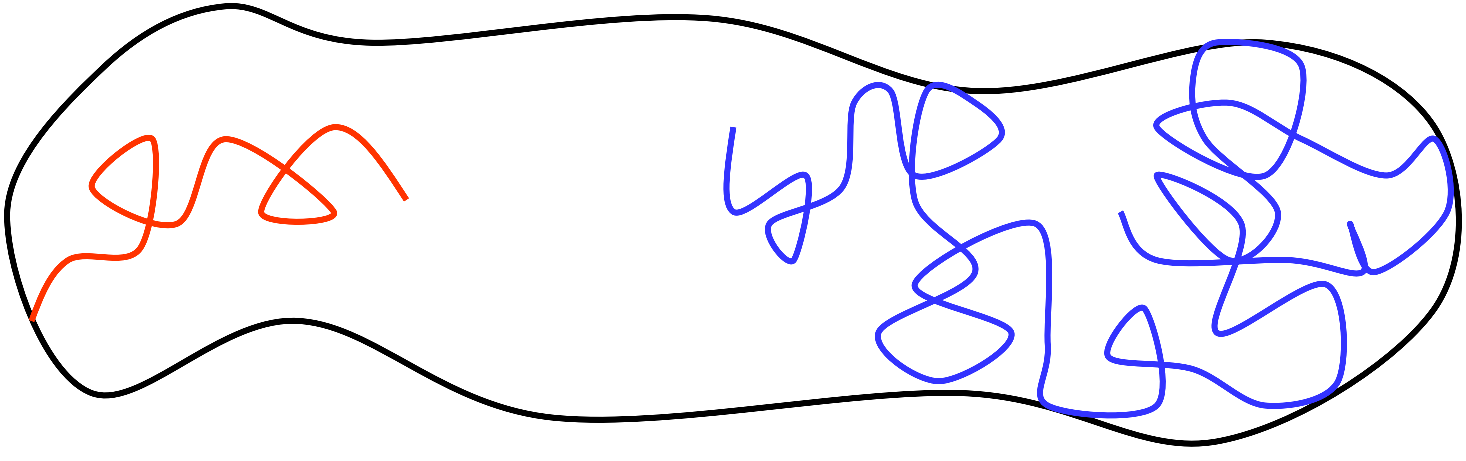
Mild modifications of BM

Mild = the new process corresponds to the Laplacian

(i) Killing – Dirichlet problem

(ii) Reflection – Neumann problem

(iii) Absorption – Robin problem



Related models – diffusions

$$dX_t = \sigma(X_t)dB_t + \mu(X_t)dt$$

(i) Markov property – yes
(ii) Martingale – only if $\mu \equiv 0$
(iii) Gaussian – no, but Gaussian tails

Related models – stable processes

Brownian motion – $dB = (dt)^{1/2}$

Stable processes – $dX = (dt)^{1/\alpha}$

(i) Markov property – yes

(ii) Martingale – yes and no

(iii) Gaussian – no

Price to pay: jumps, heavy tails, $0 < \alpha \leq 2$

$$0 < 2 \leq 2$$

Related models – fractional BM

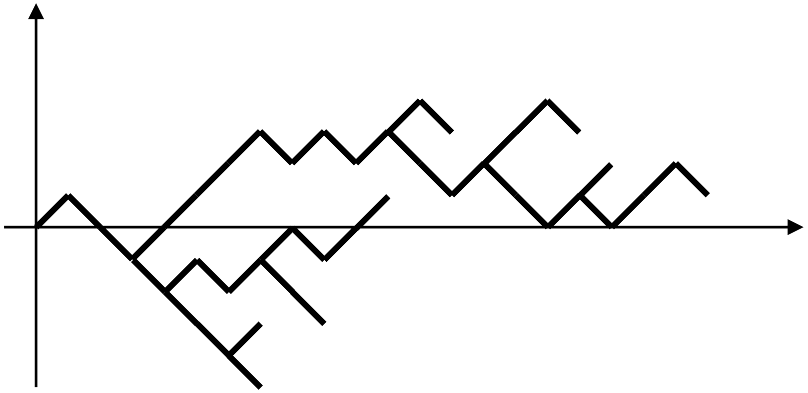
$$dX = (dt)^{1/\alpha}$$

(i) Markov property – no
(ii) Martingale – no
(iii) Gaussian – yes
(iv) Continuous

$$1 < \alpha < \infty$$

$$1 < 2 < \infty$$

Related models – super BM



Super Brownian motion is related to

$$\Delta u = u^2$$

and to a stochastic PDE.

Related models – SLE

Schramm-Loewner Evolution is a model for non-self-intersecting conformally invariant 2-dimensional paths.

Path properties

(i) $t \to B_t$ is continuous a.s.

(ii) $t \to B_t$ is nowhere differentiable a.s.

(iii) $t \to B_t$ is Holder $(1/2-\varepsilon)$

(iv) Local Law if Iterated Logarithm

$$\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{2t \log|\log t|}} = 1$$

Exceptional points

$$\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{2t \log |\log t|}} = 1$$

For any fixed s>0, a.s.,

$$\limsup_{t \downarrow s} \frac{B_t - B_s}{\sqrt{2(t-s) \log |\log(t-s)|}} = 1$$

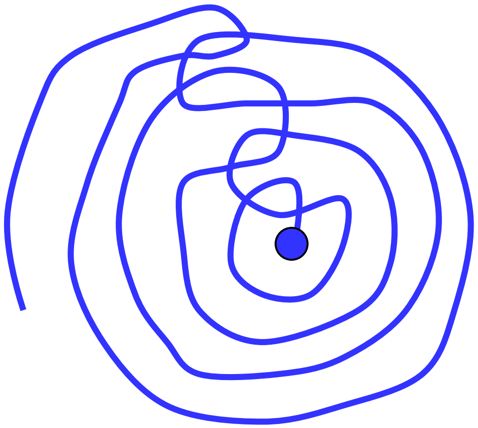
There exist s>0, a.s., such that

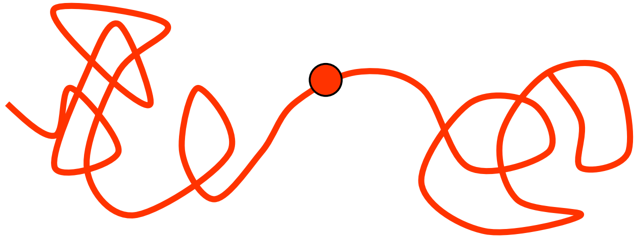
$$\limsup_{t \downarrow s} \frac{B_t - B_s}{\sqrt{2(t-s)}} \in (0, \infty)$$

Cut points

For any fixed t>0, a.s., the 2-dimensional Brownian path contains a closed loop around B_t in every interval $(t, t+\varepsilon)$

Almost surely, there exist $t \in (0,1)$ such that $B([0,t)) \cap B((t,1]) = \emptyset$





Intersection properties

$$(d=1) \quad a.s., \forall t\ \exists s \neq t \quad B_s = B_t$$

$$(d=2) \quad \forall t\ a.s. \forall s \neq t \quad B_s \neq B_t$$

$$a.s., \exists x \in R^2\ Card(B^{-1}(x)) = \infty$$

$$(d=3) \quad a.s., \exists x \in R^3\ Card(B^{-1}(x)) = 2$$

$$a.s., \forall x \in R^3\ Card(B^{-1}(x)) \leq 2$$

$$(d=4) \quad a.s., \forall x \in R^4\ Card(B^{-1}(x)) \leq 1$$

Intersections of random sets

$$\dim(A) + \dim(B) > d$$

$$\Updownarrow$$

$$A \cap B \neq \emptyset$$

The dimension of Brownian trace is 2 in every dimension.

