1. Haar measure

Let G be a locally compact group. Denote by $C_{00}(G)$ the algebra of continuous functions on G with compact support. Endow it with the Sup-norm. Let $C_{00}^{+}(G)$ denote the cone of non-negative functions. Given $f \in C_{00}(G)$ and $g \in G$, we let $R_g(f)(x) := f(xg)$ and $L_g(f)(x) := f(g^{-1}x)$, $x \in G$. Then $g \mapsto R_g$ is a homomorphism from G to the group of invertible positive isometries of $C_{00}(G)$.

Lemma 0. *(On uniform continuity) For each $f \in C_{00}(G)$, the maps $G \ni g \mapsto R_g(f) \in C_{00}(G)$ and $G \ni g \mapsto L_g(f) \in C_{00}(G)$ are continuous.*

Lemma 1. *Given $f, \phi \in C_{00}^{+}(G)$, $\phi \neq 0$, there is a finite family of elements $g_j \in G$ and reals $c_j \geq 0$ such that*

$$f \leq \sum_j c_j R_{g_j}(\phi).$$

We let $(f : \phi) := \inf \sum_j c_j$ over all such families.

Lemma 2. *(Properties of $(f : \phi)$)*

(1) $(R_g(f) : \phi) = (f : \phi)$
(2) $(cf : \phi) = c(f : \phi)$, $c \geq 0$
(3) $(f_1 + f_2 : \phi) \leq (f_1 : \phi) + (f_2 : \phi)$
(4) $(f : \psi) \leq (f : \phi)(\phi : \psi)$
(5) $(f_1 : \phi) \leq (f_2 : \phi)$ *if* $f_1 \leq f_2$
(6) $(f : \phi) \geq \|f\|/\|\phi\|$

Fix $f_0 \in C_{00}^{+}(G)$, $f_0 \neq 0$. We now let

$$I_\phi(f) = \frac{(f : \phi)}{(f_0 : \phi)}.$$

Then from (4) we deduce

$$\frac{1}{(f_0 : f)} \leq I_\phi(f) \leq (f : f_0).$$

Lemma 3. *(Properties of I_ϕ)*

(1) $I_\phi(R_g(f)) = I_\phi(f)$
(2) $I_\phi(cf) = cI_\phi(f)$
(3) $I_\phi(f_1 + f_2) \leq I_\phi(f_1) + I_\phi(f_2)$
(4) $I_\phi(f_1) \leq I_\phi(f_2)$ *if* $f_1 \leq f_2$

This is almost what we are looking for. The functionals I_ϕ are not linear, but we will show that if the support of ϕ is small enough, then I_ϕ is approximately linear.

Lemma 4. *For $f_1, f_2 \in C_{00}^{+}(G)$ and $\epsilon > 0$, there is a neighborhood U of e such that if $\operatorname{supp}(\phi) \subset U$ then*

$$I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \epsilon.$$

Proof. Let K be a compact subset of G such that $\mathrm{Supp}(f_1) \subset K$ and $\mathrm{Supp}(f_2) \subset K$. By Urysohn lemma, there is $h_0 \in C_{00}^+(G)$ such that $h_0 = 1$ on K. We let $h = f_1 + f_2 + \delta h_0$ for some $\delta > 0$. We now set

$$f_1' := \frac{f_1}{h}, \quad f_2' := \frac{f_2}{h}, \quad \text{assuming } f_1'(x) = f_2'(x) = 0 \text{ if } h(x) = 0.$$

Of course, $f_1', f_2' \in C_{00}^+(G)$ and $f_1' + f_2' \leq 1$. Choose a symmetric neighborhood U of e such that $|f_1'(x) - f_1'(y)| < \epsilon$ and $|f_2'(x) - f_2'(y)| < \epsilon$ whenever $xy^{-1} \in U$. Let $\mathrm{Supp}(\phi) \subset U$ and let $h \leq \sum_j c_j R_{g_j}(\phi)$ for some $c_j \geq 0$ and $g_j \in G$.

$$\begin{aligned} f_1(g) = f_1'(g)h(g) \leq f_1'(g)\sum_j c_j\phi(gg_j) &= \sum_{gg_j \in U} f_1'(g)c_j\phi(gg_j) \\ &\leq \sum_{gg_j \in U} (f_1'(g_j^{-1}) + \epsilon)c_j\phi(gg_j) = \sum_j (f_1'(g_j^{-1}) + \epsilon)c_j\phi(gg_j) \end{aligned}$$

Hence $(f_1 : \phi) \leq \sum_j (f_1'(g_j^{-1}) + \epsilon)c_j$ and, similarly, $(f_2 : \phi) \leq \sum_j (f_2'(g_j^{-1}) + \epsilon)c_j$ Therefore

$$(f_1 : \phi) + (f_2 : \phi) \leq \sum_j (1 + 2\epsilon)c_j.$$

It follows that

$$(f_1 : \phi) + (f_2 : \phi) \leq (1 + 2\epsilon)(h : \phi) \leq (1 + 2\epsilon)((f_1 + f_2 : \phi) + \delta(h_0 : \phi)).$$

Therefore

$$I_\phi(f_1) + I_\phi(f_2) \leq (1 + 2\epsilon)(I_\phi(f_1 + f_2) + \delta I_\phi(h_0))$$

We recall that $I_\phi(h_0) \leq (h_0 : f_0)$. Since δ is arbitrary, we conclude that

$$I_\phi(f_1) + I_\phi(f_2) \leq (1 + 2\epsilon)(I_\phi(f_1 + f_2)) \leq I_\phi(f_1 + f_2) + 2\epsilon(f_1 + g : f_0).$$

□

Theorem 5. *There is a non-trivial right invariant non-trivial integral I on $C_{00}(G)$.*

