Conley Theory: A combinatorial approach to dynamics*

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*Based on notes with W. Kalies and R. Vandervorst

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Forward. These are lecture notes for a course given at Jagiellonian University and are based in part on a book being prepared jointly with W. Kalies and R. Vandervorst. These notes are incomplete in several ways. First, because they are only intended to re-enforce the material presented in class there is no proper acknowledgement or credit given to original sources. Similarly, there is no attempt at completeness, either mathematically or grammatically. Second, many proofs are not included due to time constraints in the lectures. Third, there has been no attempt to correct errors and typos.

1 Why?

1.1 Physics

- Nicholaus Copernicus (1473-1543)
  - Heliocentric model
- Tycho Brahe (1546-1601)
  - Recognized that classical measurements were inconsistent.
  - Careful systematic measurements - goal to prove an earth centric model
- Johannes Kepler (1571-1630)
  - Continued Brahe's measurements and had access to Brahe's data.
  - Assigned to study Mars - most elliptical of orbit of planets.
  - Three laws
    1. The orbits of the planets are ellipses, with the sun at one focus of the ellipse.
    2. The line joining the planet to the Sun sweeps out equal areas in equal times as the planet travels around the ellipse.
    3. The ratio of the squares of the revolutionary periods P_i for two planets is equal to the ratio of the cubes of half of their major axes R_i:

$$\frac{P_1^2}{P_2^2} = \frac{R_1^3}{R_2^3}$$

Remark: These are descriptive laws derived from data as opposed to results obtained from a theoretical framework.

- Galileo Galilei (1564-1642)
  - Newton's First Law
- Isaac Newton (1642-1727) Principia 1687.
  - Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.

- $F = ma$
- For every action there is an equal and opposite reaction.

Remarks:

- Notice that it took approximately 100 years from collecting quality data to a mathematical model that provided a description of the phenomenon.
- Mass m is a quantity that can be measured very precisely.
- In applications to celestial mechanics approximations are made, i.e. two-body problems, point masses.
- The precession of the perihelion of Mercury does not agree with the predictions of Newtonian celestial mechanics. (one of the earliest experimental supports for general relativity)

1.2 Biology

A *parasitoid* is an insect whose females lay their eggs in or on the bodies of a host insect. Typically, hosts that have been parasitized give rise to the next generation of parasitoids. Only hosts that have not been parasitized give rise to the next generation of hosts.

We want to make a population model for the host/parasitoids interaction.

Let H denote the population of hosts and P denote the population of parasitoids.

Let $f(H, P)$ denote the fraction of hosts *not* parasitized. Then

$$\begin{aligned} f(H,P)H &= \text{number of hosts \textit{not} parasitized} \\ [1 - f(H,P)]\, H &= \text{number of hosts that are parasitized} \end{aligned}$$

Assume 1: The only control on the growth of the hosts population are the parasitoids. (this means that we are interested in the situation in which the parasites keep the host population from getting so large that they compete with themselves for resources).

$$H_{n+1} = kf(H_n, P_n)H_n$$

Observe that k is the reproductive rate for the non parasitized hosts. If $k < 1$, then the hosts will die out whether or not there are any parasitoids. Thus from now on we assume $k > 1$.

Assume 2: The average number of eggs laid in a single host that grows to be an adult parasitoid is constant.

$$P_{n+1} = c\,[1 - f(H_n, P_n)]\,H_n$$

What model shall we choose for f? It clearly depends on how the hosts and parasitoids interact.

Assume 3: The host and parasitoids meet at random and this probability is independent of whether the host is already infected.

If we use the concept from chemistry of *mass action kinetics*, then the probability of meeting should be proportional to the product of the number of hosts and parasitoids. Thus the *average* number of meetings per host is

$$\nu = \frac{aHP}{H} = aP$$

i.e. it is proportional to the number of parasitoids.

This is an average. Some hosts will have more encounters and some less. Let

$$p(i) = \text{the probability that a host meets } i \text{ parasitoids}$$

Random, independent encounters are modeled using a *Poisson distribution*

$$p(i) = \frac{\nu^i e^{-\nu}}{i!}$$

Assume 4: Assume that if the host and parasitoid meet then the host is parasitized.

$$f(H,P) = p(0) = \frac{\nu^0 e^{-\nu}}{0!} = e^{-\nu} = e^{-aP}$$

Thus the model is

$$\begin{aligned} H_{n+1} &= kH_n e^{-aP_n} \\ P_{n+1} &= cH_n\left[1 - e^{-aP_n}\right] \end{aligned} \tag{1}$$

This is the Nicholson-Bailey model.

Remarks:

- Has been used to fit data for approximately two dozen generations of populations of the greenhouse whitefly *Trialeurodes vaporairorum* and the parasitoid *Encarsia formosa* grown under laboratory conditions.
- Almost all solutions become unbounded as time goes to infinity.
- Fitting values to parameters k, c, a is not obvious.

Summary: Typical issues associated with mathematical models of biology (or many multiscale) processes:

1. Understanding the dynamics is often essential
2. Nonlinearities are often based on heuristics rather than first principles
3. Multiple, poorly measured parameters
4. Limited resolution of measurement

1.3 Data

http://www.google.org/flutrends/about/how.html

1.4 Data & Planetary Motion

Aerospace web page:

> High overhead, more than 20,000 kilometers above Earth, GPS satellites race by at speeds approaching 3800 meters per second. The movements of these spacecraft are generally described by the laws of planetary motion developed by Johannes Kepler almost 400 years agobut they are by no means certain or simple. Each satellite must contend with diverse forces that constantly nudge and pull it from its desired orbit. Yet in spite of this, the positions of GPS satellites must be known at all times with exceptional accuracy. Modeling these orbits is a complex affair...

There is no mention of Newton!

Remark: As a mathematician and scientist I do not want to give up on Newton - theory that explains why an observable phenomenon occurs.

But this takes time and what we have is lots of data and what we will have shortly is phenomenal amounts of data (Petabyte data sets). Applied science, engineering, medicine, social sciences are developing techniques based on data not theory. My opinion is that because these techniques are being used it is essential to understand them in a rigorous way, i.e. we need to have mathematical theories that can incorporate these techniques.

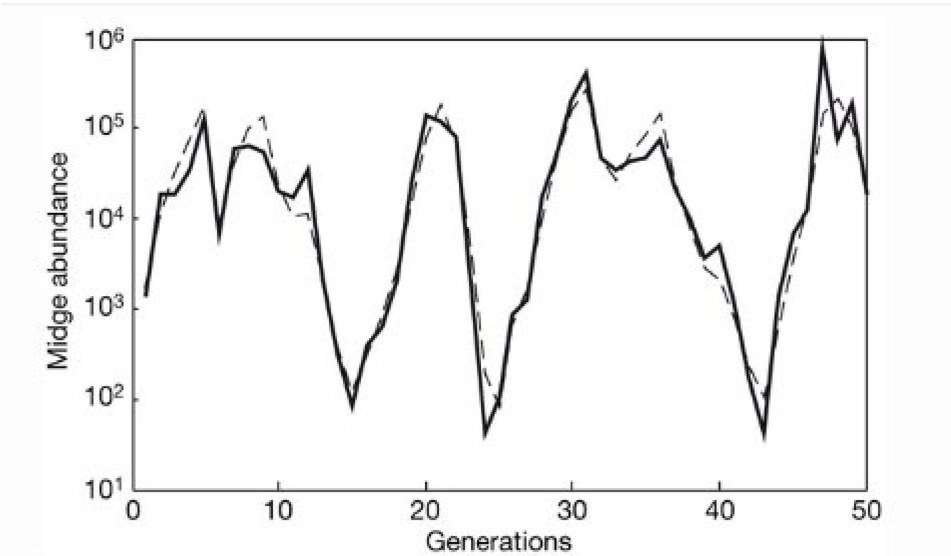
1.5 Modeling via Data

Anthony R. Ives, Árni Einarsson, Vincent A. A. Jansen and Arnthor Gardarsson, *High-amplitude fluctuations and alternative dynamical states of midges in Lake Myvatn*, Nature, vol. 452 (7183) pp. 84-87

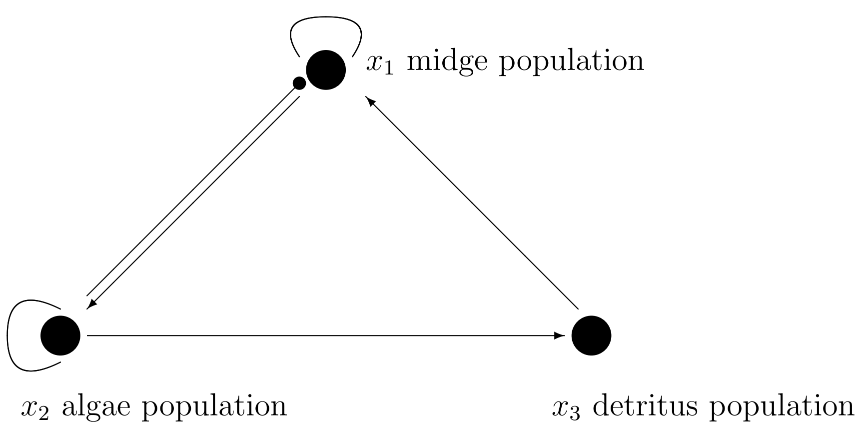
Lake Myvatn is a shallow, naturally eutrophic lake in northern Iceland.

Midges, *Tanytarsus gracilentus*, are the dominant herbivore/detritivore in Myvatn.

- Two non-overlapping generations per year (first in May, second in late July early August)
- Statistical evidence suggests that fluctuations in midge populations are driven by consumerresource interactions, with midges being the consumers and algae/detritus the resources, as opposed to predatorprey interactions with midges being the prey
- Population levels of midges have been collected since 1977. Solid line derived from measured data.



Proposed network of interactions:



It is assumed that if all the algae are consumed then the algal populations can recover through the input of small subsidies from outside the midgealgaede-tritus system. These subsidies represent small influxes of algae and detritus into the muddy midge habitat from hard-bottom areas where midges are few. There is no direct measurement of this input, but much of the algae and detritus in the lake occurs in areas inaccessible to midge larvae, and the hydrological mixing of the shallow lake makes influxes of small amounts of this material into the midge habitat a certainty.

Deterministic model

$$\begin{aligned} x_1(t+1) &= r_1 x_1(t)\left(1+\frac{x_1(t)}{x_2(t)+px_3(t)}\right)^{-q} && (2)\\ x_2(t+1) &= r_2\frac{x_2(t)}{1+x_2(t)} - \frac{x_2(t)}{x_2(t)+px_3(t)}x_1(t+1)+c && (3)\\ x_3(t+1) &= dx_3(t)+x_2(t)-\frac{px_3(t)}{x_2(t)+px_3(t)}x_1(t+1)+c && (4) \end{aligned}$$

where

- r_1 is the intrinsic population growth rate for midges
- r_2 is the intrinsic population growth rate for algae
- q density dependence parameter
- p quality of detritus for midges in relation to algae
- c is influx rate of algae from environment
- d retention rate of detritus in environment

This model is built on the interaction network information, heuristic nonlinearities, and an attempt to minimize the number of parameters.

Question: How can we justify this model?

Consider different models.

- Gompertz log-linear model. Let $u_i = \ln x_i$

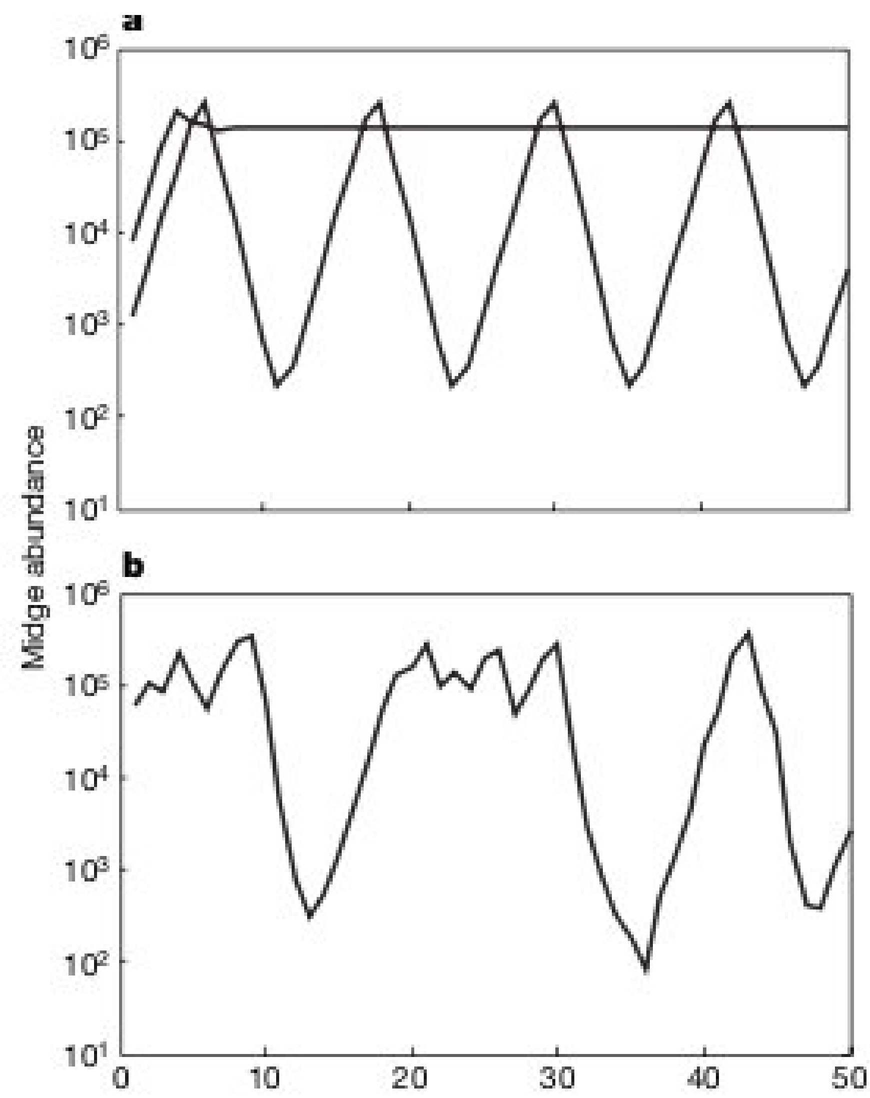
$$u_i(t+1) = \sum_{j=1}^{3} b_{ij} u_j(t)$$

- Lotka-Volterra model

$$x_i(t+1) = r_i x_i(t)\exp\left(1+\sum_{j=1}^{3} b_{ij} x_j(t)\right)$$

and compare global dynamics of these three models.

First set of results: There exist parameter values of IEJG model which exhibit multiple basins of attraction (we will give precise definitions of this later). This is not true for other models.



(a) $r_1 = 3.873$, $r_2 = 11.746$, $c = 10^{-6.435}$, $d = 0.5517$, $p = 0.06659$, $q = 0.9026$

(b) $\sigma_1 = 0.3491$, $\sigma_2 = \sigma_3 = 0.7499$

Remark: The lake Myvatn ecosystem is more complicated that that which has been modelled. This is accounted for by introducing stochastic fluctuations.

IEJG model:

$$x_1(t+1) = r_1 x_1(t)\left(1+\frac{x_1(t)}{x_2(t)+px_3(t)}\right)^{-q} e^{\epsilon_1(t)} \tag{5}$$

$$x_2(t+1) = \left[r_2\frac{x_2(t)}{1+x_2(t)} - \frac{x_2(t)}{x_2(t)+px_3(t)}x_1(t+1)+c\right] e^{\epsilon_2(t)} \tag{6}$$

$$x_3(t+1) = \left[dx_3(t)+x_2(t) - \frac{px_3(t)}{x_2(t)+px_3(t)}x_1(t+1)+c\right] e^{\epsilon_3(t)} \tag{7}$$

where the ϵ_i are normal random variables.

Gompertz log-linear model.

$$u_i(t+1) = \left(\sum_{j=1}^{3} b_{ij}u_j(t)\right) e^{\epsilon_i(t)}$$

Lotka-Volterra model

$$x_i(t+1) = r_i x_i(t)\exp\left(1+\sum_{j=1}^{3} b_{ij}x_j(t)\right) e^{\epsilon_i(t)}$$

Remark: Observe that the c is extremely small. However, dynamics is very sensitive to value of c.

> *A striking biological conclusion from the model is the sensitivity of the amplitude of midge fluctuations to very small amounts of resource input, c; the resource input sets the lower boundary of midge abundance and hence the severity of population crashes. Thus, even though resource input might be six orders of magnitude less than the abundance of resources in the lake in most years, this vanishingly small source of resources is nevertheless critical in setting the depth of the midge population nadir and the subsequent rate of recovery. This sensitivity to resource subsidies might explain changes in midge dynamics that have apparently occurred over the last decades. Although Myvatn has supported a*

local charr (salmonid) fishery for centuries, this fishery collapsed in the 1980s, coincident with particularly severe midge population crashes. Over the same period, waterbird reproduction in Myvatn was also greatly reduced during the crash years. These changes might have been caused by dredging in one of the two basins in the lake that started in 1967 to extract diatomite from the sediment. Hydrological studies indicate that dredging produces depressions that act as effective traps of organic particles, hence reducing algae and detritus inputs to the midge habitat. Our model predicts that even a slight reduction in subsidies can markedly increase the magnitude of midge fluctuations. Such slight environmental changes can then have seriously negative con- sequences for fish and bird populations.

Conclusion: Being able to derive quantitative conclusions from these data driven, heuristically generated models is important.

2 Elementary Definitions

2.1 Motivating examples

Example 2.1 The *logistic equation* is

$$\dot{x} = \frac{dx}{dt} = rx(\kappa - x) \tag{8}$$

where r is the reproduction rate and κ is the carrying capacity of the environment. The unknown function $x(t)$ describes a population density. Given an initial population x_0, this equation can be solved explicitly by

$$x(t) = \frac{x_0 \kappa e^{r\kappa t}}{\kappa - x_0 + x_0 e^{r\kappa t}}.$$

Dynamical systems approach to differential equations is to consider the collection of all solutions as a function of both time t and initial value x. In the case of the logistic equation we obtain

$$\varphi(t, x) = \frac{x \kappa e^{r\kappa t}}{\kappa - x + x e^{r\kappa t}}. \tag{9}$$

Definition 2.2 Let X be a metric space. Let $D \subset \mathbb{R} \times X$ be an open set with the property that

$$D_x := D \cap (\mathbb{R} \times \{x\}) = (\tau_x^-, \tau_x^+)$$

where $\tau_x^- \in [-\infty, 0)$ and $\tau_x^+ \in (0, \infty]$. A continuous function $\varphi : D \to X$ is a *local flow* if

(i) $\varphi(0, x) = x$ for all $x \in X$ and

(ii) $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ for all $s \in D_x$ and $t \in D_{\varphi(s,x)}$.

If $D_x = \mathbb{R}$ for all $x \in X$, then φ is a *(global) flow.*

The following exercise shows that we can reparameterize a vector field so that the local flow becomes a global flow.

Exercise 2.3 Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz vector field. Show that the differential equation

$$\dot{x} = \frac{F(x)}{1 + \|F(x)\|^2}$$

generates a global flow on $\mathbb{R}^n$ whose orbits are the same as the orbits of $\dot{x} = F(x)$.

Example 2.4 The logistic differential equation (8) models a population in which the change of the population is continuous. However, in certain populations (recall the midges of Lake Myvatn) births happen only at certain fixed times and hence are modeled by a discrete time system. In this context, one can consider the dynamics of iterating the logistic map given by

$$x_{n+1} = f(x_n) = rx_n(1 - x_n). \tag{10}$$

Orbits of this system are sequences $\{x_n\}_{n\in\mathbb{Z}}$ such that $x_{n+1} = f(x_n)$

For the logistic map with $r = 2.5$ both

$$\{x_n = 0.6\}_{n\in\mathbb{Z}} \qquad \text{and} \qquad \left\{ x_n = \begin{cases} f_-^{-n}(0.6) & \text{for } n < 0 \\ 0.6 & \text{for } n \geq 0 \end{cases} \right\}_{n\in\mathbb{Z}} \tag{11}$$

are orbits through $x_0 = 0.6$ where $f_-^{-1}(x) := (1 - \sqrt{1 - (8x)/5})/2$.

Clearly we do not have uniqueness in backward time. This is caused by the fact that f is not a monomorphism.

Example 2.5 The partial differential equation

$$u_t = u_{xx} - u, \quad u(t,0) = u(t,\pi) = 0, \quad x \in [0,\pi] \tag{12}$$

gives rise to the same phenomenon: we can solve the initial value problem

$$u_0(x) = u(0,x)$$

forward in time but not necessarily backward in time.

2.2 Definitions

Let (X, d) be a metric space X with metric d, and let $\mathbb{T}$ denote the space of time variables which is either $\mathbb{Z}$ or $\mathbb{R}$. The restriction $\mathbb{T}^+$ denotes either $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ or $\mathbb{R}^+ = [0, \infty)$.

Definition 2.6 A *dynamical system* on X is a continuous map $\varphi : \mathbb{T}^+ \times X \to X$ that satisfies the following two properties:

(i) $\varphi(0, x) = x$ for all $x \in X$, and

(ii) $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ for all $s, t \in \mathbb{T}^+$ and all $x \in X$.

Remark 2.7 Most references refer to this type of system as a *semi-dynamical system.* For them a dynamical system satisfies properties (i) and (ii) for all $t \in \mathbb{T}$. Observe that in this case we have a group action of $\mathbb{T}$ on X. As will become clear I want to de-emphasize the group action.

Though φ is only defined for nonnegative times understanding the preimages of points with respect to time is often essential. This leads to the following extension.

Definition 2.8 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. The *backward extension* $\varphi : \mathbb{T} \times X \to X$ is defined by

$$\varphi(-t, x) := \{y \in X \mid \varphi(t, y) = x\}.$$

Proposition 2.9 *Let $U \subset X$ and let $s \in \mathbb{T}$.*

(i) If $t \geq 0$, then

$$\varphi(t, \varphi(s, U)) \subset \varphi(t + s, U).$$

Moreover, if $\varphi(s, x) \neq \varnothing$ for all $x \in U$ and $t + s \geq 0$, then

$$\varphi(t, \varphi(s, U)) = \varphi(t + s, U).$$

(ii) If $t \leq 0$, then

$$\varphi(t, \varphi(s, U)) \supset \varphi(t + s, U).$$

Moreover, if $s \leq 0$, then

$$\varphi(t, \varphi(s, U)) = \varphi(t + s, U).$$

Proof is left as an exercise, the only comment is that if $\varphi(s, x) = \varnothing$, then $\varphi(t, \varphi(s, x)) = \emptyset$.

2.3 Invariant sets

The starting point for the analysis of dynamical systems is the time evolution of a single point.

Definition 2.10 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system.

The *forward orbit or trajectory through* $x \in X$ is the set $\{\varphi(t,x) \mid t \in \mathbb{T}^+\}$, denoted by γ_x^+.

A *(complete) orbit or trajectory through* $x \in X$ is the image $\{\gamma_x(t) \in X \mid t \in \mathbb{T}\}$ of a function $\gamma_x : \mathbb{T} \to X$ such that $\gamma_x(0) = x$ and $\gamma_x(t+s) = \varphi(s, \gamma_x(t))$ for all $t \in \mathbb{T}$ and $s \in \mathbb{T}^+$.

A *backward orbit or trajectory through* $x \in X$ is the image of the restriction of a complete orbit to $t \in \mathbb{T}^-$ and is denoted by γ_x^- where $\mathbb{T}^- = \{t \in \mathbb{T} \mid t \leq 0\}$.

Remark 2.11 For simplicity of notation, γ_x will be used both to denote the function $\gamma_x : \mathbb{T} \to X$ which defines a trajectory through x and its image $\{\gamma_x(t) \in X \mid t \in \mathbb{T}\}$ which is the trajectory.

Note that γ_x need not be unique nor exist for all x.

The qualitative study of dynamical systems involves the analysis of the structure of its orbits. One of the fundamental ideas in the theory of dynamical systems is that it is often useful to consider collections of orbits rather than individual orbits.

Definition 2.12 Given a dynamical system $\varphi : \mathbb{T}^+ \times X \to X$, a set $S \subset X$ is an *invariant set* if $\varphi(t,S) = S$ for all $t \in \mathbb{T}^+$. A set $S \subset X$ is called *strongly invariant* if $\varphi(t,S) = S$ for all $t \in \mathbb{T}$.

Notation: Given a dynamical system $\varphi : \mathbb{T}^+ \times X \to X$ the collection of invariant sets and strongly invariant sets in X are denoted by $\mathsf{Invset}(X,\varphi)$ and $\mathsf{SInvset}(X,\varphi)$, respectively. Clearly, $\mathsf{SInvset}(X,\varphi) \subset \mathsf{Invset}(X,\varphi)$.

The set of *forward invariant* sets is denoted by

$$\mathsf{Invset}^+(X,\varphi) := \left\{S \subset X \mid \varphi(t,S) \subset S,\ \forall t \in \mathbb{T}^+\right\}$$

and the set of *backward invariant* sets is denoted by

$$\mathsf{Invset}^-(X,\varphi) := \left\{S \subset X \mid \varphi(t,S) \subset S,\ \forall t \in \mathbb{T}^-\right\}.$$

Definition 2.13 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. The *forward image* of a set $U \subset X$ is defined by

$$\Gamma^+(U) := \bigcup_{t\geq 0} \varphi(t, U) = \varphi\big([0,\infty), U\big).$$

The *backward image* of a set $U \subset X$ is defined by

$$\Gamma^-(U) := \bigcup_{t\leq 0} \varphi(t, U) = \varphi\big((-\infty,0], U\big).$$

The *complete image* of U is $\Gamma(U) = \Gamma^+(U) \cup \Gamma^-(U)$. For $\tau \geq 0$, define $\Gamma^+_\tau(U) := \varphi\big(\tau, \Gamma^+(U)\big)$ and $\Gamma^-_{-\tau}(U) := \varphi\big(-\tau, \Gamma^+(U)\big)$.

Remark 2.14 Observe that for any dynamical system, the empty set is strongly invariant since

$$\varphi(-t, \emptyset) := \{x \in X \mid \varphi(t, x) \in \emptyset\} = \emptyset$$

Proposition 2.15 *If the dynamical system $\varphi : \mathbb{T}^+ \times X \to X$ is invertible then an invariant set is strongly invariant.*

Proof is left as an exercise.

Definition 2.16 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. An element $x \in X$ is an *equilibrium point* or *fixed point* if $\varphi(t, x) = x$ for all $t \in \mathbb{T}^+$.

For the flow φ of the logistic equation

$$\dot{x} = rx(\kappa - x)$$

$\phi(t, 0) = 0$ and $\phi(t, \kappa) = \kappa$ for all $t \in \mathbb{R}$, and thus $\{0\}$ and $\{\kappa\}$ are strongly invariant sets.

In contrast, if one lets $\psi\colon \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$ denote the dynamical system generated by the logistic map

$$f(x) = rx(1 - x)$$

for $r > 1$, then $\psi(n, 0) = 0$ and $\psi\left(n, \frac{r-1}{r}\right) = \frac{r-1}{r}$ for all $n \geq 0$. Thus, 0 and $\frac{r-1}{r}$ are invariant sets. However, $1 \in \psi(-1, 0)$ and hence 0 is not a strongly invariant set.

Definition 2.17 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. A point $x \in X$ is called a *periodic point* with *period* τ if there exists $\tau > 0$ such that $\varphi(\tau, x) = x$.

If $\varphi(t, x) \neq x$ for all $t \in (0, \tau)$, then τ is the *minimal period.*

The orbit $\{\varphi(t, x) \mid 0 \leq t \leq \tau\}$ through a periodic point x is called a *periodic orbit.*

Returning to the logistic equation

$$\dot{x} = rx(\kappa - x).$$

From the phase portrait we see that κ is an *attracting fixed point*; that is, any initial condition chosen sufficiently close to κ limits to κ in forward time.

0 is a *repelling fixed point* in that any initial condition chosen sufficiently close to 0 moves a uniform distance away from 0 in forward time.

There is one other bounded orbit $(0, \kappa)$.

Taking a more global point of view note that for each $x \in (0, \kappa)$,

$$\lim_{t \to -\infty} \varphi(x, t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \varphi(x, t) = \kappa.$$

The biological implication of this is that given an arbitrarily small but positive population, asymptotically the population will tend towards the carrying capacity. Thus, understanding the limits of orbits with respect to time is of considerable importance.

Definition 2.18 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. A *heteroclinic orbit* is a complete orbit γ_x such that

$$\lim_{t \to -\infty} \gamma_x(t) = y \quad \text{and} \quad \lim_{t \to \infty} \gamma_x(t) = z.$$

where y and z are distinct equilibria of φ.

Using the language developed so far, the structure of the dynamics of the logistic equation is as follows. The invariant set composed of bounded orbits consists of two equilibria, 0 and κ, and a unique heteroclinic orbit from 0 to κ.

Example 2.19 (**Doubling Map**) Consider the dynamical system generated by $f : S^1 \to S^1$ defined by

$$f(x) = x \mod 1$$

Write x base 2, i.e. $x = 0.x_1x_2x_3\dots$ where $x_i \in \{0, 1\}$. Then

$$f(x) = 0.x_2x_3x_4\dots$$

is the shift operator.

For every period $\tau \in \mathbb{N}$ there exist 2^τ periodic points.

Given any two periodic points we can construct a "heteroclinic" orbit from one to the other.

There is an orbit which is dense in S^1.

Given any two distinct initial conditions $x, y \in S^1$, there exists $t > 0$ such that $|\varphi(t, x) - \varphi(t, y)| > 1/4$.

The point of this example is that set of bounded orbits can be extremely complicated (this is an example of chaos)

2.4 Reality Check

Gauss fitting of logisitic map.

We are back to fundamental question of qualitative vs. quantitative or heuristic vs. predictive.

In this setting the notion of an invariant set (equilibrium point) is more of Platonic ideal than a physical reality. One of the things we want to develop is a language which allows us to replace these platonic ideals by experimentally measurable quantities.

2.5 Alpha and Omega limit sets

Of fundamental interest is what happens in future time. The following concept captures the asymptotic future.

A point $x \in X$ need not have a limit under φ as $t \to \infty$. For example points $y = \varphi(t, x)$ approaching a periodic orbit. In order to describe the limiting behavior of a point x a logical step is to consider the possible limiting behaviors for arbitrary time sequences $t_n \to \infty$.

Definition 2.20 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. A point y is called a *omega limit point* of a set $U \subset X$ under φ if there exist times $t_n \to \infty$ and points $x_n \in U$ such that $\lim_{n\to\infty} \varphi(t_n, x_n) = y$. The set of all omega limit points y is called the *omega limit set of* U and is denoted by $\omega(U, \varphi)$.

Exercise 2.21 Construct an example to show that in general $\omega(U, \varphi) \neq \bigcup_{x\in U} \omega(x, \varphi)$.

Proposition 2.22 *Let $U \subset X$. Then*

$$\omega(U) = \bigcap_{t\geq 0} \mathrm{cl}\big(\varphi([t,\infty), U)\big) = \omega\big(\Gamma^+(U)\big). \tag{13}$$

The omega limit set $\omega(U)$ is closed, forward invariant, and contained in $\mathrm{cl}(\Gamma^+(U))$. If $U \subset X$ is forward invariant, then

$$\omega(U) = \bigcap_{t\geq 0} \mathrm{cl}\big(\varphi(t, U)\big), \tag{14}$$

and $\omega(U) \subset \mathrm{cl}(U)$ (equality when U is invariant).

Proof: First prove

$$\omega(U) = \bigcap_{t\geq 0} \mathrm{cl}\big(\varphi([t,\infty), U)\big)$$

Let $y \in \omega(U)$, then

$$y \in \mathrm{cl}\big(\varphi([t,\infty), U))\big), \quad \forall\ t \in \mathbb{T}^+,$$

and thus

$$\omega(U) \subset \bigcap_{t\geq 0} \mathrm{cl}\big(\varphi\big([t,\infty), U\big)\big).$$

In the other direction, if

$$y \in \bigcap_{t\geq 0} \mathrm{cl}\big(\varphi\big([t,\infty), U\big)\big),$$

then

$$y \in \mathrm{cl}\big(\varphi([t,\infty), U))\big), \forall\ t \in \mathbb{T}^+.$$

Choose an increasing sequence of $t_n \in \mathbb{T}^+$ and $x_n \in U$ such that $d(\varphi(t_n, x_n), y) < 1/n$. Since $d(\varphi(t_n, x_n), y) \to 0$ as $t_n \to \infty$, it follows that $y \in \omega(U)$ which proves the other inclusion.

Now prove

$$\omega(U) = \bigcap_{t\geq 0} \mathrm{cl}\big(\varphi(t, U)\big).$$

Since $U \subset X$ is forward invariant, the semi-group property implies,

$$\varphi(t+s, U) = \varphi\big(t, \varphi(s, U)\big) \subset \varphi(t, U) \quad \forall\ s, t \geq 0.$$

Therefore $\varphi([t, \infty), U) = \varphi(t, U)$.

The second part of the first statement

$$\omega(U) = \omega\big(\Gamma^+(U)\big)$$

follows from the semi-group property, i.e. $\varphi([t, \infty), U) = \varphi(t, \Gamma^+(U))$, Equation (14) and the forward invariance of $\Gamma^+(U)$.

Closedness follows immediately from the definition.

As for the forward invariance of $\omega(U)$ we argue as follows. We first show forward invariance of $\omega(U)$ when U is forward invariant. For any $t \in \mathbb{T}^+$ we have that

$$\begin{aligned}
\varphi(t, \omega(U)) &= \varphi\Big(t, \bigcap_{s\geq 0} \mathrm{cl}\big(\varphi(s, U)\big)\Big) \\
&\subset \bigcap_{s\geq 0} \mathrm{cl}\Big(\varphi\big(t, \varphi(s, U)\big)\Big) \\
&= \bigcap_{s\geq 0} \mathrm{cl}\Big(\varphi\big(s, \varphi(t, U)\big)\Big) \\
&\subset \omega(U).
\end{aligned}$$

For general U, use the fact that $\Gamma^+(U)$ is forward invariant. Therefore,

$$\varphi(t, \omega(U)) = \varphi\big(t, \omega(\Gamma^+(U))\big) \subset \omega(\Gamma^+(U)) = \omega(U),$$

which proves forward invariance for general U.