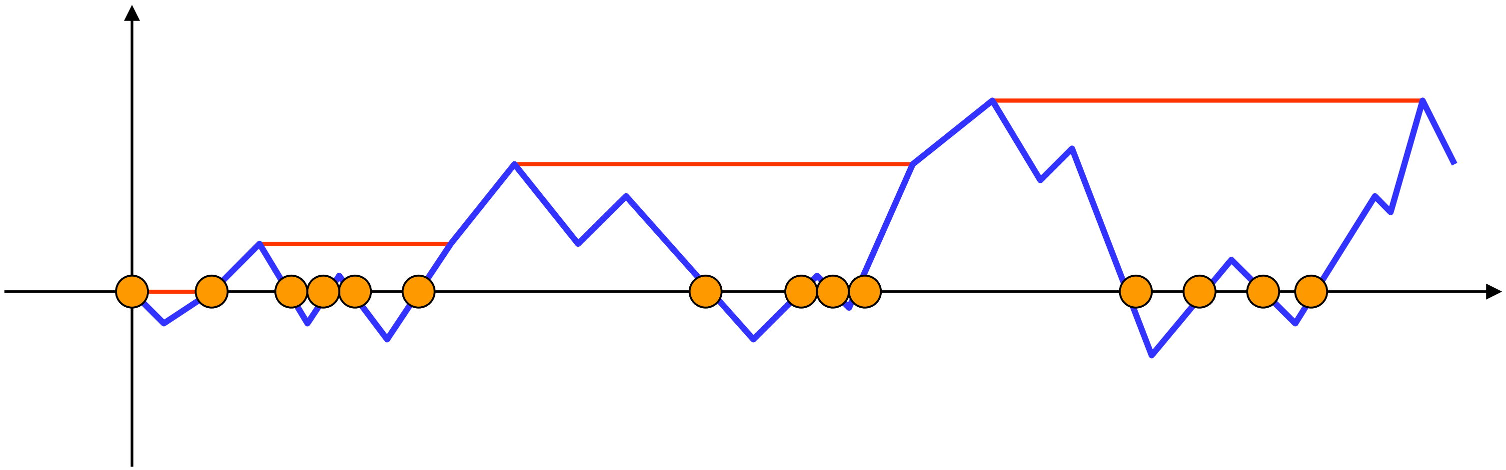
Invariance principle

(i) Random walk converges to Brownian motion (Donsker (1951))
(ii) Reflected random walk converges to reflected Brownian motion (Stroock and Varadhan (1971) - C^2 domains, B and Chen (2013) – all domains)
(iii) Self-avoiding random walk in 2 dimensions converges to SLE (20??) (open problem)

Local time

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{1}_{\{-\varepsilon < B_s < \varepsilon\}} ds$$

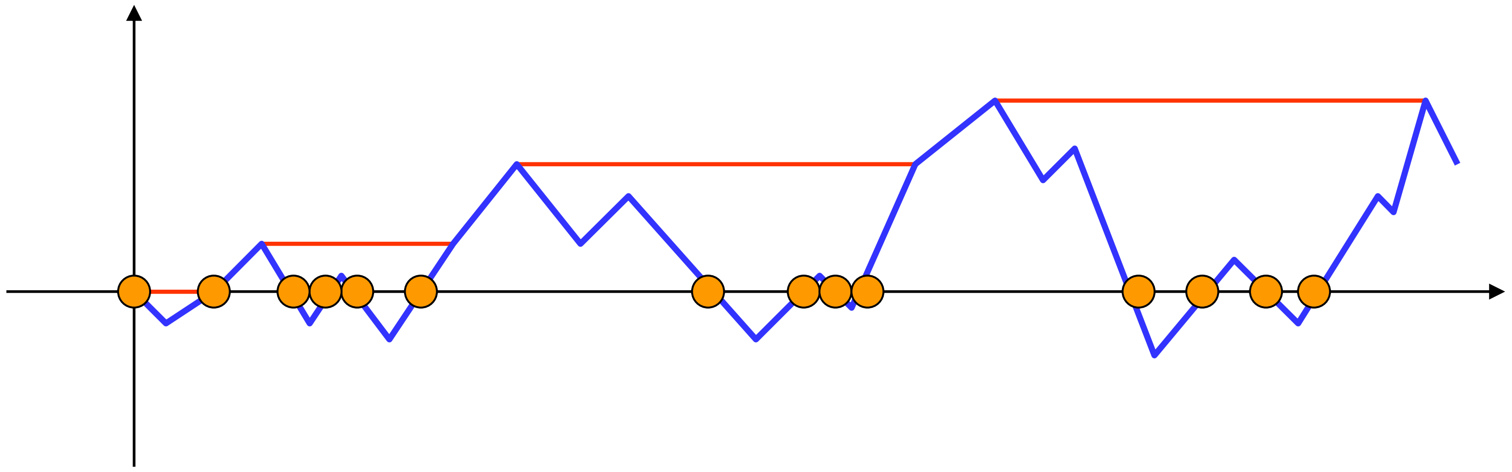
$$M_t = \sup_{s \le t} B_s$$



Local time (2)

$$\{L_t, 0 \le t \le 1\} \overset{D}{=} \{M_t, 0 \le t \le 1\}$$

$$\{M_t - B_t, 0 \le t \le 1\} \overset{D}{=} \{|B|_t, 0 \le t \le 1\}$$



Local time (3)

$$\sigma_t = \inf_{s>0}\{L_s \geq t\}$$

Inverse local time is a stable process with index ½.

References

- R. Bass *Probabilistic Techniques in Analysis*, Springer, 1995
- F. Knight *Essentials of Brownian Motion and Diffusion*, AMS, 1981
- I. Karatzas and S. Shreve *Brownian Motion and Stochastic Calculus*, Springer, 1988

Domains with moving boundaries.
The heat equation and reflected Brownian motion.