Proof. Let

$$X := \prod_{0 \neq f \in C_{00}^+(G)} [\frac{1}{(f_0 : f)}, (f, f_0)].$$

It is a compact space. For each neighborhood U of e, we set

$$M_U := \{(I_\phi(f))_{0 \neq f \in C_{00}^+(G)} \mid \mathrm{Supp}(\phi) \in U\}.$$

Then $M_U \neq \emptyset$, $M_U \subset X$ and $M_{U_1} \cap M_{U_2} = M_{U_1 \cap U_2}$. Thus the family $\{M_U \mid U$ is a neighborhood of $e\}$ is a base of a filter in X. Since X is compact, there is $I = (I(f))_{0 \neq f \in C_{00}^+(G)} \in X$ such that I belongs to the closure of each M_U. Thus, for each $\epsilon > 0$ and each finite sequence $f_1, \ldots, f_n \in C_{00}^+(G) \setminus \{0\}$ and each U, there is $\phi \in C_{00}^+(G)$ with $\mathrm{Supp}(\phi) \subset U$ such that

$$|I(f_j) - I_\phi(f_j)| < \epsilon.$$

It now follows from Lemma 3 that

(1) $I(R_g(f)) = I(f)$
(2) $I(cf) = cI(f)$
(3) $I(f_1 + f_2) \leq I(f_1) + I(f_2)$
(4) $I(f_1) \leq I(f_2)$ if $f_1 \leq f_2$.

Fix f_1, f_2. Lemma 4 yields that

$$I(f_1) + I(f_2) \le I(f_1 + f_2) + \epsilon$$

for an arbitrary $\epsilon > 0$. Therefore, jointly with (3) this gives

$$I(f_1) + I(f_2) \le I(f_1 + f_2).$$

Also, from the definition of X it follows that

$$\frac{1}{(f_0 : f)} \le I(f) \le (f : f_0).$$

Hence $I(f) > 0$ if $f \ne 0$, $f \in C_{00}^+(G)$. Now extend I from the cone $C_{00}^+(G)$ to the entire $C_{00}(G)$ in a usual way. In particular, $I(0) = 0$. □

Recall

Lemma 6. *(Riesz Representation Theorem) Let X be a locally compact Hausdorff space and let I be a positive linear functional on $C_{00}(X)$. Then there is a unique Radon measure μ (i.e. a Borel measure which is locally finite and inner regular) on X such that $I(f) = \int f \, d\mu$, for all $f \in C_{00}(X)$. Moreover, μ satisfies: $\mu(U) = \sup\{I(f) | f \in C_{00}^+(X), f \le 1_U\}$ and $\mu(K) = \inf\{I(f) | f \in C_{00}^+(X), f \ge 1_K\}$, for all open $U \subset X$ and compact $K \subset X$.*

Let μ correspond to I. Then μ is called a right-invariant Haar measure. In a similar way a left invariant Haar measure is defined. If J is the inversion in G, a measure μ is a right invariant Haar measure if and only if $\mu \circ J$ is left-invariant Haar measure.

Corollary 7. *It follows that the Haar measure of each compact set is finite and the Haar measure of each open set is strictly positive.*

Theorem 8. *(Uniqueness of Haar measure). Given two non-trivial right-invariant integrals I and I', there is $c > 0$ such that $I = cI'$.*

Proof. Let $I(f) = \int f \, d\mu$

Let I' be another right invariant integral on $C_{00}^+(G)$ and let μ' stand for the corresponding measure. Since I' is nontrivial, there is $0 \ne f_0 \in C_{00}^+(G)$ with $I'(f_0) > 0$. For each $0 \ne f \in C_{00}^+(G)$, we can find g_j and c_j with $f_0 \le \sum_j c_j R_g(f)$. Hence $I'(f_0) \le \sum_j c_j I'(f)$. It follows that $I'(f_0) \le (f_0 : f) I'(f)$. In particular, $I'(f) > 0$.

Take $f_1, f_2 \in C_{00}^+(G)$. Fix a neighborhood V of e. Select $\psi \in C_{00}^+(G)$ such that $\psi = 1$ on the union of $\mathrm{Supp}(f_1) \cdot V \cup U \cdot \mathrm{Supp}(f_1)$ and $\mathrm{Supp}(f_2) \cdot V \cup U \cdot \mathrm{Supp}(f_2)$. For each $\epsilon > 0$, there is a symmetric neighborhood $U \subset V$ of e with $|f_j(g) - f_j(g')| < \epsilon$ whenever $g' g^{-1} \in U$. Select $0 \ne \phi \in C_{00}^+$ such that $\mathrm{Supp}(\phi) \subset U$.

Then we have

$$I'(\phi) I(f_1) = \int \phi(g_1) f_1(g_2) \, d\mu'(g_1) d\mu(g_2) = \int \phi(g_1) f_1(g_2 g_1) \, d\mu'(g_1) d\mu(g_2).$$

On the other hand

$$I(\phi) I'(f_1) = \int \phi(g_1 g_2) f_1(g_2) \, d\mu(g_1) d\mu'(g_2) = \int \phi(y) f_1(g_1^{-1} y) \, d\mu(g_1) d\mu'(y).$$

Hence

$$\begin{aligned}|I'(\phi)I(f_1) - I(\phi)I'(f_1)| &\le \int \phi(g_1)|f_1(g_2g_1) - f_1(g_1^{-1}g_2)|\, d\mu'(g_1)d\mu(g_2)\\ &= \int \phi(g_1)|f_1(g_2g_1) - f_1(g_1^{-1}g_2)|\psi(g_2)\, d\mu'(g_1)d\mu(g_2)\\ &\le \epsilon I'(\phi)I(\psi)\end{aligned}$$

Therefore

$$\left|\frac{I(f_1)}{I'(f_1)} - \frac{I(\phi)}{I'(\phi)}\right| \le \epsilon\frac{I(\psi)}{I'(f_1)}$$

In a similar way,

$$\left|\frac{I(f_2)}{I'(f_2)} - \frac{I'(\phi)}{I'(\phi)}\right| \le \epsilon\frac{I(\psi)}{I'(f_2)}$$