The omega limit set is obviously contained in $\mathrm{cl}(\Gamma^+(U))$. ■

Proposition 2.23 *Suppose $\Gamma^+_\tau(U)$ is precompact for some $\tau \geq 0$, then*

(i) $\omega(U)$ is compact and invariant;

(ii) $U \neq \varnothing$*, implies* $\omega(U) \neq \varnothing$*;*

(iii) U *connected and* $\mathbb{T} = \mathbb{R}$*, implies that* $\omega(U)$ *is connected;*

(iv) for all $x \in U$*,* $d(\varphi(t,x), \omega(U)) \to 0$*, as* $t \to \infty$*.*

Proof: (Compactness) For $t \geq \tau$,

$$\mathrm{cl}\big(\varphi([t,\infty), U))\big) \subset \mathrm{cl}(\Gamma_\tau^+(U))$$

is compact, and thus

$$\omega(U) \subset \bigcap_{t\geq\tau} \mathrm{cl}\big(\varphi([t,\infty), U))\big) \subset \mathrm{cl}(\Gamma_\tau^+(U))$$

is compact. Since the latter is an intersection of nested non-empty compact sets it is non-empty (this also implies Property (ii)).

(Invariance) We need to show that

$$\varphi(t, \omega(U)) = \omega(U), \quad \forall\, t \in \mathbb{T}^+.$$

Let $y \in \omega(U)$. By definition there exists a sequence $\{(x_n, t_n)\}$, $t_n \to \infty$, such that $d(\varphi(t_n, x_n), y) \to 0$, as $n \to \infty$. Given $t > 0$, assume that $t_n > t$, then the sequence $\{\varphi(t_n - t, x_n)\} \subset \Gamma^+(U)$ is well-defined, and since $\Gamma^+(U)$ is precompact, there exists a subsequence converging to some $z \in \omega(U)$ (a limit point has to be in $\omega(U)$ by definition). By continuity $\varphi(t,z) = y$, for all $y \in \omega(U)$.

Similarly, if $z \in \omega(U)$ then there exists a sequence $\{(x_n, t_n)\}$, $t_n \to \infty$, such that $d(\varphi(t_n, x_n), z) \to 0$, as $n \to \infty$. Let $y = \varphi(t,z)$. By continuity $d(\varphi(t_n + t, x_n), y) \to 0$, as $n \to \infty$. Thus, $y \in \omega(U)$.

Therefore $\omega(U)$ is invariant.

When $\mathbb{T} = \mathbb{R}$ and U is connected, then $\varphi\big(t, \Gamma^+(U)\big)$ is connected. Using the precompactness of $\Gamma^+(U)$ we derive that $\mathrm{cl}\big(\varphi\big(t, \Gamma^+(U)\big)\big)$ is a nested sequence of compact and connected sets. Therefore, $\bigcap_{t\geq 0} \mathrm{cl}\big(\varphi\big(t, \Gamma^+(U)\big)\big)$ is connected, which proves Property (iii).

If $d(\varphi(t,x), \omega(U)) \not\to 0$ as $t \to \infty$, then $d(\varphi(t_n, x), \omega(U) \geq \delta > 0$, for some sequence $t_n \to \infty$. Since $\Gamma^+(U)$ is precompact, the sequence $\{\varphi(t_n, x)\}$ has a limit point y, with $d(y, \omega(U)) > 0$, which is a contradiction and therefore proves Property (iv). ∎

Remark 2.24 If X is compact then $\Gamma^+_\tau(U)$ is precompact for all $U \subset X$ and for all $\tau \geq 0$.

Proposition 2.25 *Let $U, V \subset X$, then the omega limit sets satisfy the following list of properties:*

(i) *if $V \subset U$, then $\omega(V) \subset \omega(U)$;*

(ii) *$\omega(U \cup V) = \omega(U) \cup \omega(V)$ and $\omega(U \cap V) \subset \omega(U) \cap \omega(V)$;*

(iii) *if $V \subset \omega(U)$, then $\omega(V) \subset \omega(U)$;*

(iv) *$\omega(U) = \omega(\mathrm{cl}(U))$, i.e. $\mathrm{cl}(\omega(U)) = \omega(\mathrm{cl}(U))$;*

(v) *$\omega(U) = \omega\big(\varphi(t, U)\big)$ for all $t \in \mathbb{T}$.*

(vi) *if there exists a backward orbit $\gamma_x^- \subset U$, then $x \in \omega(U)$;*

Describing the limiting behavior of φ as $t \to -\infty$ is more involved since φ lacks backward uniqueness as well as continuity for $t \leq 0$ in general and backward images may be empty.

Definition 2.26 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. A point y is called a *alpha limit point* of a set $U \subset X$ under φ if there exist times $t_n \to -\infty$ and points $x_n \in U$, $y_n \in \varphi(t_n, x_n)$ such that $\lim_{n\to\infty} y_n = y$. The set of all alpha limit points y is called the *alpha limit set of* U and is denoted by $\alpha(U, \varphi)$.

As for omega limit sets we have a similar characterization for alpha limit sets.

Proposition 2.27 *Let $U \subset X$. Then*

$$\alpha(U) = \bigcap_{t \leq 0} \mathrm{cl}\big(\varphi((-\infty, t], U)\big) = \alpha\big(\Gamma^-(U)\big). \tag{15}$$

The alpha limit set $\alpha(U)$ is closed, forward invariant, and contained in $\mathrm{cl}(\Gamma^-(U))$. If $U \subset X$ is backward invariant, then

$$\alpha(U) = \bigcap_{t \leq 0} \mathrm{cl}\big(\varphi(t, U)\big), \tag{16}$$

and $\alpha(U) \subset \mathrm{cl}(U)$.

The proof is left as an exercise.

Definition 2.28 If γ_x is an orbit of φ, then the *alpha limit set of* γ_x is

$$\alpha_{\mathrm{o}}(\gamma_x^-) := \alpha\big(x, \varphi\|_{\mathrm{cl}(\gamma_x)}\big) = \bigcap_{t\leq 0} \mathrm{cl}\big(\gamma_x((-\infty, t])\big). \tag{17}$$

We emphasize that $\alpha_{\mathrm{o}}(\gamma_x^-)$ is defined only for (complete) backward orbits γ_x^-.

Proposition 2.29 *Let γ_x be a complete orbit. If γ_x^- is precompact, then $\alpha_{\mathrm{o}}(\gamma_x^-)$ is non-empty, compact and invariant.*

Proposition 2.30 *If S is an invariant set, then*

(i) $\omega(S) = \mathrm{cl}(S)$ *and in particular when S is closed, $S = \omega(S)$;*

(ii) *S precompact, implies that* $\mathrm{cl}(S) = \omega(S)$ *is a compact invariant set.*

If S is a strongly invariant, then

(iii) $\alpha(S) = \mathrm{cl}(S)$ *and in particular when S is closed, $S = \alpha(S) = \omega(S)$.*

Proof: Direct consequence of Equations (14) and (16). From Equation (14) it follows that $\mathrm{cl}(S) = \omega(S)$. By Proposition 2.23, $\omega(S)$ and thus $\mathrm{cl}(S)$ is a compact invariant set. ∎

Lemma 2.31 *Let S be forward invariant. Then,* $\mathrm{cl}(S)$ *is a closed forward invariant set.*

Proof: For all $t \geq 0$, that $\varphi(t, \mathrm{cl}(S)) \subset \mathrm{cl}(\varphi(t, S)) \subset \mathrm{cl}(S)$, which proves that $\mathrm{cl}(S)$ is forward invariant. ∎

Proposition 2.32 *If S is an invariant set, then*

(i) $\omega(S) = \mathrm{cl}(S)$ *and in particular when S is closed, $S = \omega(S)$;*

(ii) *S precompact, implies that* $\mathrm{cl}(S) = \omega(S)$ *is a compact invariant set.*

If S is a strongly invariant, then

(iii) $\alpha(S) = \mathrm{cl}(S)$ *and in particular when S is closed, $S = \alpha(S) = \omega(S)$.*

Proof: Direct consequence of Equations (14) and (16). From Equation (14) it follows that $\mathrm{cl}(S) = \omega(S)$. By Proposition 2.23, $\omega(S)$ and thus $\mathrm{cl}(S)$ is a compact invariant set. ∎

2.6 Another Reality Check

Omega limit sets for the logistic map.

Discuss the period doubling bifurcation.

Feigenbaum Number. Let r_n be the value of r where the n-th period doubling bifurcation occurs.

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.66920160\ldots$$

This is a universal number (for a 1-d map under weak hypothesis on derivatives of f one always gets the same number). It implies that there are bifurcations on all scales.

Describe the global dynamics during the period doubling bifurcations in terms of graphs.

Remark: We need to have a language in which we can talk about these decompositions when we only have a finite degree of precision in our measurements both in phase space and parameter space. Conley provides such a language with the concept of a Morse decomposition

2.7 Equivalences of dynamical systems

Clearly the dynamics of the logistic maps "changes" as we change parameter values. Some changes are not important (exact value of stable fixed point) while other changes are important (period doubling bifurcation). We want to make this precise. For simplicity I will restrict my attention for the moment to discrete dynamical systems.

Definition 2.33 Two discrete dynamical systems $\varphi : \mathbb{Z}^+ \times X \to X$ and $\psi : \mathbb{Z}^+ \times Y \to Y$ are *conjugate* if there exists a homeomorphism $h : X \to Y$ such that

$$h\big(\varphi(t,x)\big) = \psi\big(t, h(x)\big), \tag{18}$$

for all $t \in \mathbb{Z}^+$ and all $x \in X$. The system φ is *semiconjugate* to ψ if h is continuous and surjective.

Observe that if $f(x) = \varphi(1,x)$ and $g(x) = \psi(1,x)$, then the above relation reduces to $h \circ f = g \circ h$. Conjugacy and semiconjugacy is captured in the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{Z}^+ \times X & \stackrel{\varphi}{\longrightarrow} & X \\
{\scriptstyle \mathrm{id}\times h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
\mathbb{Z}^+ \times Y & \stackrel{\psi}{\longrightarrow} & Y
\end{array}
$$

Observe that under conjugacy invariant sets of one dynamical system are mapped to invariant sets of the other dynamical system.

Consider a parameterized family of dynamical systems, that is a continuous map

$$\varphi\colon \mathbb{Z}^+ \times X \times \Lambda \to X$$

where Λ is a connected metric space. Let

$$\varphi_\lambda\colon \mathbb{Z}^+ \times X \to X$$

be defined by $\varphi_\lambda(t, x) = \varphi(t, x, \lambda)$.

Definition 2.34 An element $\lambda_0 \in \Lambda$ is a *bifurcation point* if for any open neighborhood $U \subset \Lambda$ of λ_0, there exists $\lambda_1 \in \Lambda$ such that φ_{λ_0} is *not* conjugate to φ_{λ_1}.

Example 2.35 The points in parameter space at which the period doubling occurs are bifurcation point for the logistic map.

Fact: There exist parameterized families of dynamical systems where the set of bifurcation points contain Cantor sets of positive measure.

This has profound implications for modeling!

3 Attractor-Repeller Decompositions

3.1 Lattice of Invariant Sets

Definition 3.1 A *partially ordered set* or *poset* $(\mathsf{P}, \leq)$ is a set P with a binary relation $\leq$, called a *partial order*, which satisfies the following axioms:

(i) (reflexivity) $p \leq p$, for all $p \in \mathsf{P}$,

(ii) (anti-symmetry) $p \leq q$, and $q \leq p$, then $p = q$,

(iii) (transitivity) if $p \leq q$, and $q \leq r$, then $p \leq r$.

A relation $<$ which satisfies only the transitivity property and $p \not< p$ for all $p \in \mathsf{P}$ is called a *strict* partial order and is denoted by $(\mathsf{P}, <)$.

Posets and strict poset are equivalent. A partial order $\leq$ defines a strict order via: $p < q$ if and only if $p \leq q$ and $p \neq q$. Similarly, a strict order $<$ defines a partial order via: $p \leq q$ if and only if $p < q$, or $p = q$. The notations $(\mathsf{P}, \leq)$ and $(\mathsf{P}, <)$ are used interchangeably to emphasize order or strict order.

Example 3.2 $\mathsf{Invset}(X, \varphi)$ is a partially ordered set with respect to inclusion $\subseteq$. This is denote this by $\big(\mathsf{Invset}(X, \varphi), \subseteq\big)$.

Definition 3.3 Let $U \subset X$, then the *maximal invariant set in* U is defined by

$$\operatorname{Inv}(U, \varphi) := \bigcup_{S \in \mathsf{Invset}(U, \varphi)} S$$

Other characterizations of the maximal invariant set include

$$\operatorname{Inv}(U, \varphi) = \{x \in U \mid \exists \gamma_x \subset U\} = \sup \{S \subset \mathsf{Invset}(X, \varphi) \mid S \subset U\}.$$

Definition 3.4 Let P and P' be posets. A mapping $f : \mathsf{P} \to \mathsf{P}'$ is called *order preserving* if $f(p) \leq f(q)$ for all $p \leq q$ and *order reversing* if $f(p) \geq f(q)$ for all $p \leq q$.

Let $\mathsf{P}(X)$ denote the *power set* of X, that is, the set of all subsets of X. Observe that for a given dynamical system φ, the maximal invariant set can be viewed as defining an order-preserving morphism

$$\begin{aligned} \mathrm{Inv}\colon \mathsf{P}(X) &\to \mathsf{Invset}(X,\varphi) \\ U &\mapsto \mathrm{Inv}(U,\varphi). \end{aligned}$$

We can regard the infimum and supremum as binary operations on a poset P if for any two elements $p,q \in \mathsf{P}$ the infimum and supremum exist:

$$p \vee q = \sup(p,q), \quad p \wedge q = \inf(p,q). \tag{19}$$

A poset for which infimum and supremum exist for all pairs $p,q \in \mathsf{P}$ is called a lattice, and introduces an algebraic structure to P. The operation $\vee$ is called 'vee' or *join* and the operation $\wedge$ is called 'wedge' or *meet*. A lattice can also be defined independently as an agebraic structure.

Definition 3.5 A *lattice* $(\mathsf{L},\vee,\wedge)$ is a set L with the binary operations $\vee,\wedge :$ $\mathsf{L}\times\mathsf{L}\to\mathsf{L}$ satisfying the following axioms:

(i) (idempotent) $a \wedge a = a \vee a = a$ for all $a \in \mathsf{L}$,

(ii) (commutative) $a \wedge b = b \wedge a$, and $a \vee b = b \vee a$ for all $a,b \in \mathsf{L}$,

(iii) (associative) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$ for all $a,b,c \in \mathsf{L}$,

(iv) (absorption) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ for all $a,b \in \mathsf{L}$.

A *distributive lattice* satisfies the additional axiom

(v) (distributive) $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$ and $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$ for all $a,b,c \in \mathsf{L}$.

A lattice is *bounded* if there exist *neutral* elements 0 and 1 with property that

(vi) $0\wedge a = 0$, $0\vee a = a$, for all $a \in \mathsf{L}$, and $1\wedge a = a$, $1\vee a = 1$, for all $a \in \mathsf{L}$.

A set $\mathsf{K} \subset \mathsf{L}$ is a *sublattice* if $a \wedge b \in \mathsf{K}$ and $a \vee b \in \mathsf{K}$ for all $a,b \in \mathsf{K}$. A sublattice that contains 0 and 1 is called a $(0,1)$-*sublattice*.

Proposition 3.6 *The set* $\big(\mathsf{Invset}(X,\varphi),\vee,\wedge\big)$ *with*

$$S \vee S' = S \cup S', \quad S \wedge S' = \mathrm{Inv}(S\cap S',\varphi), \tag{20}$$

is a bounded distributive lattice. The sets $\varnothing$ *and* $S = \mathrm{Inv}(X,\varphi)$ *are the neutral elements.*

Remark: $S, S' \in \mathsf{Invset}(X,\varphi)$ does not imply that $S\cap S' \in \mathsf{Invset}(X,\varphi)$.

Lemma 3.7 *For any pair* $S, S' \in \mathsf{Invset}^-(X,\varphi)$*, or* $S, S' \in \mathsf{Invset}^+(X,\varphi)$

$$\begin{aligned}\mathrm{Inv}(S\cup S') &= \mathrm{Inv}(S)\cup\mathrm{Inv}(S') = \mathrm{Inv}(S)\vee\mathrm{Inv}(S');\\ \mathrm{Inv}(S\cap S') &= \mathrm{Inv}\big(\mathrm{Inv}(S)\cap\mathrm{Inv}(S')\big) = \mathrm{Inv}(S)\wedge\mathrm{Inv}(S').\end{aligned}$$

Proof: Let $S, S' \in \mathsf{Invset}^-(X,\varphi)$, or $S, S' \in \mathsf{Invset}^+(X,\varphi)$, then

$$\mathrm{Inv}(S)\cup\mathrm{Inv}(S') \subset \mathrm{Inv}(S\cup S').$$

As for the reversed inclusion we argue as follows. Let $x \in \mathrm{Inv}(S\cup S')$ and $\gamma_x \subset \mathrm{Inv}(S\cup S')$ a complete orbit. Since S and S' are both forward or backward invariant sets it follows that for all $y\in\gamma_x$ we have that $\gamma_y^\pm \subset S$ when $y\in S$, and $\gamma_y^\pm\subset S'$, when $y \in S'$. This implies that $y\in S$, or $y \in S'$ for all $y\in\gamma_x$, and thus $\gamma_x \in \mathrm{Inv}(S)$, or $\gamma_x\in\mathrm{Inv}(S')$. Consequently,

$$\mathrm{Inv}(S\cup S') \subset \mathrm{Inv}(S)\cup\mathrm{Inv}(S').$$

Combining these two inclusions we obtain $\mathrm{Inv}(S)\cup\mathrm{Inv}(S') = \mathrm{Inv}(S\cup S')$. In terms of join-operation $\vee$:

$$\mathrm{Inv}(S\cup S') = \mathrm{Inv}(S)\vee\mathrm{Inv}(S').$$

As for intersections we have that $\mathrm{Inv}(S)\cap\mathrm{Inv}(S')\subset S\cap S'$, and therefore

$$\mathrm{Inv}\big(\mathrm{Inv}(S)\cap\mathrm{Inv}(S')\big)\subset\mathrm{Inv}(S\cap S').$$

On the other hand $\mathrm{Inv}(S\cap S')\subset\mathrm{Inv}(S)$, and $\mathrm{Inv}(S\cap S')\subset\mathrm{Inv}(S')$, which implies $\mathrm{Inv}(S\cap S')\subset\mathrm{Inv}(S)\cap\mathrm{Inv}(S')$ and $\mathrm{Inv}(S\cap S')\subset\mathrm{Inv}\big(\mathrm{Inv}(S)\cap\mathrm{Inv}(S')\big)$. Combining the inclusions yields

$$\mathrm{Inv}(S\cap S') = \mathrm{Inv}\big(\mathrm{Inv}(S)\cap\mathrm{Inv}(S')\big),$$

and in terms of the meet-operation $\wedge$:

$$\mathrm{Inv}(S \cap S') = \mathrm{Inv}(S) \wedge \mathrm{Inv}(S'),$$

which proves the lemma. ■

Proof: That $\mathsf{Invset}(X, \varphi)$ is a bounded lattice is obvious.

To prove distributivity we use the fact that Inv is a lattice homomorphism. For $S, S', S'' \in \mathsf{Invset}(X, \varphi)$, then sets $S \cap S'$ and $S \cap S''$ are forward invariant. Then by Lemma 3.7

$$\begin{aligned}
(S \wedge S') \vee (S \wedge S'') &= \mathrm{Inv}(S \cap S', \varphi) \cup \mathrm{Inv}(S \cap S'', \varphi) \\
&= \mathrm{Inv}((S \cap S') \cup (S \cap S''), \varphi) \\
&= \mathrm{Inv}(S \cap (S' \cup S''), \varphi) = \mathrm{Inv}(S, \varphi) \wedge \mathrm{Inv}(S' \cup S'', \varphi) \\
&= S \wedge (S' \vee S''),
\end{aligned}$$

which completes the proof. ■

3.2 Attractors and repellers

From now on we assume that X is a compact, or locally compact metric space.

Definition 3.8 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. A compact set $N \subset X$ is a *trapping region* if N is forward invariant and there exists a $T > 0$ such that $\varphi(T, N) \subset \mathrm{int}(N)$.

Not an asymptotic definition.

Consider a parameterized family of dynamical systems

$$\varphi : \mathbb{T}^+ \times X \times \Lambda \to X$$

For applications we probably want to choose $T = 1$. Let $f \colon X \times \Lambda \to X$ be defined by $f_\lambda(x) = \varphi_\lambda(1, x)$.

Discuss robustness of trapping regions with respect to errors and perturbations in parameters! Assme N is a trapping region for f_{λ_0}. Then there exists $\epsilon > 0$ such that

$$B_\epsilon(f_{\lambda_0} N) \subset N$$

There exists $\delta > 0$ such that if $\|\lambda - \lambda_0\| < \delta$ then $\|f_\lambda(x) - f_{\lambda_0}(x)\| < \epsilon$. In which case N remains a trapping region.

Trapping regions are what are observable with respect to data!

Definition 3.9 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. A set $A \subset X$ is an *attractor* if there exists a trapping region $N \subset X$ such that

$$A = \mathrm{Inv}(N, \varphi).$$

The set of attractors is denoted by $\mathsf{Att}(X, \varphi)$.

The concept of a repeller is not a matter of time reversal and therefore requires a slight modification in its definition.

Definition 3.10 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. A compact set $N \subset X$ is a *repelling region* if N is backward invariant and there exists a $T < 0$ such that $\varphi(T, N) \subset \mathrm{int}(N)$.

Definition 3.11 A set $R \subset X$ is an *repeller* if there exists a repelling region $N \subset X$ such that

$$R = \mathrm{Inv}^+(N, \varphi).$$

The set of repellers is denoted by $\mathsf{Rep}(X, \varphi)$.

Example 3.12 The empty set can serve as a trapping region or as a repelling region, and the corresponding attractor or repeller is $\varnothing$.

Remark 3.13 Let $Y \subset X$ be a compact forward invariant set. The restriction

$$\varphi\|_Y : \mathbb{T}^+ \times Y \to Y,$$

is a well-defined dynamical system.

When considering this restricted system, interiors are taken relative to Y in determining trapping and repelling regions, and the corresponding attractors and repellers are denoted by $\mathsf{Att}(Y, \varphi\|_Y)$ and $\mathsf{Rep}(Y, \varphi\|_Y)$.

By Proposition 2.23 we have $\omega(Y, \varphi\|_Y) = \mathrm{Inv}(Y, \varphi\|_Y)$, which is an attractor in $\mathsf{Att}(Y, \varphi\|_Y)$ since Y is a trapping region. Note that since Y is forward invariant, the attractor of $\varphi\|_Y$ are exactly the attractors of φ restricted to Y. Therefore we write $\mathsf{Att}(Y, \varphi) = \mathsf{Att}(Y, \varphi\|_Y)$.

It is important to note that $A \in \mathsf{Att}(Y,\varphi)$ does not necessarily imply that $A \in \mathsf{Att}(X,\varphi)$. Consider the dynamical system $\varphi : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by the differential equation

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= y. \end{aligned}$$

Observe that $Y = [-1,1] \times \{0\}$ is a compact forward invariant set under φ and the point $(0,0) \in Y$ is an attractor for $\varphi\|_Y$. However, $\{(0,0)\} \notin \mathsf{Att}(\mathbb{R}^2,\varphi)$.

Proposition 3.14 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system.*

(i) Attractors are compact, invariant sets.

(ii) If N is a trapping region with corresponding attractor $A = \mathrm{Inv}(N,\varphi)$, then $A = \omega(N)$ and $A \subset \mathrm{int}(N)$. If $N \neq \varnothing$, then $A \neq \varnothing$.

Proof: Since attractors are defined as maximal invariant sets of a compact set, Proposition 2.32 implies that they are compact and invariant.

Let N be a trapping region with corresponding attractor $A = \mathrm{Inv}(N,\varphi)$. If $N = \varnothing$, by Proposition 2.23, $\omega(N)$ is nonempty and $\omega(N) = \mathrm{Inv}(N,\varphi) = A$. Since N is a trapping region, there exists $T > 0$ such that $\varphi(T,N) \subset \mathrm{int}(N)$. By the invariance of $A \subset N$ we have $A = \varphi(T,A) \subset \varphi(T,N) \subset \mathrm{int}(N)$. ■

3.2.1 Attracting neighborhoods

An attractor is defined as the maximal invariant set inside a trapping region. However, the forward invariance required for a trapping region can sometimes be difficult to establish. It is often useful to have a weaker condition which guarantees that a region contains an attractor.

Definition 3.15 Suppose $\varphi : \mathbb{T}^+ \times X \to X$ is a dynamical system. A compact set N is an *attracting neighborhood* if $\omega(N) \subset \mathrm{int}(N)$.

The following proposition, which is of interest in its own right, is used to establish that an attracting neighborhood must contain an attractor.

Proposition 3.16 *Suppose $\varphi : \mathbb{T}^+ \times X \to X$ is a dynamical system. A compact set N is an attracting neighborhood if and only if there exists $T > 0$ such that $\varphi(t,N) \subset \mathrm{int}(N)$ for all $t \geq T$.*

Proof: Suppose N is an attracting neighborhood so that $\omega(N) \subset \text{int}(N)$. Fix $x \in N$. We claim that there exists $\delta_x > 0$ and $T_x > 0$ such that $\varphi(t, y) \in \text{int}(N)$ for all $t \geq T_x$ and all $y \in B_{\delta_x}(x)$. Suppose not. Then we can choose $t_n \to \infty$ and $y_n \to x$ such that $\varphi(t_n, y_n) \notin \text{int}(N)$ and $\varphi(t_n, y_n) \to z \in \omega(N) \subset \text{int}(N)$. This is a contradiction since $N \setminus \text{int}(N)$ is closed. Therefore, by the compactness of N, we can choose $T > 0$ such that $\varphi(t, N) \subset \text{int}(N)$ for all $t \geq T$.

Now suppose there exists $T > 0$ such that $\varphi(t, N) \subset \text{int}(N)$ for all $t \geq T$. Fix $x \in \omega(N)$. Since $\omega(N)$ is invariant by Lemma 2.23, there exists an orbit $\gamma_x : \mathbb{T} \to \omega(N)$. Then $x = \varphi(T, \gamma_x(-T)) \in \text{int}(N)$. Thus $\omega(N) \subset \text{int}(N)$. ■

Theorem 3.17 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. If N is an attracting neighborhood, then $A = \omega(N)$ is an attractor. In particular, there exists a trapping region $N' \subset N$ such that $A = \omega(N')$. Moreover, if $N \neq \varnothing$, then $A \neq \varnothing$.*

Proof: Since A is compact by Proposition 2.23, there exists $\epsilon_0 > 0$ such that $\text{cl}(B_\epsilon(A)) \subset \text{int}(N)$ for all $0 < \epsilon < \epsilon_0$. Fix $0 < \epsilon < \epsilon_0$. Proposition **??** implies that $\omega(N) = \omega(\omega(N)) = \omega(A) \subset \omega(\text{cl}(B_\epsilon(A))) \subset \omega(N)$, which further implies that $\omega(\text{cl}(B_\epsilon(A))) = A \subset B_\epsilon(A)$. Thus $\text{cl}(B_\epsilon(A))$ is an attracting neighborhood.

By Proposition 3.16 there exists $T_\epsilon > 0$ such that $\varphi(t, \text{cl}(B_\epsilon(A))) \subset B_\epsilon(A)$ for all $t \geq T_\epsilon$. Define $N'_\epsilon = \varphi([0, T_\epsilon], \text{cl}(B_\epsilon(A)))$. By definition N'_ϵ is forward invariant and compact, but N'_ϵ may not be a subset of N.

We claim that ϵ can be chosen small enough so that $N'_\epsilon \subset N$. Suppose not and choose $\epsilon_n \to 0$ such that $N'_{\epsilon_n} \not\subset N$. Then there exists $x_n \in B_{\epsilon_n}(A)$ and $t_n \leq T_{\epsilon_n}$ such that $x_n \to x \in A$ and $\varphi(t_n, x_n) \notin N$ with $\varphi(t_n, x_n) \to z \notin \text{int}(N)$ since $S \setminus \text{int}(N)$ is closed. Passing to a subsequence either $t_n \to \tau < \infty$ or $t_n \to \infty$. In the former case $\varphi(\tau, x) = z$, which contradicts the invariance of A. In the latter case $z \in \omega(N) \subset \text{int}(N)$, a contradiction.

Choose $\epsilon > 0$ so that $N'_\epsilon \subset N$. By construction,

$$\begin{aligned}
\varphi(T_\epsilon, N'_\epsilon) &= \varphi\Big(T_\epsilon, \bigcup_{t \in [0, T_\epsilon]} \varphi\big(t, \text{cl}(B_\epsilon(A))\big)\Big) = \bigcup_{t \in [0, T_\epsilon]} \varphi\big(T_\epsilon, \varphi\big(t, \text{cl}(B_\epsilon(A))\big)\big) \\
&= \bigcup_{t \in [0, T_\epsilon]} \varphi\big(T_\epsilon + t, \text{cl}(B_\epsilon(A))\big) \subset B_\epsilon(A) \subset \text{int}(N'_\epsilon),
\end{aligned}$$

so that N'_ϵ is a trapping region, and

$$A = \omega(A) \subset \omega(N'_\epsilon) \subset \omega(N) = A.$$

By Proposition 2.23 , we have $A = \text{Inv}(N'_\epsilon)$ so that A is an attractor and if $N \neq \varnothing$, then $A \neq \varnothing$. ∎

The following corollary gives the characterization that is often used as a definition of attractor in Conley theory.

Corollary 3.18 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. A set A is an attractor if and only if there exists a precompact neighborhood U of A such that $A = \omega(U)$.*

Corollary 3.19 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. A set A is an attractor if and only if there exists $\epsilon_0 > 0$ such that $A = \omega(B_\epsilon(A))$ for every $0 < \epsilon < \epsilon_0$.*

Proof: Fix an attracting neighborhood N of A, then by compactness there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(A) \subset N$. Then $A = \omega(A) \subset \omega(B_\epsilon(A)) \subset \omega(N) = A$. ∎

Remark 3.20 As the following exercise demonstrates the existence of a compact neighborhood N of an invariant set A such that for all $x \in N$, $\omega(x) \subset A$ is not sufficient to imply that A is an attractor.

Example 3.21 Consider the flow on $\mathbb{R}$ generated by the differential equation $\dot{x} = x - x^3$. Let $A = \{-1, 0, 1\}$ and $N = [-1.1, -0.9] \cup [-0.1, 0.1] \cup [0.9, 1.1]$. Observe that N is a compact neighborhood of A such that $\omega(x) \subset A$ for all $x \in N$, but A is not an attractor.

3.2.2 Attractor within an attractor

If Y is a compact forward invariant set for $\varphi : \mathbb{T}^+ \times X \to X$, then Remark 3.13 points out that $A \in \mathsf{Att}(Y, \varphi\|_Y) = \mathsf{Att}(Y, \varphi)$ does not imply that $A \in \mathsf{Att}(X, \varphi)$ in general. However, the result is true in the special case that $Y \in \mathsf{Att}(X, \varphi)$, as the following theorem indicates.

Theorem 3.22 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. If $A \in \mathsf{Att}(X, \varphi)$ and $A' \in \mathsf{Att}(A, \varphi\|_A)$, then $A' \in \mathsf{Att}(X, \varphi)$.*

The proof of Theorem 3.22 is based on an alternative characterization of an attractor as an invariant set A which has a neighborhood N with the property that all backward orbits outside of A must leave N. A consequence of this property is that a relative attractor inside an attractor must also be an attractor in the larger system. First we need a few technical results.

Definition 3.23 A compact set $N \subset X$ is an isolating neighborhood if $S = \text{Inv}(N) \subset \text{int}(N)$.

Lemma 3.24 *Let N be an isolating neighborhood. For $T > 0$ define $N_T^+ = \{x \in N \mid \varphi([0,T],x) \subset N\}$. Then there exists $\delta_T > 0$ such that $B_{\delta_T}(S) \subset N_T^+$, i.e. $\varphi([0,T], B_{\delta_T}(S)) \subset N$.*

Proof: By continuity and the invariance of S, for each $x \in S$ there exists $\delta_x > 0$ such that $\varphi([0,T], B_{\delta_x}(x)) \subset N$. By compactness there exists finitely many such balls $B_{\delta_{x_i}}(x_i)$ such that $S \subset \cup_i B_{\delta_{x_i}}(x_i)$. Since this union is open, there exists $\delta > 0$ such that

$$B_\delta(S) \subset \cup_i B_{\delta_{x_i}}(x_i) \subset N_T^+.$$

∎

Lemma 3.25 *Let N be an isolating neighborhood. Assume that for all $x \in N \setminus \text{Inv}(N)$ there are no backward orbits $\gamma_x^- : \mathbb{T}^- \to N$. Then for every $\epsilon > 0$ there exists $T > 0$ such that there are no backward orbit segments $\gamma_x^- : [-T, 0] \to N$ through $x \in N \setminus B_\epsilon(\text{Inv}(N))$.*

Proof: Define

$$N_T^- = \left\{x \in N \,\middle|\, \exists \text{ a backward orbit segment } \gamma_x^- : [-T,0] \to N\right\}$$

for $T > 0$, and let

$$\epsilon_T^- := \sup_{x \in N_T^-} \{d(x,S)\}.$$

First we show that $\epsilon_T^- \to 0$ as $T \to \infty$. Suppose not. Then there exists a sequences $T_n \to \infty$ and $x_n \in N$ with $x_n \to x \in N \setminus \text{Inv}(N)$ and $\gamma_{x_n}^- : [-T_n, 0] \to N$. Observe that this implies that there exists a backward orbit $\gamma_x^- \subset N$, which is a contradiction. Finally choose $T > 0$ large enough so that $\epsilon_T^- < \epsilon$. ∎

Proposition 3.26 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. An invariant set A is an attractor if and only if there exists a compact neighborhood N of A such that there are no backward orbits $\gamma_x^- : \mathbb{T}^- \to N$ through $x \in N \setminus A$.*

Proof: The necessity of the condition is straightforward. Suppose A is an attractor and N is a trapping region so that $A = \omega(N)$. If there is a backward orbit $\gamma_x^- \subset N$ with $x \in N \setminus A$, then the complete orbit γ_x lies in N, which implies $x \in \text{Inv}(N) = A$, a contradiction.

Now suppose N is a neighborhood with the stated property. Since there are no backward orbits $\gamma_x^- : \mathbb{T}^- \to N$ with $x \in N \setminus A$ and N is a neighborhood of A, we have $A = \text{Inv}(N) \subset \text{int}(N)$. By Lemma 3.24, there exists $\delta_1 > 0$ such that $B_{\delta_1}(A) \subset N$ and $\varphi([0,1], B_{\delta_1}(A)) \subset N$. Fix $\epsilon < \delta_1$ and consider the neighborhood $B_\epsilon(A)$. By Lemma 3.25, there exists $T > 0$ such that there are no backward orbit segments $\gamma_x^- : [-T, 0] \to N$ for $x \in N \setminus B_\epsilon(A)$.