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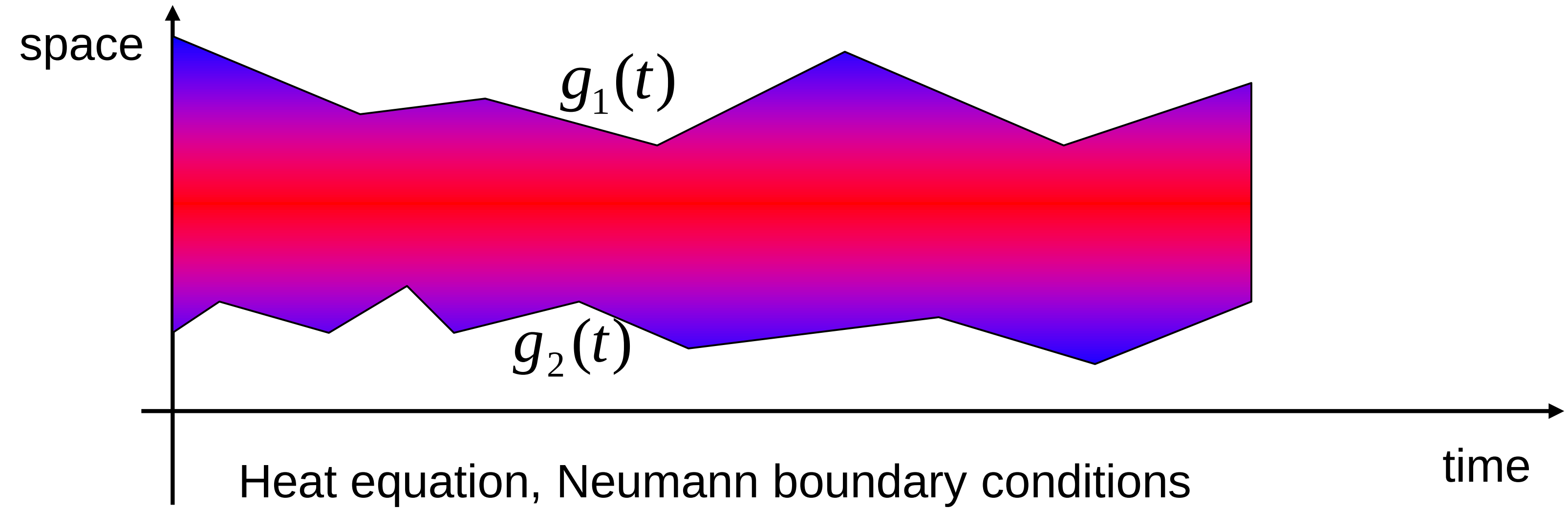
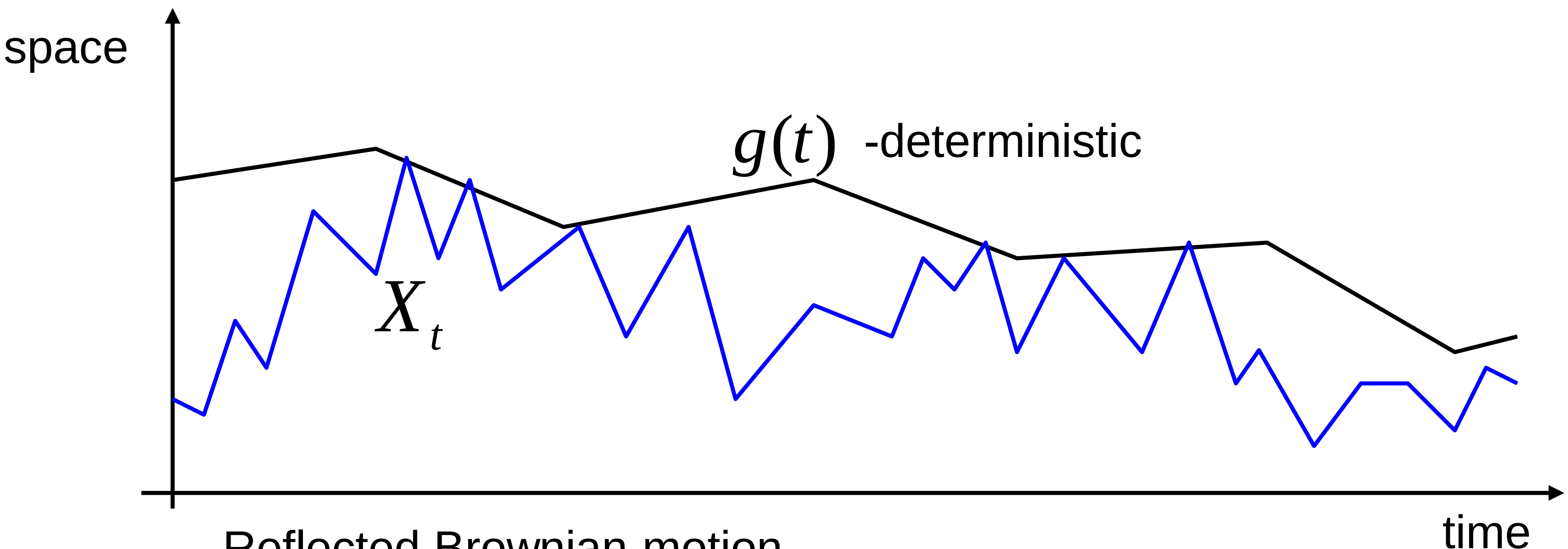
Time dependent domains



Heat equation, Neumann boundary conditions



Reflected Brownian motion

Reflected Brownian motion in time dependent domains

- **Cranston and Le Jan (1989)**
- **Knight (2001)**
- **Soucaliuc, Toth and Werner (2000)**
- **Zheng (1996)**
- **Bass and B (1999)**
- **Lewis and Murray (1995)**
- **Hofmann and Lewis (1996)**
- **Lepeltier and San Martin (2004)**
- **B, Chen and Sylvester (2003, 2004, 2004)**
- **B and Nualart (2002)**

Heat equation

$u(t,x)$ - temperature at time t at point x

$$\begin{cases} \frac{1}{2}\Delta_x u(t,x) = u_t(t,x), \quad x < g(t),\ t > 0, \\ \int_{-\infty}^{g(t)} u(t,x)dx = 1, \quad t \geq 0, \\ u(0,x) = u_0(x). \end{cases} \qquad (1)$$

$$\begin{cases} \frac{1}{2}\Delta_x u(t,x) = u_t(t,x), \quad x < g(t),\ t > 0, \\ u_x(t,x) = -g'(t)u(t,x), \quad x = g(t), \\ u(0,x) = u_0(x). \end{cases} \qquad (2)$$

Heat equation solutions – existence and uniqueness

Theorem. If $g(t)$ is C^3 then solutions to (1) and (2) exist, are unique and equal to each other.

$$\begin{cases} \frac{1}{2}\Delta_x u(t,x) = u_t(t,x), \quad x < g(t),\ t > 0, \\ \int_{-\infty}^{g(t)} u(t,x)dx = 1, \quad t \geq 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

$$\begin{cases} \frac{1}{2}\Delta_x u(t,x) = u_t(t,x), \quad x < g(t),\ t > 0, \\ u_x(t,x) = -g'(t)u(t,x), \quad x = g(t), \\ u(0,x) = u_0(x). \end{cases} \tag{2}$$