Therefore

$$\left|\frac{I(f_1)}{I'(f_1)} - \frac{I(f_2)}{I'(f_2)}\right| \le \epsilon\left(\frac{I(\psi)}{I'(f_1)} + \frac{I(\psi)}{I'(f_2)}\right).$$

Hence

$$\frac{I(f_1)}{I'(f_1)} = \frac{I(f_2)}{I'(f_2)},$$

i.e. $I = cI'$. □

Theorem 9. *Let G be a locally compact group G and λ be a left Haar measure on it. Then G is compact if and only if $\lambda(G) < \infty$.*

Proof. If G is compact then the assertion of the theorem is trivial. If G is not compact, fix U, a neighborhood of e with compact closure. There is an infinite sequence $(g_n)_{n=1}^{\infty}$ such that $g_n \notin \bigcup_{j<n} g_jU$. Take a symmetric neighborhood V of e with $VV \subset U$. Then g_jV, $j = 1, 2, \ldots$ are pairwise disjoint. Hence $\lambda(G) \ge \sum_j \lambda(g_jV) = \sum_j \lambda(V) = \infty$. □

Modular homomorphism. Let I be a right Haar integral. Then $I \circ L_g$ is also a right Haar integral. Hence there is a number $\Delta(g) > 0$ such that $I \circ L_g = \Delta(g)I$. It is easy to see that $\Delta(g)$ does not depend on the choose of I. It is easy to verify that $\Delta : g \mapsto \Delta(g)$ is a group homomorphism from G to $\mathbb{R}_+^*$.

Theorem 10. *Δ is continuous.*

Proof. Take $0 \ne \phi \in C_{00}^+(G)$. Select a symmetric precompact neighborhood U of e Let $\psi \in C_{00}^+(G)$ be such that $\psi = 1$ on $U \cdot \mathrm{Supp}(\phi)$.

Fix $\epsilon > 0$. Select a symmetric precompact neighborhood $V \subset U$ of e such that $\|L_g(\phi) - \phi\| < \epsilon$. Then

$$|I(L_g(\phi)) - I(\phi)| \le I(|L_g(\phi) - \phi|) = I(|L_g(\phi) - \phi|\psi) \le \epsilon I(\psi).$$

Thus $|\Delta(g) - 1| \le \epsilon\frac{I(\psi)}{I(\phi)}$. □

G is called unimodular if Δ is trivial.

Proposition 11. *Each locally compact Abelian group is unimodular. Each compact group is unimodular.*

Exercise 12. Let I be a right Haar integral. (1)Show that $I(f) = I(J(f) \cdot \Delta)$. (2)As a corollary: G is unimodular iff $I \circ J = I$. Hint: a) Show that the right-hand-side is also a right Haar integral. b) Prove that the corresponding proportional coefficient is 1.

Examples. 1) G is a finite group.

2) G is infinite discrete group.

3) $G = \mathbb{R}^n$.

4) $G = \mathbb{T}^n$

5) Suppose now that G is an open subset in $\mathbb{R}^n$. Suppose that the multiplication is given by the formula

$$(x * y)_j = \langle A^{(j)} x, y \rangle + d_j \text{ for all } x = (x_j)_j, y = (y_j)_j,$$

for some matrices $A^{(j)}$ and reals d_j. We let $r(g) := |\det R'_g|$ and $l(g) := |\det L'_{g^{-1}}|$. Then

$$I_r(f) := \int_G \frac{f(x)}{r(x)}\, dx \quad \text{and} \quad I_l(f) := \int_G \frac{f(x)}{l(x)}\, dx$$

are right and left Haar integrals. Indeed,

$$I_r(R_g(f)) = \int_G \frac{f(x * g)}{r(x)}\, dx = \int_G \frac{f(y)}{r(y * g^{-1})} \frac{1}{r(g)}\, dy = \int_G \frac{f(y)}{r(y)}\, dy = I_r(f).$$

Moreover,

$$I_r(L_g f) = \frac{l(g)}{r(g)} I_r(f).$$

Hence $\Delta(g) = \frac{l(g)}{r(g)}$.

5a) $G = \mathbb{R}^*$. $I_r(f) = \int \frac{f(x)}{|x|}\, dx$.

5b) $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x \neq 0 \right\}$. Then

$$I_r(f) = \int \frac{f(x,y)}{|x|}\, dxdy, \quad I_l(f) = \int \frac{f(x,y)}{x^2}\, dxdy, \quad \Delta(x,y) = \frac{1}{|x|}.$$

5c) $G = GL_2(\mathbb{R})$. Then $r(A) = l(A) = (\det A)^2$.

6) Haar measures for product of l.c.g.

Literature.

1. A. Weil, Integration in topological groups and its applications
2. Hewitt, Ross, Abstract harmonic analysis, vol 1.
3. Naimark, Normed rings

2. Amenability for discrete groups

Invariant means on discrete groups. Let G be a group endowed with discrete topology. Denote by $BC(G)$ the algebra of bounded continuous (real) functions on G.

Definition 1. A left invariant mean (LIM) on G is a functional $M : BC(G) \to \mathbb{R}$ such that

$$\inf f \le M(f) \le \sup f \text{ for each } f \in BC(G) \text{ and} \tag{1}$$

$M \circ L_g = M$ for all $g \in G$. In this case G is called *left amenable*. In a similar way we can define a right invariant mean, 2-side invariant mean, right amenability and (two-sided) amenability.

(1) is equivalent to $M(f) \ge 0$ if $f \ge 0$ plus $M(1) = 1$.

Proposition 2. *G is left amenable iff G is right amenable iff G is amenable.*

Proof. Indeed, if M is LIM then $M \circ J$ is RIM. Moreover, given $f \in BC(B)$, let $f'(g) := M(R_{g^{-1}}(f))$. Then $f' \in BC(G)$ and $f \mapsto M(J(f'))$ is a 2-sided invariant mean. □

Of course, every finite group is amenable.