Now we construct an attracting neighborhood for A. By Lemma 3.24, there exists $\delta_{2T} > 0$ such that the neighborhood $U = B_{2T}(A)$ satisfies $\varphi([0, 2T], U) \subset N$. We show that U is an attracting neighborhood. Let $V_0 = \varphi(T, U)$. For each $x \in V_0$ there exists a backward orbit segment $\gamma_x^- : [-T, 0] \to N$. The definition of T from Lemma 3.25 implies that that $V_0 \subset B_\epsilon(A)$. Moreover, by our choice of $\epsilon < \delta_1$ we have $\varphi([0,1], B_\epsilon(A)) \subset N$ so that $V_1 = \varphi([0,1], V_0) \subset N$. Thus for each $x \in V_1$ there exists a backward orbit segment $\gamma_x^- : [-T-1, 0] \to N$ which implies that $\gamma_x^-([-T, 0]) \subset N$ so that $V_1 \subset B_\epsilon(A)$. We can repeat this argument inductively to prove that

$$V_k = \varphi([k-1, k], V_0) \subset B_\epsilon(A) \subset N, \quad \forall\, k > 0.$$

Therefore $\varphi([0,\infty), U) \subset N$. Finally, $A \subset U$ implies that $A = \omega(A) \subset \omega(U)$, and $\varphi([0,\infty), U) \subset N$ implies that $\omega(U) \subset \text{Inv}(N) = A$. Therefore $A = \omega(U)$, and A is an attractor by Theorem 3.17. ∎

Proof of Theorem 3.22. Let N be an attracting neighborhood for A in X. Choose a neighborhood $N' \subset N$ of A' in X such that $N' \cap A$ is an attracting neighborhood for A' in A. Suppose that γ_x^- is a backward orbit through $x \in N' \setminus A'$ such that $\gamma_x^- \subset N'$. Then $\gamma_x^- \subset N$ so that Proposition 2.25 implies $x \in \omega(N) = A$. Indeed the entire backward orbit γ_x^- is contained in $\omega(N) = A$ by a similar argument. Thus $\gamma_x^- \subset N' \cap A$, and again applying Proposition 2.25 gives $x \in \omega(N' \cap A) = A'$, a contradiction. The criterion in Proposition 3.26 now yields that A' is an attractor for φ in X. ∎

Corollary 3.27 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. If $N \subset X$ is a trapping region and $A' \in \mathsf{Att}(N, \varphi\|_N)$, then $A' \in \mathsf{Att}(X, \varphi)$.*

Proof: By definition there exists a trapping region $N' \subset N$ for $\varphi\|_N$, such that $A' = \mathrm{Inv}(N', \varphi\|_N)$. Since N is a trapping region for φ we set $A = \mathrm{Inv}(N, \varphi)$ and since $N' \subset N$, we have that $A' \subset A$. Indeed

$$A' = \mathrm{Inv}(N' \cap A, \varphi\|_N) = \mathrm{Inv}(N' \cap A, \varphi\|_A).$$

Moreover, since N' and A are forward invariant, it follows that $N' \cap A$ is forward invariant because $\mathsf{Invset}^+(X, \varphi)$ is a lattice with respect to $\cap$ and $\cup$. Therefore $N' \cap A$ is also forward invariant for $\varphi\|_A$. Now

$$\begin{aligned} \varphi(T, N' \cap A) &\subset \varphi(T, N') \cap \varphi(T, A) \\ &\subset \mathrm{int}_N(N') \cap A \\ &= \mathrm{int}_N(N') \cap \mathrm{int}_A(A) \\ &\subset \mathrm{int}_{N \cap A}(N' \cap A) \\ &= \mathrm{int}_A(N' \cap A), \end{aligned}$$

which shows that $N' \cap A$ is a trapping region for A' with respect to $\varphi\|_A$. Thus, $A' \in \mathsf{Att}(A, \varphi\|_A)$. From Theorem 3.22 it follows that $A' \in \mathsf{Att}(X, \varphi)$. ∎

3.3 Attractor-Repeller pairs

Definition 3.28 Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X. Let $A \in \mathsf{Att}(X, \varphi)$. The *dual repeller* of A is defined by

$$A^* := \mathrm{Inv}^+(X \setminus N, \varphi)$$

where $N \subset X$ is a trapping region for A. A pair (A, A^*) is called an *attractor-repeller pair in X*.

Example 3.29 Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X. Let $S = \omega(X)$, then $S \in \mathsf{Att}(X, \varphi)$. If $x \in X \setminus S$, then $\omega(x) \cap S \neq \varnothing$. Thus, $S^* = \varnothing$.

One of the themes of Conley theory is that it is easier to identify regions than invariant sets. As the following result, which is a restatement of the definition, indicates our characterization of attractor-repeller pairs emphasizes this philosophy.

Proposition 3.30 *Let X be a compact metric space and let $N \subset Y$ be a trapping region for $\varphi\colon \mathbb{T}^+ \times X \to X$. Then*

$$\big(\mathrm{Inv}(N,\varphi), \mathrm{Inv}^+(X \setminus N, \varphi)\big)$$

is an attractor-repeller pair in X.

We now show that dual repellers and dual attractors are well-defined by providing a characterization of these sets which is independent of the choice of trapping or repelling region.

For $A \subset X$ define

$$A^\oplus := \{x \in X \mid \omega(x) \cap A = \varnothing\}.$$

Lemma 3.31 *If A is invariant, then $A^\oplus \in \mathsf{Invset}^+(X,\varphi)$.*

Proposition 3.32 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact metric space X. For an attractor A, the dual repeller A^* is well-defined, compact, and characterized by*

$$A^* = \{x \in X \mid \omega(x) \cap A = \varnothing\}.$$

Moreover, if X is invariant, then A^* is strongly invariant.

Proof: Let N be a trapping region for A with $A^* = \mathrm{Inv}(X \setminus N)$. We first show that $A^\oplus \subset A^*$. If $x \in N$, then since N is a trapping region, $\omega(x) \subset \omega(N) = A$, and therefore $A^\oplus \subset X \setminus N$. Moreover, $A^\oplus$ is forward invariant by Lemma 3.31 so that $A^\oplus \subset \mathrm{Inv}^+(X \setminus N) = A^*$ since $\mathrm{Inv}^+(X \setminus N)$ is the maximal forward invariant set in $X \setminus N$.

To see that $A^* \subset A^\oplus$ we argue as follows. Since $A^* \subset X \setminus N$, it holds that $\mathrm{cl}(A^*) \subset \mathrm{cl}(X \setminus N)$, and $\mathrm{cl}(A^*)$ is forward invariant by Lemma 2.31. We now show that $\mathrm{cl}(A^*) \subset X \setminus N$. Indeed, if $x \in \mathrm{cl}(A^*) \cap N$, then since N is a trapping region, $\varphi(T,x) \in \mathrm{int}(N)$ for some $T > 0$. However, $\varphi(T,x) \in \mathrm{cl}(A^*) \subset \mathrm{cl}(X \setminus N)$, which is a contradiction. Therefore $A^* \subset \mathrm{cl}(A^*) \subset \mathrm{Inv}^+(X \setminus N) = A^*$ so that $A^* = \mathrm{cl}(A^*)$ and hence is compact. If $x \in A^*$, then $\omega(x) \subset \omega(A^*) \subset \mathrm{cl}(A^*) = A^* \subset X \setminus N$, which implies $x \in A^\oplus$. Therefore $A^* \subset A^\oplus$. Combining these inclusions gives that $A^* = A^\oplus$. ■

The following proposition provides a correspondence between the attractor-repeller pairs in a forward invariant set X and the attractor-repeller pairs in $S = \mathrm{Inv}(X,\varphi)$. Therefore, to develop the theory of attractor-repeller pairs in a dynamical system, it is sufficient to consider systems defined on invariant sets, i.e. surjective systems.

Proposition 3.33 *Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X. Let $S = \omega(X, \varphi)$ be the global attractor. Then there is a bijective correspondence between the attractor-repeller pairs in S and X. Specifically, a set $A \subset X$ is an attractor in X if and only if $A \subset S$ and is an attractor in S. Moreover, if A^*_S is the dual repeller of A in S and A^* is the dual repeller of A in X, then*

$$A^*_S = A^* \cap S. \tag{21}$$

Theorem 3.34 *Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X. Let $A, R \subset X$, then the following statements are equivalent.*

(i) *(A, R) is an attractor-repeller pair.*

(ii) *A and R are disjoint, compact sets with A invariant and R both forward and backward invariant such that for every $x \in S \setminus (A \cup R)$ and every (complete) orbit γ_x through x we have $\alpha_o(\gamma_x^-) \subset R$ and $\omega(x) \subset A$.*

Proof: (i)⇒(ii) Suppose (A, R) is an attractor-repeller pair. Proposition 3.32 implies that A, R are compact and disjoint. By Corollary 3.19, there exists $\epsilon_0 > 0$ such that $\omega(B_\epsilon(A)) = A$ for all $0 < \epsilon < \epsilon_0$. By Proposition 3.32 if $x \notin A \cup R$, then $\omega(x) \cap A \neq \varnothing$, which implies that there exists $T > 0$ such that $d(\varphi(T, x), A) < \epsilon_0$. Hence

$$\omega(x) = \omega(\varphi(T, x)) \subset \omega(B_{\epsilon_0}(A)) = A.$$

We also need to show that for every backward orbit γ_x^- we have $\alpha_o(\gamma_x^-) \subset R$. By Theorem 3.17, there exists a trapping region N for A with $N \subset B_{\min\{\epsilon_0, d(x,A)\}}(A)$. Since N is forward invariant, $X \setminus N$ is backward invariant, and hence $x \in X \setminus N$ implies $\gamma_x^- \subset X \setminus N$ whenever such a backward orbit exists. By Proposition 2.29, $\alpha_o(\gamma_x^-)$ is invariant. Therefore

$$\alpha_o(\gamma_x^-) \subset \operatorname{Inv}(X \setminus N) \subset \operatorname{Inv}^+(X \setminus N) = R,$$

since by Proposition 3.32 the dual repeller is well-defined independent of the choice of trapping region.

(i)⇒(ii) Suppose A and R are disjoint, compact sets in S as in statement (ii). Note that the conditions on A, R imply that $X = R \cup W^s(A, \varphi)$ and $R \cap W^s(A, \varphi) = \varnothing$. We begin by showing that $R = A^\oplus$. Indeed, if $x \in R$, then by forward invariance $\omega(x) \subset \omega(R) = R$ so that $\omega(x) \cap A = \varnothing$ which implies $R \subset A^{\oplus}$. Conversely, if $x \in A^{\oplus}$, then $x \notin W^s(A, \varphi) = S \setminus R$, and thus $x \in R$ and $A^{\oplus} \subset R$.

Now the goal is to show that A is an attractor. If so, then by Proposition 3.32 $R = A^{\oplus} = A^*$ is its dual repeller, and hence (A, R) is an attractor-repeller pair. Fix a compact neighborhood U of A such that $A \subset \mathrm{int}(U)$ and $U \cap R = \varnothing$. The rest of the proof argues that U is an attracting neighborhood of A.

Since R is both forward and backward invariant it follows that $X \setminus R$ is also both forward and backward invariant. Therefore, $\omega(U) \subset \mathrm{cl}(X \setminus R)$. We divide the proof into two steps. First we show that $\omega(U) \cap R = \varnothing$. The second step is to show that $\omega(U) = A$.

Suppose $y \in \omega(U) \cap R$. By definition, there exists a sequence (t_n, x_n) with $x_n \in U$ such that $y_n = \varphi(t_n, x_n) \to y$ and $t_n \to \infty$ as $n \to \infty$. The forward invariance of $X \setminus R$ implies $V = \varphi([0,1], U) \subset X \setminus R$. Since V and R are disjoint, compact sets, we can assume $y_n \notin V$ for all $n \geq 1$. Let $T_n = \sup\{t \in [0, t_n] \mid \varphi(t, x_n) \in U\}$. The compactness of V and $y_n = \varphi(t_n, x_n) \notin V$ imply $T_n + 1 < t_n$. Define $w_n = \varphi(T_n, x_n)$ and $z_n = \varphi(T_n + 1, x_n)$. Since U is compact, by the definition of T_n, we have $w_n \in U$. Also, by definition of T_n, we have $z_n \notin U$ so that $z_n \in \mathrm{cl}(U^c) \cap V$. Passing to a subsequence if necessary, $w_n \to w \in U$ as $n \to \infty$. The uniform continuity of φ implies $\varphi(t, w_n)$ converges to $\varphi(t, w)$ uniformly on $[0, 1]$. Let $z = \lim_{n\to\infty} z_n = \varphi(1, w) \in V \subset X \setminus R$. The uniform continuity of φ again implies $\varphi(t, z_n) \to \varphi(t, z)$ for all $t \geq 0$, and the convergence is uniform on compact intervals $[0, T]$.

Define $\tau_n = t_n - T_n - 1 > 0$. We claim that $\tau_n \to \infty$ as $n \to \infty$. Suppose not. Then there exists a subsequence such that $\tau_{n_k} \to \tau$ as $k \to \infty$. Thus $\varphi(\tau_{n_k}, z_{n_k}) = \varphi(t_{n_k}, x_{n_k}) \to \varphi(\tau, z) = y \in R$, which contradicts the forward invariance of $X \setminus R$. Thus $\tau_n \to \infty$ as $n \to \infty$, and by passing to a subsequence if necessary, we can assume that τ_n is increasing. By definition of T_n, we have $\varphi(t, x_n) \in U^c$ for $t \in (T_n, t_n]$, and hence $\varphi(t, z_n) \in U^c$ for $t \in [0, \tau_n]$. Therefore $\varphi(\tau_i, z_n) \in U^c$ for all $n \geq i$, which implies that $\varphi(\tau_i, z) = \lim_{n\to\infty} \varphi(\tau_i, z_n) \in \mathrm{cl}(U^c)$ for all $i \geq 1$.

Since $\varphi(\tau_i, z) \in \mathrm{cl}(U^c)$ for all $i \geq 1$, there exists a subsequence i_k such that $\varphi(\tau_{i_k}, z) \to p \in \mathrm{cl}(U^c)$ as $k \to \infty$. Since $\tau_{i_k} \to \infty$ as $k \to \infty$, we have $p \in \omega(z)$. Therefore $p \in \omega(z) \cap \mathrm{cl}(U^c)$. By assumption $\omega(z) \subset A$ for any point $z \in X \setminus R$, and hence $\omega(z) \cap \mathrm{cl}(U^c) = \varnothing$, which is a contradiction. This proves the first claim that $\omega(U) \subset X \setminus R$.

The second step uses the invariance of $\omega(U)$. Let $x \in \omega(U) \cap (U \setminus A)$, then by the previous step $x \in X \setminus (A \cup R)$. Invariance of $\omega(U)$ implies that there exists a complete orbit $\gamma_x \subset \omega(U)$ so that $\alpha_{\mathrm{o}}(\gamma_x^-) \subset \omega(U)$. However, by assumption, every orbit through $x \in X \setminus (A \cup R)$ has the property that $\alpha_{\mathrm{o}}(\gamma_x^-) \subset R$ with $\alpha_{\mathrm{o}}(\gamma_x^-) \neq \varnothing$ by Proposition 2.29. Therefore $\alpha_{\mathrm{o}}(\gamma_x^-) \subset R \cap \omega(U)$ contradicts the fact that $\omega(U) \subset X \setminus R$. This shows that $\omega(U) \cap (U \setminus A) = \varnothing$, and thus $\omega(U) = A$. ∎

Corollary 3.35 *Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X and and let $A, R \subset X$. If φ is a surjective system (i.e. X is invariant), then the following statements are equivalent.*

(i) *(A, R) is an attractor-repeller pair in X.*

(ii) *A and R are disjoint, compact invariant sets in X such that for every $x \in X \setminus (A \cup R)$ and every orbit γ_x through x we have $\alpha_{\mathrm{o}}(\gamma_x^-) \subset R$ and $\omega(x) \subset A$.*

Remark 3.36 In general, the existence of a decomposition $X = A \cup R \cup C(R, A)$ where A, R are disjoint, compact, and A invariant and R both forward and backward invariant, implies $C(R, A) \cap R = \varnothing$. However, the existence of such a decomposition is not equivalent to an attractor-repeller pair since it can happen that $C(A, A) \not\subset A$. This distinction arises from the fact that $x \in C(R, A)$ implies only the existence an orbit connecting R to A through x, whereas Theorem 3.34 asserts that for an attractor-repeller pair every orbit outside of $A \cup R$ connects R to A. If the system is invertible, then $C(A, A) = A$, $C(R, A) \cap A = \varnothing$, and (A, R) is an attractor-repeller pair.

Corollary 3.37 *Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X and let $A \in \mathsf{Att}(X, \varphi)$. A compact set $N \subset X$ is an attracting neighborhood for A if and only if N is a neighborhood of A and $N \cap A^* = \varnothing$.*

Proposition 3.38 *Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X. Given an attractor-repeller pair (A, A^*) the following statements hold.*

(i) *A and A^* are disjoint.*

(ii) *$A = \mathrm{Inv}(X, \varphi) = \omega(X)$ if and only if $A^* = \varnothing$, and $A = \varnothing$ if and only if $A^* = X$.*

(iii) *If* $x \in X$ *such that* $\omega(x) \cap A \neq \varnothing$*, then* $\omega(x) \subset A$*.*

(iv) *If* $x \in X$ *and* γ_x *is a complete orbit in* X *such that* $\alpha_o(\gamma_x^-) \cap A^* \neq \varnothing$*, then* $\alpha_o(\gamma_x^-) \subset A^*$*.*

(v) *If* A' *is an attractor with* $A \subsetneq A'$*, then* $A' \cap A^* \neq \varnothing$*.*

Give combinatorial representation of AR pair.

3.4 Lyapunov functions

Definition 3.39 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact metric space X and let $\{U_i\}$ be a collection of pairwise disjoint subsets of X. A continuous function $V : X \to \mathbb{R}$ is a *Lyapunov function* for $\big(X, \{U_i\}\big)$ if

(i) for each forward orbit $\gamma_x^+ \subset X$ the function $V \circ \gamma_x^+ \colon \mathbb{T}^+ \to \mathbb{R}$ is nonincreasing,

(ii) V is constant on each U_i, and

(iii) $V(\gamma_x^+(t)) < V(x)$ for all $t > 0$ and $x \in X \setminus (\bigcup_i U_i)$.

If a Lyapunov function exists, the dynamics is said to be *gradient-like* on $X \setminus (\bigcup_i U_i)$.

Remark 3.40 In the above definition the space X may be replaced by an arbitrary set $N \subset X$. Under the same conditions (i)-(iii), the notion of Lyapunov function for $\big(N, \{U_i\}\big)$ is well-defined.

Proposition 3.41 *Suppose* V *is a Lyapunov function for* $\big(X, \{U_i\}\big)$ *and* $\gamma_x \colon \mathbb{T} \to X$ *is a complete orbit. Then* $V \circ \gamma_x \colon \mathbb{T} \to \mathbb{R}$ *is nonincreasing. Moreover,* $V \circ \gamma_x$ *is strictly decreasing on any interval* $I \in \mathbb{T}$ *with* $\gamma_x^+(I) \subset X \setminus (\bigcup_i U_i)$*.*

Proof: By part (i) of the definition, $V \circ \gamma_x$ is decreasing on $[T, \infty)$ for all $T \in \mathbb{T}$, and hence decreasing on all of $\mathbb{T}$. Suppose $I = [a, b]$ and choose $t_0, t_1 \in [a, b]$. Consider the orbit $\gamma_{\gamma_x(t_0)}$. Then $V \circ \gamma_x$ on $[t_0, t_1]$ is the same as $V \circ \gamma_{\gamma_x(t_0)}$ on $[0, t_1 - t_0]$. By part (iii) of the definition,

$$V \circ \gamma_x(t_0) = V \circ \gamma_{\gamma_x(t_0)}(0) > V \circ \gamma_{\gamma_x(t_0)}(t_1 - t_0) = V \circ \gamma_x(t_1),$$

which completes the proof. ■

For arbitrary $(X, \{U_i\})$ a Lyapunov function need not exist in general, since there need not be any order relation between the sets U_i in the dynamics. A primary goal is to establish the equivalence between an attractor-repeller pair and a Lyapunov function for a pair of sets. First we prove a general theorem that explicitly describes the order characterized by a Lyapunov function.

Theorem 3.42 (LaSalle's Invariance Principle) *Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X. Let $\{U_i\}$ be a collection of pairwise disjoint subsets of X. Suppose V is a Lyapunov function for $(X, \{U_i\})$ and $\sigma_i = V(U_i)$. Let $x \in X$.*

(i) There exists p such that $\omega(x) \subset V^{-1}(\sigma_p)$. For every orbit γ_x there exists q such that $\alpha_o(\gamma_x^-) \subset V^{-1}(\sigma_q)$.

(ii) If the numbers σ_i are distinct, then p, q are unique with $\alpha_o(\gamma_x^-) \subset \mathrm{Inv}(U_q, \varphi)$ and $\omega(x) \subset \mathrm{Inv}(U_p, \varphi)$, and if $x \in X \setminus (\bigcup_i \mathrm{Inv}(U_i, \varphi))$, then $V(U_q) > V(U_p)$.

Proof: Let γ_x^+ be the forward orbit through x. Since $V \circ \gamma_x^+$ is monotone and V is bounded, $V(\gamma_x^+(t)) \to C \in \mathbb{R}$ as $t \to \infty$. If $y \in \omega(x)$, then there exists $t_n \to \infty$ as $n \to \infty$ such that $\gamma_x^+(t_n) \to y$ and hence $V(\gamma_x^+(t_n)) \to V(y) = C$. Therefore, V is constant on $\omega(x)$. By part (iii) of Definition 3.39, we have $\omega(x) \subset \bigcup_i U_i$, which implies $V(\omega(x)) \in V(\bigcup_i U_i) = \{\sigma_i\}$. If γ_x is a complete orbit through x, then $V \circ \gamma_x$ is decreasing on $\mathbb{T}$ by Proposition 3.41. Now a similar argument proves that $V(\alpha_o(\gamma_x^-)) \in \{\sigma_i\}$.

If the numbers σ_i are distinct, then $\omega(x) \subset \bigcup_i U_i$ and $\omega(x) \subset V^{-1}(\sigma_p)$ imply that $\omega(x) \subset U_p$ and by invariance $\omega(x) \subset \mathrm{Inv}(U_p)$. A similar argument implies $\alpha_o(\gamma_x^-) \subset \mathrm{Inv}(U_q)$. Finally, if $x \in X \setminus (\bigcup_i \mathrm{Inv}(U_i))$, then there exists $t \in \mathbb{T}$ such that $\gamma_x(t) \in X \setminus (\bigcup_i U_i)$. By definition, $\sigma_q = V(\alpha_o(\gamma_x^-)) \geq V(\gamma_x(t)) > V(\gamma_x(t+1)) \geq V(\omega(x)) = \sigma_p$. ■

3.4.1 Lyapunov functions for attractor-repeller pairs

Theorem 3.43 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact metric space X. If (A, A^*) is an attractor-repeller pair in X, then there exists a Lyapunov function $V : X \to [0, 1]$ for $\big(X, \{A, A^*\}\big)$.*

Theorem 3.44 *Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X and let $A, R \subset X$. Then the following statements are equivalent.*

(i) (A, R) *is an attractor-repeller pair in X.*

(ii) There exists a Lyapunov function $V : X \to [0, 1]$ for $\big(X, \{A, R\}\big)$, where $V^{-1}(0) = A$ and $V^{-1}(1) = R$.

(iii) There exists a Lyapunov function $V : X \to [0, 1]$ for $\big(X, \{U_A, U_R\}\big)$, where $V^{-1}(0) = U_A$ and $V^{-1}(1) = U_R$ with $\mathrm{Inv}(U_A) = A$ and $\mathrm{Inv}(U_R) = R$.

Proof: By Theorem 3.43, (i) implies (ii), and (ii) trivially implies (iii). So we must prove (iii) implies (i). Since A, R are disjoint, compact invariant sets, by Theorem 3.34, we need to show that if γ_x is an orbit through $x \in X \setminus (A \cup R)$ then $\alpha_{\mathrm{o}}(\gamma_x^-) \subset R$ and $\omega(x) \subset A$. By Theorem 3.42, we have that either $\omega(x) \subset V^{-1}(0)$ or $\omega(x) \subset V^{-1}(1)$. The invariance of $\omega(x)$ implies $\omega(x) \subset \mathrm{Inv}(V^{-1}(0)) = A$ or $\omega(x) \subset \mathrm{Inv}(V^{-1}(1)) = R$. Since V is decreasing and $V(x) < 1$, we have $\omega(x) \subset A$. A similar argument proves that $\alpha_{\mathrm{o}}(\gamma_x^-) \subset R$. ■

3.5 Attracting blocks

An alternative method of proving Theorem 3.44 is to use the Lyapunov function to construct a trapping region of the form $N = V^{-1}([0, \epsilon])$. From the definition of Lyapunov function, N satisfies a property that is stronger than a trapping region; it is mapped immediately into its interior for all positive times. Likewise, we can construct a repelling region that maps immediately into its interior for all negative times.

Definition 3.45 Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system. A compact set N is an *attracting block* if $\varphi(t, N) \subset \mathrm{int}(N)$ for all $t > 0$. A compact set $N \subset X$ is an *repelling block* if $\varphi(t, N) \subset \mathrm{int}(N)$ for all $t < 0$.

The set of attracting blocks is denoted by $\mathsf{ABlock}(X, \varphi)$ and the set of repelling blocks is denoted by $\mathsf{RBlock}(X, \varphi)$.

Proposition 3.46 *Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X. Suppose $V : X \to [0, 1]$ is a Lyapunov function for the attractor-repeller pair (A, A^*).*

(i) Then for every $\epsilon \in (0,1)$ the set $V^{-1}([0,\epsilon])$ is an attracting block for A and $V^{-1}([\epsilon,1])$ is a repelling block for A^*.

(ii) For any neighborhood U of A there exists an attracting block $N \subset U$. Similarly, for any neighborhood U of A^* there exists a repelling block $N \subset U$.

Proof: Let $N = V^{-1}([0,\epsilon])$. For $x \in N \setminus A$ we have $V(\varphi(t,x)) < V(x) \leq \epsilon$ for all $t > 0$, and therefore $\varphi(t,x) \subset V^{-1}([0,\epsilon))$. This yields

$$\varphi(t,N) \subset V^{-1}([0,\epsilon)) \subset \mathrm{int}(N) \text{ for all } t > 0,$$

which proves that N is an attracting block. The proof that $V^{-1}([\epsilon,1])$ is a repelling block is analogous.

To prove (ii) note that we can choose $\epsilon > 0$ such that $N = V^{-1}([0,\epsilon]) \subset U$. Indeed, if not, then there exists a sequence of points $x_n \in (S \setminus U) \cap V^{-1}([0,\epsilon_n])$ with $\epsilon_n \to 0$ as $n \to \infty$. By definition $V(x_n) \to 0$, and by the compactness of S, and passing to a subsequence if necessary, $x_n \to x$ with $x \in S \setminus U$ so that $x \notin A$, which contradicts $V(x) = 0$. ■

Corollary 3.47 *Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X. If $A \in \mathsf{Att}(X,\varphi)$, then its dual repeller A^* is in $\mathsf{Rep}(X,\varphi)$. Likewise, if $R \in \mathsf{Rep}(X,\varphi)$, then its dual attactor R^* is in $\mathsf{Att}(X,\varphi)$.*

Proof: By Theorem 3.44 there exists a Lyapunov function V for the pair (A, A^*), and Proposition 3.46 implies that $V^{-1}([1/2,1])$ is a repelling block. Since repelling blocks are repelling regions and $A^* = \mathrm{Inv}(V^{-1}([1/2,1])) = \mathrm{Inv}(V^{-1}(1)) = V^{-1}(1)$, we have that A^* is a repeller by Definition 3.9. The analogous argument implies R^* is an attractor. ■

4 Lattice structures for attractor-repeller decompositions

Definition 4.1 Do example of dynamics on square. Discuss set of attractor-repeller pairs.

4.1 Lattices of attractors and repellers

Proposition 4.2 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact metric space X.*

(i) The collection of attracting blocks $(\mathsf{ABlock}(X,\varphi), \cup, \cap)$ is a bounded, distributive sublattice of $\mathsf{P}(X)$ with neutral elements $\varnothing$ and X.

(ii) The collection of repelling blocks $(\mathsf{RBlock}(X,\varphi), \cup, \cap)$ is a bounded, distributive sublattice of $\mathsf{P}(X)$ with neutral elements $\varnothing$ and X.

(iii) If $N \in \mathsf{ABlock}(X,\varphi)$, then $\mathrm{cl}(X \setminus N) \in \mathsf{RBlock}(X,\varphi)$, and if $K \in \mathsf{RBlock}(X,\varphi)$, then $\mathrm{cl}(X \setminus K) \in \mathsf{ABlock}(X,\varphi)$.

Proof: To proof (i) observe that $\mathsf{ABlock}(X,\varphi) \subset \mathsf{P}(X)$ and $\varnothing, X \in \mathsf{ABlock}(X,\varphi)$ so that it is sufficient to demonstrate that $\mathsf{ABlock}(X,\varphi)$ is closed with respect to $\cup$ and $\cap$. Let $N, K \in \mathsf{ABlock}(X,\varphi)$ and $t > 0$. Then

$$\varphi(t, N \cap K) \subset \varphi(t, N) \cap \varphi(t, K)$$

so that

$$\varphi(t, N \cap K) \subset \mathrm{int}(N) \cap \mathrm{int}(K) = \mathrm{int}(N \cap K).$$

Hence $N \cap K$ is an attracting block. Likewise

$$\varphi(t, N \cup K) = \varphi(t, N) \cup \varphi(t, K) \subset \mathrm{int}(N) \cup \mathrm{int}(K) \subset \mathrm{int}(N \cup K),$$

and $N \cup K$ is an attracting block.

The proof of (ii) is similar.

To prove (iii) let $N \in \mathsf{ABlock}(X,\varphi)$. Then $\varphi(s, N) \subset \mathrm{int}(N)$ for all $s > 0$. Suppose $K = \mathrm{cl}(X \setminus N)$ is not a repelling block. Then there exists $x \in K$, $t < 0$, and $y \in \varphi(t, x)$ such that $y \notin \mathrm{int}(K)$. Hence

$$y \in \mathrm{cl}(X \setminus K) = \mathrm{cl}(X \setminus \mathrm{cl}(X \setminus N)) = \mathrm{cl}(\mathrm{int}(N)) \subset N.$$

This implies $x \in \varphi(|t|, y) \subset \mathrm{int}(N)$ which contradicts $x \in K = \mathrm{cl}(X \setminus N)$. The proof starting with a repelling block is analogous. ■

Lemma 4.3 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact metric space X. Let $A, B \in \mathsf{Att}(X, \varphi)$ with attracting blocks $N, K \subset X$, respectively. Then*

$$A \vee B = \mathrm{Inv}(N \cup K, \varphi) \quad \text{and} \quad A \wedge B = \mathrm{Inv}(N \cap K, \varphi).$$

Let $R, S \in \mathsf{Rep}(X, \varphi)$ with repelling blocks $L, M \subset X$, respectively. Then

$$R \vee S = \mathrm{Inv}^+(L \cup M, \varphi) \quad \text{and} \quad R \wedge S = \mathrm{Inv}^+(L \cap M, \varphi).$$

Proof: Lemma 3.7 implies that

$$A \vee B = A \cup B = \mathrm{Inv}(N, \varphi) \cup \mathrm{Inv}(K, \varphi) = \mathrm{Inv}(N \cup K, \varphi) \quad \text{and}$$

$$A \wedge B = \mathrm{Inv}(A \cap B, \varphi) = \mathrm{Inv}(\mathrm{Inv}(N, \varphi) \cap \mathrm{Inv}(K, \varphi), \varphi) = \mathrm{Inv}(N \cap K, \varphi).$$

The argument for repellers is similar. ■

Proposition 4.4 *Let $\varphi : \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact metric space X.*

(i) The collection of attractors $(\mathsf{Att}(X, \varphi), \vee, \wedge)$ is a bounded, distributive sublattice of $(\mathsf{Invset}(X, \varphi), \vee, \wedge)$ with neutral elements $\varnothing$ and $\mathrm{Inv}(X, \varphi)$.

(ii) The collection of repellers $(\mathsf{Rep}(X, \varphi), \vee, \wedge)$ is a bounded, distributive sublattice of $(\mathsf{Invset}(X, \varphi), \vee, \wedge)$ with neutral elements $\varnothing$ and X.

(iii) The mapping $\mathsf{Att}(X, \varphi) \to \mathsf{Rep}(X, \varphi)$ defined by $A \mapsto A^*$ is a $(0, 1)$-anti-isomorphism.

Proof: Let $A, B \in \mathsf{Att}(X, \varphi)$, then by Proposition 3.46 there exist attracting blocks $N, K \subset X$ for A, B respectively.

To prove (i) observe that $\mathsf{Att}(X, \varphi) \subset \mathsf{Invset}(X, \varphi)$ and $\varnothing, \mathrm{Inv}(X, \varphi) \in \mathsf{Att}(X, \varphi)$ so that it is sufficient to prove that $\mathsf{Att}(X, \varphi)$ is closed with respect to $\vee$ and $\wedge$. This follows from Lemma 4.3. The proof of (ii) is similar using Lemma 4.3 for repellers.