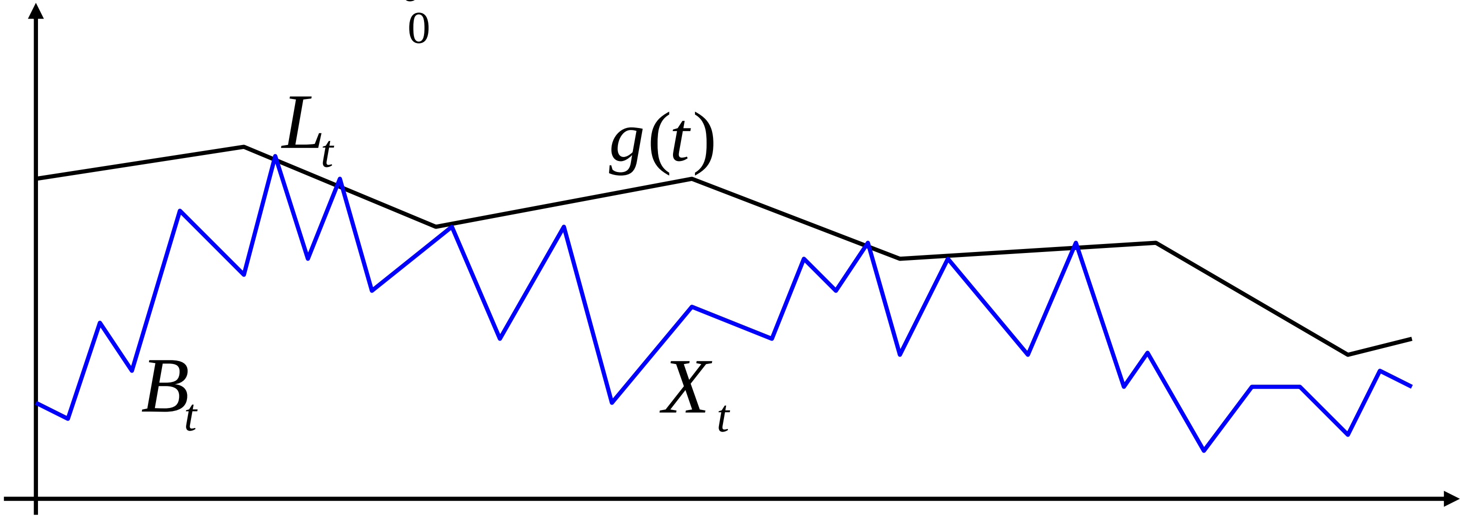
Lewis and Murray (1995), Hofmann and Lewis (1996)

Skorohod Lemma

$g(t), B_t$ - continuous functions

Lemma. There exists a unique continuous non-decreasing function L_t such that $X_t = B_t - L_t \le g(t)$ for every t and L_t does not increase when $X_t < g(t)$, i.e.,

$$\int_0^\infty 1_{(-\infty, g(t))}(X_s)\, dL_s = 0.$$



Heat equation solution via reflected Brownian motion

$g(t)$ - continuous function

B_t - Brownian motion

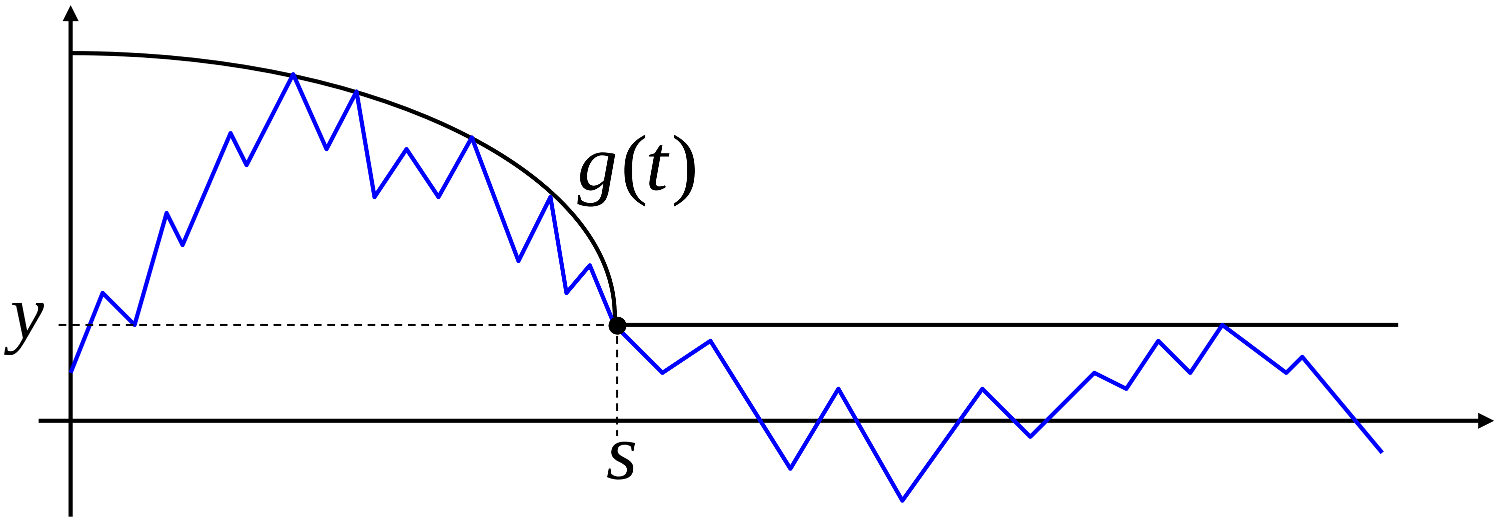
$X_t = B_t - L_t$

Theorem. The function $u(t,x)dx = P(X_t \in dx)$ solves (1).

$$\begin{cases} \frac{1}{2}\Delta_x u(t,x) = u_t(t,x), \quad x < g(t),\ t > 0, \\ \int\limits_{(-\infty, g(t)]} u(t,x)dx = 1, \quad t \geq 0, \\ u(0,x) = u_0(x). \end{cases} \qquad (1)$$

Lewis and Murray (1995), Hofmann and Lewis (1996)

Heat atoms

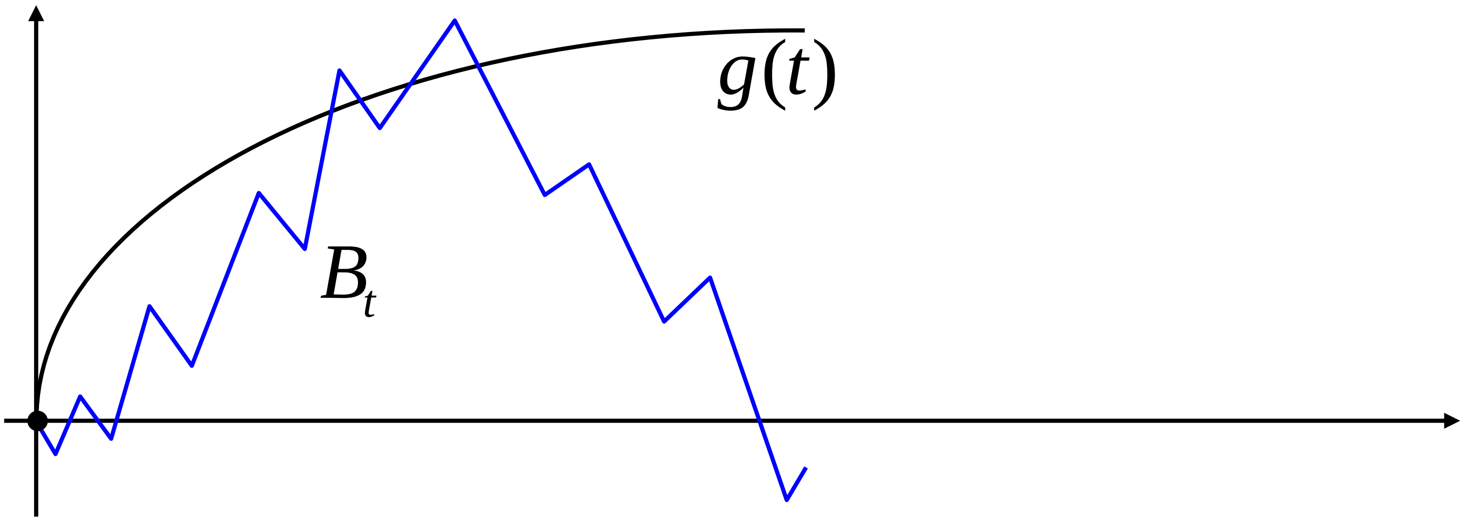


$$\int_{(-\infty,y)} u(s,x)\,dx < 1$$

$$P(X_s = y) > 0$$

Theorem. Heat atoms exist for some $g(t)$.