Theorem 3 (Dixmier criterium). *Let $\mathcal{S}_l := \{\sum_j (L_{g_j}(f_j) - f_j) \mid f_1, \dots, f_n \in BC(X), g_1, \dots, g_n \in G\}$ and $\mathcal{S}_{l,r} := \{f_1 + f_2 \circ J \mid f_1, f_2 \in \mathcal{S}_l\}$. Then*

(1) A left invariant mean on $BC(G)$ exists iff $\sup f \ge 0$ for each $f \in \mathcal{S}_l$.

(2) A 2-side invariant mean on $BC(G)$ exists iff $\sup f \ge 0$ for each $f \in \mathcal{S}_{l,r}$.

Proof. (1) Let $p(f) := \sup f$. Then p is a sub linear function on $BC(X)$. Suppose (1) holds. We put $M(f) := 0$ on $\mathcal{S}_l$. Then $p \ge M$ on $\mathcal{S}_l$. By Hahn-Banach theorem, M extends as a linear functional to $BC(X)$ with $p \ge M$. For $f \in BC(G)$, $\inf f = -\sup(-f) \le -M(-f) = M(f)$. And since $M(f - L_g f) = 0$, M is left invariant. □

Definition 4. Let every finite subset of G generate a finite subgroup. Then G is called locally finite.

Theorem 5. *Every locally finite group is amenable.*

Proof. Take $f \in \mathcal{S}_l$. Then $f = \sum_j (L_{g_j}(f_j) - f_j)$ for some g_j, f_j. Then there is a finite subgroup $H \subset G$ such that $g_j \in H$. Let $f'_j := f_j \restriction H$, $f' := f \restriction H$. Then $f' = \sum_j (L_{g_j}(f'_j) - f'_j)$. By Theorem 2(i), $\sup(f \restriction H) \ge 0$. It follows that $\sup f \ge 0$. Hence G is amenable by Theorem 2. □

In a similar way one can prove the following

Theorem 6. *If $G = \operatorname{inj\,lim}_\alpha G_\alpha$ and each G_α is amenable then G is amenable. In particular, G is amenable if and only if every finitely generated subgroup of it is amenable.*

In particular, a group G is amenable off each finitely generated subgroup of G is amenable. This means that the amenability is a "local" concept.

Markov-Kakutani theorem. *Let V be a locally convex vector space. Let K be a compact convex subset in V. Let $(T_x)_{x\in X}$ be a family of pairwise commuting affine transformations of K. Then there is a common fixed point for $(T_x)_{x\in X}$.*

Proof. Let $F_{T_x} := \{k \in K \mid T_x k = k\}$. Let us show that $F_{T_x} \neq \emptyset$. Set

$$T_{x,n} := \frac{1}{n}\sum_{j=1}^{n} T_x^j.$$

Of course, $T_{x,n} : K \to K$. Fix $k \in K$ and put $k_n := T_{x,n}k$. Let k_0 be a limit point of $(k_n)_{n=1}^{\infty}$. Then $k_0 \in K$. It is easy to check that $k_0 \in F_{T_x}$. Next, we note that F_{T_x} is a compact convex set. Moreover, $T_y F_{T_x} \subset F_{T_x}$ for each $y \in G$. By induction, for each finite subset $S \subset G$, $\bigcap_{x\in S} F_{T_x} \neq \emptyset$. Therefore, because K is compact, $\bigcap_{x\in G} F_{T_x} \neq \emptyset$. $\square$

Theorem 7. *If G is Abelian then it is amenable.*

Proof. a) $BC(G)$ is a Banach space.

b) Endow $BC(G)$ with the *-weak topology. Then the subset

$$\mathcal{M} := \{F \in BC(G)' \mid F(f) \geq 0 \text{ for each } f \geq 0, F(1) = 1\}$$

is compact (follows from the Banach-Alaoglu theorem) and convex.

c) $T_g F := F \circ L_g$ is affine.

It remains to apply Markov-Kakutani. $\square$

Remark 8. Let $G = \mathbb{Z}$. Put $F_n := \{-n, \dots, n\}$ and

$$M_n(f)(k) := \frac{1}{2n+1}\sum_{j\in F_n} f(k+j).$$

Then $M_n \in \mathcal{M}$. Hence there is a subsequence M_{n_m} that $*$-weakly converges to a LIM M. It is interesting to note that if f is a converging sequence of reals then $f \in BC(\mathbb{Z})$ and $\lim_{n\to\infty} f(n) = M(f)$. However if f is bounded but it does not converge then $M(f)$ is still well defined. Thus $M(f)$ can be considered as a "generalized" (so-called Banach) limit of f.

Theorem 9. *If H is a subgroup in G and G is amenable then H is amenable.*

Proof. Let M be LIM on G. Fix a subset $X \subset G$ such that $Hx \cap Hy = \emptyset$ and $\bigcup_{x\in X} Hx = G$. then every element $g \in G$ can be represented uniquely as $g = hx$ for some $h \in H$ and $h \in H$. Given $f \in BC(H)$, we let $f'(g) := f(h)$. Then $f' \in BC(G)$. We now set $M'(f) := M(f')$. Of course, $M'(f) \geq 0$ if $f \geq 0$ and $M'(1) = 1$. $(L_h(f))' = L_h(f')$. Hence M' is LIM on H. $\square$

Theorem 10. *Let H be a normal subgroup in G. Then G is amenable if and only if H and G/H are both amenable.*

Proof. ($\Rightarrow$) By Theorem 9, H is amenable. Let $\pi : G \to G/H$ be the natural projection and let π^* denote the corresponding map $f \to f \circ \pi$ from $BC(H)$ to $BC(G)$. Let M be LIM on G. Then $M \circ \pi^*$ is LIM on $BC(G)$.