Turning to the demonstration of (iii), Proposition 4.2 and Definition 3.28 followed by Lemma 3.7 leads to following identities. Let $A, B \in \mathsf{Att}(X, \varphi)$ be attractors with isolating blocks N and K respectively. Then by Lemma 4.3, $N \cup K$ is an attracting block for $A \vee B$ and the dual repeller $(A \vee B)^*$ is given by

$$\begin{aligned}(A \vee B)^* &= \mathrm{Inv}^+(X \setminus (N \cup K)) \\ &= \mathrm{Inv}^+((X \setminus N) \cap (X \setminus K)) \\ &= \mathrm{Inv}^+(\mathrm{Inv}^+(X \setminus N) \cap \mathrm{Inv}^+(X \setminus K)) \\ &= \mathrm{Inv}^+(A^* \cap B^*) = A^* \cap B^* = A^* \wedge B^*.\end{aligned}$$

Similarly, $N \cup K$ is an isolating block for $A \wedge B$. The dual repeller $(A \wedge B)^*$ is given by

$$\begin{aligned}(A \wedge B)^* &= \mathrm{Inv}^+(X \setminus (N \cap K)) \\ &= \mathrm{Inv}^+((X \setminus N) \cup (X \setminus K)) \\ &= \mathrm{Inv}^+(X \setminus N) \cup \mathrm{Inv}^+(X \setminus K) = A^* \cup B^* = A^* \vee B^*.\end{aligned}$$

Therefore, $A \mapsto A^*$ is a lattice anti-homomorphism. Similarly, $R \mapsto R^*$ is a lattice anti-homorphism from $\mathsf{Rep}(X, \varphi)$ to $\mathsf{Att}(X, \varphi)$. Since applying the duality anti-homomorphism twice is the identity, we conclude that its is a lattice anti-isomorphism. ■

Proposition 4.5 *Let* $\varphi : \mathbb{T}^+ \times X \to X$ *a dynamical system on a compact metric space* X*. The lattices of attractors* $\mathsf{Att}(X, \varphi)$ *and repellers* $\mathsf{Rep}(X, \varphi)$ *are at most countable.*

Proof: Since X is compact it has a countable basis for the metric topology. In Corollary 3.37 we show that any neighborhood U of an attractor A is an attracting neighborhood if $\mathrm{cl}(U) \cap A^* = \varnothing$. Therefore we can choose as a finite union of elements of the countable basis. Since different attractors cannot have the same finite union of basis elements, the set of attractors is at most countable. The lattice $\mathsf{Rep}(X, \varphi)$ is isomorphic to $\mathsf{Att}(S, \varphi)$ and therefore also at most countable. ■

Proposition 4.6 *Let* $\varphi : \mathbb{T}^+ \times X \to X$ *be a dynamical system on a compact metric space* X*.*

(i) The collection of attracting neighborhoods $(\mathsf{ANbhd}(X,\varphi),\cup,\cap)$ *is a bounded, distributive sublattice of* $\mathsf{P}(X)$ *with neutral elements* $\varnothing$ *and* X*.*

(ii) The collection of repelling neighborhoods $(\mathsf{RNbhd}(X,\varphi),\cup,\cap)$ *is a bounded, distributive sublattice of* $\mathsf{P}(X)$ *with neutral elements* $\varnothing$ *and* X*.*

(iii) If $N \in \mathsf{ANbhd}(X,\varphi)$*, then* $\mathrm{cl}(X \setminus N) \in \mathsf{RNbhd}(X,\varphi)$*, and if* $K \in \mathsf{RNbhd}(X,\varphi)$*, then* $\mathrm{cl}(X \setminus K) \in \mathsf{ANbhd}(X,\varphi)$*.*

Proof: Recall that a neighborhood N of an attractor A is an attracting neighborhood if $\omega(N) = A$, see Corollary 3.18. Let N, N' be attracting neighborhoods for attractors A, A' respectively. Then $N \cup N'$ and $N \cap N'$ are neighborhoods of $A \cup A'$ and $A \cap A'$ respectively. By Proposition 2.25(ii) we have that

$$\omega(N \cup N') = \omega(N) \cup \omega(N') = A \cup A',$$

which shows that $N \cup N' \in \mathsf{ANbhd}(X,\varphi)$. For the intersection we argue as follows. It holds that $\omega(\omega(N \cap N')) = \omega(N \cap N') \supset A \wedge A'$. By Proposition 2.25(ii) we have that

$$\omega(N \cap N') \subset \omega(N) \cap \omega(N') = A \cap A',$$

and thus $\omega(N \cap N') \subset A \wedge A'$. Combining these inclusions gives $\omega(N \cap N') = \omega(N) \cap \omega(N') = A \wedge A'$, which proves that $N \cap N' \in \mathsf{ANbhd}(X,\varphi)$.

Let $N^\dagger$ and $N'^\dagger$ be repelling neighborhoods for A^* and A'^* respectively. It holds that

$$N^\dagger \cup N'^\dagger = \mathrm{cl}(X \setminus N) \cup \mathrm{cl}(X \setminus N') = \mathrm{cl}(X \setminus N \cup X \setminus N') = \mathrm{cl}(X \setminus N \cap N'),$$

which shows that $N^\dagger \cup N'^\dagger = (N \cap N')^\dagger$ is a repelling neighborhood of $(A \wedge A')^* = A^* \cup A'^*$. The latter is the maximal forward invariant set in $N^\dagger \cup N'^\dagger$. On the other hand

$$(N^\dagger \cap N'^\dagger) \cap (A \cup A') = (N^\dagger \cap N'^\dagger \cap A) \cup (N^\dagger \cap N'^\dagger \cap A') = \varnothing,$$

which shows that $N^\dagger \cap N'^\dagger$ is a repelling neighborhood. The maximal invariant set inside is $(A \cup A')^* = A \wedge A'^*$. ∎

Proposition 4.7 *Let* $\varphi : \mathbb{T}^+ \times X \to X$ *a dynamical system on a compact metric space* X*.*

(i) The mapping

$$\begin{array}{rcl} \mathrm{Inv}\colon \mathsf{ABlock}(X,\varphi) & \to & \mathsf{Att}(X,\varphi) \\ N & \mapsto & \mathrm{Inv}(N,\varphi) \end{array}$$

is a surjective, $(0,1)$*-homomorphism.*

(ii) The mapping

$$\begin{array}{rcl} \mathrm{Inv}^+\colon \mathsf{RBlock}(s,\varphi) & \to & \mathsf{Rep}(X,\varphi) \\ N & \mapsto & \mathrm{Inv}^+(N,\varphi) \end{array}$$

is a surjective, $(0,1)$*-homomorphism.*

The same mapping apply to the lattices $\mathsf{ANbhd}(X,\varphi)$ *and* $\mathsf{RNbhd}(X,\varphi)$*.*

Proof: To prove (i) let $N, K \in \mathsf{ABlock}(X,\varphi)$. From Lemma 4.3 it follows that

$$\begin{array}{rcl} \mathrm{Inv}(N\cup K) & = & \mathrm{Inv}(N\cup K,\varphi) = \mathrm{Inv}(N)\cup \mathrm{Inv}(K) \\ \mathrm{Inv}(N\cap K) & = & \mathrm{Inv}(N\cap K,\varphi) = \mathrm{Inv}\big(\mathrm{Inv}(N)\cap \mathrm{Inv}(K)\big), \end{array}$$

which completes the proof of (i). The proof of (ii) is analogous. The proof for $\mathsf{ANbhd}(X,\varphi)$ and $\mathsf{RNbhd}(X,\varphi)$ follows from the fact that $\mathrm{Inv} = \omega$ and the proof of Proposition 4.6. ∎

4.2 Lattices of Lyapunov functions

Let $\mathsf{ARLyap}(X,\varphi)$ be the set of all Lyapunov functions $V\colon X \to [0,1]$ for a pair of sets $\{V^{-1}(0), V^{-1}(1)\}$. The set $\mathsf{ARLyap}(X,\varphi) \subset C(X,[0,1])$, and the continuous functions have the natural structure of a lattice under the operations

$$V \vee W = \max\{V,W\} \text{ and } V\wedge W = \min\{V,W\}$$

where the maximum and minimum are defined pointwise on X.

Proposition 4.8 *Let* $\varphi : \mathbb{T}^+ \times X \to X$ *a dynamical system on a compact metric space* X*. The collection of all Lyapunov functions for attractor-repeller pairs* $(\mathsf{ARLyap}(X,\varphi),\vee,\wedge)$ *is a* $(0,1)$*-sublattice of* $\big(C(X,[0,1]),\vee,\wedge\big)$*.*

Proof: We must show closedness with respect to the binary operations $\vee$ and $\wedge$. Let $V, W \in \mathsf{ARLyap}(X,\varphi)$, and γ_x^+ be a forward orbit in X. Then for $0 \le s \le t$

$$\begin{aligned}(V \vee W)(\gamma_x^+(s)) &= \max\{V(\gamma_x^+(s)), W(\gamma_x^+(s))\}\\ &\le \max\{V(\gamma_x^+(t)), W(\gamma_x^+(t))\} = (V \vee W)(\gamma_x^+(t))\end{aligned}$$

so that $(V \vee W) \circ \gamma_x^+$ is decreasing.

Note that $U_0 = (V \vee W)^{-1}(0) = V^{-1}(0) \cap W^{-1}(0)$ and $U_1 = (V \vee W)^{-1}(1) = V^{-1}(1) \cup W^{-1}(1)$. Thus we need to show that $(V \vee W)(\gamma_x^+(t)) < (V \vee W)(x)$ for $t > 0$ and $x \in X \setminus (U_0 \cup U_1)$. Since $x \notin U_1$, we have $x \notin V^{-1}(1)$ and $x \notin W^{-1}(1)$. Since $x \notin U_0$, we have $x \notin V^{-1}(0)$ or $x \notin W^{-1}(0)$. Hence by Definition 3.39, either $V(\gamma_x^+(t)) < V(x)$ or $W(\gamma_x^+(t)) < W(x)$. If both inequalities are strict, then

$$\begin{aligned}(V \vee W)(\gamma_x^+(t)) &= \max\{V((\gamma_x^+(t)), W(\gamma_x^+(t))\}\\ &< \max\{V(x), W(x)\} = (V \vee W)(x).\end{aligned}$$

If $V(\gamma_x^+(t)) = V(x)$, then $V(\gamma_x^+(t)) = V(x) = 0 < W(x)$ so that

$$\begin{aligned}(V \vee W)(\gamma_x^+(t)) &= \max\{V((\gamma_x^+(t)), W(\gamma_x^+(t))\} = W(\gamma_x^+(t))\\ &< W(x) = \max\{V(x), W(x)\} = (V \vee W)(x),\end{aligned}$$

and the same argument holds when $W(\gamma_x^+(t)) = W(x)$. Hence $V \vee W \in \mathsf{ARLyap}(X,\varphi)$. A similar proof shows $V \wedge W \in \mathsf{ARLyap}(X,\varphi)$. ■

Theorem 4.9 *Let $\varphi : \mathbb{T}^+ \times X \to X$ a dynamical system on a compact metric space X.*

(i) *The map c*

$$\begin{aligned} l_{\mathsf{AB}}\colon \mathsf{ARLyap}(X,\varphi) &\to \mathsf{ABlock}(X,\varphi)\\ V &\mapsto V^{-1}([0,c])\end{aligned}$$

is a $(0,1)$-anti-homomorphism for every $0 < c < 1$.

(ii) *The map*

$$\begin{aligned} l_{\mathsf{Att}}\colon \mathsf{ARLyap}(X,\varphi) &\to \mathsf{Att}(X,\varphi)\\ V &\mapsto \mathrm{Inv}(V^{-1}([0,c]))\end{aligned}$$

is a surjective $(0,1)$-anti-homomorphism which is independent of $c \in [0,1)$.

(iii) *The map*

$$\begin{aligned} l_{\mathsf{RB}} \colon \mathsf{ARLyap}(X,\varphi) &\to \mathsf{RBlock}(X,\varphi) \\ V &\mapsto V^{-1}([c,1]) \end{aligned}$$

is a $(0,1)$*-homomorphism for every* $0 < c < 1$*.*

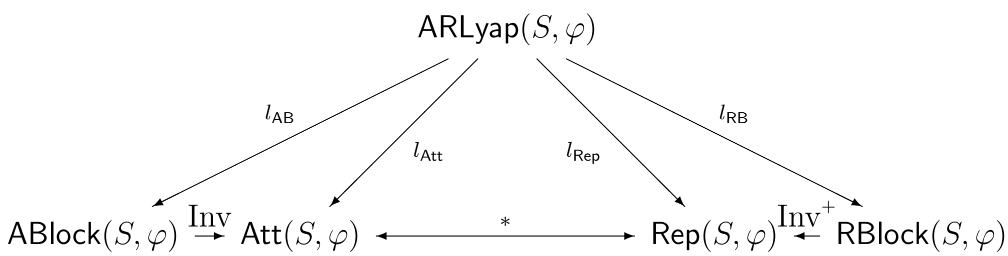
(iv) *The map*

$$\begin{aligned} l_{\mathsf{Rep}} \colon \mathsf{ARLyap}(X,\varphi) &\to \mathsf{Rep}(X,\varphi) \\ V &\mapsto \mathrm{Inv}^{+}(V^{-1}([c,1])) \end{aligned}$$

is a surjective $(0,1)$*-homomorphism which is independent of* $c \in (0,1]$*.*

Proof: By Picture ∎

We can summarize our lattice results up to now by



(22)

What is computable? What can be approximated?

4.3 Filtrations

Proposition 4.10 *Let* $\varphi : \mathbb{T}^{+} \times X \to X$ *a dynamical system on a compact metric space* X *and let* $A \in \mathsf{Att}(X,\varphi)$*. Then given any* $\epsilon > 0$*, there exist* $N_i \in \mathsf{ABlock}(X,\varphi)$*,* $i = 0,1$ *such that* $\omega(N_i,\varphi) = A$ *and furthermore* $N_0 \subset B_\epsilon(A)$ *and* $X \setminus N_1 \subset B_\epsilon(A^*)$*.*

Proof: Since $l_{\mathsf{Att}} \colon \mathsf{ARLyap}(X,\varphi) \to \mathsf{Att}(X,\varphi)$ is surjective, there exists $V \colon X \to [0,1]$ such that $A = V^{-1}(0)$. Furthermore, since V is continuous and monotone there exists $\delta \in (0,1)$ such that

$$N_0 := \{x \in X \mid V(x) \le \delta\} \subset B_\epsilon(A).$$

Since V is strictly decreasing on $V^{-1}(\delta)$, $N_0 \in \mathsf{ABlock}(X,\varphi)$. The construction of N_1 is similar. ■

Observe that this proposition implies that any individual attractor and furthermore its basin of attraction within the invariant set X can be arbitrarily well approximated by attracting blocks.

Proposition 4.10 is somewhat unsatisfactory in the sense that, while it applies to any given attractor, it does not incorporate the lattice structure of the set of all attractors. This raises the question of whether there is a consistent means of choosing attracting blocks with respect to intersections and unions. The following definition is used to make this precise.

Definition 4.11 A *filtration over a lattice* K is a finite lattice L for which there exists a lattice (anti-)embedding of L into K.

Definition 4.12 Let H,K be distributive lattices. A *lift* of a finite sublattice $\mathsf{L} \subset \mathsf{K}$ into H, is a finite sublattice $\mathsf{L}' \subset \mathsf{H}$ such that L' is $(0,1)$-(anti)-isomorphic to L. In particular, L' is a filtration over K.

Theorem 4.13 *Let* H *and* K *be distributive lattices and* $h\colon \mathsf{H} \to \mathsf{K}$ *is lattice (anti-)homomorphism. Let* $\varnothing \neq \mathsf{L} \subset h(\mathsf{K})$ *be a finite sublattice. If there exists a finite sublattice* $\mathsf{H}' \subset \mathsf{H}$ *and an order-surjection* $\theta : \mathsf{J}^{\vee}(\mathsf{H}') \to \mathsf{J}^{\vee}(\mathsf{L})$, *such that*

(i) $h : \mathsf{H}' \to \mathsf{L}$ *is a surjective* $(0,1)$*-lattice homomorphism, and*

(ii) $\mathsf{J}^{\vee}(h) \subset \theta^{-1}$, *i.e.* $\mathsf{J}^{\vee}(h)(\alpha) \subset \theta^{-1}(\alpha)$ *for all* $\alpha \in \mathsf{J}^{\vee}(\mathsf{L})$,

then there exists a lift $\mathsf{L}' \subset \mathsf{H}' \subset \mathsf{H}$, *such that* $h : \mathsf{L}' \to \mathsf{L}$ *is a lattice (anti-)isomorphism.*

The lift can be obtained in a algorithmic manner.

An example of a how this theorem can be used.

Theorem 4.14 *For any finite sublattice of attractors* $\mathsf{L} \subset \mathsf{Att}(X,\varphi)$ *there exists a lift* $\mathsf{V} = \{V_A\}_{A\in\mathsf{L}} \subset \mathsf{ARLyap}(X,\varphi)$ *to the lattice of Lyapunov functions, i.e.* V *is anti-isomorphic to* L, *with* $A = \mathrm{Inv}(V_A^{-1}(0))$ *and*

$$V_{A\vee A'} = V_A \wedge V_{A'}, \quad V_{A\wedge A'} = V_A \vee V_{A'},$$

for all $A, A' \in \mathsf{L}$. *The lattice* V *is filtration of Lyapunov functions over* $\mathsf{Att}(X,\varphi)$ *with image* L.

We need to make use of the theory of lattices and posets.

A poset $(\mathsf{P}, \leq)$ with $\vee$ and $\wedge$ as defined in equation 19 satisfies (i)-(iv) of Definition 3.5. Conversely, a lattice L has a naturally induced partial order as follows. Given $a, b \in \mathsf{L}$ define $a \leq b \Leftrightarrow a \wedge b = a$ or equivalently $a \leq b \Leftrightarrow a \vee b = b$.

Lemma 4.15 *If $a = \bigvee_{i=1}^{n} b_i$, then $b_i \leq a$ for all $i = 1, \ldots, n$. Likewise if $a = \bigwedge_{i=1}^{n} b_i$, then $a \leq b_i$ for all $i = 1, \ldots, n$*

Definition 4.16 Let $(\mathsf{L}, \wedge, \vee)$ and $(\mathsf{L}', \wedge', \vee')$ be lattices. A function $h : \mathsf{L} \to \mathsf{L}'$ is a *lattice homomorphism* if

$$h(a \wedge b) = h(a) \wedge' h(b) \text{ and } h(a \vee b) = h(a) \vee' h(b),$$

a *lattice anti-homomorphism*
if

$$h(a \wedge b) = h(a) \vee' h(b) \text{ and } h(a \vee b) = h(a) \wedge' h(b).$$

The set of lattice homomorphisms is denoted by $\text{Hom}(\mathsf{L}, \mathsf{L}')$ and the set of lattice anti-homomorphisms is denoted by $\text{Hom}^*(\mathsf{L}, \mathsf{L}')$. Lattice homomorphisms for which $h(0) = 0$ and $h(1) = 1$, or $h(0) = 1$ and $h(1) = 0$, are called $(0,1)$-*(anti-) homomorphisms*. These sets are denoted by $\text{Hom}_{0,1}(L, L')$ and $\text{Hom}^*_{0,1}(L, L')$ respectively.

Definition 4.17 Let $(\mathsf{P}, \leq)$ be a poset. A *down-set* I in P is characterized by the property:

$$\text{if } p \in \mathsf{P} \text{ and } p \leq q \text{ for some } q \in I \text{ implies } p \in I.$$

The set of down-sets in P is denoted by $\mathsf{O}(\mathsf{P})$. Similarly, an *up-set* I in P is characterized by the property:

$$\text{if } p \in \mathsf{P} \text{ and } p \geq q \text{ for some } q \in I \text{ implies } p \in I.$$

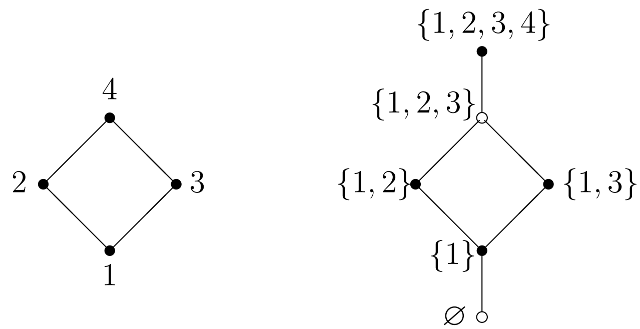
The set of up-sets in P is denoted by $\mathsf{U}(\mathsf{P})$. An *convex set* I in P is characterized by the property:

$$\text{if } p \in \mathsf{P} \text{ and } q \leq p \leq q' \text{ for some } q, q' \in I \text{ implies } p \in I.$$

The set of convex sets in P is denoted by $\mathsf{Convex}(\mathsf{P})$.

Lemma 4.18 *The sets* $\mathsf{O}(\mathsf{P})$ *and* $\mathsf{U}(\mathsf{P})$ *are bounded distributive lattices with* $\wedge = \cap$ *and* $\vee = \cup$*, and the partial order is given by inclusion* $\subseteq$*. The sets* $\varnothing$ *and* P *serve as the* $0, 1$ *elements of* $\mathsf{O}(\mathsf{P})$ *and the reverse for* $\mathsf{U}(\mathsf{P})$*.*

Consider the following figure that shows a poset and the collection of down sets. Observe that the down sets form a lattice.



Definition 4.19 Let $(\mathsf{L}, \vee, \wedge)$ be a lattice. An element $c \in \mathsf{L}$ is *join-irreducible* if

(i) $c \neq 0$ and

(ii) $c = a \vee b$ implies $c = a$ or $c = b$ for all $a, b \in \mathsf{L}$.

The set of join-irreducible elements in L is denoted by $\mathsf{J}^{\vee}(\mathsf{L})$. Similarly, an element $x \in \mathsf{L}$ is if (i)-(ii) are satisfied with $\vee$ replaced by $\wedge$. The set of meet-irreducible elements in L is denoted by $\mathsf{J}^{\wedge}(\mathsf{L})$.

An element $b \in \mathsf{L}$ is join-irreducible if and only if it has a unique predecessor a in the covering relation and hence we can define the unique predecessor map

$$\text{pred}\colon \mathsf{J}^{\vee}(\mathsf{L}) \to \mathsf{L}. \tag{23}$$

An element $a \in \mathsf{L}$ is meet-irreducible if and only if it has a unique successor b in the covering relation which leads the unique successor map

$$\text{succ}\colon \mathsf{J}^{\wedge}(\mathsf{L}) \to \mathsf{L}. \tag{24}$$

Every element $a \in \mathsf{L}$ of a finite distributive lattice L has a representation as a join of join-irreducible elements, i.e. $a = \bigvee_i c_i$, $c_i \in \mathsf{J}^{\vee}(\mathsf{L})$. A representation $a = \bigvee_i c_i$ is *redundant* if $a = \bigvee_{i \neq j} c_i$ for some j, otherwise the the representation is *irredundant*.

We will use the following result from Lattice theory.

Proposition 4.20 *Let* L *be a finite distributive lattice. For every* $a \in \mathsf{L}$ *there exists a unique set of irredundant join-irreducible elements* $\iota^{\vee}(a) \subset \mathsf{J}^{\vee}(\mathsf{L})$ *and a unique set of irredundant meet-irreducible elements* $\iota^{\wedge}(a) \subset \mathsf{J}^{\wedge}(\mathsf{L})$ *such that*

$$a = \bigvee_{b \in \iota^{\vee}(a)} b = \bigwedge_{c \in \iota^{\wedge}(a)} c.$$

4.4 Birkhoff's Representation Theorem

Proposition 4.21 *Let* P *be a finite poset. The function*

$$\begin{array}{rcl} \downarrow : (\mathsf{P}, \leq) & \to & (\mathsf{J}^{\vee}(\mathsf{O}(\mathsf{P})), \subseteq) \\ p & \mapsto & \downarrow(p) = \{q \in \mathsf{P} \mid q \leq p\} \end{array}$$

is a poset isomorphism.

Lemma 4.22 *If* $I \in \mathsf{O}(\mathsf{P})$, *then*

$$I = \bigvee_{p \in \max(I)} \downarrow(p).$$

Proof of Proposition 4.21. We first show that $\downarrow(\mathsf{P}) \subset \mathsf{J}^{\vee}(\mathsf{O}(\mathsf{P}))$. Let $p \in \mathsf{P}$ and assume $\downarrow(p) = \mathsf{B} \vee \mathsf{C} = \mathsf{B} \cup \mathsf{C}$ where $\mathsf{B}, \mathsf{C} \in \mathsf{O}(\mathsf{P})$. Then either $p \in \mathsf{B}$ or $p \in \mathsf{C}$ which implies that either $\mathsf{B} = \downarrow(p)$ or $\mathsf{C} = \downarrow(p)$. Therefore $\downarrow(p) \in \mathsf{J}^{\vee}(\mathsf{O}(\mathsf{P}))$. The opposite inclusion, $\mathsf{J}^{\vee}(\mathsf{O}(\mathsf{P})) \subset \downarrow(\mathsf{P})$, follows from Lemma 4.22. ∎

Theorem 4.23 (Birkhoff's Representation Theorem) *Let* L *be a finite distributive lattice and let* P *be finite partially ordered set. Then*

$$\downarrow : \mathsf{P} \to \mathsf{J}^{\vee}(\mathsf{O}(\mathsf{P}))$$

is a poset isomorphism and

$$\begin{array}{rcl} \downarrow^{\vee} : \mathsf{L} & \to & \mathsf{O}\big(\mathsf{J}^{\vee}(\mathsf{L})\big) \\ x & \mapsto & \downarrow^{\vee}(x) := \{y \in \mathsf{J}^{\vee}(\mathsf{L}) \mid y \leq x\} \end{array}$$

is a lattice isomorphism.

In purely combinatorial setting, Birkhoff's Representation Theorem provides a precise description of the relation between posets and lattices. Our interest in this theorem is that it provides a tool by which we can relate objects of dynamical interest. To maintain this clarity when working with objects of different types we adopt the language of category theory.

Let $\mathfrak{Lat}_D^F$ denote the category of finite distributive lattices, whose morphisms are $(0,1)$-homomorphisms, and $\mathfrak{Poset}_F$ denote the category of finite posets, whose morphisms are order-preserving mappings.

The following are standard results from Lattice theory. See Davey and Priestly.

Proposition 4.24 *Let* L *and* K *be objects in* $\mathfrak{Lat}_D^F$ *and let* $f\colon \mathsf{L} \to \mathsf{K}$ *be a* $(0,1)$*-lattice morphism. Then*

$$\begin{aligned} \mathsf{J}^\vee(f)\colon \mathsf{J}^\vee(\mathsf{K}) &\to \mathsf{J}^\vee(\mathsf{L}) \\ x &\mapsto \min\{y \in L \mid x \in f(\downarrow(y))\} \end{aligned}$$

is a poset morphism.

Proposition 4.25 *Let* P *and* Q *be objects in* $\mathfrak{Poset}_F$ *and let* $f\colon \mathsf{P} \to \mathsf{Q}$ *be a poset morphism. Then*

$$\begin{aligned} \mathsf{O}(f)\colon \mathsf{O}(\mathsf{Q}) &\to \mathsf{O}(\mathsf{P}) \\ x &\mapsto f^{-1}(x) = \bigcup_{q\in x} f^{-1}(q) \end{aligned}$$

is a $(0,1)$*-lattice morphism.*

Proposition 4.26 $\mathsf{J}^\vee$ *defines a contravariant functor from* $\mathfrak{Lat}_D^F$ *to* $\mathfrak{Poset}_F$*.*

Proof: Apply Proposition 4.24 to show that composition properties are satisfied. ∎

Proposition 4.27 O *defines a contravariant functor from* $\mathfrak{Poset}_F$ *to* $\mathfrak{Lat}_D^F$*.*

5 Morse Decompositions

Definition 5.1 Let S be a compact invariant set for φ. A *Morse decomposition in* S consists of a finite poset $(\mathsf{P}, \leq)$ and a finite collection of compact, pairwise disjoint, invariant sets $\{M(p) \in \mathsf{Invset}(S,\varphi) \mid p \in \mathsf{P}\}$ such that for every $x \in S \setminus \left(\bigcup_{p\in\mathsf{P}} M(p)\right)$ and every orbit $\gamma_x \subset S$ there exist $p, q \in \mathsf{P}$, with $q < p$, such that

$$\alpha_{\mathrm{o}}(\gamma_x^-) \subset M(p), \quad \text{and} \quad \omega(x) \subset M(q).$$

The Morse decomposition is denoted by the set of pairs

$$\mathsf{M} = \{(p, M(p)) \in \mathsf{P} \times \mathsf{Invset}(S,\varphi)\}\,.$$

and is a poset with the ordering inherited from P; that is,

$$(p, M(p)) \leq (q, M(q)) \quad \text{if and only if} \quad p \leq q.$$

Each set $M(p)$ is called a *Morse set*, P is referred to as the *labelling* of M, and the order $\leq$ is called an *admissible order*. The set of all Morse decompositions of S under φ is denoted by $\mathsf{MD}(S,\varphi)$.

Example 5.2 Consider a gradient flow $\varphi\colon \mathbb{R} \times S \to S$ defined on a smooth compact invariant manifold generated by the differential equation $\dot{x} = -\nabla V(x)$. Assume that there are a finite set of critical points $\mathcal{E} = \{x_i \mid i = 1,\dots,n\}$. Let $\mathsf{P} := \{p \mid i = 1,\dots,n\}$ with partial order defined by $p < q$ if and only if $V(x_p) < V(x_p)$. Then $\mathsf{M} := \{(p, M(p)) \mid M(p) := x_p)\}$ is a Morse decomposition.

Example 5.3 Let $S = \varnothing$. Observe that for any dynamical system φ this is a compact invariant set. Let $(\mathsf{P}, \leq)$ be any finite poset. Then

$$\mathsf{M} := \{(p, M(p)) \mid p \in \mathsf{P},\; M(p) := \varnothing\}$$

is a Morse decomposition. This example demonstrates two important points. First, there is no contradiction in choosing a Morse decomposition in which some of the Morse sets are the emptyset. Second, Morse decompositions of a compact invariant set are not unique.

Definition 5.4 Let S be a compact invariant set for φ. A Morse decomposition in S,

$$\mathsf{M} = \{(p, M(p))\},$$

is a *p-Morse decomposition* if $M(p) \neq \varnothing$ for all $p \in \mathsf{P}$. If we are working with a p-Morse decomposition we will frequently write

$$\mathsf{M} = \{M(p) \mid p \in (\mathsf{P}, \leq)\}.$$

Observe that in a p-Morse decomposition the Morse sets themselves can be used as an indexing set for the partial order, that is; $M(p) \leq M(q)$ if and only if $p \leq q$. The set of all p-Morse decompositions in S under φ is denoted by $\mathsf{pMD}(S, \varphi)$.

Morse decompositions need not be unique both in terms of the Morse sets and in terms of the partial order.

While the set of possible admissible orders for a collection of Morse sets $\{M(p)\}$ may contain more than one element there is always unique admissible order with the minimal number of relations. This is called the *minimal* or *flow defined order*.

Example 5.5 Let $\varphi\colon \mathbb{R} \times [-1, 1] \to [-1, 1]$ be the flow generated by the differential equation

$$\dot{x} = x(1 - x^2).$$

The collection of equilibria $\{0, \pm 1\}$ can be viewed as Morse sets. Observe that there are three possible admissible orders: $-1 < 1 < 0$; $1 < -1 < 0$; and $-1 < 0$, $1 < 0$. By definition different choices of admissible orders result in different Morse decompositions.

Explain the importance of admissible order.

Remark 5.6 The terminology Morse decomposition is somewhat misleading since in general $S \neq \bigcup_{p \in \mathsf{P}} M(p)$. In fact,

$$S = \left(\bigcup_{p \in \mathsf{P}} M(p) \right) \bigcup \left(\bigcup_{\substack{p,q \in \mathsf{P} \\ p < q}} C\big(M(q), M(p)\big) \right). \tag{25}$$

Observe, however, that the relevant data for (25), i.e. $\{M(p)\}$ and $(\mathsf{P}, \leq)$ is encoded in M.

Remark 5.7 A Morse decomposition uses a poset P to label and order the Morse sets. Since a poset can be represented as an acyclic directed graph, this language can be used to describe a Morse decomposition. The vertices of the graph of P represent the Morse sets and lack of a directed path between two Morse sets implies that there is no connecting orbit between the Morse sets. This representation of a Morse decomposition is called a *Morse graph*.

Proposition 5.8 *Let S be a compact invariant set and M a p-Morse decomposition for S, labeled by $(\mathsf{P}, \leq)$. Then, for any compact invariant subset $S' \subset S$, there exists a p-Morse decomposition*

$$\mathsf{M}' = \{M'(p) := \mathrm{Inv}(M(p) \cap S', \varphi) \mid p \in \mathsf{P}'\},$$

with labeling set

$$\mathsf{P}' = \{p \in \mathsf{P} \mid M(p) \cap S' \neq \varnothing\}$$

and partial order induced by $\leq$.

Proof: Since M is a p-Morse decomposition, $\{M'(p)\}$ is a collection of compact, pairwise disjoint, non-empty invariant subsets of S' labelled by P'. It remains to be shown that the restriction of $\leq$ to P' induces an admissible order. Consider $x \in S'$. Let $\gamma_x \subset S'$ be a complete orbit through x which exists because S' is assumed to be invariant. By the compactness of S', $\omega(x) \subset S'$ and $\alpha_{\mathrm{o}}(\gamma_x^-) \subset S'$. Now assume $x \in S' \setminus \left(\bigcup_{p \in \mathsf{P}} M(p)\right)$. The invariance of S' and the fact that M is a p-Morse decomposition for S, implies that there exist $p, q \in \mathsf{P}'$ with $q < p$ such that

$$\alpha_{\mathrm{o}}(\gamma_x^-) \subset M(p) \cap S' = M'(p) \quad \text{and} \quad \omega(x) \subset M(q) = M'(q) \cap S'.$$

This insures that $\leq$ is an admissible order on P'. ■

Given a Morse decomposition $\mathsf{M} = \{M(p) \mid p \in (\mathsf{P}, \leq)\}$, define

$$\mathsf{sP} := \{p \in \mathsf{P} \mid M(p) \neq \varnothing\}.$$

Since $\mathsf{sP} \subset \mathsf{P}$ it inherits the poset structure from P. Define the *associate p-Morse decomposition* as

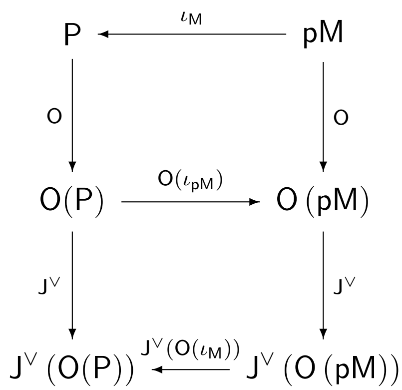
$$\mathsf{pM} := \{M(p) \mid p \in \mathsf{sP}\}$$

Define

$$\begin{array}{rcl} \iota_{\mathsf{M}} \colon \mathsf{pM} & \to & \mathsf{P} \\ M(p) & \mapsto & p. \end{array}$$

Observe that if M is a p-Morse decomposition, then ι_{M} is a bijection and the inverse to the labeling map. In general, $\iota_{\mathsf{M}}(\mathsf{pM}) = \mathsf{sP}$. The partial order on pM is defined to be the minimal partial order on pM such that ι_{M} is an poset isomorphism onto sP.