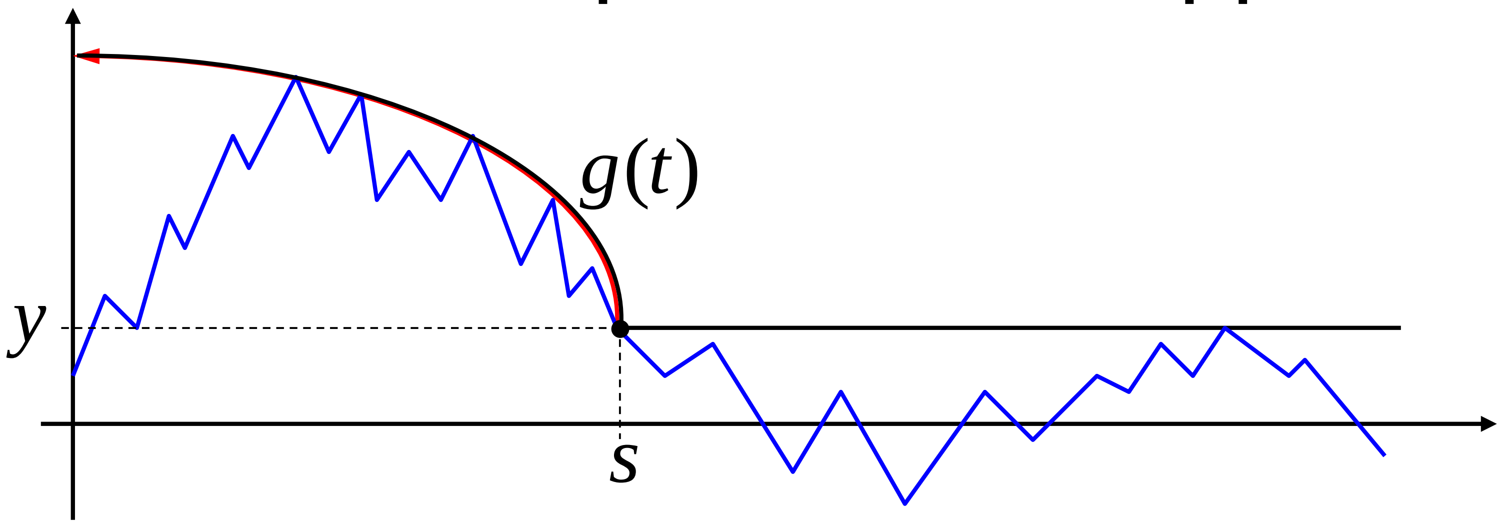
Upper functions for Brownian motion



$$P(\inf\{t>0 : B_t = g(t)\} = 0) = 0$$

B_t - Brownian motion

Heat atoms – probabilistic approach



Theorem. $g(s)$ is a heat atom if an only if $f(t) = g(s-t) - g(s)$ is an upper function.

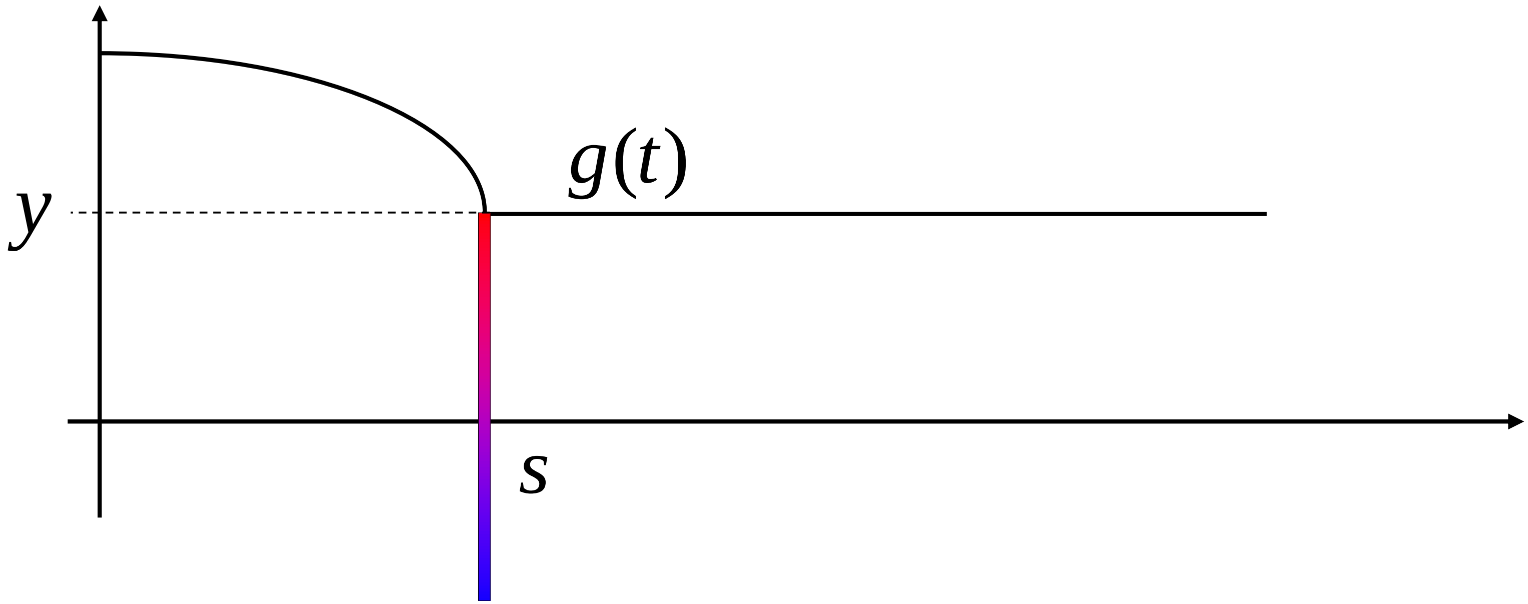
Kolmogorov's criterion: $f(t)$ is upper class if and only if

$$\int_0^1 t^{-3/2} f(t) \exp(-f^2(t)/(2t))\, dt < \infty$$

Example (LIL): $f(t) = (1+\varepsilon)\sqrt{2t \log|\log t|}$

$f(t)$ is upper class if and only if $\varepsilon > 0$

Singularities



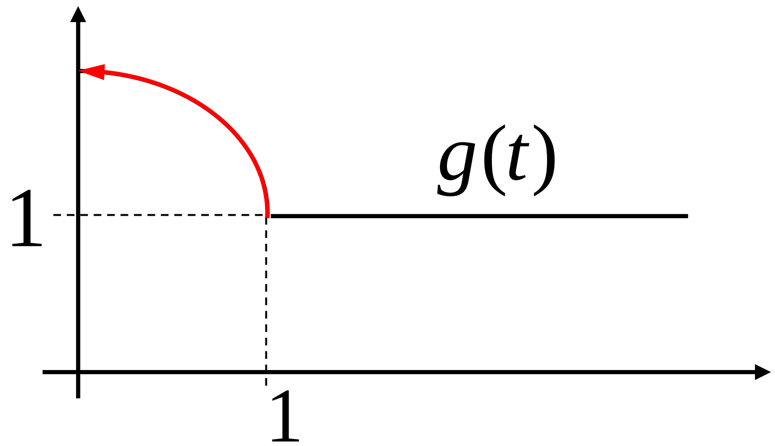
$$\limsup_{x \uparrow y} u(s,x) = \infty$$