(⇐) Let M_H and $M_{G/H}$ be RIM on H and G/H respectively. Let $s : G/H \to H$ be a map with $Hs(x) = x$ and $Hs(G/H) = G$. Given $f \in BC(G)$, we let

$$f'_x(h) := f(hs(x)), \quad x \in G/H, h \in H.$$

Let $f''(x) := M_H(f'_x)$. Finally we let $M(f) := M_{G/H}(f'')$. Then M is RIM on G. Let $g = hs(x)$, $g_0 = h_0 s(x_0)$. Then

$$gg_0 = hs(x)h_0 s(x)^{-1} s(x)s(x_0) = hh_1h_2 s(xx_0).$$

$$(R_{g_0} f)'_x(h) = (R_g f)(hs(x)) = f(hh_1h_2 s(xx_0)) = f'_{xx_0}(hh_1h_2) = (R_{h_1h_2} f'_{xx_0})(h).$$

$$(R_{g_0} f)''(x) = M_H(f'_{xx_0}) = f''(xx_0) = (R_{x_0}(f''))(x).$$

Hence M is right invariant. □

Finitely additive invariant measures. Let M be LIM on G. For each subset $A \subset G$, we set $\mu(A) := M(1_A)$. Then

(1) $0 \leq \mu(A) \leq 1$,
(2) $\mu(A) + \mu(B) = \mu(A \cup B)$ if $A \cap B = \emptyset$.
(3) $\mu(gA) = \mu(A)$
(4) $\mu(\emptyset) = 0$, $\mu(X) = 1$.

Conversely,

Proposition 11. *If μ is a finitely additive left invariant normalized measure on the set of all subsets of G then G is amenable.*

Proof. (a) Given $f \in BC(G)$, there is a sequence of functions f_n on G such that f_n takes finitely many values for each n and $\|f - f_n\| \to 0$.

(b) Of course, one can define naturally $M(f_n)$. Now define $M(f) := \lim_n M(f_n)$. It is well defined.

(c) M is LIM on $CB(G)$. □

Example 12. Let $G = F_2$ (free group with 2 generators). Then G is not amenable. Let a, b be the free generators of F_2. Consider a partition $F_2 = \bigsqcup_{n \in \mathbb{Z}} B_n$, where $w \in B_n$ if $w = a^n b^j \cdots$, $j \neq 0$ unless $w = a^n$. Suppose that F_2 is amenable. Let μ be the left invariant finitely invariant normalized measure on F_2. Since $aB_n = B_{n+1}$, $\mu(B_n) = \mu(B_{n+1})$. For each n, $1 \geq \mu(\bigsqcup_{1 \leq j \leq n} B_j) = n\mu(B_0)$. Hence $\mu(B_0) = 0$. On the other hand, let $B = F_2 \setminus B_0$. Then $bB \subset B_0$.

$$1 = \mu(B \sqcup B_0) = \mu(B) + \mu(B_0) = \mu(B) = \mu(bB) \leq \mu(B_0) = 0,$$

a contradiction.

Corollary 13. *If G contains F_2 as a subgroup then G is not amenable.*

There was a conjecture that every non-amenable group contains F_2. It is not true. A counterexample was constructed by Olshansky. However the conjecture is true within the class of linear groups due to Tits alternative.

Tits alternative. *(without proof) Let V be a finite dimensional vector space over a field $\mathbb{F}$ of characteristic 0. Let G be a subgroup of $GL(V)$. Then G is amenable if and only if it does not contain F_2 as a a subgroup. If G is amenable then G contains a normal solvable subgroup of finite index.*

Definition 14. Let EG be the smallest class of groups such that

(1) EG contains the abelian groups and the finite groups and
(2) EG is closed under operation of taking subgroups, quotient groups, group extensions and inductive limits.

In particular all nilpotent, all solvable groups are elementary. Every elementary group is amenable. The converse is true within the class of linear groups (follows from Tits alternative). However, in general there are non-elementary amenable groups (Grigorchuk).

Example 15. The group $PSL_2(\mathbb{R})$ is non-amenable. We first note that each matrix $A := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{R})$ generates a linear fractional transformation Q_A of $\mathbb{C}$ by

$$Q_A(z) := \frac{\alpha z + \beta}{\gamma z + \delta}.$$

The map $A \mapsto Q_A$ is a homomorphism from $SL_2(\mathbb{R})$ and its kernel is $\{I, -I\}$. Hence this map identifies $PSL_2(\mathbb{R})$ with the group of linear fractional transformation of $\mathbb{C}$ preserving the upper half plane.

We show that $PSL_2(\mathbb{R})$ contains F_2. Consider 4 circles in $\mathbb{C}$: $C_1 := \{z \mid |z+2| = 1\}$, $C_1' := \{z \mid |z-2| = 1\}$, $C_2 := \{z \mid |z+5| = 1\}$, $C_2' := \{z \mid |z-5| = 1\}$. Consider two linear fractional transformations of $\mathbb{C}$:

$$T_1(z) := \frac{2z+3}{z+2}, \qquad T_2(z) := \frac{5z+24}{z+5}.$$

Extend them to a homomorphism T of F_2 into $PSL_2(\mathbb{R})$ in a natural way: if $w = a^{j_1} b^{j_2} a^{j_3} \cdots \in F_2$ is a nontrivial reduced word, i.e. each $j_k \neq 0$, then $T_w := T_1^{j_1} T_2^{j_2} T_1^{j_3} \cdots$. In a similar way we define T_w for a word w starting with b. Our purpose to prove that the kernel of T is trivial, i.e. if w is a nontrivial word then $T_w \neq I$.

It is easy to see that $T_1(C_1) = C_1'$ and $T_2(C_2) = C_2'$. Moreover,

$$T_1(e(C_1)) = i(C_1') \text{ and } T_2(e(C_2)) = i(C_2').$$

Let $z_0 = 4i \in \mathbb{C}$. Then $T_w z_0$ is inside the interior of the 4 circles. Hence $T_w z_0 \neq z_0$. Therefore $T_w \neq I$.