Example 5.9 Let M be a Morse decomposition labelled by the finite poset P. Then pM is a poset and we can draw the following diagram.

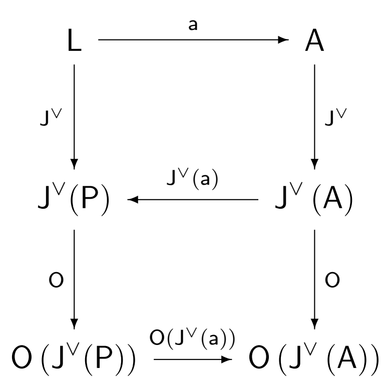


Birkhoff's representation theorem implies that

$$\downarrow \colon \mathsf{pM} \to \mathsf{J}^{\vee}(\mathsf{O}(\mathsf{pM})) \tag{26}$$

is an isomorphism.

Example 5.10 Let A be a finite $(0,1)$-lattice of attractors indexed by the lattice L, that is $\mathsf{a} \colon \mathsf{L} \to \mathsf{A}$ is a lattice isomorphism. We can draw the following diagram



Birkhoff's representation theorem implies that

$$\downarrow^{\vee} \colon \mathsf{A} \to \mathsf{O}(\mathsf{J}^{\vee}(\mathsf{A})) \tag{27}$$

is an isomorphism.

5.1 Attractor lattices and Morse decompositions

Goal: Show that the structure present in a p-Morse decomposition is equivalent to the structure present in a finite, $(0,1)$-sublattice of attractors.

Definition 5.11 Let $S \subset X$ be an invariant set, then

$$\begin{aligned} W^s(S,\varphi) &= \{x \in X \mid \lim_{t\to\infty} d(\gamma_x^+(t), S) = 0\} \\ W^u(S,\varphi) &= \{x \in X \mid \exists \gamma_x^- \text{ such that } \lim_{t\to-\infty} d(\gamma_x^-(t), S) = 0\}, \end{aligned}$$

are called the *stable* and *unstable sets* of S respectively.

Observe that

$$S \subset W^s(S,\varphi) \quad \text{and} \quad S \subset W^u(S,\varphi)$$

Define

$$\begin{aligned} \nu : \mathsf{Invset}(S,\varphi) &\to \mathsf{Invset}(S,\varphi) \\ K &\mapsto \nu(K) := W^u(K,\varphi) \end{aligned} \tag{28}$$

Recall that Morse decompositions are posets and attractors form lattices. Thus, in light of Birkhoff's Representation Theorem 4.23 it is reasonable to consider attracting sets of Morse sets, i.e. $\mathsf{O}(\mathsf{pM})$. The following result, which is proved below, confirms this.

Proposition 5.12 *Let* $\mathsf{M} = \{M(p) \subset S \mid p \in (\mathsf{P}, \leq)\}$ *be a Morse decomposition of* S*, a compact invariant set under* φ*. Then*

$$\nu : \mathsf{O}(\mathsf{pM}) \to \mathsf{Att}(S,\varphi)$$

is a $(0,1)$*-lattice morphism. If* M *is a p-Morse decomposition, then* ν *is a monomorphism.*

As is indicated in Proposition 4.4 $\mathsf{Att}(S,\varphi)$ is a $(0,1)$-bounded, distributive lattice that can, in general, have an infinite number of elements. Since Morse decompositions are finite, Proposition 5.12 suggests that we can restrict our attention to finite sublattices of $\mathsf{Att}(S,\varphi)$. This leads to the following notation. Let

$$\mathrm{sub}^F_{0,1}\mathsf{Att}(S,\varphi) := \{L \subset \mathsf{Att}(S,\varphi) \mid L \text{ a } (0,1)\text{-finite sublattice of } \mathsf{Att}\}\,. \quad (29)$$

Proposition 5.12 associates a particular $(0,1)$-finite lattice of attractors to a Morse decomposition. Because of its importance we label it as follows.

Definition 5.13 Let S be a compact invariant set for a dynamical system φ with a Morse decomposition M labelled by $(\mathsf{P},\leq)$. The *associated lattice of attractors* of M is defined by

$$\mathsf{A}(\mathsf{pM}) := \nu(\mathsf{O}(\mathsf{pM})) \in \mathrm{sub}^F_{0,1}\mathsf{Att}(S,\varphi).$$

The following lemma which is of interest in its own right is used to set up an induction argument for the proof of Proposition 5.12.

Lemma 5.14 *Let S be a compact invariant set with Morse decomposition M labeled by the poset $(\mathsf{P},\leq)$. If p_0 is a maximal element of sP, then the pair (A,R) defined by*

$$A = \nu(\mathsf{O}(\mathsf{pM} \setminus \{M(p_0)\})) \quad \textit{and} \quad R = M(p_0)$$

is an attractor-repeller pair in S.

Proof: The set A is invariant. To show that (A,R) is an attractor-repeller pair in S we apply Theorem 3.44. Since M is a Morse decomposition, the orbits outside of the Morse sets must be connecting orbits between Morse sets, and therefore $S = A\cup R\cup C(R,A)\cup C(A,R)$. By assumption $M(p_0)$ is compact, hence by Theorem 3.44(ii) it is sufficient to prove that A is closed, $A\cap R=\varnothing$, and $C(A,R)=\varnothing$.

We begin by showing that p_0 maximal implies that $W^s\big(M(p_0)\big) = M(p_0)$, from which it follows that $A\cap R=\varnothing$ and $C(A,R)=\varnothing$. Suppose $x \in W^s\big(M(p_0)\big)\setminus M(p_0)$. Since $M(p)$ is invariant for every $p\in\mathsf{P}$, we have $x\in S\setminus\bigcup_{p\in\mathsf{P}} M(p)$. Furthermore, $\omega(x)\subset M(p_0)$ (see Section **??**), and hence there exists $q\in\mathsf{P}$ such that $x\in C\big(M(q),M(p_0)\big)$, which contradicts the maximality of p_0.

Now we have left to show that A is closed. For every element $x_n \in A$, there exists an orbit γ_{x_n} such that $\alpha_o(\gamma_{x_n}^-) \subset M(q)$ with $q \in \mathsf{P} \setminus \{p_0\}$. Suppose A is not closed. Then there exists $x_n \in A$ such that $x_n \to x \notin A$. The compactness of S implies that for any $t < 0$ there exists a subsequence of $\gamma_{x_n}(t)$ which converges to x_t and by continuity $\varphi(-t, x_t) = x$. Since $x \notin A$ it holds at for every γ_x we have $\alpha_o(\gamma_x^-) \subset M(p_0)$. Since the sets $M(p)$ are compact and disjoint, we can choose an open neighborhood U of $R = M(p_0)$ such that $\mathrm{cl}(U) \cap \bigcup_{p \neq p_0} M(p) = \varnothing$. The compactness of S now implies that there exists $s \in \mathbb{T}$ such that for any solution γ_x we have $\gamma_x(t) \in U$ for all $t \leq s$.

Let $\{\gamma_{x_{n,0}}\} \subset \{\gamma_{x_n}\}$ be a subsequence of orbits such that $\gamma_{x_{n,0}}(s) \to \gamma_x(s)$ and $\{\gamma_{x_{n,0}}(s)\} \subset U$. Inductively, choose further subsequences $\{\gamma_{x_{n,l}}\} \subset \{\gamma_{x_{n,l-1}}\}$ such that $\gamma_{x_{n,l}}(s-l) \to \gamma_x(s-l)$ and $\{\gamma_{x_{n,l}}(s-l)\} \subset U$. Define $r_n = \inf_{t \in \mathbb{T}} \{\gamma_{x_{n,n}}(t) \in U\}$. By assumption $x_n \in A$, thus $-\infty < r_n < s - n$. Let y be the limit of a convergent subsequence of elements of $\{\gamma_{x_{n,n}}(r_n)\}$. Then $y \in \mathrm{cl}(U)$. Moreover, since $r_n \to -\infty$, we have $\varphi(t, y) \in \mathrm{cl}(U)$ for all $t \geq 0$. Therefore, $\omega(y) \subset \mathrm{cl}(U)$, and since $M(p_0) = \mathrm{Inv}(\mathrm{cl}(U), \varphi)$, we have $\omega(y) \subset M(p_0)$, which contradicts the fact that $W^s\big(M(p_0)\big) = M(p_0)$. Therefore A is closed. ∎

A technical lemma.

Lemma 5.15 *Let $I_1, I_2 \in \mathsf{O}(\mathsf{pM})$, and suppose that $\nu(I_1), \nu(I_2) \in \mathsf{Att}(S, \varphi)$, then*

$$\nu(I_1) \vee \nu(I_2) = \nu(I_1 \vee I_2) \in \mathsf{Att}(S, \varphi)$$

and

$$\nu(I_1) \wedge \nu(I_2) = \nu(I_1 \wedge I_2) \in \mathsf{Att}(S, \varphi).$$

Proof:

$$\begin{aligned}
\nu(I_1 \vee I_2) &= \bigcup_{p \in I_1 \vee I_2} W^u(M_p) = \bigcup_{p \in I_1 \cup I_2} W^u(M(p)) \\
&= \left(\bigcup_{p \in I_1} W^u(M(p)) \right) \bigcup \left(\bigcup_{p \in I_2} W^u(M(p)) \right) \\
&= \nu(I_1) \cup \nu(I_2) = \nu(I_2) \vee \nu(I_2).
\end{aligned}$$

Moreover, if $p \in I_i \in \mathsf{O}(\mathsf{pM})$, $i = 1, 2$, then $W^u(M(p)) \subset \nu(I_i)$, since $\nu(I_i)$ is an attractor by assumption. Therefore,

$$\begin{aligned}\nu(I_1 \wedge I_2) &= \bigcup_{p \in I_1 \wedge I_2} W^u(M(p)) = \bigcup_{p \in I_1 \cap I_2} W^u(M(p)) \\ &\subset \operatorname{Inv}(\nu(I_1) \cap \nu(I_2)) = \nu(I_1) \wedge \nu(I_2)\end{aligned}$$

(unions of invariant sets are invariant). To obtain the opposite inclusion, suppose that γ_x is an orbit in $\nu(I_1) \wedge \nu(I_2) = \operatorname{Inv}(\nu(I_1) \cap \nu(I_2))$. If $\gamma_x \in M(p)$ for some $p \in I_1 \cap I_2$, then $\gamma_x \subset \operatorname{Inv}(\nu(I_1) \cap \nu(I_2))$ since $M(p) \subset \nu(I_1) \cap \nu(I_2)$. If $\gamma_x \notin M(p)$ for all $p \in I_1 \cap I_2$, then there exist $p, q \in I_1 \cap I_2$ with $p < q$ such that $\omega(x) \subset M(p)$ and $\alpha_{\mathrm{o}}(\gamma_x^-) \subset M(q)$ since M is a Morse decomposition and $\nu(I_1) \wedge \nu(I_2)$ is an attractor containing no Morse sets with labels outside of $I_1 \cap I_2$. Therefore $\gamma_x \subset W^u(M(q))$ for some $q \in I_1 \cap I_2$, and hence $\nu(I_1) \wedge \nu(I_2) = \operatorname{Inv}(\nu(I_1) \cap \nu(I_2)) \subset \nu(I_1 \wedge I_2)$. ■

Proof of Proposition 5.12. We begin with some simple observations.

To see that ν is a $(0, 1)$-map observe that by definition $\nu(\varnothing) = \varnothing$ and $\nu(\mathsf{P}) = \cup_{p \in \mathsf{P}} W^u(M(p)) = S$.

If $I_1 \neq I_2$ and $M(p) \neq \varnothing$ for all $p \in \mathsf{P}$, then $\nu(I_1) \neq \nu(I_2)$ so that $\nu : \mathsf{A}(\mathsf{P}) \to \mathsf{Att}(S, \varphi)$ is a monomorphism.

If we show that $\nu(I) \in \mathsf{Att}(S, \varphi)$ for all $I \in \mathsf{A}(\mathsf{P})$ then by Lemma 5.15 we are done.

Birkhoff's theorem with up sets implies that $\mathsf{J}^\wedge\big(\mathsf{A}(\mathsf{pM})\big)$ is isomorphic to sP. For any maximal element $I \in \mathsf{J}^\wedge(\mathsf{P})$ we have $I = \uparrow\!(p_0)^c = \{p \in \mathsf{P} \mid p \neq p_0\}$, where p_0 is the corresponding maximal element $p_0 \in \mathsf{P}$. From Lemma 5.14, $\nu(I)$ is an attractor in S. Set $\mathsf{P}' = \mathsf{P} \backslash \{p_0\}$, then $\mathsf{M}' = \{M(p) \mid p \in \mathsf{P}'\}$ is a Morse decomposition $\nu(I)$ and the argument can be repeated. This yields an attractor $\nu(I')$ in $\nu(I)$. By Theorem 3.22, $\nu(I')$ is an attractor in S and thus $\nu(I) \in \mathsf{Att}(S, \varphi)$ for all $I \in \mathsf{J}^\wedge(\mathsf{P})$. All elements in $\mathsf{A}(\mathsf{P})$ can be uniquely represented as an irredundant meet, which establises that $\nu\big(\mathsf{A}(\mathsf{P})\big)$ is a $(0, 1)$-sublattice of $\mathsf{Att}(S, \varphi)$. ■

The fact that ν is a monomorphism leads to the following proposition. Heuristically it states that given an attractor, if one strips away the appropriate families of connecting orbits then one is left with a collection of Morse sets.

Proposition 5.16 *For any Morse decomposition* M *of a compact invariant set* S *labelled by* $(\mathsf{P}, \leq)$ *we have the following diagram*

$$\begin{array}{ccc} \mathsf{O}(\mathsf{pM}) & \xleftarrow{\nu^{-1}} & \mathsf{A}(\mathsf{pM}) \\ \Big\downarrow{\scriptstyle \mathsf{J}^{\vee}} & & \Big\downarrow{\scriptstyle \mathsf{J}^{\vee}} \\ \mathsf{J}^{\vee}(\mathsf{O}(\mathsf{pM})) & \xrightarrow{\mathsf{J}^{\vee}(\nu^{-1})} & \mathsf{J}^{\vee}(\mathsf{A}(\mathsf{pM})) \end{array}$$

where $\mathsf{J}^{\vee}(\nu^{-1}) = \mathsf{J}^{\vee}(\nu)^{-1} = \nu$ *restricted to* $\mathsf{J}^{\vee}(\mathsf{A}(\mathsf{pM}))$

Proof: Observe that $x \in \mathsf{J}^{\vee}(\mathsf{O}(\mathsf{pM}))$ implies that $x = \downarrow(\xi)$ for some $\xi \in \mathsf{pM}$. Recall that

$$\mathsf{J}^{\vee}(\nu^{-1})(x) = \min \left\{ y \in \mathsf{A}(\mathsf{pM}) \mid x \in \nu^{-1}(\downarrow(y)) \right\}$$

But the min is achieved with $\downarrow(\xi)$. Thus $y = \xi$, but $\xi = \nu(x)$. ∎

5.2 Constructing a p-Morse decomposition from a lattice of attractors

Let S be a compact invariant set under the dynamical system φ and consider the following pairs of attractors,

$$\Delta(\mathsf{Att}(S,\varphi)) := \{(A_1, A_0) \in \mathsf{Att}(S,\varphi) \times \mathsf{Att}(S,\varphi) \mid A_0 \subset A_1\}.$$

Define

$$\begin{aligned} \mathsf{m}\colon \Delta(\mathsf{Att}(S,\varphi)) &\to \mathsf{Invset}(S,\varphi) \\ (A_1, A_0) &\mapsto \mathsf{m}(A_1, A_0) := A_1 \wedge A_0^* \end{aligned} \tag{30}$$

For the purposes of the current discussion we restrict our attention to a finite lattice of attractors $\mathsf{A} \in \mathrm{sub}_{0,1}^{F}\mathsf{Att}(S,\varphi)$ of S. Birkhoff's Representation Theorem 4.23 motivates restricting our attention further to $\mathsf{J}^{\vee}(\mathsf{A})$. Recall that one of the principle properties of a join-irreducible element is that it has a unique predecessor, which leads to the following definition

$$\begin{aligned} \mu : \mathsf{J}^{\vee}(\mathsf{A}) &\to \mathsf{Invset}(S,\varphi) \\ A &\mapsto \mu(A) := \mathsf{m}\left(\mathrm{id}_{\mathsf{Att}(S,\varphi)}(A), \mathrm{pred}(A)\right) = A \wedge \mathrm{pred}(A)^* \end{aligned} \tag{31}$$

Example 5.17 Let S be a compact invariant set for φ. Consider $\mathsf{A} \in \mathrm{sub}^F_{0,1}\mathsf{Att}(S,\varphi)$ such that $\mathsf{A} = \{\emptyset, A, S\}$. Observe that $S \in \mathsf{J}^\vee(\mathsf{A})$ and $A = \mathrm{pred}(S)$. Thus

$$\mu(S) = S \wedge A^* = S \cap A^* = A^*.$$

As is shown below this function is a poset monomorphism that maps join-irreducible attractors to Morse sets. Key to this is the following result which is of interest in its own right.

Lemma 5.18 *Consider a compact invariant set S and $\mathsf{A} \in \mathrm{sub}^F_{0,1}\mathsf{Att}(S,\varphi)$. Given an orbit $\gamma_x \subset S$ define*

$$\begin{aligned} A_1 &:= \min\{B \in \mathsf{A} \mid \gamma_x \subset B\} \\ A_0 &:= \min\{B \in \mathsf{A} \mid \omega(x) \subset B\}. \end{aligned}$$

Then, $A_0, A_1 \in \mathsf{J}^\vee(\mathsf{A})$ and $A_0 \subset A_1$. Furthermore,

$$\alpha_o(\gamma_x^-) \subset \mu(A_1) \quad \textit{and} \quad \omega(x) \subset \mu(A_0). \tag{32}$$

Proof: Since $A_i \in \mathsf{Att}(S,\varphi)$, $i = 0,1$ they are compact forward invariant sets. In particular, if $x \in A_i$, then $\omega(x,\varphi) \subset A_i$. Thus, $A_0 \subset A_1$.

To show that $A_0, A_1 \in \mathsf{J}^\vee(\mathsf{A})$, it suffices to show that if $A_i = A_i^1 \wedge A_i^2$ for $A_i^J \in \mathsf{A}$, $j = 1,2$ then either $A_i^1 = A_i$ or $A_i^2 = A_i$. Observe that $A_i^j \subset A_i$.

Consider the case of $i = 1$. Since A_1^j, $j = 1,2$ are invariant if $\gamma_x(s) \in A_1^j$, then $\gamma_x(t) \in A_1^j$ for all $t \geq s$. However, $\gamma_x \subset A_1^1 \cup A_1^2$ and thus without loss of generality we can assume that $\gamma_x \subset A_1^1$. Since A_1 is the minimal element in A with this property, $A_1 \subset A_1^1$ which implies $A_1 = A_1^1$.

Now consider the case $i = 0$. Since $\omega(x,\varphi) \subset A_0$, it can without loss of generality be assumed that $\omega(x,\varphi) \cap A_0^1 \neq \varnothing$. By Proposition 3.38(iii) this implies that $\omega(x,\varphi) \subset A_0^1$. Thus, $A_0 \subset A_0^1$ which implies $A_0 = A_0^1$.

To characterize the alpha and omega limit sets observe that $(\mathrm{pred}(A_1), \mu(A_1))$ is an attractor-repeller pair in A_1. Assuming that $\gamma_x \subset \mathrm{pred}(A_1)$ contradicts the definition of A_1 since $\mathrm{pred}(A_1) \subset A_1$. If $\omega(x,\varphi) \subset \mu(A_1)$, then $\omega(x,\varphi) \cap \pi(A_1) = \varnothing$. This implies that $A_0 = A_1$ and hence (32) is satisfied. Thus without loss of generality it can be assumed that $x \in A_1 \setminus (\mathrm{pred}(A_1) \cup \mu(A_1))$ in which case Theorem 3.34 guarantees (32). ∎

Proposition 5.19 *Let S be a compact invariant set under φ and let $\mathsf{A} \in \mathrm{sub}^F_{0,1}\mathsf{Att}(S,\varphi)$. Then*

$$\mu\colon \mathsf{J}^\vee(\mathsf{A}) \to \mathsf{pMD}(S,\varphi)$$

is a poset monomorphism. In particular,

$$\mu\big(\mathsf{J}^\vee(\mathsf{A})\big) = \{\mu(A) \mid A \in \mathsf{J}^\vee(\mathsf{A})\},$$

is a p-Morse decomposition of S labelled by $(\mathsf{J}^\vee(\mathsf{A}), \leq)$.

This proposition provides a well defined means of obtaining a Morse decomposition from a particular lattice of attractors. Because of its importance we label it as follows.

Definition 5.20 Let S be a compact invariant set for a dynamical system φ. Given $\mathsf{A} \in \mathrm{sub}^F_{0,1}\mathsf{Att}(S,\varphi)$ the *associated Morse decomposition of S* is defined by

$$\mathsf{M}(\mathsf{A}) := \mu(\mathsf{J}^\vee(\mathsf{A})) \in \mathsf{MD}(S,\varphi).$$

Proof: Proving that $\mathsf{M}(\mathsf{A})$ is p-Morse decomposition requires demonstrating that the sets $\mu(A)$ are disjoint, nonempty compact invariant sets, and that $\mathsf{J}^\vee(\mathsf{A})$ gives an admissible order on $\mathsf{M}(\mathsf{A})$. The latter follows from Lemma 5.18.

As indicated in the definition of μ (see(31)), $\mu(A)$ is a nonempty compact invariant set. Consider $A_i \in \mathsf{J}^\vee(\mathsf{A})$, $i = 1, 2$, with $A_1 \neq A_2$. It needs to be shown that $\mu(A_1) \cap \mu(A_2) = \varnothing$. So assume the contrary. Observe that $\mu(A_1) \cap \mu(A_2)$ is forward invariant. Since it is compact $\varnothing \neq \mathrm{Inv}\,(\mu(A_1) \cap \mu(A_2))$. Observe that

$$\mathrm{Inv}\,(\mu(A_1) \cap \mu(A_2)) \subset \mathrm{Inv}(A_1 \cap A_2) = A_1 \wedge A_2 \in \mathsf{A}$$

which follows from the fact that A is a lattice. Since A_1 and A_2 are join-irreducible $A_1 \wedge A_2 \subset \mathrm{pred}(A_1)$. Recall that $\mu(A_1) = \mathrm{pred}(A_1)^*$ with respect to $\varphi\|_A$ and thus

$$\mathrm{Inv}\big(\mu(A_1) \cap \mu(A_2)\big) \cap \mu(A_1) \subset (A_1 \wedge A_2) \cap \mu(A_1) \subset \mathrm{pred}(A_1) \cap \mathrm{pred}(A_1)^* = \varnothing.$$

This is the desired contradiction. ∎

Proposition 5.21 *Let S be a compact invariant set for φ. For any $\mathsf{A} \in \mathrm{sub}^F_{0,1}\mathsf{Att}(S,\varphi)$ we can draw the following diagram*

$$
\begin{array}{ccc}
\mathsf{J}^\vee(\mathsf{A}) & \xleftarrow{\mu^{-1}} & \mathsf{M}(\mathsf{A}) \\
\downarrow{\scriptstyle \mathsf{O}} & & \downarrow{\scriptstyle \mathsf{O}} \\
\mathsf{O}(\mathsf{J}^\vee(\mathsf{A})) & \xrightarrow{\mathsf{O}(\mu^{-1})} & \mathsf{O}(\mathsf{M}(\mathsf{A}))
\end{array}
$$

where $\mathsf{O}(\mu^{-1}) = \mathsf{O}(\mu)^{-1} = \mu$.

5.3 A Dynamics version of Birkhoff's Representation Theorem

The goal is this section is to prove the following theorem.

Theorem 5.22 *Let S be a compact invariant set for φ.*

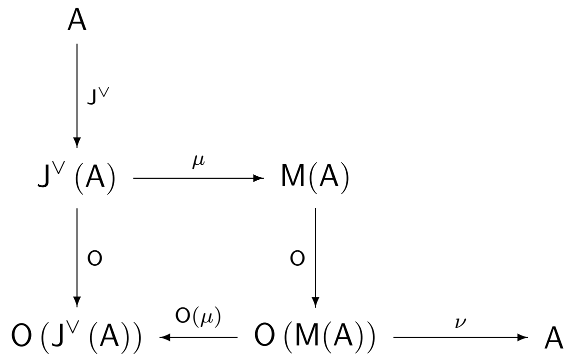
(i) If M is a Morse decomposition in S labelled by the finite poset $(\mathsf{P},\le)$ then we have the following diagram,

$$
\begin{array}{ccccc}
\mathsf{pM} & & & & \\
\downarrow{\scriptstyle \mathsf{O}} & & & & \\
\mathsf{O}(\mathsf{pM}) & \xrightarrow{\nu} & \mathsf{A}(\mathsf{pM}) & & \\
\downarrow{\scriptstyle \mathsf{J}^\vee} & & \downarrow{\scriptstyle \mathsf{J}^\vee} & & \\
\mathsf{J}^\vee(\mathsf{O}(\mathsf{pM})) & \xleftarrow{\mathsf{J}^\vee(\nu)} & \mathsf{J}^\vee(\mathsf{A}(\mathsf{pM})) & \xrightarrow{\mu} & \mathsf{pM}
\end{array}
$$

and

$$\mu \circ \mathsf{J}^\vee(\nu)^{-1} \circ \downarrow = \mathrm{id}_{\mathsf{pM}}\,.$$

(ii) If $\mathsf{A} \in \mathrm{sub}_{0,1}^{F}\mathsf{Att}(S,\varphi)$ *then we have the following diagram,*



and

$$\nu \circ \mathsf{O}(\mu)^{-1} \circ \downarrow^{\vee} = \mathrm{id}_{\mathsf{A}}\,.$$

Proof: (i) By Proposition 5.16, $\mathsf{J}^{\vee}(\nu)^{-1} = \nu$. Thus, by Birkhoff's Representation Theorem 4.23 and Example 5.9 it is sufficient to prove that $\mu \circ \nu \circ \downarrow = \mathrm{id}_{\mathsf{pM}}$ which is the content of Lemma 5.23.

(ii) By Proposition 5.21, $\mathsf{O}(\mu)^{-1} = \mu$. Thus, by Birkhoff's Representation Theorem 4.23 and Example 5.10 it is sufficient to prove that $\nu \circ \mu \circ \downarrow^{\vee} = \mathrm{id}_{\mathsf{A}}$ which is the content of Lemma 5.24. ∎

Lemma 5.23 $\mu \circ \nu \circ \downarrow = \mathrm{id}_{\mathsf{pM}}$

Proof: Consider $M(p) \in \mathsf{pM}$. We begin by showing that

$$\{M(q) \mid q \in \downarrow(p)\}$$

is a Morse decomposition of $\nu\left(\downarrow(M(p))\right) \in \mathsf{Att}(S,\varphi)$.

Since $\downarrow(M(p)) \in \mathsf{J}^{\vee}(\mathsf{O}(\mathsf{pM}))$, by Proposition 5.16 $\nu\left(\downarrow(M(p))\right) \in \mathsf{J}^{\vee}(\mathsf{A}(\mathsf{pM}))$.

Observe that

$$\nu\left(\downarrow(M(p))\right) = \bigcup_{q \in \downarrow(p)} W^{u}(M(q),\varphi).$$

Thus for all $q \in \downarrow(p)$, $M(q) \subset \nu\big(\downarrow(M(p))\big)$.

Let

$$x \in \nu\big(\downarrow(M(p))\big) \setminus \bigcup_{q \in \downarrow(p)} M(q)$$

and consider $\gamma_x \subset \nu\big(\downarrow(M(p))\big) \subset S$. Since we have a Morse decomposition for S, there exists $M(r) \in \mathsf{pM}$ such that $\alpha_o(\gamma_x^-) \in M(r)$. Since $M(r) \subset \nu\big(\downarrow(M(p))\big)$, $r \in \downarrow(p)$. Furthermore, $\omega(x) \subset M(r')$ such that $r' < r$ and hence $r' \in \downarrow(p)$. Thus,

$$\{M(q) \mid q \in \downarrow(p)\}$$

is the desired Morse decomposition.

Since

$$\{M(q) \mid q \in \downarrow(p)\}$$

is a Morse decomposition by Lemma 5.14

$$\left(\bigcup_{q \in \mathrm{pred}(\downarrow(p))} W^u\big(M(q)\big), M(p)\right)$$

is an attractor-repeller pair. Observe that $\mathrm{pred}(\downarrow(\mathrm{M(p)})) \in \mathsf{O}(\mathsf{pM})$. Thus $\nu(\mathrm{pred}(\downarrow(\mathrm{M(p)}))) \in \mathsf{Att}(\mathrm{S}, \varphi)$ and is the unique attractor preceeding $\nu\big(\downarrow(M(p))\big)$.

Using the attractor-repeller structure we obtain

$$\begin{aligned}
\mu\Big(\nu\big(\downarrow(M(p))\big)\Big) &= \mathsf{m}\left(\nu\big(\downarrow(M(p))\big), \nu(\mathrm{pred}(\downarrow(\mathrm{M(p)})))\right) \\
&= \nu\big(\downarrow(M(p))\big) \wedge \nu(\mathrm{pred}(\downarrow(\mathrm{M(p)})))^* \\
&= M(p)
\end{aligned}$$

which completes the proof. ■

Lemma 5.24 $\nu \circ \mu \circ \downarrow^\vee = \mathrm{id}_{\mathsf{A}}$

Proof: Let $a \in \mathsf{A}$. By definition

$$\nu \circ \mu \circ \downarrow^\vee(a) = \bigcup_{\{b \in \mathsf{J}^\vee(\mathsf{A}) \mid b \subset a\}} W^u(b \wedge \mathrm{pred(b)}^*, \varphi) = \bigcup_{\{\mathrm{b} \in \mathsf{J}^\vee(\mathsf{A}) \mid \mathrm{b} \subset \mathrm{a}\}} \mathrm{W^u}(\mu(\mathrm{b}), \varphi)$$

We need to show that $a = \nu \circ \mu \circ \downarrow^\vee(a)$. Consider $x \in \nu \circ \mu \circ \downarrow^\vee(a)$. Then

$$\begin{aligned}
x \in W^u(b \wedge \mathrm{pred(b)}^*, \varphi) &\subset W^u(b, \varphi) \\
&\subset W^u(a, \varphi)
\end{aligned}$$

Thus, $\nu \circ \mu \circ \downarrow^\vee(a) \subset a$.

The proof of the reverse inclusion is slightly more complicated. By Proposition 4.20

$$a = \bigvee_{b \in \iota^\vee(a)} b = \bigcup_{b \in \iota^\vee(a)} b$$

where $\iota^\vee(a) \subset \mathsf{J}^\vee(\mathsf{A})$. Thus if $x \in a$ then $x \in b \subset a$ for some $b \in \iota^\vee(a)$.

Observe that if $b \in \mathsf{J}^\vee(\mathsf{A})$, then $\{\emptyset, \mathrm{pred(b)}, \mathrm{b}\} \in \mathrm{sub}^F_{0,1}\mathsf{Att}(S, \varphi)$. Thus $(\mathrm{pred(b)}, \mu(\mathrm{b}))$ is an attractor repeller pair in b. Furthermore,

$$x \in b = \mathrm{pred(b)} \cup \mu(\mathrm{b}) \cup \mathrm{C}(\mu(\mathrm{b}), \mathrm{pred(b)}).$$

If $x \in \mu(b) \cup C(\mu(b), \mathrm{pred(b)})$, then $x \in W^u(\mu(b), \varphi)$ and hence

$$x \in \nu \circ \mu \circ \downarrow^\vee(a).$$

If $x \in \mathrm{pred(b)}$, then consider

$$\mathrm{pred(b)} = \bigvee_{\mathrm{c} \in \iota^\vee(\mathrm{pred(b)})} \mathrm{c}$$

As above since $x \in \mathrm{pred(b)}$, there exists $c \in \iota^\vee(\mathrm{pred(b)})$ such that $x \in c$ and repeat the argument. Now observe that we have strict containment $c \subset b \subset a$, thus by induction

$$x \in \nu \circ \mu \circ \downarrow^\vee(a).$$

■

5.4 Lyapunov Functions for Morse Decompositions

Theorem 5.25 *Let S be a compact invariant set. For any Morse decomposition $\mathsf{M} \in \mathsf{MD}(S, \varphi)$ for S, there exists a Lyapunov function $V : S \to [0, 1]$ for (S, M).*

Let $\mathsf{MDLyap}(S, \varphi)$ denote the set of all Lyapunov functions $V : S \to [0, 1]$ for any Morse decomposition $\mathsf{M} \in \mathsf{MD}(S, \varphi)$.

Define

$$\mathsf{M} \wedge \mathsf{M}' := \{M(p, q) := \mathrm{Inv}(M(p) \cap M'(q), \varphi) \mid M(p) \cap M'(q) \neq \emptyset,\ M \in \mathsf{M}, M' \in \mathsf{M}'\}$$

where $(p, q) \leq (p', q')$ if $p \leq p'$ or $q \leq q'$.

Exercise: show that $\mathsf{M} \wedge \mathsf{M}' \in \mathsf{MD}(S, \varphi)$.

Proposition 5.26 *Let* $\mathsf{M}, \mathsf{M}' \in \mathsf{MD}(S,\varphi)$ *be Morse decompositions and let* $V, V' \in \mathsf{MDLyap}(S,\varphi)$ *be Lyapunov functions for* M *and* M' *respectively. Then* $V + V' \in \mathsf{MDLyap}(S,\varphi)$ *is a Lyapunov function for* $\mathsf{M} \wedge \mathsf{M}'$.

Proof: good exercise in understanding how two different Morse decompositions of S are related. ∎

Proof of Theorem 5.25. Consider the p-Morse decomposition pM. Observe that a Lyapunov function for pM defines a Lyapunov function for M.

By Birkhoff for each $M(p) \in \mathsf{pM}$ there exists $A_p \in \mathsf{J}^{\vee}(\mathsf{A}(\mathsf{pM})$ such that $\mu(A_p) = M(p)$. Let $V_p \colon S \to [0,1]$ be a Lyapunov function for the attractor-repeller pair (A_p, A_p^*).