Heat atoms and singularities

Theorem: There exist g_1, g_2, g_3, g_4 such that

	Singularity	Heat atom
g_1	No	No
g_2	Yes	No
g_3	Yes	Yes
g_4	No	Yes

Heat atoms and singularities - examples



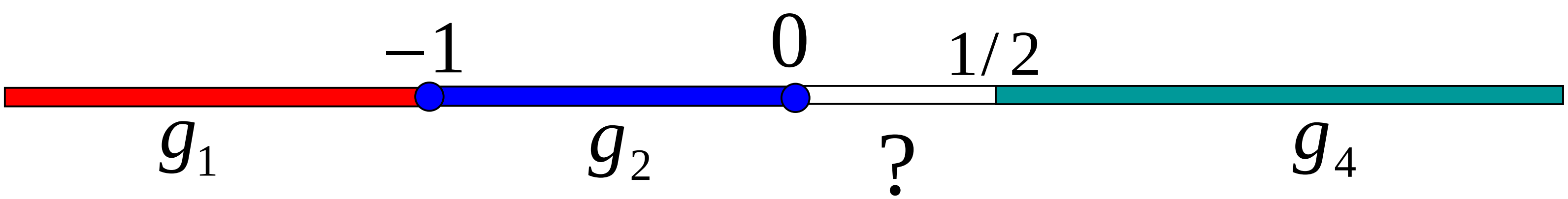
$$g(1-t) = 1 + \sqrt{t}\,|\log t|^{\beta}$$

$$\beta \in (-\infty, -1) \Rightarrow g_1$$

$$\beta \in [-1, 0] \Rightarrow g_2$$

$$??? \Rightarrow g_3$$

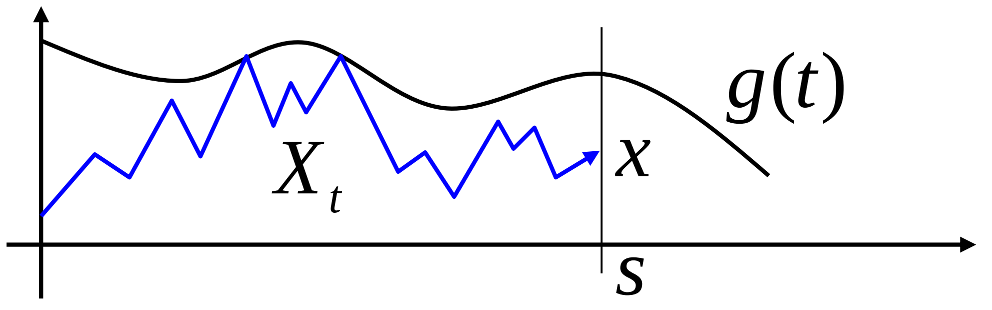
$$\beta \in (1/2, \infty) \Rightarrow g_4$$



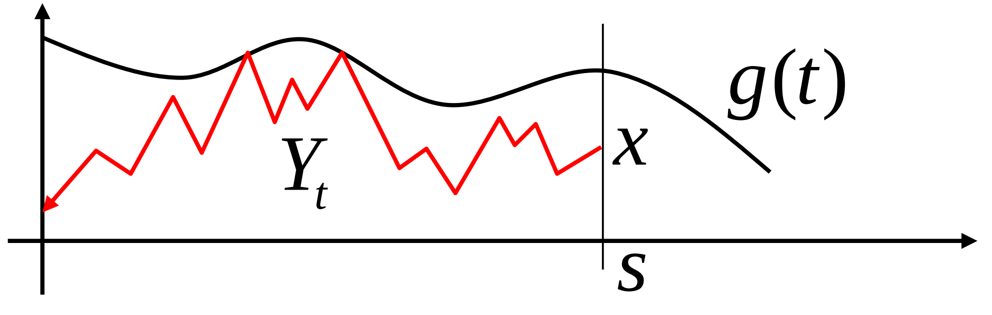
Conjecture: $\beta \in (0, 1/2] \Rightarrow g_4$