Corollary 16. $SL_2(\mathbb{R})$ *is not amenable.*

Literature.

1. Hewitt, Ross, Abstract harmonic analysis, vol 1.
2. Greenleaf, Invariant means on topological groups and their applications
3. Paterson, Amenability.

3. Invariant mean on almost periodic functions

As before, G is a group with discrete topology. Given $f \in BC(G)$ and $g \in G$, we define $D_g : G \times G \to \mathbb{C}$ by $D_g f(x,y) = f(xgy)$.

Definition 1. A function $f \in BC(G)$ is called almost periodic (AP) if of of the following is satisfied:

(1) $\{L_g(f) \mid g \in G\}$ is relatively compact in $BC(G)$.
(2) $\{R_g(f) \mid g \in G\}$ is relatively compact in $BC(G)$.
(3) $\{R_h L_g(f) \mid h, g \in G\}$ is relatively compact in $BC(G)$.
(4) $\{D_g f \mid g \in G\}$ is relative compact in $BC(G^2)$.

Proposition 2. *The conditions (1)–(4) are equivalent.*

Proof. $BC(G)$ is a Banach space. A subset of it is relatively compact off it is totally bounded.

(1) $\Rightarrow$ (3) Given $\epsilon > 0$, there are $a_1, \dots, a_n \in G$ such that $L_{a_1}(f), \dots, L_{a_n}(f)$ is an ϵ-net in $\{L_g(f) \mid g \in G\}$. Since $L_{a_j}(f)$ is AF, there is an ϵ-net $R_{b_{j,k}}$, $k = 1, \dots K_j$, in $\{R_h(L_{a_j}(f)) \mid h \in G\}$ by (ii). Then $R_{b_{j,k}}(L_{a_j})$ is 2ϵ-net in $\{R_h L_g(f) \mid h, g \in G\}$. Thus (1), (2), (3) are equivalent. Of course, (4) implies (1) and (2).

Given ϵ, we find $a_1, \dots, a_n \in G$ such that $R_{a_1}(f), \dots, R_{a_n}(f)$ is ϵ-net in $\{R_g(f) \mid g \in G\}$. Next, let $A_j := \{g \in G \mid \|R_g(f) - R_{a_j}(f)\| < \epsilon\}$. Then $\bigcup_j A_j = G$. Consider the family of sets $\bigcap_j A_{l_j} a_j^{-1}$ when $l_1, \dots, l_n$ runs $1, \dots, n$. Enumerate them (only those which are nonempty) as $B_1, \dots, B_m$. Then $\bigcup_{k=1}^m B_m = G$. Select $b_k \in B_k$ for each $k = 1, \dots, m$. Now given $c \in G$, take k_0 such that $c \in B_{k_0}$. Let $(x,y) \in G^2$. Take j_0 such that $y \in A_{j_0}$. Then

$$
\begin{aligned}
|f(xcy) - f(xb_{k_0}y)| &\leq |f(xcy) - f(xca_{j_0})| \\
&\quad + |f(xca_{j_0}) - f(xb_{k_0}a_{j_0})| + |f(xb_{k_0}a_{j_0}) - f(xb_{k_0}y)| \\
&\leq \|R_{cy}f - R_{ca_{j_0}}f\| + \|R_{ca_{j_0}}f - R_{b_{k_0}a_{j_0}}f\| + \|R_{b_{k_0}a_{j_0}}f - R_{b_{k_0}y}f\| \\
&\leq \|R_y f - R_{a_{j_0}}f\| + \|R_{ca_{j_0}}f - R_{b_{k_0}a_{j_0}}f\| + \|R_{a_{j_0}}f - R_y f\| < 4\epsilon
\end{aligned}
$$

because ca_{j_0} and $b_{k_0}a_{j_0}$ are in the same set $A_{l_{j_0}}$. □

Denote by $AP(G)$ the set of all AP-functions on G.

Proposition 3. *$AP(G)$ is a closed subalgebra of $BC(G)$. It contains constants. It is invariant under L_g and R_g, $g \in G$. If $f \in AP(G)$ then $Re(f)$, $Im(f)$ and $\overline{f}$ are in $AP(G)$.*

Marriage Lemma. *Let X, Y be two nonempty sets. Let F be a map from X to the set of all non-empty finite subsets of Y. Suppose that for each $X_1 \subset X$,*

$$\#\Big(\bigcup_{x \in X_1} F(x)\Big) \geq \#(X_1)$$

then there is a 1-to-1 map $s : X \to Y$ such that $s(x) \in F(x)$.

Proof. Suppose that X is finite. Apply an induction argument.

— If $\#(X) = 1$ then the claim is trivial.

— Let $\#(X) = n$. Fix $x_0 \in X$. Choose $y_0 \in F(x_0)$. If for each $X_1 \subset X \setminus \{x_0\}$,

$$\#(\bigcup_{x \in X_1} F(x)) > \#(X_1)$$

then we put $F'(x) := F(x) \setminus \{y_0\}$ and apply the inductive assumption: there is a 1-to-1 map $s : X \setminus \{x_0\} \to Y \setminus \{y_0\}$ with $s(x) \in F'(x)$, which we extend to x_0 as $s(x_0) := y_0$.