Define

$$V = \sum_{p \in \mathsf{sP}} V_p$$

By Proposition 5.26 the function $\sum_i V_i$ is a Lyapunov function for (S, M). To make it a function taking values in $[0,1]$ we normalize the Lyapunov function.
∎

6 Directed Graphs

6.1 Definitions

A *directed graph* or *digraph* consists of a non-empty finite set of elements $\mathcal{X}$ called *vertices* and a finite set of ordered pairs of vertices $\mathcal{E} \subset \mathcal{X} \times \mathcal{X}$ called *edges*. Vertices $G, H \in \mathcal{X}$ are *adjacent* if (G, H) or (H, G) is an edge. The edge $(G, H) \in \mathcal{E}$ which is often denoted by $G \to H$ has vertex G as its *tail* and vertex H as its *head*. Observe that this definition of a digraph explicitly allows for the existence of *loops*, that is edges of the form (G, G).

It is convenient to emphasize the fact that a directed graph is a relation. With this in mind,

$$\mathcal{F} \colon \mathcal{X} \rightrightarrows \mathcal{X}$$

denotes a digraph with vertices $\mathcal{X}$ such that given $G \in \mathcal{X}$,

$$\mathcal{F}(G) := \{H \in \mathcal{X} \mid (G, H) \in \mathcal{E}\}$$

denotes the set of heads of edges with tail G. We will often refer to $\mathcal{F}$ as a *multivalued map* on $\mathcal{X}$ and given $\mathcal{Y} \subset \mathcal{X}$

$$\mathcal{F}(\mathcal{Y}) := \bigcup_{G \in \mathcal{Y}} \mathcal{F}(G).$$

The *converse* of a digraph $\mathcal{F}$, denoted by $\mathcal{F}^{-1}$, is the digraph obtained by reversing all edges, that is

$$G \in \mathcal{F}^{-1}(H) \quad \text{if and only if} \quad H \in \mathcal{F}(G).$$

A digraph $\mathcal{F}' \colon \mathcal{Y} \rightrightarrows \mathcal{Y}$ is a *subdigraph* of $\mathcal{F}$ if $\mathcal{Y} \subset \mathcal{X}$ and $\mathcal{F}'(G) \subset \mathcal{F}(G) \cap \mathcal{Y}$ for all $G \in \mathcal{Y}$. Given $\mathcal{Y} \subset \mathcal{X}$ the *restriction* of $\mathcal{F}$ to $\mathcal{Y}$ is the subdigraph $\mathcal{F}\|_{\mathcal{Y}} \colon \mathcal{Y} \rightrightarrows \mathcal{Y}$ defined by $\mathcal{F}\|_{\mathcal{Y}}(G) = \mathcal{F}(G) \cap \mathcal{Y}$ for all $G \in \mathcal{Y}$. To maintain simplicity of notation unless explicitly stated otherwise, given a digraph $\mathcal{F} \colon \mathcal{X} \rightrightarrows \mathcal{X}$ and $\mathcal{Y} \subset \mathcal{X}$, $\mathcal{F} \colon \mathcal{Y} \rightrightarrows \mathcal{Y}$ will always denote the restriction of $\mathcal{F}$ to $\mathcal{Y}$.

A *walk* in $\mathcal{F} \colon \mathcal{X} \rightrightarrows \mathcal{X}$ of *length* K is a sequence $\{G_k \in \mathcal{X} \mid k = 0, \ldots, K\}$ such that $G_{k+1} \in \mathcal{F}(G_k)$ for all $k = 0, \ldots, k-1$. A walk is *closed* if $G_k = G_0$ and *open* otherwise. The walk is *from* G *to* G' if $G_0 = G$ and $G_K = G'$, and *passes through* G'' if $G_k = G''$ for some $k = 0, \ldots, K$. The notation $G_0 \rightsquigarrow G_1$

indicates the existence of a walk from G_0 to G_1 and more generally if $\mathcal{K}_i \subset \mathcal{X}$, $i = 0, 1$, then $\mathcal{K}_0 \rightsquigarrow \mathcal{K}_1$ implies $G_0 \rightsquigarrow G_1$ for some $G_i \subset \mathcal{K}_i$.

Our interest in dynamics leads to the following extended notions. An *orbit* through $G \in \mathcal{X}$ is a sequence $\gamma_G = \{G_k\}_{k\in\mathbb{Z}}$ satisfying $G = G_0$ and $G_{k+1} \in \mathcal{F}(G_k)$ for every $k \in \mathbb{Z}$. A *forward orbit* (*backward orbit*) through G is a sequence $\gamma_G^+ = \{G_k \mid k \geq 0\}$ $\left(\gamma_G^- = \{G_k \mid k \leq 0\}\right)$ such that $G = G_0$ and $G_{k+1} \in \mathcal{F}(G_k)$ for all $k \geq 0$ ($k < 0$).

A restriction $\mathcal{F}\colon \mathcal{M} \rightrightarrows \mathcal{M}$ of a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is *strongly connected* if for every pair of **distinct** vertices $G, H \in \mathcal{M}$ there exists a walk from G to H and a walk from H to G in $\mathcal{M}$. It is *strongly path connected* if there exists a walk from G to H and a walk from H to G in $\mathcal{M}$ for any pair of vertices. Observe that any strongly connected digraph with more than one vertex is strongly path connected. A strongly connected digraph with a unique vertex is path connected if and only if there exists a loop at the vertex.

Definition 6.1 A *strongly connected (path) component* of a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is a maximal set of vertices $\mathcal{M} \subset \mathcal{X}$ such that the restriction $\mathcal{F}\colon \mathcal{M} \rightrightarrows \mathcal{M}$ is a strongly connected (path) component.

Observe that given a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$, the set of strongly connected components $\{\mathcal{D}_k \mid k = 0, \ldots, K\}$ defines an equivalence relation on $\mathcal{X}$.

Given a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ and a strongly connected component $\mathcal{D}$ the *contraction* of $\mathcal{F}$ by $\mathcal{D}$ is the digraph $\mathcal{F}'\colon \mathcal{X}' \rightrightarrows \mathcal{X}'$ defined as follows. The set of vertices is given by $\mathcal{X}' := \{D\} \cup (\mathcal{X} \setminus \mathcal{D})$ where D is a new vertex; that is $D \notin \mathcal{X}$. The edges are defined as follows. Given $G, H \in \mathcal{X} \setminus \mathcal{D}$, if $H \in \mathcal{F}(G)$ then $H \in \mathcal{F}'(G)$, and if $H \in \mathcal{F}^{-1}(\mathcal{D})$ then $D \in \mathcal{F}'(H)$. Furthermore, $\mathcal{F}'(D) := \{H \in \mathcal{X} \setminus \mathcal{D} \mid H \in \mathcal{F}(\mathcal{D})\}$.

Proposition 6.2 *Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be different strongly connected components of a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$. Let $\mathcal{F}_i\colon \mathcal{X}_i \rightrightarrows \mathcal{X}_i$ be the contraction of $\mathcal{F}$ by $\mathcal{D}_i$ where $\mathcal{X}_i = \{D_i\} \cup (\mathcal{X} \setminus \mathcal{D}_i)$. Let $\mathcal{F}_{i,j}\colon \mathcal{X}_{i,j} \rightrightarrows \mathcal{X}_{i,j}$ be the contraction of $\mathcal{F}_i$ by $\mathcal{D}_j$ where $i \neq j$. Then $\mathcal{F}_{0,1} = \mathcal{F}_{1,0}$ where*

$$\mathcal{X}_{0,1} = \mathcal{X}_{1,0} = \{D_0\} \cup \{D_1\} \cup (\mathcal{X} \setminus (\mathcal{D}_0 \cup \mathcal{D}_1)).$$

Proposition 6.2 implies that the following concept is well defined.

Definition 6.3 Given a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ let $\{\mathcal{D}_k \mid k = 0, \dots, K\}$ be the collection of strongly connected components. The *order graph* of $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is the contraction of $\mathcal{F}$ by $\{\mathcal{D}_k \mid k = 0, \dots, K\}$ and denoted by

$$\bar{\mathcal{F}}\colon \bar{\mathcal{X}} \rightrightarrows \bar{\mathcal{X}}$$

where $\bar{\mathcal{X}} := \{D_k \mid k = 0, \dots, K\}$. The associated *order contraction map* is defined by

$$\begin{array}{rcl} \rho_{\mathcal{F}}\colon \mathcal{X} & \to & \bar{\mathcal{X}} \\ G & \mapsto & \rho_{\mathcal{F}}(G) \end{array}$$

where $\rho_{\mathcal{F}}(G) = D_k$ if $G \in \mathcal{D}_k$.

Proposition 6.4 *For any digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ the associated order graph $\bar{\mathcal{F}}\colon \bar{\mathcal{X}} \rightrightarrows \bar{\mathcal{X}}$ is an acyclic directed graph and hence a poset.*

Proof: The proof is by contradiction. Assume that there exists a closed walk $\{D_k \mid k = 0, \dots, K\}$ in $\bar{\mathcal{F}}$. The fact that $D_{k+1} \in \bar{\mathcal{F}}(D_k)$ implies that there exist $G_k \in \rho_{\mathcal{F}}^{-1}(D_k)$ and $H_{k+1} \in \rho_{\mathcal{F}}^{-1}(D_{k+1})$ such that $H_{k+1} \in \mathcal{F}(G_k)$. Since $G_k, H_k \in \mathcal{D}_k = \rho_{\mathcal{F}}^{-1}(D_k)$, either $G_k = H_k$ or $H_k \rightsquigarrow G_k$ under $\mathcal{F}$. This implies the existence of a single closed walk passing through each of the strongly connected components $\{\mathcal{D}_k \mid k = 0, \dots, K\}$ contradicting the definition of a strongly connected component. ∎

6.2 Digraph Dynamics

Definition 6.5 Let $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ be a digraph. An *invariant* digraph in $\mathcal{F}$ is a restricted digraph $\mathcal{F}\colon \mathcal{S} \rightrightarrows \mathcal{S}$ such that $\mathcal{F}(G) \cap \mathcal{S} \neq \emptyset$ and $\mathcal{F}^{-1}(G) \cap \mathcal{S} \neq \emptyset$ for every $G \in \mathcal{S}$. $\mathcal{S} \subset \mathcal{X}$ is an *invariant set* for $\mathcal{F}$ if the restricted digraph $\mathcal{F}\colon \mathcal{S} \rightrightarrows \mathcal{S}$ is invariant.

Let $\mathsf{Invset}(\mathcal{X}, \mathcal{F})$ denote the set of invariant digraphs in $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$. Observe that $\mathsf{Invset}(\mathcal{X}, \mathcal{F})$ is a poset with the ordering given by inclusion.

Proposition 6.6 *Let $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ be a digraph, then the following statements are equivalent:*

(i) $\mathcal{S} \subset \mathcal{X}$ is an invariant set for $\mathcal{F}$;

(ii) for all $G \in \mathcal{S}$ there exists an orbit $\gamma_G \subset \mathcal{S}$.

(iii) $\mathcal{S} \subset \mathcal{F}(\mathcal{S})$ and $\mathcal{S} \subset \mathcal{F}^{-1}(\mathcal{S})$.

Proof: (i) $\Rightarrow$ (ii) Given $G \in \mathcal{S}$ an orbit through G can be constructed inductively as follows. Define $G_0 = G$. Assume G_k has been chosen. The assumption that $\mathcal{S}$ is invariant implies that $\mathcal{F}(G_k) \cap \mathcal{S} \neq \varnothing$ and $\mathcal{F}^{-1}(G_k) \cap \mathcal{S} \neq \varnothing$. If $k \geq 0$, choose $G_{k+1} \in \mathcal{F}(G_k) \cap \mathcal{S}$ and if $k \leq 0$, choose $G_{k-1} \in \mathcal{F}^{-1}(G_k) \cap \mathcal{S}$.

(ii) $\Rightarrow$ (iii) For any $G_1 \in \mathcal{S}$ there exists a $G \in \mathcal{F}^{-1}(G_1) \cap \mathcal{S}$. Moreover $\mathcal{F}(G) \subset \mathcal{F}(\mathcal{S})$ for all $G \in \mathcal{S}$ so that $G_1 \in \mathcal{F}(\mathcal{S})$, which proves that $\mathcal{S} \subset \mathcal{F}(\mathcal{S})$. The property for $\mathcal{F}^{-1}$ follows similarly.

(iii) $\Rightarrow$ (i) We prove the contrapositive. If $\mathcal{F}(G) = \varnothing$ for some $G \in \mathcal{S}$, then $G \notin \mathcal{F}^{-1}(S)$. Similarly, if $\mathcal{F}^{-1}(G) = \varnothing$ for some $G \in \mathcal{S}$, then $G \notin \mathcal{F}(S)$. ∎

Corollary 6.7 *Let $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ be a digraph. If $\mathcal{P} \subset \mathcal{X}$ is a strongly connected path component, then $\mathcal{P} \in \mathsf{Invset}(\mathcal{X}, \mathcal{F})$.*

Proposition 6.8 *$\mathcal{M}$ is a strongly connected path component of the digraph $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ if and only if it is a strongly connected component and invariant.*

Proof: Assume $\mathcal{M}$ is a strongly connected path component. The fact that there exist walks to and from any vertex in a strongly connected path component implies that it is a strongly connected component. Given $G \in \mathcal{M}$ there exists a closed walk beginning and ending at G. Concatenation of this walk produces an orbit $\gamma_G \subset \mathcal{M}$. Proposition 6.6 implies that $\mathcal{M}$ is invariant.

Now assume that $\mathcal{M}$ is a strongly connected component and invariant. If $\mathcal{M}$ contains more than one element, then the fact that it is a strongly connected component implies that it is a strongly connected path component. If $\mathcal{M}$ contains a unique element G, then the fact that $\mathcal{M}$ is invariant implies that there is a loop $G \to G$. ∎

As the following example indicates it is not necessarily true that the intersection of invariant sets for combinatorial multivalued maps is invariant.

Example 6.9 Let $\mathcal{X} = \{G_0, G_1, G_2\}$. Let $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ be given by the adjacency matrix

$$\mathcal{F} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then $\{G_0, G_1\}$ and $\{G_0, G_2\}$ are invariant sets, but $G_0 = \{G_0, G_1\} \cap \{G_0, G_2\}$ is not an invariant set.

Proposition 6.10 *If $\mathcal{F}\colon \mathcal{S} \rightrightarrows \mathcal{S}$ is an invariant digraph, then $IS(\mathcal{X}, \mathcal{F})$ is a $(0,1)$-lattice where*

$$\begin{aligned} \mathcal{K}_0 \vee \mathcal{K}_1 &:= \mathcal{K}_0 \cup \mathcal{K}_1 \\ \mathcal{K}_0 \wedge \mathcal{K}_1 &:= \max\{\mathcal{K} \in IS(\mathcal{X}, \mathcal{F}) \mid \mathcal{K} \subset \mathcal{K}_i,\ i = 0, 1\}. \end{aligned}$$

Proof: ■

Let $\mathcal{F}\colon \mathcal{S} \rightrightarrows \mathcal{S}$ be an invariant digraph. Define

$$\begin{aligned} \nu\colon IS(\mathcal{X}, \mathcal{F}) &\to IS(\mathcal{X}, \mathcal{F}) \\ \mathcal{K} &\mapsto \nu(\mathcal{K}) := \{G \in \mathcal{S} \mid \mathcal{K} \rightsquigarrow G\} \end{aligned}$$

Proposition 6.11 *If $\mathcal{F}\colon \mathcal{S} \rightrightarrows \mathcal{S}$ is an invariant digraph, then $\nu(\mathcal{K}) \in \mathsf{Invset}(\mathcal{X}, \mathcal{F})$.*

Proof: By Proposition 6.6 it is sufficient to show that there exists an orbit $\gamma_G \subset \nu(\mathcal{K})$ for all $G \in \nu(\mathcal{K})$. Since $\mathcal{K} \in \mathsf{Invset}(\mathcal{X}, \mathcal{F})$, this is true if $G \in \mathcal{K}$. So assume $G \in \nu(\mathcal{K}) \setminus \mathcal{K}$. Since $G \in \mathcal{S}$ there exists a forward orbit $\gamma_G^+ \subset \mathcal{S}$. By definition $\mathcal{K} \rightsquigarrow G$. Thus there exists $G_0 \in \mathcal{K}$ such that $G_0 \rightsquigarrow G$. Since $G_0 \in \mathcal{K}$ there exists a backward orbit $\gamma_G^- \subset \mathcal{K}$. Concatenating the walk from G_0 to G with γ_G^+ produces a forward orbit contained in $\nu(\mathcal{K})$. Thus, concatenating this with $\gamma_G^- \subset \mathcal{K}$ produces an orbit $\gamma_G \subset \nu(\mathcal{K})$. ■

Definition 6.12 The *Morse graph* of a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is the poset

$$\bar{\mathsf{M}}_{\mathcal{F}} := \left\{M_p \in \bar{\mathcal{X}} \mid \mathcal{M}_p = \rho_{\mathcal{F}}^{-1}(M_p) \text{ is a strongly connected path component}\right\}$$

where the order $\leq$ is inherited from $\bar{\mathcal{X}}$.

Definition 6.13 Consider a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ and its Morse graph $\bar{\mathsf{M}}_{\mathcal{F}} = \{D_p \mid p \in (\mathsf{P}, \leq)\}$. The *finest Morse decomposition* of $\mathcal{F}$ is the poset

$$\mathsf{M}_{\mathcal{F}} := \left\{\mathcal{M}(p) := \rho_{\mathcal{F}}^{-1}(D_p) \mid D_p \in \bar{\mathsf{M}}_{\mathcal{F}},\ p \in (\mathsf{P}, \leq)\right\}$$

where the order $\leq$ is inherited from $\bar{\mathsf{M}}_{\mathcal{F}}$.

Definition 6.14 The *attractor structure* of a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is the lattice $\mathsf{O}(\bar{\mathsf{M}}_{\mathcal{F}})$.

Definition 6.15 An *attractor* in a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is a set of vertices $\mathcal{A} \subset \mathcal{X}$ such that

$$\mathcal{A} \in \nu\left(\rho_{\mathcal{F}}^{-1}\left(\mathsf{O}(\bar{\mathsf{M}}_{\mathcal{F}})\right)\right).$$

We denote the set of attractors in $\mathcal{F}$ by $\mathsf{Att}(\mathcal{X}, \mathcal{F})$.

Proposition 6.16 *If* $\mathsf{A} \in \mathsf{O}\left(\bar{\mathsf{M}}_{\mathcal{F}}\right)$*, then* $\rho_{\mathcal{F}}^{-1}(\mathsf{A}) \in \mathsf{Invset}(\mathcal{X}, \mathcal{F})$ *and hence an attractor is an invariant set.*

Proof: This follows from Corollary 6.7 and Propositions 6.10 and 6.11. ■

Proposition 6.17 *Given an invariant digraph* $\mathcal{F}\colon \mathcal{S} \rightrightarrows \mathcal{S}$*,* $\mathcal{A} \subset \mathcal{S}$ *is an attractor if and only if* $\mathcal{F}(\mathcal{A}) = \mathcal{A}$*.*

Proof: Let $\mathcal{A}$ be an attractor. By Proposition 6.16 $\mathcal{A}$ is invariant and hence by Proposition 6.6 $\mathcal{A} \subset \mathcal{F}(\mathcal{A})$. Consider $G_1 \in \mathcal{F}(\mathcal{A})$. Then there exists $G_0 \in \mathcal{A}$ such that $G_1 \in \mathcal{F}(G_0)$. Since $\mathcal{A} \in \nu\left(\rho_{\mathcal{F}}^{-1}\left(\mathsf{O}(\bar{\mathsf{M}}_{\mathcal{F}})\right)\right)$, there exists a Morse set $\mathcal{M}(p)$ such that $\mathcal{M}(p) \rightsquigarrow G_0$. This implies that $\mathcal{M}(p) \rightsquigarrow G_1$ and hence $G_1 \in \mathcal{A}$.

Assume $\mathcal{F}(\mathcal{A}) = \mathcal{A}$. Consider $\rho_{\mathcal{F}}(\mathcal{A}) \subset \bar{\mathcal{S}}$. Since $\bar{\mathcal{S}}$ is a finite poset, the set of maximal elements $\{D_k \mid k = 1, \ldots, K\}$ in $\rho_{\mathcal{F}}(\mathcal{A})$ is well defined. Let $\mathcal{D}_k = \rho_{\mathcal{F}}^{-1}(D_k)$. If $\mathcal{D}_k$ is not a strongly connected path component, then there exists a unique element $G_k \in \mathcal{D}_k$. Furthermore, if $H \in \mathcal{F}^{-1}(G)$, then $H \notin \mathcal{D}_k$. Since D_k is maximal, $H \notin \mathcal{A}$. Thus, $G \notin \mathcal{F}(\mathcal{A})$ contradicting the assumption that $\mathcal{F}(\mathcal{A}) = \mathcal{A}$. Thus for all $k = 1, \ldots, K$, $\mathcal{D}_k$ is a strongly connected path component and hence $D_k \in \bar{\mathsf{M}}_{\mathcal{F}}$. Observe that

$$\mathcal{A} = \nu\left(\rho_{\mathcal{F}}^{-1}\left(\bigcup_{k=1}^{k} \downarrow(D_k)\right)\right)$$

and hence $\mathcal{A} \in \mathsf{Att}(\mathcal{X}, \mathcal{F})$. ■

7 Combinatorialization

7.1 Time Discretization

Definition 7.1 Two dynamical systems $\varphi\colon \mathbb{R}^+ \times X \to X$ and $\psi\colon \mathbb{R}^+ \times Y \to Y$ are *topologically equivalent* if there exists a homeomorphism $h\colon X \to Y$ and a continuous reparametrization $r : \mathbb{R}^+ \times X \to \mathbb{R}^+$, with $r(0, x) = 0$ and monotone in r for all x, such that

$$h\big(\varphi(r(t, x), x)\big) = \psi\big(t, h(x)\big). \tag{33}$$

Definition 7.2 Let $\varphi\colon \mathbb{R}^+ \times X \to X$ be a continuous dynamical system. A *faithful time reparametrization* of φ is a continuous function

$$r\colon \mathbb{R}^+ \times X \to \mathbb{R}^+$$

that satisfies the following two conditions:

(i) $r(t, x) > 0$ for all $x \in X$,

(ii)
$$r(t + t', x) = r(t, \varphi(r(t', x), x)) + r(t', x) \tag{34}$$
for all $x \in X$.

Proposition 7.3 *Let $\varphi : \mathbb{R}^+ \times X \to X$ be a continuous dynamical system with a faithful time reparametrization $r : \mathbb{R}^+ \times X \to X$. Then φ is topologically equivalent to the dynamical system $\psi : \mathbb{R}^+ \times X \to X$ defined by*

$$\psi(t, x) := \varphi(r(t, x), x).$$

The function $r(\cdot, x)$ is a homeomorphism for all $x \in X$.

Proof: From (34) and the fact that $r(t, x) > 0$ for $t > 0$ it follows that $r(\cdot, x)$ is a strictly increasing function and that $r(0, x) = 0$ for all $x \in X$. Therefore, $r(\cdot, x)$ a homeomorphism in the t-variable for all $x \in X$. Observe that (33) is satisfied if $h = \mathrm{id}_X$.

It remains to show that ψ is a dynamical system. Observe that

$$\psi(0, x) = \varphi)(r(0, x), x) = \varphi(0, x) = x.$$

Furthermore,

$$\begin{aligned}
\psi(t+t',x) &= \varphi(r(t+t',x),x) \\
&= \varphi(r(t,\varphi(r(t',x),x)) + r(t',x),x) \\
&= \varphi(r(t,\varphi(r(t',x),x)),\varphi(r(t',x),x)) \\
&= \varphi(r(t,\psi(t',x)),\psi(t',x)) \\
&= \psi(t,\psi(t',x))
\end{aligned}$$

■

Definition 7.4 Let $\varphi : [0,\infty) \times X \to X$ be a continuous dynamical system. A *time selection for* φ is a continuous map $\tau : X \to (0,\infty)$ such that $\tau(x) = r(1,x)$ for some faithful time reparametrization r. A *τ-discretization of* φ is the map $f : X \to X$ defined by

$$f(x) = \varphi(\tau(x),x).$$

If τ is constant, then f is called the *time-τ map*.

Proposition 7.5 *If S is an invariant set for a continuous time dynamical system φ, then S is also an invariant set for a τ-discretization f of φ.*

Proof: Exercise ■

Remark: The converse need not be true.

Theorem 7.6 *A sets S is an isolated invariant set of φ if and only if S is an isolated invariant set for f, where f is any τ-discretization of φ.*

Proof will follow from the following two propositions.

Proposition 7.7 *If S is an isolated invariant set for a τ-discretization f of φ, with isolating neighborhood N, i.e. $S = \mathrm{Inv}(N,f)$, then N is also an isolating neighborhood for φ and $S = \mathrm{Inv}(N,\varphi)$.*

Proof: We first show that there exists an $\epsilon > 0$ such that $\varphi(t,S) \subset \mathrm{int}(N)$ for all $0 \le t \le \epsilon$. If not, there exists a sequence $t_n \to 0$ and a sequence of points $x_n \in S$ such that $\varphi(t_n,x_n) \in X \backslash \mathrm{int}(N)$. Since S is compact there is a subsequence $x_{n_k} \to x \in S$ and by the continuity of φ, $\varphi(t_{n_k},x_{n_k}) \to x \in S \subset$

$\text{int}(N)$ by isolation, which contradicts the fact that $\varphi(t_{n_k}, x_{n_k}) \in X\backslash\text{int}(N)$. Now let $\gamma_x = \{x_n \mid x_{n+1} = f(x_n)\}$ be an orbit for f in S. By the semi-group property $f^n(\varphi(t,x)) = \varphi(t, f_\tau^n(x))$ we obtain that $\varphi(t, x_{n+1}) = \varphi(t, f(x_n)) = f(\varphi(t, x_n))$. If $0 \le t \le \epsilon$ this implies that $\varphi(t, \gamma_x) = \{\varphi(t, x_n)\} \subset \text{int}(N)$ are orbits of f and thus in S. Repeat the argument using $\gamma_{x'}$, with $x' = \varphi(\epsilon, x)$. This produces orbits $\varphi(t, \gamma_x) \subset S$ for all $0 \le t \le 1$ after fintely many steps. Since by the semi-group property, $x_{n+2} = \varphi(1, x_{n+1}) = \varphi(1, f(x_n)) = f(\varphi(1, x_n)) = f(x_{n+1})$, it follows that $\{\varphi([0,1], \gamma_x)\} \subset S$ is a complete orbit for φ through x, which proves that $S = \text{Inv}(N, f) \subset \text{Inv}(N, \varphi)$.

On the other hand, by definition $\text{Inv}(N, \varphi)$ is a compact invariant set for φ contained in N. By Proposition 7.5 we have that $\text{Inv}(N, \varphi)$ is an invariant set for f. Hence $\text{Inv}(N, \varphi) \subset \text{Inv}(N, f)$. Therefore $S = \text{Inv}(N, f) = \text{Inv}(N, \varphi) = \text{Inv}(N, \varphi)$ which completes the proof. ∎

Proposition 7.8 *If S be an isolated invariant set for φ, then S is an isolated invariant set for f.*

Proof: Let N be an isolating neighborhood for S under φ, i.e. $S = \text{Inv}(N, \varphi)$. Define

$$N' = \{x \in N \mid \varphi([0,1], x) \subset N\} \subset N.$$

The continuity of φ implies that N' is closed and hence compact. Also by continuity $S \subset N'$ and $S \cap \partial N' = \varnothing$.

By Proposition 7.5, $f(S) = S$ and hence $S \subset \text{Inv}(N', f)$. On the other hand, if $x \in \text{Inv}(N', f)$, then $\varphi([0,1], x) \subset N$ and since $x \in \text{Inv}(N', f)$ it holds that $x' = \varphi(1, x) = f(x) \in \text{Inv}(N', f) \subset N$. Consequently, if $\gamma_x \subset \text{Inv}(N', f)$, then $\varphi([0,1], \gamma_x) \subset N$ and as before $\{\varphi([0,1], \gamma_x)\}$ gives a complete orbit through x contained in N. Therefore, $x \in S = \text{Inv}(N, \varphi)$ and $\text{Inv}(N', f) \subset S$. Combining both inclusions proves that $S = \text{Inv}(N, \varphi) = \text{Inv}(N, \varphi) = \text{Inv}(N', f)$, which completes the proof. ∎

7.2 Space Discretization

Example 7.9 Let $f : \left[0, \frac{5}{8}\right] \to \left[0, \frac{5}{8}\right]$ be the logistic map given by $f(x) = \frac{5}{2}x(1-x)$. Consider the dynamical system defined by $x_{n+1} = f(x_n) = \frac{5}{2}x_n(1 - x_n)$. This defines a surjective dynamical system. To provide a purely

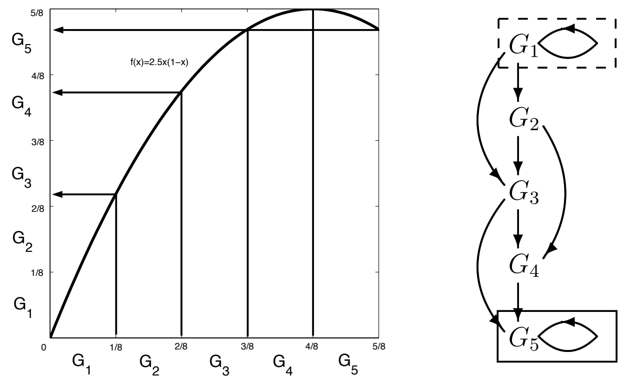


Figure 1: The discretization of $\left[0, \frac{5}{8}\right]$, the definition of the multivalued map for $f(x) = \frac{5}{2}x(1-x)$, and the representation of $\mathcal{F}$ as a directed graph.

combinatorial representation of f, the phase space $X = \left[0, \frac{5}{8}\right]$ is partitioned as follows

$$X = \bigcup_{i=1}^{5} G_i \quad \text{where} \quad G_i = \left[\frac{i-1}{8}, \frac{i}{8}\right].$$

There are two equivalent ways to view the dynamics on this partition. The first is to consider $\mathcal{X} = \{G_1, \cdots, G_5\}$ as a finite set with a multivalued map defined by the relation

$$\mathcal{F}(G_i) = \{G_j \mid f(G_i) \cap G_j \neq \varnothing\}.$$

For example $f(G_1) = \left[0, \frac{35}{128}\right] \subset \{G_1, G_2, G_3\}$, so that $\mathcal{F}(G_1) = \{G_1, G_2, G_3\}$. In the alternative interpretation, each subinterval is viewed as a vertex in a directed graph where there exists an edge from vertex G_i to vertex G_j if $f(G_i) \cap G_j \neq \varnothing$, as shown in Figure 1.

The directed graph $\mathcal{F}$ respects the dynamics of f in the sense that for any orbit $\gamma = \{x_n\}$ of f and any sequence $\{G_n\}$ such that $x_n \in G_n$ we have $G_n \in \mathcal{F}(G_{n-1})$.

Example 7.10 Let $f : [0,1] \to [0,1]$ be given by $f(x) = x^2$ and consider the dynamical system defined by $x_{n+1} = f(x_n) = x_n^2$. The phase space $X = [0,1]$ is partitioned as follows

$$X = \bigcup_{i=1}^{6} G_i \quad \text{where} \quad G_i = \left[\frac{i-1}{6}, \frac{i}{6}\right].$$

The associated multivalued map is defined by the relation

$$\mathcal{F}(G_i) = \{G_j \mid f(G_i) \cap G_j \neq \varnothing\}.$$

Observe that an edge $G_5 \to G_5$ suggests the occurrence of a fixed point in the interval G_5, but this is not the case for the map f. The discretization $\mathcal{X} = \{G_1, \ldots, G_6\}$ is quite coarse. However, if a finer decomposition

$$\mathcal{X} = \left\{ G_i = \left[\frac{i-1}{n}, \frac{i}{n}\right] \mid i = 1, \ldots, n \right\}$$

is chosen, the problem is not resolved. It is left to the reader to check that for all $n > 4$ there are self-loops precisely at the vertices G_1, G_{n-1}, and G_n.

7.2.1 Grids and covering grids

Definition 7.11 A *grid* on a metric space X is a collection $\mathcal{X}$ of nonempty compact subsets of X with the following properties:

(i) $X = \bigcup_{G \in \mathcal{X}} G$

(ii) $G = \operatorname{cl}(\operatorname{int}(G))$ for all $G \in \mathcal{X}$

iii) $G \cap \operatorname{int}(H) = \varnothing$ for all $G \neq H \in \mathcal{X}$.

(iv) If $K \subset X$ is compact, then $\{G \in \mathcal{X} \mid G \cap K \neq \varnothing\}$ is a finite set.

Observe that if X is compact, then $\mathcal{X}$ is finite. The *diameter* of a grid is defined by

$$\operatorname{diam}(\mathcal{X}) := \sup_{G \in \mathcal{X}} \operatorname{diam}(G).$$

The *realization map* $|\cdot|$ is a mapping from subsets of $\mathcal{X}$ to subsets of X, and is defined as $|\mathcal{A}| := \cup_{A \in \mathcal{A}} A \subset X$. Notice that given a fixed grid not every subset of X can be realized. On locally compact spaces, establishing the existence of grids of abitrarily small diameter is straightforward.

Theorem 7.12 *For every separable, locally compact subset N of a metric space X and every $\epsilon > 0$, there exists a grid $\mathcal{N}$ on N with* $\operatorname{diam}(\mathcal{N}) \leq \epsilon$.

Proof: The construction involves an induction argument. Start with the case that N is compact. By compactness we can choose a finite subcover $\{U_i \mid i = 1, \ldots, n\}$ from the cover $\{B_\epsilon(x) \mid x \in N\}$ with $U_i = B_\epsilon(x_i)$ for some $x_i \in N$. Let $W^0 = \{V_i^0 = \mathrm{cl}(U_i) \mid i = 1, \ldots, n\}$. Set $G_1 = V_1^0$. In general define recursively

$$W^k = \left\{V_i^k = \mathrm{cl}(V_i^{k-1} \setminus G_k) \mid i = k+1, \ldots, n\right\}$$

for $k = 1, ..., n-1$ with $G_k := V_k^{k-1}$. After $n-1$ steps we let $\mathcal{N} := \{G_i \mid i = 1, \ldots, n\}$. Condition (i) of Definition 7.11 is satisfied. Condition (ii) follows from $\mathrm{cl}(\mathrm{int}(V_i^0)) = V_i^0$. Condition (iii) follows from the fact that for $i > 1$

$$\mathrm{int}(G_i) \;=\; \mathrm{int}(V_i^{i-2} \setminus G_{i-1}) = \left\{x \;\middle|\; x \in \mathrm{int}(V_i^{i-2}) \setminus G_{i-1}\right\}.$$

Also $\mathrm{diam}(G_k) \leq \epsilon$ for all $k = 1, \ldots, n$ so that $\mathrm{diam}(\mathcal{N}) = \mathrm{diam}(G_1) = \epsilon$.