Probabilistic representations of heat equation solutions

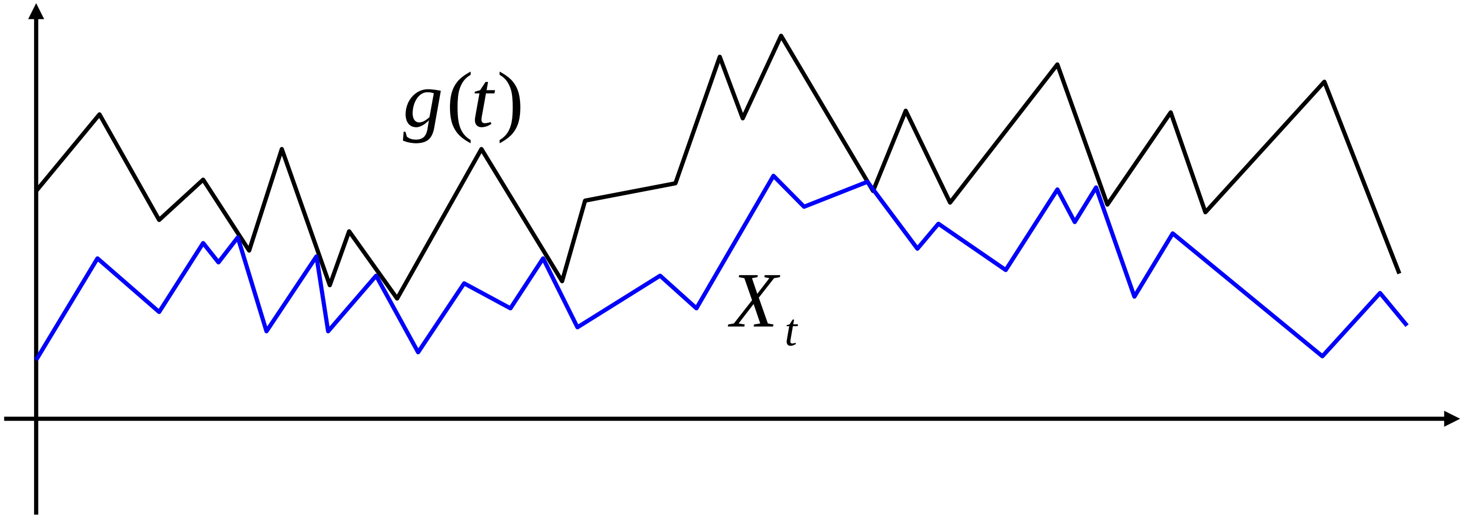
$$u(s,x)dx = P(X_s \in dx)$$



$$u(s,x) = E^{0,x}\left[\exp\left(-\int_0^s 2g'(t)dL_t^Y\right)u(0,Y_s)\right]$$



The set of heat atoms



$$A(g) = \{t : g(t) \text{ is a heat atom } \}$$

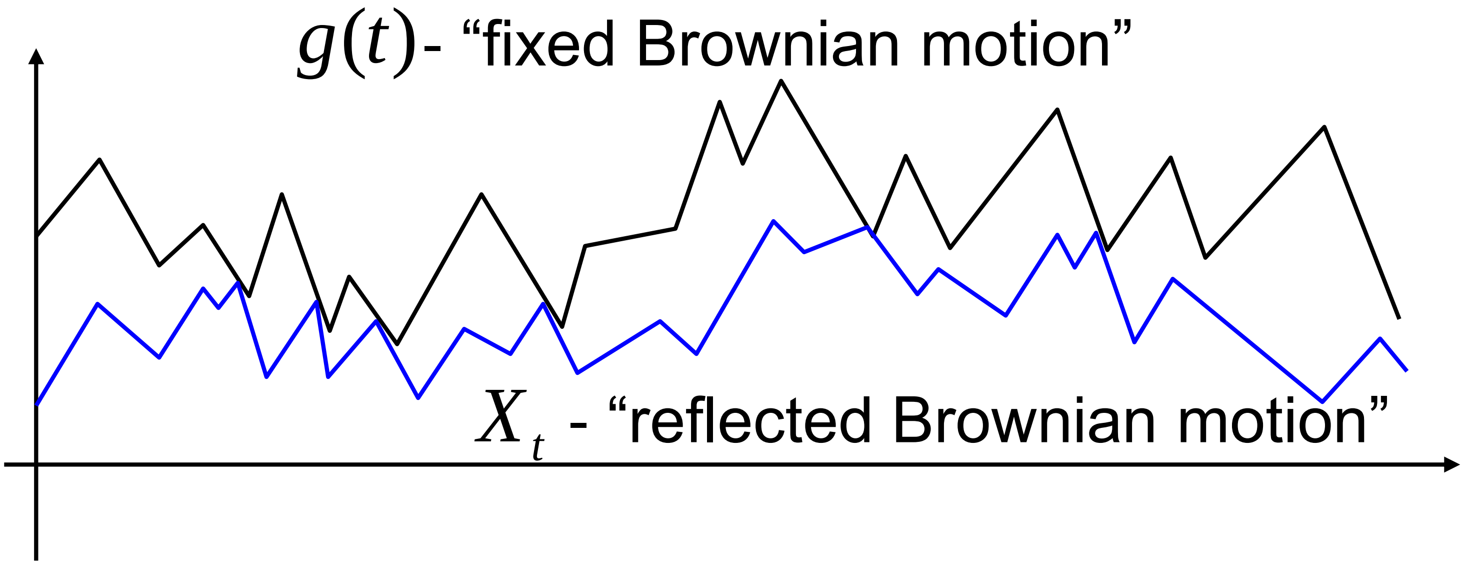
Theorem:

(i) $\forall g \quad \dim A(g) \leq 1/2$

(ii) $\exists g \quad \dim A(g) = 1/2$

Corollary: Lebesgue$(A(g)) = 0$

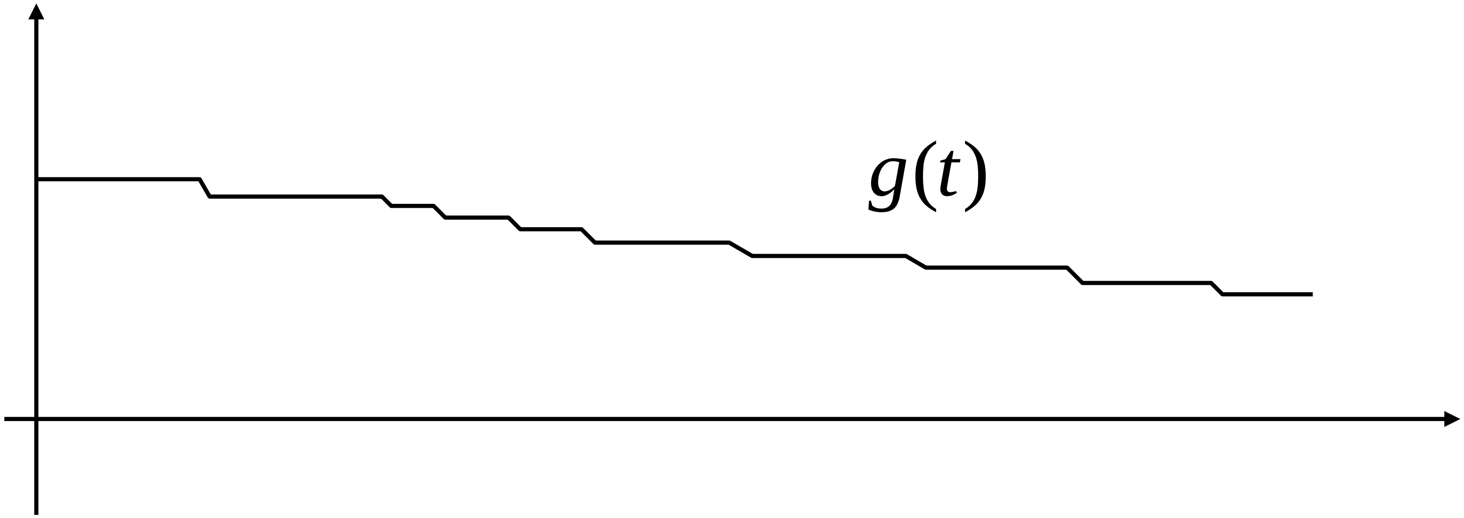
Brownian motion reflected on Brownian motion



Soucaliuc, Toth and Werner (2000)

Theorem: There are no heat atoms on Brownian path.