If there is $X_1 \subset X$, $\#(X_1) < n$ with $\#(\bigcup_{x \in X_1} F(x)) = \#(X_1)$, we put $Y_1 := \bigcup_{x \in X_1} F(x)$. Then by inductive assumption, there is a 1-to-one map $s : X_1 \to Y_1$ with $s(x) \in F(x) \subset Y_1$ for each $x \in X_1$. We now set $X_2 := X \setminus X_1$, $Y_2 := Y \setminus Y_1$, $F_2(x) := F(x) \setminus Y_1$. Let us verify that the conditions of the lemma are satisfied for X_2, Y_2, F_2. If not, there is a non-empty subset $X_3 \subset X_2$ with $\#(\bigcup_{x \in X_3} F_2(x)) < \#(X_3)$. However then

$$\#\left(\bigcup_{x \in X_3} F_2(x) \sqcup \bigcup_{x \in X_1} F(x)\right) < \#(X_3) + \#(X_1), \text{ i.e.}$$

$$\#\left(\bigcup_{x \in X_3 \sqcup X_1} F(x)\right) < \#(X_3 \sqcup X_1).$$

Therefore by inductive assumption, there is a 1-to-1 map $r : X_2 \to Y_2$ with $r(x) \in F_2(x)$. It remains to concatenate s and r.

Consider now the case of infinite X. The infinite product $Z := \prod_{x \in X} F(x)$ is a compact space. For each finite subset $S \subset X$, let Z_S be the subset of all $z \in Z$ such that $z \upharpoonright S$ is one-to-one. By the above argument, $Z_S \neq \emptyset$ and it is closed. Moreover, $Z_{S_1} \cap Z_{S_2} \cap \dots \cap Z_{S_n} \supset Z_{S_1 \cap \dots S_n}$. Hence these finite intersections are non-empty. Therefore

$$\bigcap_{S \text{ is a finite subset of X}} Z_S \neq \emptyset.$$

□

Lemma 4. *Let X be a mertic space and let $x_1, \dots, x_n$ be an ϵ-net of smallest possible cardinality (for this ϵ). Let Y be an ϵ-net in X. Then there is a 1-to-1 map $s : \{x_1, \dots, x_n\} \to Y$ with x_j and $s(x_j)$ being inside the same ϵ-ball for each j.*

Proof. Given $j \leq n$, let $F(x_j) := \{y \in Y \mid x_j \text{ and } y \text{ are inside the same } \epsilon\text{-ball}\}$. All that we need to prove is that $\#(\bigcup_{j \in I} F(x_j)) \geq \#(I)$ for each nonempty subset $I \subset \{1, \dots, n\}$. Indeed, assume that $\#(\bigcup_{j \in I} F(x_j)) < \#(I)$ for some I. We claim that the set $A := \bigcup_{i \in I} F(x_i) \cup \{x_j \mid j \notin I\}$ is a ϵ-net. Take $z \in X$. If z is ϵ-far from each x_j, $j \notin I$, then there is $j_0 \in I$ such that z and x_{j_0} are ϵ-close. On the other hand, there is $y \in Y$ which is ϵ-close to z. Hence $y \in \bigcup_{j \in I} F(x_j)$. Thus A is ϵ-net and hence $\#(A) \geq n$. □

Corollary 5. *Let $f \in AP(G)$, $\epsilon > 0$ and $D_{a_1} f, \dots, D_{a_n} f$ be an ϵ-net in $\{D_g f \mid g \in G\}$ of minimal cardinality among the ϵ-nets. Then*

$$\left\| \frac{1}{n} \sum_{j=1}^{n} f(a_j) - \frac{1}{n} \sum_{j=1}^{n} D_{a_j} f \right\| \leq 2\epsilon.$$

Proof. Let $D_{b_1}, \dots, D_{b_n}$ be another ϵ-net in $\{D_g f \mid g \in G\}$. Then there is a bijection σ of $\{1, \dots, n\}$ such that $\|D_{a_j} f - D_{b_{\sigma(j)}} f\| < 2\epsilon$. Therefore

$$\left\| \frac{1}{n} \sum_{j=1}^{n} D_{a_j} f - \frac{1}{n} \sum_{j=1}^{n} D_{b_j} f \right\| \le \frac{1}{n} \sum_{j=1}^{n} \left\| D_{a_j} f - D_{b_{\sigma(j)}} f \right\| < 2\epsilon. \tag{1}$$

Let $u, v \in G$. Then $D_{ua_1 v}, \dots, D_{ua_n v}$ is an ϵ-net in $\{D_g f \mid g \in G\}$. Indeed, given $g \in G$, we have $\|D_{u^{-1}gv^{-1}} f - D_{a_j} f\| < \epsilon$ for some j. Hence $\|D_g f - D_{ua_j v} f\| < \epsilon$. It now follows from (1) with $b_j := ua_j v$ that

$$2\epsilon > \left| \frac{1}{n} \sum_{j=1}^{n} D_{a_j} f(e,e) - \frac{1}{n} \sum_{j=1}^{n} D_{b_j} f(e,e) \right| = \left| \frac{1}{n} \sum_{j=1}^{n} f(a_j) - \frac{1}{n} \sum_{j=1}^{n} f(ua_j v) \right|.$$

It remains to take supremum over $u, v \in G$. $\square$

Theorem 6 (Existence of 2-sided means on the AP-functions). *There is a 2-sided invariant mean M on $AP(G)$. Moreover, M is strictly positive, i.e. if $f \ge 0$ and $f \ne 0$ then $M(f) > 0$.*

Proof. Take $f \in AP(G)$ and $\epsilon > 0$. We put

$$E_\epsilon := \left\{ z \in \mathbb{R} \mid \left\| z - \frac{1}{\#A} \sum_{a \in A} D_a f \right\| < \epsilon \text{ for a finite } A \subset G \right\}.$$