For N locally compact and separable, we argue as follows. There exists a sequence of open, precompact sets $U_n \subset N$ such that $\mathrm{cl}(U_n) \subset U_{n+1}$ for all $n \geq 0$, and $N = \bigcup_n U_n$. Define $N_n = \mathrm{cl}(U_n \backslash U_{n-1})$ for $n \geq 1$, and $N_0 = \mathrm{cl}(U_0)$. For each N_n the above proof of the compact case provides a finite grid $\mathcal{N}_n$ with $\mathrm{diam}(\mathcal{N}_n) = \epsilon$. Therefore $\mathcal{N} = \bigcup_n \mathcal{N}_n$ is a grid for N with $\mathrm{diam}(\mathcal{N}) = \epsilon$. ∎

7.2.2 Multivalued mappings for grids and covering grids

Definition 7.13 Let $\mathcal{X}$ and $\mathcal{X}'$ be grids or covering grids. An associated *combinatorial multivalued map* $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}'$ assigns to each element $G \in \mathcal{X}$ a finite (possibly empty) subset $\mathcal{F}(G)$ of $\mathcal{X}'$. The *inverse* of $\mathcal{F}$ is defined by $\mathcal{F}^{-1}(H) := \{G \in X \mid H \in \mathcal{F}(G)\}$.

Remark: A combinatorial multivalued map is a digraph. We will use the two terms interchangeably depending on the perspective we want to emphasize.

Definition 7.14 Let f be a τ-discretization of φ on X and let N be a compact set with a grid, or grid covering $\mathcal{N}$ on N. The *τ-minimal multivalued mapping* of φ on $\mathcal{N}$ is defined by $\widetilde{\mathcal{F}}(G) = \{H \in \mathcal{N} \mid H \cap f(G) \neq \varnothing\}$. If $\tau = 1$, then $\widetilde{\mathcal{F}}$ is referred to as the *canonical minimal multivalued mapping* of φ on N.

Definition 7.15 Let $N \subset X$ be a compact set and let $\mathcal{N}$ be a grid. A combinatorial multivalued map $\mathcal{F} : \mathcal{N} \rightrightarrows \mathcal{N}$ is called an *outer approximation* of φ on N, if there exists a time selection $\tau : X \to (0, 1]$ such that the τ-discretization $f = \varphi(\tau(\cdot), \cdot)$ of φ satisfies

$$f(G) \cap N \subset \mathrm{int}_N\big(|\mathcal{F}(G)|\big), \tag{35}$$

for all $G \in \mathcal{N}$. The multivalued mapping $\mathcal{F} : \mathcal{N} \rightrightarrows \mathcal{N}$ is called a *combinatorialization* of the dynamical system φ on N.

Proposition 7.16 *A τ-minimal multivalued map $\widetilde{\mathcal{F}} : \mathcal{N} \rightrightarrows \mathcal{N}$ for φ on $\mathcal{N}$ is an outer approximation of φ on N, and*

$$f^{-1}(G) \cap N \subset \mathrm{int}_N\big(|\widetilde{\mathcal{F}}^{-1}(G)|\big), \tag{36}$$

for all $G \in \mathcal{N}$. In particular, every dynamical system allows a combinatorialization.

Proof: Define $f = \varphi(\tau(\cdot), \cdot)$, and let $y \in f(G) \cap N$. If $y \in \mathrm{int}_N(H)$ for some $H \in \mathcal{N}$, then $H \in \widetilde{\mathcal{F}}(G)$, and hence $y \in \mathrm{int}_N(|\widetilde{\mathcal{F}}(G)|)$. Suppose y does not lie in the interior of any $H \in \mathcal{N}$. Then $y \in \partial H$ for finitely many $H \in \mathcal{N}$, all of which are by definition in $\widetilde{\mathcal{F}}(G)$. If $y \notin \mathrm{int}_N(|\widetilde{\mathcal{F}}(G)|)$, then there exists a sequence $y_n \to y$ which can be chosen to lie in a single grid element $H_0 \notin \widetilde{\mathcal{F}}(G)$. However, in this case, $y \in \mathrm{cl}(H_0) = H_0$ so that $H_0 \in \widetilde{\mathcal{F}}(G)$, which is a contradiction. This proves that $f(G) \cap N \subset \mathrm{int}_N(|\widetilde{\mathcal{F}}(G)|)$ and $\widetilde{\mathcal{F}}$ is an outer approximation.

By definition $\widetilde{\mathcal{F}}^{-1}(G) = \{H \mid G \in \widetilde{\mathcal{F}}(H)\} = \{H \mid G \cap f(H) \neq \varnothing\}$ and $f^{-1}(G) \cap N = \{x \in N \mid f(x) \in G\} \subset |\widetilde{\mathcal{F}}^{-1}(G)|$. By same reasoning as above it follows that $f_\tau^{-1}(G) \cap N \subset \mathrm{int}_N(|\widetilde{\mathcal{F}}^{-1}(G)|)$. ∎

Determining the τ-minimal multivalued map for a dynamical system φ requires complete knowledge of the image of $f = \varphi(\tau(\cdot), \cdot)$ on $G \in \mathcal{X}$. From a computational perspective, this definition is too restrictive, and a method for constructing larger outer approximations is necessary.

Definition 7.17 Let $\mathcal{F}, \mathcal{F}' : \mathcal{X} \rightrightarrows \mathcal{X}'$ be two combinatorial multivalued mappings. The map $\mathcal{F}'$ is an *enclosure* of the map $\mathcal{F}$ if $\mathcal{F}(G) \subset \mathcal{F}'(G)$ for every $G \in \mathcal{X}$.

From the definition of $\mathcal{F}^{-1}$ it follows that $\mathcal{F}'$ encloses $\mathcal{F}$ if and only if $(\mathcal{F}')^{-1}$ encloses $\mathcal{F}^{-1}$. Enclosures of τ-minimal multivalued maps provide a method for obtaining outer approximations.

Proposition 7.18 *If $\mathcal{F} : \mathcal{N} \rightrightarrows \mathcal{N}$ encloses a τ-minimal multivalued map of φ on N, then $\mathcal{F}$ is an outer approximation of φ on N.*

Proof: By Proposition 7.16 $\widetilde{\mathcal{F}}$ is an outer approximation and therefore $f(G) \cap N \subset \operatorname{int}_N\big(|\widetilde{\mathcal{F}}(G)|\big) \subset \operatorname{int}_N\big(|\mathcal{F}(G)|\big)$, for all $G \in \mathcal{N}$, where the latter inclusion follows from enclosure. This proves that $\mathcal{F}$ is an outer approximation. ∎

Note that by Propositions 7.16 and 7.18 it holds that

(i) $f^k(G) \cap N \subset \operatorname{int}_N\big(|\mathcal{F}^k(G)|\big)$,

(ii) $f^{-k}(G) \cap N \subset \operatorname{int}_N\big(|\mathcal{F}^{-k}(G)|\big)$,

for all $k \geq 1$ and for all $G \in \mathcal{N}$.

In the numerical study of dynamical systems, uncertainty is introduced in the form of round-off errors in floating-point arithmetic and truncation errors due to the fact that only a finite number of operations are possible. However, it is often possible to bound these errors by some constant $\epsilon > 0$. In this case the multivalued mapping

$$\mathcal{F}_\epsilon(G) := \{H \mid B_\epsilon(f(G)) \cap H \neq \varnothing\},$$

where f is a τ-discretization, encloses the τ-minimal multivalued map and thus is an outer approximation.

7.3 Digraph Representation of Dynamics

We start investigating the relation between combinatorial invariant sets for a multivalued mapping of dynamics of φ restricted to a compact invariant set S. Let $\varphi\|_S : \mathbb{T}^+ \times S \to S$ be the dynamical system restricted to a compact invariant set S, which we will agin denote by φ. Let $\mathcal{S}$ be a grid, or covering grid for S and φ is a τ-discretization of φ.

Proposition 7.19 *The τ-minimal multivalued mapping $\widetilde{\mathcal{F}} : \mathcal{S} \rightrightarrows \mathcal{S}$ of φ on $\mathcal{S}$, defined by $\widetilde{\mathcal{F}}(G) := \{H \in \mathcal{S} \mid H \cap f(G) \neq \varnothing\}$, is an invariant digraph. In particular, any outer approximation of φ on S is invariant.*

Proof: By Proposition 7.5, S is invariant for any τ-discretization f and thus $f(S) = S$. In particular $f(G) \subset S$ for any $G \in \mathcal{S}$, which implies that $\widetilde{\mathcal{F}}(G) = \{H \in \mathcal{S} \mid H \cap f(G) \neq \varnothing\} \neq \varnothing$ for all $G \in \mathcal{S}$.

As before $\widetilde{\mathcal{F}}^{-1}(G) = \{H \in \mathcal{S} \mid G \cap f(H) \neq \varnothing\} \neq \varnothing$, which shows that $\widetilde{\mathcal{F}}^{-1}(G) \neq \varnothing$ for all $G \in \mathcal{S}$. Since any outer approximation $\mathcal{F}$ encloses $\widetilde{\mathcal{F}}$ we conclude that outer approximations are closed. ∎

Proposition 7.20 *If $\mathcal{A} \subset \mathcal{S}$ such that $\mathcal{F}(\mathcal{A}) \subset \mathcal{A}$, then $|\mathcal{A}|$ is a trapping region for f and an attracting neighborhood for φ.*

Proof: Because $\mathcal{F}$ is an outer approximation we have

$$\begin{aligned} f(|\mathcal{A}|) &= \bigcup_{G \in \mathcal{A}} f(G) \\ &\subset \bigcup_{G \in \mathcal{A}} \operatorname{int}_S\big(|\mathcal{F}(G)|\big) \\ &\subset \operatorname{int}_S\big(|\mathcal{F}(\mathcal{A})|\big) \\ &\subset \operatorname{int}_S\big(|\mathcal{A}|\big). \end{aligned}$$

Therefore $f^k(|\mathcal{A}|) \subset \operatorname{int}_S(|\mathcal{A}|, f)$ for all $k \geq 1$, which proves that $|\mathcal{A}|$ is a trapping region for f.

The associated attractor

$$A = \operatorname{Inv}(|\mathcal{A}|, f) = \omega(|\mathcal{A}|, f)$$

for f is non-trivial. By Proposition 7.7 A is also an attractor for φ, with $A = \omega(|\mathcal{A}|, \varphi)$ and $|\mathcal{A}|$ is a attractor neighborhood for φ. ∎

Corollary 7.21 *Assume $\mathcal{A} \subset \mathcal{S}$ such that $\mathcal{F}(\mathcal{A}) \subset \mathcal{A}$. There exists $T_0 \geq 0$ such that if $x \in \partial |\mathcal{A}|$ then $\varphi(t, x) \subset \operatorname{int} |\mathcal{A}|$ for all $t > T_0$.*

Corollary 7.22 *If $\mathcal{A} \subset \mathcal{S}$ is an attractor of $\mathcal{F}$, then $|\mathcal{A}|$ is a trapping region for f and an attracting neighborhood for φ.*

Proposition 7.23 *If $\mathcal{M}$ is a strongly connected component of $\mathcal{F}$, then $|\mathcal{M}|$ is an isolating neighborhood for f and φ*

Proof: By Proposition 7.7 if $|\mathcal{M}|$ is an isolating neighborhood for f, then $|\mathcal{M}|$ is an isolating neighborhood for φ. Thus it is sufficient to prove that $|\mathcal{M}|$ is an isolating neighborhood for f. This is equivalent to showing that if $x \in \partial |\mathcal{M}|$ then $x \notin \mathrm{Inv}(|\mathcal{M}|, f)$.

Observe that if $x \in \partial |\mathcal{M}|$ then

$$x \in \partial |\nu(\mathcal{M})| \cup \partial |\nu(\mathcal{M}) \setminus \mathcal{M}| .$$

By definition $\mathcal{F}(\nu(\mathcal{M})) \subset \nu(\mathcal{M})$. Thus, by Proposition 7.20 $|\nu(\mathcal{M})|$ is a trapping region for f. This implies that $\mathrm{Inv}(|\nu(\mathcal{M})|, f) \subset \mathrm{int}(|\nu(\mathcal{M})|)$. In particular, if $x \in \partial |\nu(\mathcal{M})|$ then $x \notin \mathrm{int}(|\nu(\mathcal{M})|, f)$.

We also claim that

$$\mathcal{F}(\nu(\mathcal{M}) \setminus \mathcal{M}) \subset \nu(\mathcal{M}) \setminus \mathcal{M}.$$

Clearly, $\mathcal{F}(\nu(\mathcal{M}) \setminus \mathcal{M}) \subset \nu(\mathcal{M})$. So assume there exists $G \in \mathcal{M}$ such that $G \in \mathcal{F}(\nu(\mathcal{M}) \setminus \mathcal{M})$. In particular, there exists $H \in \nu(\mathcal{M}) \setminus \mathcal{M}$ such that $G \in \mathcal{F}(H)$, i.e. $H \rightsquigarrow G$. Since $H \in \nu(\mathcal{M})$ there exists $G' \in \mathcal{M}$ such that $G' \rightsquigarrow H$. Since $G' \in \mathcal{M}$, $G \rightsquigarrow G'$. Thus $H \in \mathcal{M}$ a contradiction.

If $x \in \partial |\nu(\mathcal{M}) \setminus \mathcal{M}|$, then by Proposition 7.20 $\omega(x, f) \subset \mathrm{int}(|\nu(\mathcal{M}) \setminus \mathcal{M}|)$ and hence $x \notin \mathrm{Inv}(|\nu(\mathcal{M})|, f)$. ∎

The following Corollary justifies our focus on strongly connected path components.

Corollary 7.24 *If $\mathcal{M}$ be a strongly connected component of $\mathcal{F}$, but not a strongly connected path component of $\mathcal{F}$, then* $\mathrm{Inv}(\mathcal{M}, f) = \emptyset$.

Theorem 7.25 *Let $\mathsf{M}_{\mathcal{F}} = \{\mathcal{M}(p) \mid p \in (\mathsf{P}, \leq)\}$ be the finest Morse decomposition of $\mathcal{F} \colon \mathcal{S} \rightrightarrows \mathcal{S}$. Then*

$$\mathsf{M} := \{M(p) := \mathrm{Inv}(\mathcal{M}(p), f) \mid p \in (\mathsf{P}, \leq)\}$$

is a Morse decomposition for S under φ.

Proof: By construction $\{M(p) \mid p \in \mathsf{P}\}$ is a finite collection of compact pairwise disjoint invariant sets. So consider $x \in S \setminus \bigcup_{p \in \mathsf{P}} M(p)$ and an orbit $\gamma_x \colon \mathbb{T} \to S$ under φ. Observer that $\gamma_x \colon \mathbb{Z} \to S$ is an orbit under f. Define $\bar{\gamma}_x \colon \mathbb{Z} \to \mathcal{S}$ such that $\gamma_x(n) \in \bar{\gamma}_x(n)$. Observe that there exist $\mathcal{M}(q)$ and

$\mathcal{M}(p)$ with $p < q$ and $T^- < T^+$ such that $\bar{\gamma}_x(n) \in \mathcal{M}(q)$ for all $n \leq T^-$ and $\bar{\gamma}_x(n) \in \mathcal{M}(p)$ for all $n \geq T^+$. This implies that

$$\alpha_o(\bar{\gamma}_x^-, f) \subset M(q) \quad \text{and} \quad \omega(x, f) \subset M(p)$$

and hence

$$\alpha_o(\gamma_x^-, \varphi) \subset M(q) \quad \text{and} \quad \omega(x, \varphi) \subset M(p).$$

■

8 Convergence

Consider a dynamical system $\varphi\colon \mathbb{T}^+ \times X \to X$ where X is compact. For a fixed grid $\mathcal{X}$ on X we can construct an outer approximation $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ from which we can recover attractors, repellers and a Morse decomposition. We need to show that this procedure can find $\mathsf{Att}(X,\varphi)$ and $\mathsf{MD}(X,\varphi)$. Of course the collection of attractors, repellers and Morse sets is limited by the resolution of $\mathcal{F}$ and thus we need to consider refinements of the grid.

Throughout this section we will let f denote a τ-discretization of φ. (If $\mathbb{T} = \mathbb{Z}$, then $\tau = 1$). For a given grid $\mathcal{X}$, $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_{\mathcal{X}}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ denotes the canonical minimal multivalued map for f.

Definition 8.1 The *recurrent set* of a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is defined by

$$\mathcal{R}(\mathcal{F}) := \{G \in \mathcal{X} \mid G \rightsquigarrow G\}$$

Observe that

$$\mathcal{R}(\mathcal{F}) = \bigcup_{\mathcal{M} \in \mathsf{M}_{\mathcal{F}}} \mathcal{M}.$$

By Theorem 7.25 if $x \in X \setminus \mathcal{R}(\mathcal{F})$ then each γ_x is a connecting orbit between different Morse sets. This raises the question of characterizing the dynamics of the elements that lie within $\mathcal{R}(\mathcal{F})$.

It should be clear that given $\widetilde{\mathcal{F}}\colon \mathcal{X} \rightrightarrows \mathcal{X}$, $\left|\mathcal{R}(\widetilde{\mathcal{F}})\right|$ depends on $\operatorname{diam}(\mathcal{X})$.

Definition 8.2 The *chain recurrent set* for φ is defined as

$$\mathcal{R}(X,\varphi) := \bigcap_{n \in \mathbb{Z}} \left|\mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}_n})\right|$$

where $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ is any sequence of grids for which $\lim_{n\to\infty} \operatorname{diam}(\mathcal{X}_n) = 0$.

We need to show that $\mathcal{R}(X,\varphi)$ is well defined.

Proposition 8.3 *Consider a grid $\mathcal{X}$. Then there exists $\delta > 0$ with the following property. For any grid $\mathcal{X}'$ such that $0 < \operatorname{diam}(\mathcal{X}') < \delta$ and for any $G \in \mathcal{X}$ and $G' \in \mathcal{X}'$ such that $G \cap G' \neq \emptyset$,*

$$\widetilde{\mathcal{F}}_{\mathcal{X}'}(G') \subset \widetilde{\mathcal{F}}_{\mathcal{X}}(G).$$

Moreover, if $G' \in \mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}'})$ *and* $G \cap G' \neq \emptyset$ *for some* $G \in \mathcal{X}$*, then* $G \in \mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}})$*. Therefore*

$$\left|\mathcal{R}\left(\widetilde{\mathcal{F}}_{\mathcal{X}'}\right)\right| \subset \left|\mathcal{R}\left(\widetilde{\mathcal{F}}_{\mathcal{X}}\right)\right|.$$

Proof: By Proposition 7.16 $\widetilde{\mathcal{F}}$ is an outer approximation for f. Since there are finitely many grid elements there exists $\epsilon > 0$ such that $B_\epsilon(f(G)) \subset \operatorname{int}\left(\left|\widetilde{\mathcal{F}}(G)\right|\right)$ for all $G \in \mathcal{X}$.

Since X is compact, f is uniformly continuous and hence there exists $\rho > 0$ such that if $\operatorname{diam}(G') < \rho$ then $\operatorname{diam}(f(G')) < \epsilon/2$.

Choose $\delta = \min\{\rho, \epsilon/2\}$. This implies that $f(G') \subset B_{\epsilon/2}(f(G))$. However, since $\operatorname{diam}(\mathcal{X}') < \delta$, if $H \in \mathcal{X}'$ and $H \not\subset \left|\widetilde{\mathcal{F}}_{\mathcal{X}'}(G)\right|$, then $H \cap B_{\epsilon/2}(f(G)) \cap \emptyset$, a contradiction. Therefore, $\widetilde{\mathcal{F}}_{\mathcal{X}'}(G') \subset \widetilde{\mathcal{F}}_{\mathcal{X}}(G)$.

Assume $G' \in \mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}'})$. Let $\{G'_i \mid i = 0, \dots n\}$ be a closed walk in $\widetilde{\mathcal{F}}_{\mathcal{X}'}$ starting at G'. Let $G \in \mathcal{X}$ such that $G \cap G' \neq \emptyset$. Set $G_0 = G$. By the first part of the proposition,

$$\widetilde{\mathcal{F}}_{\mathcal{X}'}(G'_0) \subset \widetilde{\mathcal{F}}_{\mathcal{X}}(G_0).$$

Choose $G_1 \in \widetilde{\mathcal{F}}_{\mathcal{X}}(G_0)$ such that $G_1 \cap G'_1 \neq \emptyset$. Repeating this procedure we can choose $\{G \mid i = 0, \dots n-1\}$ such that $G_i \in \widetilde{\mathcal{F}}_{\mathcal{X}}(G_i)$ and $G_i \cap G'_i \neq \emptyset$. Observe that $G' \in \widetilde{\mathcal{F}}_{\mathcal{X}'}(G'_{n-1})$. Since $G \cap G' \neq \emptyset$, $G \in \widetilde{\mathcal{F}}_{\mathcal{X}}(G_{n-1})$ and hence there exists a closed walk in $\widetilde{\mathcal{F}}_{\mathcal{X}}$ starting at G. Therefore, $G \in \mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}})$. ∎

Proposition 8.4 *Let* $\{\mathcal{X}^i_n\}_{n\in\mathbb{N}}$ *be a sequences of grids such that* $\lim_{n\to\infty} \operatorname{diam}(\mathcal{X}^i_n) = 0$ *for* $i = 0, 1$*. Then*

$$\bigcap_{n\in\mathbb{Z}} \left|\mathcal{R}\left(\widetilde{\mathcal{F}}_{\mathcal{X}^0_n}\right)\right| = \bigcap_{n\in\mathbb{Z}} \left|\mathcal{R}\left(\widetilde{\mathcal{F}}_{\mathcal{X}^1_n}\right)\right|$$

Hence, $\mathcal{R}(X, \varphi)$ *is well-defined.*

Proof: As a preliminary step consider a fixed grid $\mathcal{X}$ and $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ a sequence of grids with $\lim_{n\to\infty} \operatorname{diam}(\mathcal{X}_n) = 0$. We claim that

$$\bigcap_{n\in\mathbb{Z}} \left|\mathcal{R}\left(\widetilde{\mathcal{F}}_{\mathcal{X}_n}\right)\right| \subset \left|\mathcal{R}(\mathcal{F}_{\mathcal{X}})\right|.$$

Choose $n > 0$ so that $\mathrm{diam}(\mathcal{X}_n) < \delta$ where $\delta > 0$ is associated to $\mathcal{X}$ by Proposition 8.3. Let $x \in \cap_n |\mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}_n})|$. Then there exists $H \in \mathcal{X}_n$ such that $x \in H \in \mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}_n})$. Choose $G \in \mathcal{X}$ so that $x \in G$. By Proposition 8.3, we have $G \in \mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}})$, and hence $x \in \left|\mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}})\right|$.

Observe that this shows that

$$\bigcap_{n\in\mathbb{Z}} \left|\mathcal{R}\left(\widetilde{\mathcal{F}}_{\mathcal{X}_n^0}\right)\right| \subset \left|\mathcal{R}\left(\widetilde{\mathcal{F}}_{\mathcal{X}_m^1}\right)\right|$$

for every $m \in \mathbb{Z}$. Interchanging the roles of $\mathcal{X}_n^0$ and $\mathcal{X}_n^1$ gives the desired equality. ■

Our next goal is the proof of the following alternative characterization of the chain recurrent set of φ.

Theorem 8.5 *Let X be a compact invariant set under φ. Then*

$$\mathcal{R}(X,\varphi) = \bigcap_{A\in\mathsf{Att}(X,\varphi)} A \cup A^*$$

We begin with two results of interest in and of themselves. The first is a description of attractor-repeller pairs in digraphs and the second is an approximation result.

Recall that by definition an attractor in a digraph $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is of the form

$$\mathcal{A} \in \nu\left(\rho_{\mathcal{F}}^{-1}\left(\mathsf{O}(\bar{\mathsf{M}}_{\mathcal{F}})\right)\right)$$

where $\bar{\mathsf{M}}_{\mathcal{F}}$ is the Morse graph for $\mathcal{F}$. This implies that there exists $\mathsf{P}_{\mathcal{A}} \subset \mathsf{O}(\mathsf{P})$ such that

$$\mathcal{A} = \nu\left(\rho_{\mathcal{F}}^{-1}\left(\bigcup_{p\in\mathsf{P}_{\mathcal{A}}} M_p\right)\right).$$

Furthermore,

$$\mathcal{A}^* = \nu^*\left(\rho_{\mathcal{F}}^{-1}\left(\bigcup_{p\in\mathsf{P}\setminus\mathsf{P}_{\mathcal{A}}} M_p\right)\right).$$

where

$$\nu^*(\mathcal{K}) := \{G \in \mathcal{X} \mid G \rightsquigarrow \mathcal{K}\}$$

Proposition 8.6 *Let $\epsilon > 0, N > 0$, and $x \in X$. Then there exists a $\delta > 0$ such that for any grid $\mathcal{X}$ with* $\operatorname{diam}(\mathcal{X}) < \delta$, *we have $|\widetilde{\mathcal{F}}^n(G)| \subset B_\epsilon(f^n(G))$ for all $G \in \mathcal{X}$ and $0 < n \leq N$.*

Proof: Let $\delta_0 = \epsilon/2$. Since $f^{n-1}(G)$ is compact and f is uniformly continuous, there exists $\delta_1 > 0$ such that $f(B_{\delta_1}(f^{n-1}(G))) \subset B_{\delta_0}(f^n(G))$. Furthermore, for each $1 < i < n$ there exists $\delta_i > 0$ such that $f(B_{\delta_i}(f^{n-i}(G))) \subset B_{\delta_{i-1}/2}(f^{n-i+1}(G))$. Let $\delta = \min_{0 \leq i \leq N-1} \delta_i/2$.

If $\operatorname{diam}(\mathcal{X}) < \delta$, then $|\mathcal{F}(G)| \subset B_{\delta_{n-1}}(f(G))$. For any $H \in \mathcal{F}(G)$, we have $f(H) \subset B_{\delta_{n-2}/2}(f^2(G))$. Hence $|\mathcal{F}^2(G)| \subset B_{\delta_{n-2}}(f^2(G))$. Repeating this argument for each $n \leq N$, we obtain

$$|\mathcal{F}^n(G)| \subset B_{\delta_0}(G) = B_\epsilon(f^n(G)).$$

■

Proposition 8.7 (i) *For every attractor-repeller pair $(\mathcal{A}, \mathcal{A}^*)$ for $\mathcal{F}$, there exists a unique attractor-repeller pair (A, A^*) for f such that $A \subset |\mathcal{A}|$ and $A^* \subset |\mathcal{A}^*|$.*

(ii) *Let (A, A^*) be an attractor-repeller pair for f. For every $0 < \epsilon <$* $\operatorname{dist}(A, A^*)/2$, *there exists $\delta > 0$ such that if $\mathcal{X}$ is a grid with* $\operatorname{diam}(\mathcal{X}) < \delta$, *then there exists a unique attractor-repeller pair $(\mathcal{A}, \mathcal{A}^*)$ for $\widetilde{\mathcal{F}}$ with the property that $A \subset |\mathcal{A}| \subset B_\epsilon(A)$ and $A^* \subset |\mathcal{A}^*| \subset B_\epsilon(A^*)$. Any other attractor-repeller pair $(\mathcal{B}, \mathcal{B}^*)$ for $\widetilde{\mathcal{F}}$ has the property that*

$$|\mathcal{B}| \cap B_\epsilon(A^*) \neq \emptyset \quad \text{or} \quad |\mathcal{B}^*| \cap B_\epsilon(A) \neq \emptyset.$$

Proof: (i) Since $\mathcal{F}(\mathcal{A}) = \mathcal{A}$, we have $f(|\mathcal{A}|) \subset \operatorname{int}(|\mathcal{A}|)$ so that $|\mathcal{A}|$ is an attracting neighborhood. Then $A = \omega(|\mathcal{A}|) \subset |\mathcal{A}|$ is the maximal attractor in $|\mathcal{A}|$, and

$$A^* = \operatorname{Inv}\left(\operatorname{cl}(X \setminus |\mathcal{A}|)\right) \subset |\operatorname{Inv}(\mathcal{X} \setminus \mathcal{A})| = |\mathcal{A}^*|\,.$$

Moreover, any other attractor $A' \subset |\mathcal{A}|$ is not maximal, and hence $A'^* \cap |\mathcal{A}| \neq \varnothing$ so that $A'^* \not\subset |\mathcal{A}^*|$, which implies uniqueness.

(ii) A is an attractor and hence there exists a compact neighborhood $V \subset B_\epsilon(A)$ of A with the property that $f(V) \subset \operatorname{int}(V)$. Similarly, there exists a compact neighborhood $V^* \subset B_\epsilon(A^*)$ of A^* such that $f^{-1}(V^*) \subset \mathrm{int}(V^*)$. Since $\mathrm{dist}(A, A^*) > 2\epsilon$, we have $V \cap V^* = \varnothing$.

By definition of an attractor-repeller pair and compactness, there exists $N > 0$ such that for all $x \in X \setminus f^{-1}(V^*)$, $f^N(x) \in V$. Proposition 8.6 provides δ such that if $\mathrm{diam}(\mathcal{X}) < \delta$, then $|\mathcal{F}^N(G)| \subset V$ for any G such that $|G| \subset X \setminus f^{-1}(V^*)$. We can furthermore assume that δ is sufficiently small such that $B_\delta(f(V)) \subset V$ and $B_\delta(f^{-1}(V*)) \subset V*$.

Let
$$\mathcal{C} = \mathrm{cov}(f(V)) := \{G \in \mathcal{X} \mid f(V) \cap G \neq \emptyset\}.$$
Since $f(V) \subset V$, we have $\mathcal{F}(\mathcal{C}) \subset \mathcal{C}$. Now let

$$\mathcal{A} = \mathrm{Inv}(\downarrow(\mathcal{C}), \mathcal{F}) \subset \mathcal{C}$$

and $\mathcal{A}^*$ be its dual repeller. By construction, $|\mathcal{A}| \subset B_\epsilon(A)$. Moreover, since $f(A) = A$, we have $\mathrm{cov}(A) \subset \mathcal{F}(\mathrm{cov}(A))$, and hence $\mathrm{cov}(A) \subset \mathcal{F}^k(\mathrm{cov}(A))$ for all $k > 0$. Thus

$$\mathrm{cov}(A) \subset \mathrm{Inv}(\downarrow(\mathcal{A}), \mathcal{F}) \subset \mathrm{Inv}(\downarrow(\mathcal{C}), \mathcal{F}) = \mathcal{A}$$

so that $A \subset |\mathcal{A}|$.

Suppose G is an element which does not intersect $f^{-1}(V^*)$. Then by the above construction, $\mathcal{F}^{N+1}(G) \subset \mathcal{C}$ so that $\mathcal{A}^*$ does not contain G. Thus $|\mathcal{A}^*| \subset V^* \subset B_\epsilon(A^*)$. Moreover, $A^* \subset f^{-1}(A^*)$ implies that $\mathrm{cov}(A^*) \subset \mathcal{F}^{-1}(\mathrm{cov}(A^*))$, and hence $\mathrm{cov}(A^*) \subset \mathcal{F}^{-k}(\mathrm{cov}(A^*))$ for all $k > 0$. Thus

$$\mathrm{cov}(A^*) \subset \mathrm{Inv}(\uparrow(\mathrm{cov}(A^*)), \mathcal{F}) \subset \mathrm{Inv}(\uparrow(\mathcal{X} \setminus \mathcal{C}), \mathcal{F}) = \mathcal{A}^*$$

so that $A^* \subset |\mathcal{A}^*|$.

Suppose $(\mathcal{B}, \mathcal{B}^*)$ is any other attractor-repeller pair for $\mathcal{F}$. We can characterize $\mathcal{A}$ and $\mathcal{B}$ by $\mathsf{P}_\mathcal{A}$ and $\mathsf{P}_\mathcal{B}$ respectively.

Since $\mathsf{P}_\mathcal{A} \neq \mathsf{P}_\mathcal{B}$, then there exists $q \in (\mathsf{P} \setminus \mathsf{P}_\mathcal{A}) \cap \mathsf{P}_\mathcal{B}$ and $\mathcal{M}(q) \subset \mathcal{A}^* \cap \mathcal{B}$. Similarly, there exists $q' \in (\mathsf{P} \setminus \mathsf{P}_\mathcal{B}) \cap \mathsf{P}_\mathcal{A}$ and $\mathcal{M}(q') \subset \mathcal{A} \cap \mathcal{B}^*$. ∎

Proof of Theorem 8.5: Let (A, A^*) be an attractor-repeller pair for f. Suppose $\{\mathcal{X}_n\}$ is a sequence of grids with $\lim_{n\to\infty} \mathrm{diam}(\mathcal{X}_n) = 0$. Let $\epsilon > 0$. By Proposition 8.7, there exists $N > 0$ and an attractor-repeller pair $(\mathcal{A}_N, \mathcal{A}_N^*)$ for $\widetilde{\mathcal{F}}_{\mathcal{X}_N}$ such that $|\mathcal{A}_N \cup \mathcal{A}_N^*| \subset B_\epsilon(A \cup A^*)$. By definition of attractor-repeller pair $|\mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}_N})| \subset |\mathcal{A}_N \cup \mathcal{A}_N^*|$, and thus $\mathcal{R}(X,f) = \cap_n |\mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}_n})| \subset B_\epsilon(A \cup A^*)$. Since $\epsilon > 0$ was arbitrary, $\mathcal{R}(X,f) \subset A \cup A^*$ for every attractor-repeller pair; hence $\mathcal{R}(X,f) \subset \cap(A \cup A^*)$.

Let $x \in \bigcap_{A \in \mathsf{Att}(X,\varphi)} A \cup A^*$, but assume $x \notin \mathcal{R}(X,\varphi)$. Then there exists a grid $\mathcal{X}_n$ such that $x \in G \in \mathcal{X}_n$ and $G \notin \mathcal{R}(\widetilde{\mathcal{F}}_{\mathcal{X}_n})$.

Define

$$\mathcal{A} := \operatorname{Inv}(\nu(G), \widetilde{\mathcal{F}}_{\mathcal{X}_n})$$

Observe that $\mathcal{A}$ is an attractor and $x \notin |\mathcal{A}| \cap |\mathcal{A}^*|$. This implies that $x \notin \operatorname{Inv}(|\mathcal{A}|, f) \cap \operatorname{Inv}(|\mathcal{A}^*|, f)$ a contradiction. ∎

Two theorems presented without proof.