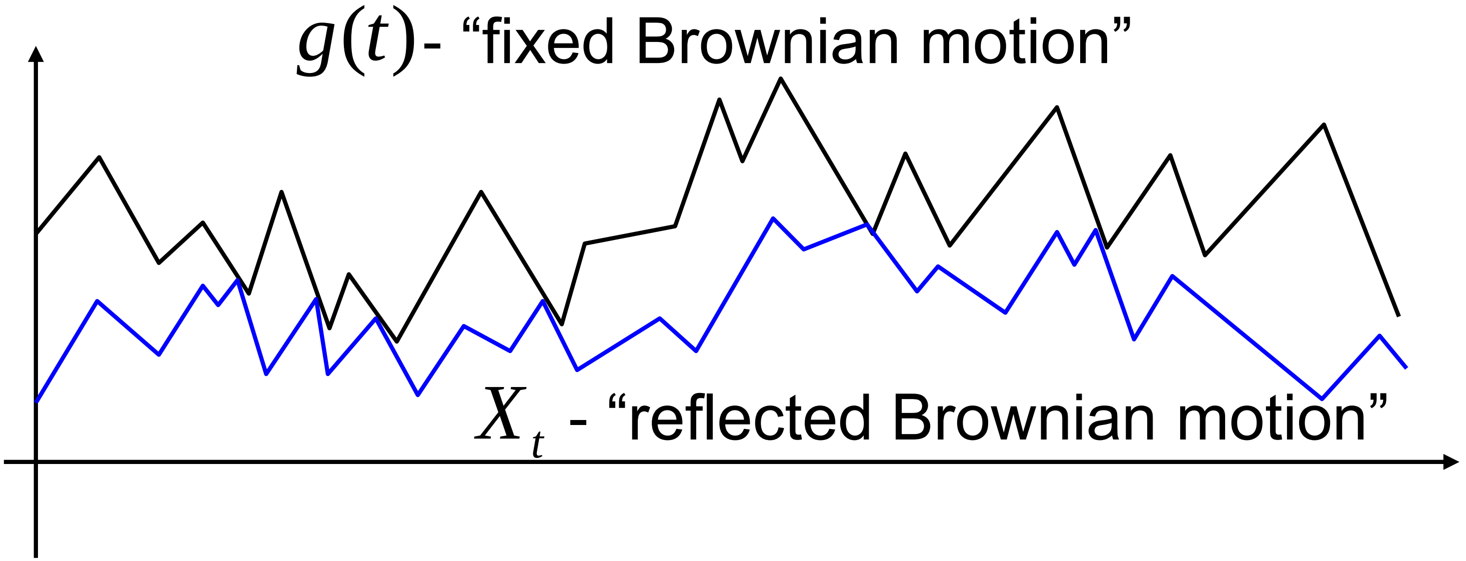
Stable boundary



$g(t)$ - inverse of a stable subordinator

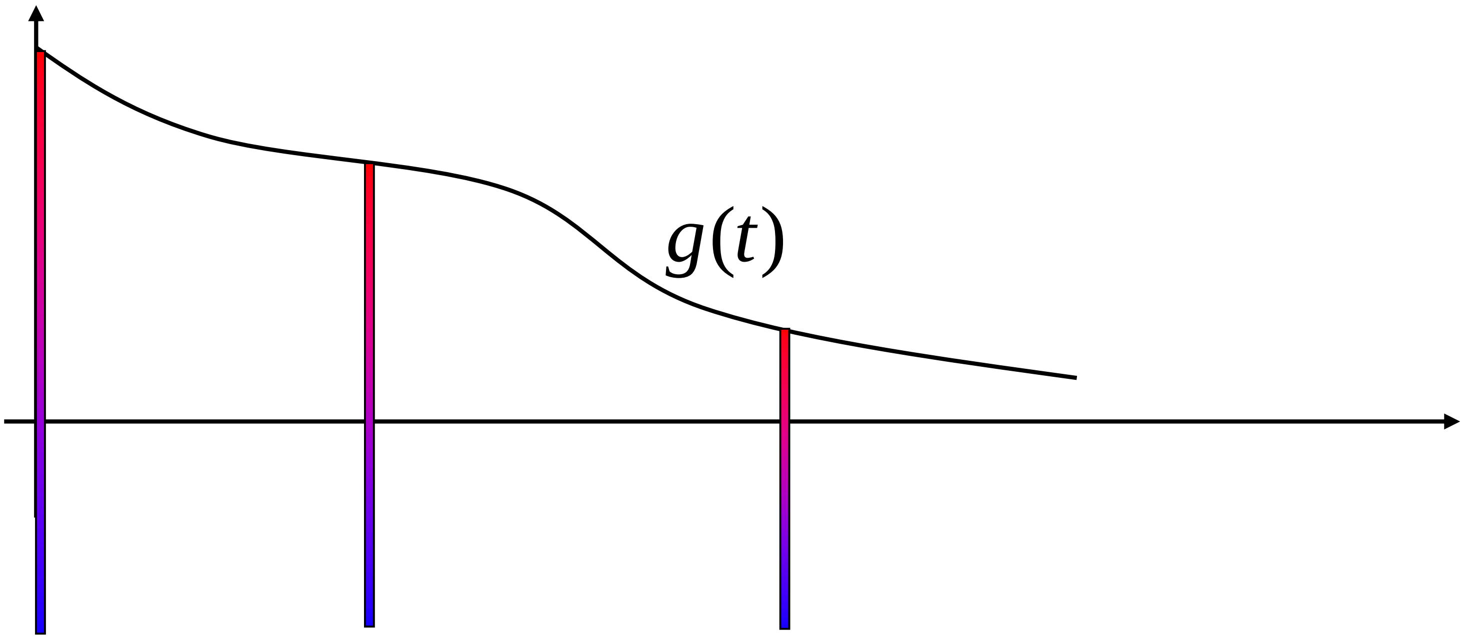
$$\dim A(g) = 1/2$$

Set of singularities



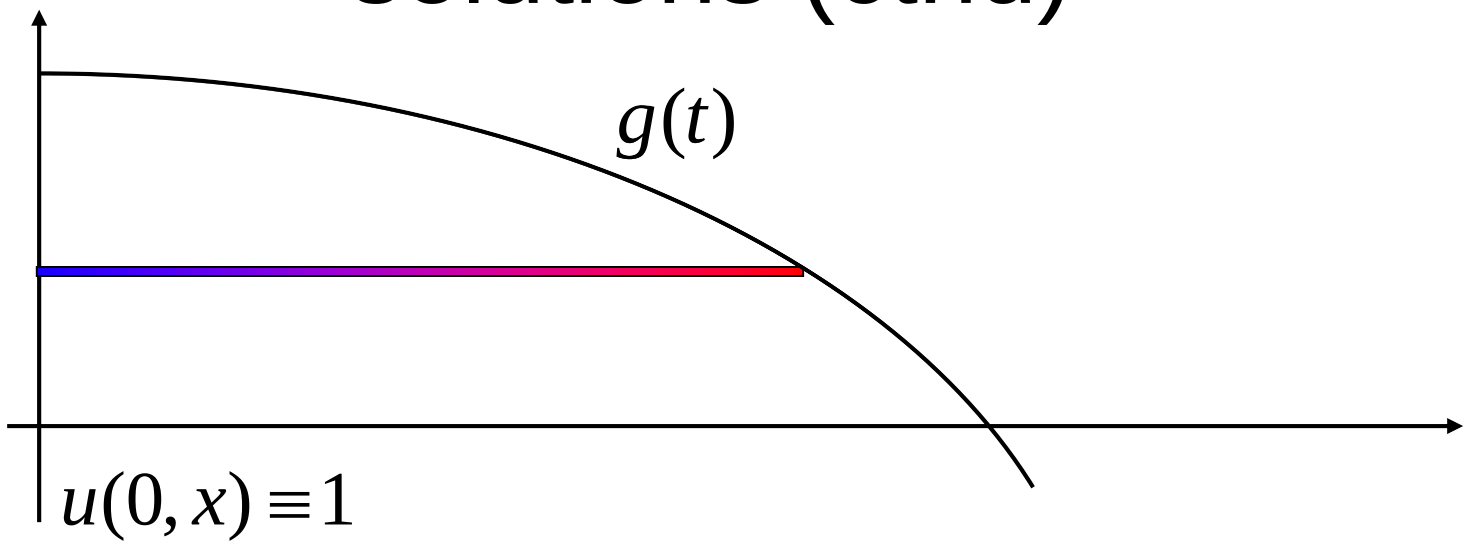
Theorem: Singularities are dense on a Brownian path.

Monotonicity of heat equation solutions



Theorem: If $t \to g(t)$ is decreasing and $x \to u(0,x)$ is increasing then for any $t > 0$, the function $x \to u(t,x)$ is increasing.

Monotonicity of heat equation solutions (ctnd)



Theorem: If $t \to g(t)$ is decreasing and concave and $u(0,x) \equiv 1$ then for any x, the function $t \to u(t,x)$ is increasing.

Monotonicity- probabilistic proof

$$u(s,x) = E^{0,x}\left[\exp\left(-\int_0^s 2g'(t)\,dL_t^Y\right)u(0,Y_s)\right]$$