Theorem 8.8 *Let $\varphi\colon \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact set X. Let $S = \operatorname{Inv}(S,\varphi)$. Then there exists a Lyapunov function $V\colon S \to [0,1]$ such that*

1. *if $x \in \mathcal{R}(X,\varphi)$, then $V(\varphi(t,x)) = V(x)$ for all $t \geq 0$;*
2. *if $x \in S \setminus \mathcal{R}(X,\varphi)$, then $V(\varphi(t,x)) < V(x)$ for all $t > 0$.*

Theorem 8.9 *Let $\varphi\colon \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact set X. Then the restriction*

$$\varphi\colon \mathbb{T}^+ \times \mathcal{R}(X,\varphi) \to \mathcal{R}(X,\varphi)$$

is a dynamical system. Furthermore,

$$\mathcal{R}(\mathcal{R}(X,\varphi),\varphi) = cR(X,\varphi).$$

This theorem implies that all the nongradient like dynamics lies in the chain recurrent set.

A final result that has to do with identifying isolated invariant sets.

Definition 8.10 Let $\mathcal{X}$ be a grid and let $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ be an outer approximation of f. A subset $\mathcal{S} \subset \mathcal{X}$ is an *isolated invariant set* for $\mathcal{F}$ if

$$\mathcal{S} = \operatorname{Inv}(\operatorname{cov}(|\mathcal{S}|), \mathcal{F}).$$

Proposition 8.11 *Let $\mathcal{X}$ be a grid and let $\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ be an outer approximation of f. If $\mathcal{S}$ is an isolated invariant set for $\mathcal{F}$, then*

$$\partial(|\mathcal{S}|) \cap f^{-1}(|\mathcal{S}|) \cap f(|\mathcal{S}|) = \emptyset$$

Theorem 8.12 *Let S be an isolated invariant set under φ with isolating neighborhood N. There exists an $\epsilon > 0$ such that if $\mathcal{X}$ is a grid with $\operatorname{diam}(\mathcal{X}) < \epsilon$, then there exists an invariant set $\mathcal{S}$ for $\widetilde{\mathcal{F}}$ such that*

$$S = \operatorname{Inv}(|\mathcal{S}|, \varphi)$$

and $|\mathcal{S}| \subset N$.

9 Reduced Dynamics

Given a pair of compact sets $P = (P_1, P_0)$ with $P_0 \subset P_1$ define an equivalence relation on P_1 as follows:

$$x \sim y \quad \text{if either} \quad x = y \text{ or } x, y \in P_0.$$

The equivalence classes $[x]$ are single points $x \in P_1 \backslash P_0$, or $[P_0]$ for all points in P_0. The space of equivalence classes is denoted by P_1/P_0 — the *quotient space*. The metric properties of (X, d) are inherited by the quotient space P_1/P_0. On P_1/P_0 define the function

$$d_P([x],[y]) = \begin{cases} d(x,y) & \text{for } x, y \in P_1 \backslash P_0 \\ d(x, P_0) & \text{for } x \in P_1 \backslash P_0,\ y \in P_0 \\ 0 & \text{for } x, y \in P_0. \end{cases}$$

Lemma 9.1 *For a compact pair $P = (P_1, P_0)$ in a metric space (X, d) the function d' is a metric on P_1/P_0 and $(P_1/P_0, d')$ is a compact metric space.*

A topological space X with a designated point x_0 or *basepoint* is called a *pointed space* and denoted by (X, x_0). A morphism between two pointed spaces (X, x_0) and (Y, y_0) is a continuous map $f\colon X \to Y$ such that $f(x_0) = y_0$.

We will be interested in pointed spaces $(P_1/P_0, [P_0])$ that arise from compact pairs.

9.1 Continuity

Definition 9.2 Let $\varphi\colon \mathbb{T}^+ \times X \to X$ be a dynamical system. Let $P = (P_1, P_0)$ be a pair of compact subsets of X. The *reduced* dynamical system is $\varphi_P\colon T^+ \times (P_1/P_0, [P_0]) \to (P_1/P_0, [P_0])$ given by

$$\varphi_P(t,[x]) = \begin{cases} [\varphi(t,x)] & \text{if } \varphi([0,t],x) \subset P_1 \backslash P_0, \\ [P_0] & \text{otherwise.} \end{cases}$$

Lemma 9.3 $\varphi_P\colon T^+ \times (P_1/P_0, [P_0]) \to (P_1/P_0, [P_0])$ *satisfies*

1. $\varphi(0,[x]) = [x]$ *for all* $[x] \in P_1/P_0$

2. $\varphi(t+s,[x]) = \varphi(t,\varphi(s,[x]))$ *for all* $[x] \in P_1/P_0$ *and all* $t, s \geq 0$.

Remark: In general φ_P is *not* continuous.

Remark: Observe that in general φ_P is not surjective.

The following results provide conditions on the compact pair $P = (P_1, P_0)$ such that φ_P is continuous.

Theorem 9.4 *Let* $\varphi : \mathbb{R}^+ \times X \to X$ *be a flow on* X *and let* $P = (P_1, P_0)$ *be a compact pair.* φ_P *is continuous if and only if*

(i) P_0 *is* positively invariant *in* P_1, *i.e. if* $x \in P_0$ *and* $\varphi([0,t],x) \subset P_1$ *then* $\varphi([0,t],x) \subset P_0$, *and*

(ii) P_0 *is an* exit set *for* P_1, *i.e. if* $x \in P_1$ *and* $\varphi([0,t],x) \not\subset P_1$ *then there exists* $0 \leq \tau \leq t$ *such that* $\varphi(x,\tau) \in P_0$.

Proof: Suppose that property (i) fails so that there exists $x \in P_0$ and $t > 0$ such that $\varphi([0,t],x) \subset P_1$ but $\varphi(t,x) \notin P_0$. Let

$$s_0 = \sup\{s \in [0,t] \mid \varphi(s,x) \in P_0\}.$$

Then by replacing x by $\varphi(s_0,x)$ and t by $t - s_0$ we can assume without loss of generality that $\varphi(s,x) \in P_1 \backslash P_0$ for all $0 < s \leq t$. We now show that φ_P is not continuous at $(t,[P_0].)$ Let $x_n = \varphi(1/n,x)$ and $t_n = 1 - 1/n$. Then $(t_n,[x_n]) \to (t,[P_0])$ as $n \to \infty$, but $\varphi_P(t_n,[x_n]) = [\varphi(t,x)] \to [\varphi(t,x)] \neq [P_0] = \varphi_P(t,[P_0])$, hence φ_P is not continuous at $(t,[P_0])$.

Suppose that property (ii) fails so that there there exists $x \in P_1$ with $\varphi([0,\infty),x) \not\subset P_1$ but $\varphi(t,x) \notin P_0$ for all $t \geq 0$. Let

$$s = \sup\{t \in [0,\infty) \mid \varphi([0,t],x) \subset P_1\}.$$

Then there is a sequence $s_n \to s$ with $\varphi(s_n,x) \notin P_1$ so that $\varphi_P(s_n,[x]) = [P_0]$. However, $\varphi([0,s],x) \subset P_1 \backslash P_0$ so that $\varphi_P(s,[x]) = [\varphi(s,x)] \neq [P_0]$. Therefore φ_P is not continuous at $(s,[x])$.

Finally, suppose that properties (ii) and (iii) hold but φ_P is not continuous. Then there exist sequences $x_n \in P_1 \backslash P_0$ converging to $x \in P_1$ and $t_n \in \mathbb{R}^+$ converging to $t \in \mathbb{R}^+$ such that $\varphi_P(t_n,[x_n])$ does not converge to $\varphi_P(t,[x])$.

We consider two cases: $\varphi_P(t,[x]) = [P_0]$ and $\varphi([0,t],x) \subset P_1\backslash P_0$. In the first case, by passing to a subsequence, we can assume that $\varphi([0,t_n],x_n) \subset P_1\backslash P_0$ and $\varphi(t,x) \notin P_0$, as otherwise we would have convergence. Since P_1 is compact, $\varphi([0,t_n],x_n) \subset P_1\backslash P_0 \subset P_1$ implies $\varphi([0,t],x) \subset P_1$. Since $\varphi_P(t,[x]) = [P_0]$, we must have $\varphi([0,t],x) \cap P_0 \neq \varnothing$ which together with $\varphi(t,x) \notin P_0$ contradicts the fact that P_0 is positively invariant in P_1. In the second case, by passing to a subsequence, we may assume that $\varphi_P(t_n,[x_n]) = [P_0]$ for all $n > 0$ so that $\varphi([0,t_n],x_n) \not\subset P_1\backslash P_0$. Using the fact that P_0 is an exit set, there must exist $s_n \in (0,t_n]$ such that $\varphi(s_n,x_n) \in P_0$. The points $\varphi(s_n,x_n)$ accumulate at a point $\varphi(s,x) \in P_0$ for some $s \in [0,t]$, since P_0 is compact. This contradicts the assumption that $\varphi([0,t],x) \subset P_1\backslash P_0$. ∎

Theorem 9.5 *Let $\varphi : \mathbb{Z}^+ \times X \to X$ be a dynamical system with $f(\cdot) = \varphi(1,\cdot)$. Let $P = (P_1, P_0)$ be a compact pair. Then φ_P is continuous if and only if the following conditions are satisfied.*

(i) Every $x \in P_0 \cap f^{-1}(P_1\backslash P_0)$ has a neighborhood U in X with

$$f(U \cap P_1\backslash P_0) \subset X \setminus P_1.$$

(ii) Every $x \in (P_1\backslash P_0) \cap f^{-1}((\partial P_1)\backslash P_0)$ has a neighborhood V in X with

$$f(V \cap P_1\backslash P_0) \subset P_1.$$

Proof: Assume condition (i) fails. This implies the existence of $x \in P_0$ such that $f(x) \in P_1 \setminus P_0$ but for every neighborhood U of x, there exists $y \in f(U \cap P_1\backslash P_0)$ such that $y \in P_1$. In particular, there exists a sequence $x_n \in P_1\backslash P_0$ converging to x such that $f(x_n) \in P_1$. P_0 is closed and hence for n sufficiently large, $f(x_n) \in P_1 \setminus P_0$. By definition $f_P(x_n) = f(x_n)$. If f_P is continuous, then

$$f_P(x) = \lim_{n\to\infty} f_P(x_n) = \lim_{n\to\infty} f(x_n) = f(x) \in P_1 \setminus P_0$$

but by definition $f_P(x) = [P_0]$, a contradiction.

Assume condition (ii) fails. Then there exists $x \in P_1 \setminus P_0$ such that $f(x) \in P_1 \setminus P_0$, but for every neighborhood V of x there exists $y \in f(V \cap P_1 \setminus P_0)$ such that $y \in X \setminus P_1$. Again, this implies the existence of a sequence $x_n \in P_1 \setminus P_0$

converging to x such that $f(x_n) \in X \setminus P_1$. Observe that $f_P(x_n) = [P_0]$, but $f_P(x) = f(x) \in P_1 \setminus P_0$ and hence f_P is not continuous at x.

Now assume that (i) and (ii) are satisfied. It needs to be shown that f_P is continuous. There are four cases to consider.

1. Let $x \in (P_1 \setminus P_0) \cap f^{-1}(P_0)$. Observe that $f_P(x) = [P_0]$. Consider a sequence x_n converging to x. By continuity of f, $\lim_{n\to\infty} d(f(x_n), P_0) = 0$ and hence
$$\lim_{n\to\infty} d_P(f_P(x_n), [P_0]) = 0.$$

2. Let $x \in (P_1 \setminus P_0) \cap f^{-1}(X \setminus P_1)$. Let $W \subset X \setminus P_1$ be a neighborhood of $f(x)$. Consider a sequence x_n converging to x. For n sufficiently large $f(x_n) \in W$ and hence
$$\lim_{n\to\infty} f_P(x_n) = [P_0] = f_P(x).$$

3. Let $x \in (P_1 \setminus P_0) \cap f^{-1}(P_1 \setminus P_0)$. Observe that there exists and open neighborhood W in P_1/P_0 for x. Thus, if $f(x) \in \text{int}(P_1 \setminus P_0)$, then continuity of f implies continuity of f_P. Condition (ii) insures that the same argument applies if $f(x) \in (\partial(P_1) \setminus P_0)$.

4. Let $x = [P_0]$. The proof is by contradiction, so assume f_P is not continuous at $[P_0]$. Consider a sequence $x_n \in P_1 \setminus P_0$ such that $lim_{n\to\infty} d(x_n, P_0) = 0$ but $f(x_n) \in P_1 \setminus P_0$ and bounded away from P_0. Without loss of generality, i.e. by choosing a convergent subsequence, we can assume x_n converges to $x \in P_0$. By continuity of f, $f(x) \in P_1 \setminus P_0$. However condition (i) implies that for all n sufficiently large $f(x_n) \in X \setminus P_1$, a contradiction.

∎

Definition 9.6 Consider a compact set $N \subset X$. The *immediate exit set* for N under f is
$$N^- := \{x \in N \mid f(x) \notin N\}.$$
The *weak exit set* of N under f is
$$N^{w-} := \{x \in N \mid f(x) \notin \text{int}(N)\}.$$

Clearly $N^- \subset N^{w-}$

Definition 9.7 Let $P = (P_1, P_0)$ be a compact pair. P_0 is *positively invariant* in P_1 under f if

$$f(P_0) \cap P_1 \subset P_0.$$

P_0 is an *exit neighborhood* in P_1 under f if

$$P_1^- \subset P_0$$

The following results follow from Theorem 9.5

Proposition 9.8 *Let $\varphi : \mathbb{Z}^+ \times X \to X$ be a dynamical system with $f(\cdot) = \varphi(1, \cdot)$. If $P = (P_1, P_0)$ is a compact pair for which P_0 is a positively invariant exit neighborhood for P_1, then φ_P is continuous.*

Proposition 9.9 *Let $\varphi : \mathbb{Z}^+ \times X \to X$ be a dynamical system with $f(\cdot) = \varphi(1, \cdot)$. If $P = (P_1, P_0)$ is a compact pair for which P_0 a positively invariant in P_1 and*

$$\mathrm{cl}(f(P_1) \backslash P_1) \cap P_1 \subset P_0,$$

then φ_P is continuous.

Proposition 9.10 *Let $\varphi : \mathbb{Z}^+ \times X \to X$ be a dynamical system with $f(\cdot) = \varphi(1, \cdot)$. Let $P_1 \subset X$ be a compact set such that*

$$\partial(P_1) \cap f^{-1}(P_1) \cap f(P_1) = \emptyset.$$

Define

$$P_{\max} := \mathrm{cl}\left(P_1 \setminus f^{-1}(P_1)\right) \quad \textit{and} \quad P_{\min} := f(P_1) \cap \partial(P_1).$$

If P_0 is a compact set satisfying $P_{\min} \subset P_0 \subset P_{\max}$, then for the pair $P = (P_1, P_0)$, φ_P is continuous.

9.2 Index Pairs

Definition 9.11 A compact pair $P = (P_1, P_0)$ is an *index pair* if

1. $\mathrm{cl}(P_1 \backslash P_0)$ is an isolating neighborhood and
2. φ_P is continuous.

Remark 9.12 *If $P = (P_1, P_0)$ is an index pair, then*

$$\varphi_P \colon T^+ \times (P_1/P_0, [P_0]) \to (P_1/P_0, [P_0])$$

is a dynamical system.

Definition 9.13 A compact pair $P = (P_1, P_0)$ is a *filtration pair* if

1. $\mathrm{cl}(\mathrm{int}(P_i)) = P_i$, $i = 0, 1$,
2. $\mathrm{cl}(P_1 \setminus P_0)$ is an isolating neighborhood,
3. P_0 is a neighborhood of P_1^{w-} in P_1, and
4. $f(P_0) \cap \mathrm{cl}(P_1 \setminus P_0) = \emptyset$.

Proposition 9.14 *A filtration pair is an index pair.*

Proof: This follows directly from Proposition 9.8. ∎

9.3 Index Pairs from Digraph Dynamics

Let $\mathcal{S}$ be an isolated invariant set for $\mathcal{F}$ an outer approximation for f. Let $S := \mathrm{Inv}(|\mathcal{S}|, \varphi)$. Our goal is to construct an index pair $P = (P_1, P_0)$ such that $S = \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_0), \varphi)$ using $\mathcal{F}$.

Definition 9.15 Let $\mathcal{K} \subset \mathcal{X}$. The *wrap* of $\mathcal{K}$ is

$$\mathcal{W} := \{G \in \mathcal{X} \mid G \cap |\mathcal{K}| \neq \emptyset\}.$$

The *collar* of $\mathcal{K}$ is given by

$$\mathcal{W} \setminus \mathcal{K}.$$

Algorithm 9.16 Combinatorial index pair

```
function indexPair(cubicalInvariantSet S, combinatorialMap F)
W := Wrap(S);
C := Collar(S);
P0 := intersection(evaluate(F, S), C);
repeat
   lastP0 := P0;
```

```
    P0 := intersection(evaluate(F, P0), C);
    P0 := union(P0, lastP0);
until (P0 = lastP0);
P1 := union(S, P0);
return (P1, P0);
```

Proposition 9.17 *Let $P_i := |\mathcal{P}_i|$ where $\mathcal{P}_i$ is the output of the algorithm. Then $P = (P_1, P_0)$ is an index pair and $S = \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_0), \varphi)$.*

Proof: Observe that by Theorem 8.12

$$S = \mathrm{Inv}(|\mathcal{S}|, \varphi) = \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_0), \varphi).$$

We will use Proposition 9.9 to prove that $P = (P_1, P_0)$ is an index pair.

The first step is to show that P_0 is positively invariant in P_1. This follows from the fact that $\mathcal{S}$ is an isolated invariant set. In particular, consider $G \in \mathcal{P}_0$. By construction $\mathcal{S} \rightsquigarrow G$ via a path in $\mathcal{C}$. If $\mathcal{F}(G) \cap \mathcal{S} \neq \emptyset$ then $G \in \mathrm{Inv}(\mathcal{W}, \mathcal{F})$ a contradiction since $G \in \mathcal{C}$. Since $\mathcal{F}$ is an outerapproximation $f(P_0) \cap |S| = \emptyset$.

It remains to show that

$$\mathrm{cl}(f(P_1) \setminus P_1) \cap P_1 \subset P_0.$$

Observe that $\mathcal{F}(\mathcal{P}_1) \cap \mathcal{C} \subset \mathcal{P}_0$ and $\partial P_1 \subset |\mathcal{C}|$. Thus,

$$\begin{aligned} \mathrm{cl}(f(P_1) \setminus P_1) \cap P_1 &\subset f(P_1) \cap \partial P_1 \\ &\subset \mathrm{int}\,|\mathcal{F}(\mathcal{P}_1)| \cap |\mathcal{C}| \\ &\subset |\mathcal{F}(\mathcal{P}_1) \cap \mathcal{C}| \subset |\mathcal{P}_0| = P_0. \end{aligned}$$

■

Examples of index pairs.

10 Conley Index

We are interested in the following question. Given an isolated invariant set S for $\varphi\colon \mathbb{T}^+ \times X \to X$ we can find index pairs $P^i = (P_1^i, P_0^i)$ such that $S = \mathrm{Inv}(\mathrm{cl}(P_1^i \setminus P_0^i), \varphi)$. What is the relationship between f_{P^i}?

10.1 Shift Equivalence

Definition 10.1 Fix a category. Two endomorphisms $f : X \to X$ and $g : Y \to Y$ are *shift equivalent* if there exist morphisms $r : X \to Y$ and $s : Y \to X$ and an integer $m \geq 0$ such that $r \circ f = g \circ r$, $s \circ g = f \circ s, r \circ s = g^m$, and $s \circ r = f^m$. The integer m is called the *lag*.

We denote shift equivalence by

$$f \sim_s g.$$

Proposition 10.2 *Shift equivalence is an equivalence relation.*

Proposition 10.3 *Let $f\colon X \to X$ and $g\colon Y \to Y$ be isomorphisms. Then $f \sim_s g$ if and only if they are conjugate. Furthermore, a conjugacy is given by*

$$h = r \circ f^{-m} = g^{-m} \circ r$$

and $s = h^{-1}$.

Proof: By definition f and g are conjugate if there exists an isomorphism $h\colon X \to Y$ such that

$$f = h^{-1} \circ g \circ h.$$

In this case we can choose $r = h$, $s = h^{-1}$, and $m = 1$.

So assume $f \sim_s g$ with a given r, s, and m. If r and s are isomorphisms, then a conjugacy is given in the statement of the proposition. Observe that $s \circ r = f^m$ implies that r is a monomorphism and s is an epimorphism. Similarly $r \circ s = g^m$ implies that r is an epimorphism and s is a monomophism. Thus, r and s are isomorphisms. ∎

Corollary 10.4 *Let $f : X \to X$ and $g : Y \to Y$ be linear isomorphisms on vector spaces. $f \sim_s g$ if and only if f and g have the same Jordan canonical form.*

Proof: The Jordan canonical form is unique up to conjugacy. ∎

Proposition 10.5 *Let $f : V \to V$ and $g : W \to W$ be linear maps that are shift equivalent. Then the non-zero eigenvalues of f and g coincide.*

Proof: Assume λ is a non-zero eigenvalue of f. Let v be a corresponding eigenvector. Since $f \sim_s g$, there exists linear maps $r : V \to W$, $s : W \to V$ and a positive integer m such that $r \circ f = g \circ r$ and $r \circ s = f^m$. Since $r \circ s(v) = f^m(v) = \lambda^m v$, v is not in the kernel of r. Observe that

$$g \circ r(v) = r \circ f(v) = \lambda r(v).$$

Thus, $r(v)$ is an eigenvector for g with eigenvalue λ. ∎

Example 10.6 The linear maps

$$f = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{and} \quad g = \begin{bmatrix} \lambda \end{bmatrix} : \mathbb{R} \to \mathbb{R}$$

are shift equivalent. To see this choose

$$r = \begin{bmatrix} 0 & 1 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R} \quad \text{and} \quad s = \begin{bmatrix} 0 \\ \lambda \end{bmatrix} : \mathbb{R} \to \mathbb{R}^2$$

and $m = 1$.

An open problem: Let $f : X \to X$ and $g : Y \to Y$ be endomorphisms in the category of finitely generate abelian groups. Provide an algorithm (if it exists) that determines if $f \sim_s g$.

10.2 Conley index

Theorem 10.7 *If $P = (P_1, P_0)$ and $Q = (Q_1, Q_0)$ are filtration pairs such that*

$$S = \operatorname{Inv}(\operatorname{cl}(P_1 \setminus P_0), \varphi) = \operatorname{Inv}(\operatorname{cl}(Q_1 \setminus Q_0), \varphi),$$

then

$$f_P \sim_s f_Q$$

in the category of pointed topological spaces.

We begin with two lemmas.

Lemma 10.8 *Assume $P = (P_1, P_0)$ and $Q = (P_1 \cup Q_0, Q_0)$ are filtration pairs for S and that $P_0 \subset Q_0$ and $f(Q_0) \subset \mathrm{int}(Q_0)$, then $f_P \sim_s f_Q$.*

Proof: Let $Q_1 := P_1 \cup Q_0$. Let $\pi_P \colon P_1 \to P_1/P_0$ and $\pi_Q \colon Q_1 \to Q_0$ be the projection maps. Define $r \colon P_1/P_0 \to Q_1/Q_0$ by

$$r(x) := \begin{cases} [Q_0] & \text{if } x = [P_0], \\ \pi_Q(x) & \text{otherwise.} \end{cases}$$

Observe that r is continuous and $r \circ f_P = f_Q \circ r$.

Since we are working with filtration pairs for S, $P_1^{w-} \subset P_0$ and $S \cap Q_0 = \emptyset$. Thus, there exists $n > 0$ such that if $x \in P_1 \cap Q_0$ then $f^k(x) \in P_0$ for some $k < n$.

Define $s \colon Q_1/Q_0 \to P_1/P_0$ by

$$s(x) := \begin{cases} [P_0] & \text{if } x = [Q_0], \\ f_P^n(\pi_P(x)) & \text{otherwise.} \end{cases}$$

Observe that $s \circ f_Q = f_P \circ s$. That s is continuous on $Q_1/Q_0 \setminus \{[Q_0]\}$ follows directly from the definition. To check continuity at $[Q_0]$ recall that $f(Q_0) \subset \mathrm{int}(Q_0)$ and hence there exists a neighborhood V of Q_0 such that $f(V) \subset \mathrm{int}(Q_0)$. Thus for any sequence $x_n \in Q_1$ such that $d(x_n, Q_0) \to 0$, $\lim_{n\to\infty} s(x_n) = [Q_0]$ which implies that s is continuous at $[Q_0]$.

Finally observe that $s \circ r = f_P^n$ and $r \circ s = f_Q^n$. ∎

Lemma 10.9 *Assume $P = (P_1, P_0)$ and $Q = (Q_1, P_0)$ are filtration pairs for S and that $P_1 \setminus P_0 \subset Q_1 \setminus P_0$ and $f(P_0) \subset \mathrm{int}(P_0)$, then $f_P \sim_s f_Q$.*

Proof: To simplify the notation set $Q_0 = P_0$. Define $r \colon P_1/P_0 \to Q_1/Q_0$ by

$$r(x) := \begin{cases} [Q_0] & \text{if } x = [P_0], \\ \pi_Q(x) & \text{otherwise.} \end{cases}$$

Observe that r is continuous and $r \circ f_P = f_Q \circ r$.

Since $cl(Q_1 \setminus Q_0)$ is an isolating neighborhood and $f(Q_0) \subset \operatorname{int}(Q_0)$, there exists $n > 0$ such that $f^n(Q_1) \subset \operatorname{int}(P_1)$. Define $s\colon Q_1/Q_0 \to P_1/P_0$ by

$$s(x) := \begin{cases} [P_0] & \text{if } x = [Q_0], \\ \pi_P(f^n(x)) & \text{otherwise.} \end{cases}$$

Observe that $s \circ f_Q = f_P \circ s$. Again continuity for s only needs to be checked at $[Q_0]$ and the argument is similar to the previous lemma. ∎

Proof of Theorem 10.7: We will construct a filtration pair $R = (R_1, R_0)$ such that $f_R \sim_s f_P$ and $f_R \sim_s f_Q$.

Let $N := \operatorname{cl}(P_1 \setminus P_0) \cap \operatorname{cl}(Q_1 \setminus Q_0)$. This is an isolating neighborhood for S. Let $\mathcal{X}$ be a grid on X. If $\operatorname{diam}(\mathcal{X})$ is sufficiently small then there exists an isolated invariant set $\mathcal{S} \subset \mathcal{X}$ for $\widetilde{\mathcal{F}}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ such that $\operatorname{Inv}(|\mathcal{S}|, \varphi) = S$ and $|\mathcal{W}^2| \subset \operatorname{cov}(N)$ where $\mathcal{W}^2$ is the wrap of the wrap of $\mathcal{S}$. Choose $\mathcal{S}$ to be the maximal invariant set satisfying these properties.

Apply the index pair algorithm to $\mathcal{S}$ and label the output as $\mathcal{R}_1$ and $\mathcal{R}_0$. $R_i = |\mathcal{R}_i|$. Let $x \in R_0$ and let $G \in \mathcal{X}$ such that $x \in G$. The maximality of $\mathcal{S}$ implies that there is no walk from G to $\mathcal{S}$ in $\operatorname{cov}(N)$. Since $\gamma_x^+ \subset \nu(G)$, this implies that there exists $n > 0$ such $f^n(x) \in P_0$. Since there are a finite number of grid elements in $\mathcal{R}_0$ we can assume that this value of n is valid for all $x \in R_0$.

Define $K_0 \subset P_1/P_0$ by $K_0 := \operatorname{cl}\left(f_P^{-n}(P_0)\right)$. Observe that $f_P(K) \subset \operatorname{int}(K_0)$.

The compact pairs $K = (K_1, K_0)$ where $K_1 = P_1/P_0$ and $L = (L_1, L_0)$ where $L_1 = R_1 \cup K_0$ and $L_0 = K_0$ are filtration pairs for S.

By Lemma 10.8, $f_R \sim_s f_L$ and by Lemma 10.9, $f_L \sim_s f_K$ and finally by Lemma 10.8, $f_L \sim_s f_P$. Therefore $f_R \sim_s f_P$.

The same argument shows that $f_R \sim f_Q$. ∎

Working with continuous functions on pointed spaces is difficult. By passing to homology we can use algebra.

Definition 10.10 Let S be an isolated invariant set under $\varphi\colon \mathbb{T}^+ \times X \to X$. The homological Conley index $Con_*(S, \varphi)$ is the shift equivalence class of $f_{P*}\colon H_*(P_1/P_0, [P_0]) \to H_*(P_1/P_0, [P_0])$ obtained from a filtration pair (P_1, P_0) for S.

Theorem 10.11 *The homological Conley index of $\emptyset$ is the trivial map.*

Corollary 10.12 *Let N is an isolating neighborhood. Let $P = (P_1, P_0)$ be a filtration pair such that* $\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_0), \varphi) = \mathrm{Inv}(N, \varphi)$. *If $f_P \not\sim_s 0$, then* $\mathrm{Inv}(N, \varphi) \neq \emptyset$.

Theorem 10.13 *Let S be an isolated invariant set under $\varphi \colon \mathbb{R}^+ \times X \to X$. Let $P = (P_1, P_0)$ be a filtration pair for S. Then $f_{P*} \sim_s \mathrm{id}_{P_1/P_0*}$.*

Proof: f_P is homotopic to the identy map on $P_1 \setminus P_0$. ■

This implies that in the case of continuous time, the Conley index is determined by the topology of the quotient space.

Examples of Conley index

- fixed points
- periodic orbits
- normally hyperbolic invariant manifolds.
- horseshoe
- g-horseshoe
- connecting orbit

10.3 Extracting Dynamics

The fundamental result has already been stated.

Corollary 10.14 *Let N is an isolating neighborhood. Let $P = (P_1, P_0)$ be a filtration pair such that* $\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_0), \varphi) = \mathrm{Inv}(N, \varphi)$. *If $f_P \not\sim_s 0$, then* $\mathrm{Inv}(N, \varphi) \neq \emptyset$.

A simple but I think extremely practical result for applications is based on following idea.

Definition 10.15 Let $f\colon X \to X$ be a continuous map. An isolated invariant set S is a *T-cycle set* if there exist T disjoint, compact regions $N_1, \ldots, N_T$ such that $S = \operatorname{Inv}(N, f)$, where $N := \bigcup_{i=1}^{T} N_i$ is an isolating neighborhood, and

$$f(N_i) \cap N \subset N_{i+1}, \quad i = 0, \ldots T-1,$$

where $N_0 = N_T$. Moreover, S is an *attracting* T-cycle set if $f(N_i) \subset N_{i+1}$ for $i = 0, \ldots, T-1$.

Proposition 10.16 *Let $\mathcal{F}\colon \mathcal{X} \to \mathcal{X}$ be an outer approximation for $f\colon X \to X$. Let $\mathcal{M}$ be a Morse set for $\mathcal{F}$. Let $M = \operatorname{Inv}(|\mathcal{M}|, f)$. Then the set of nontrivial eigenvalues of the index map on the 0th level of homology either is*

$$\emptyset \quad or \quad \left\{ e^{2\pi i \frac{k}{T}} \mid k = 0, \ldots, T-1 \right\}.$$

In the latter case, M is an attracting T-cycle set.

Proof: Since $|\mathcal{M}|$ is an isolating block Proposition 9.10 implies that there exists an index pair for f of the form $P = (P_1, P_0)$, where $P_1 = |\mathcal{M}|$.

Let $\mathcal{M} = \bigcup_{j=1}^{J} \mathcal{N}_j$, where $|\mathcal{N}_j|$ are the disjoint connected components of P_1. Let $\mathcal{J} = \{1, \ldots, J\}$. Let $\mathcal{I} = \{j \in \mathcal{J} \mid |\mathcal{N}_j| \cap P_0 = \emptyset\}$. Observe that

$$H_0(N_1, N_0) \cong \bigoplus_{j \in \mathcal{J}} H_0\big(|\mathcal{N}_j|, |\mathcal{N}_j| \cap P_0\big) \cong \bigoplus_{j \in \mathcal{I}} H_0\big(|\mathcal{N}_j|, |\mathcal{N}_j| \cap P_0\big) \cong \bigoplus_{j \in \mathcal{I}} \mathbb{Z}[\xi_j],$$

where ξ_j is the generator of $H_0\big(|\mathcal{N}_j|, |\mathcal{N}_j| \cap P_0\big)$.

Consider $j \in \mathcal{I}$. Since (P_1, P_0) is an index pair and $|\mathcal{N}_j|$ has no exit set associated with it, this implies that $f\big(|\mathcal{N}_j|\big) \subset |\mathcal{N}_\ell|$ for some $\ell \in \mathcal{J}$. If $\ell \in \mathcal{I}$, then $f_{P*}(\xi_j) = \xi_\ell$. If $\ell \notin \mathcal{I}$, then $f_{P*}(\xi_j) = 0$.

Since $\mathcal{M}$ is a combinatorial Morse set, it is an equivalence class of the recurrent set. Thus, if $\mathcal{I} \neq \mathcal{J}$, then F_{Q*} is nilpotent on the 0th level. In this case the set of nonzero eigenvalues is $\emptyset$. If $\mathcal{I} = \mathcal{J}$, then F_{QP*} restricted to the 0th level is a permutation matrix and the associated eigenvalues are roots of unit with $T = J$. ∎

11 Parameterized Dynamics

For applications we are interested in multiparameter families of dynamical systems

$$\varphi\colon \mathbb{T}^+ \times X \times \Lambda \to X$$

where we will assume that X is compact metric space and Λ is a compact connected metric space.

Definition 11.1 Given a multiparameter dynamical system $\varphi\colon \mathbb{T}^+ \times X \times \Lambda \to X$, the associated *parameterized dynamical system* is given by

$$\begin{array}{rcl} \Phi\colon \mathbb{T}^+ \times X \times \Lambda & \to & X \times \Lambda \\ (t, x, \lambda) & \mapsto & (\varphi(t, x, \lambda), \lambda) = (\varphi_\lambda(t, x), \lambda) \end{array}$$

Let $\Lambda_0 \subset \Lambda$. The restricted parameterized dynamics is denoted by

$$\Phi_{\Lambda_0}\colon \mathbb{T}^+ \times X \times \Lambda_0 \to X \times \Lambda_0.$$

In particular, $\Phi_{\lambda_0} = \varphi_{\lambda_0}$.

We will denote a time discretization of Φ by

$$\begin{array}{rcl} F\colon X \times \Lambda & \to & X \times \Lambda \\ (x, \lambda) & \mapsto & (f_\lambda(t, x), \lambda) \end{array}$$

Similarly, given $\Lambda_0 \subset \Lambda$ the restricted time discretization of F is denoted by F_{Λ_0}.