By Corollary 5, $E_\epsilon \ne \emptyset$. Let us verify that the diameter of E_ϵ is less than 2ϵ. Indeed, if $z_1, z_2 \in E_\epsilon$, then

$$|z_1 - \frac{1}{\#A} \sum_{a \in A} f(xay)| < \epsilon, \tag{2}$$

$$|z_2 - \frac{1}{\#B} \sum_{b \in B} f(xby)| < \epsilon \tag{3}$$

for all $x, y \in G$ and some finite subsets A, B in G. From (2) and (3) we deduce

$$|z_1 - \frac{1}{\#A \#B} \sum_{a \in A, b \in B} f(ab)| < \epsilon \quad \text{and}$$

$$|z_2 - \frac{1}{\#A \#B} \sum_{a \in A, b \in B} f(ab)| < \epsilon$$

respectively. Hence

$$|z_1 - z_2| \le 2\epsilon. \tag{4}$$

Moreover, E_ϵ is bounded. Indeed, $|z| < \|f\| + \epsilon$ for each $z \in E_\epsilon$. Since $E_{\epsilon_1} \cap \dots \cap E_{\epsilon_k} \supset E_{\min(\epsilon_1, \dots, \epsilon_k)}$, it follows that $\bigcap_{\epsilon > 0} \overline{E_\epsilon} \ne \emptyset$. From (4) we deduce that $\bigcap_{\epsilon > 0} \overline{E_\epsilon}$

is a singleton. Denote it by $M(f)$. Thus $M(f)$ is the only number such that for each $\epsilon > 0$, there is a finite $A \subset G$ with

$$\left\| M(f) - \frac{1}{\#A} \sum_{a \in A} D_a f \right\| < \epsilon.$$

It it easy to see that $f \mapsto M(f)$ is a 2-sided invariant functional, $M(\alpha f) = M(f)$ for each number α, M is non-negative and $M(1) = 1$. To show additivity, let $\|M(f) - \frac{1}{\#A} \sum_{a \in A} D_a f\| < \epsilon$ and $\|M(f_1) - \frac{1}{\#B} \sum_{b \in B} D_b f_1\| < \epsilon$ for some $f, f, \in AP(G)$ and $A, B \subset G$. Then we obtain easily that $\|M(f) - \frac{1}{\#A\#B} \sum_{a \in A, b \in B} D_{ab} f\| < \epsilon$ and $\|M(f_1) - \frac{1}{\#B\#A} \sum_{b \in B, a \in A} D_{ab} f_1\| < \epsilon$ and hence

$$\|M(f) + M(f_1) - \frac{1}{\#B\#A} \sum_{b \in B, a \in A} D_{ab}(f + f_1)\| < \epsilon.$$

Since ϵ is arbitrary, $M(f + f_1) = M(f) + M(f_1)$. It remains to prove that M is strictly positive. Let $f \in AP(G)$, $f \geq 0$ and $f(g_0) > 0$ for some $g_0 \in G$. Let $A \subset G$ be an $f(g_0)/2$-net in $\{D_g f \mid g \in G\}$. Then for $x, y, g \in G$, we have

$$\sum_{a \in A} f(xay) \geq \max_{a \in A} D_a f(x, y) > D_g f(x, y) - \frac{f(g_0)}{2} = f(xgy) - \frac{f(g_0)}{2}.$$

Therefore

$$\sum_{a \in A} f(xa) > \frac{f(g_0)}{2}$$

for each $x \in G$. Therefore $\sum_{a \in A} R_a(f) > f(g_0)/2$ and hence $\#(A)M(f) \geq f(g_0)/2$. $\square$

Theorem 7 (Uniqueness of RIM on $AP(G)$). *Let M' be a RIM on $AP(G)$. Then $M' = M$.*

Proof. Given $f \in AP(G)$ and $\epsilon > 0$, we have

$$-\epsilon < M(f) - \frac{1}{n} \sum_{a \in A} f(xay) < \epsilon$$

for all $x, y \in G$ and a finite subset $A \subset G$. Hence $-\epsilon < M(f) - \frac{1}{n} \sum_{a \in A} R_a(f) < \epsilon$. Therefore $-\epsilon \leq M(f) - M'(f) \leq \epsilon$. $\square$

Remark 8. If G is locally compact (or Polish) then every bounded function f with relatively compact $\{L_g(f) \mid g \in G\}$ is (uniformly) continuous. Indeed, for each $\epsilon > 0$, there are subsets $A_j \subset G$ and elements $a_j \in G$ with $\|L_g f - L_{a_j} f\| \leq \epsilon$ for all $g \in A_j$. It follows from our assumption on G that there is j_0 such that A_{j_0} has the Baire property. Then $\|L_b(f) - f\| \leq 2\epsilon$ for all $b \in A_{j_0}^{-1} A_{j_0}$. By the Pettis theorem, there is an open neighborhood U of e with $U \subset A_{j_0}^{-1} A_{j_0}$.

Literature.

1. Hewitt, Ross, Abstract harmonic analysis, vol 1.

4. Amenability for locally compact groups

Let G be a locally compact group. Fix a left Haar measure λ on G. Denote by $L^\infty(X)$ the space of essentially bounded functions. Recall that $L^\infty(G) = L^1(G)'$. It is a Banach space when endowed with the norm

$$\|f\|_\infty := \text{vraisup}|f| = \inf\left\{ \sup_{g\in G\setminus N} |f(g)| \mid N \text{ is a } \lambda\text{-locally 0 subset of } G \right\}.$$

We consider the following subspaces of $L^\infty(G)$:

— $BC(X)$,

— $BUC_l(G)$, the left uniformly continuous bounded functions on G, i.e.

$$\{f \in BC(G) \mid \text{ the map } G \ni g \mapsto L_g(f) \text{ is continuous}\}.$$

— $BUC_r(G)$, the right uniformly continuous bounded functions on G,

— $BUC(G)$, the 2-sided uniformly continuous bounded functions on G, i.e. $BUC_l(G) \cap BUC_r(G)$.

Lemma 1. *$BUC_l(G)$, $BUC_r(G)$, $BUC(G)$ are 2-sided invariant closed subspaces of $BC(G)$.*

In each of these spaces (and also in $L^\infty(G)$) one can define concepts of LIM, RIM and 2-sided invariant mean.

Definition 2. A locally compact group G is called amenable if there is a LIM (or, equivalently, RIM or, equivalently, 2-sided mean) on $L^\infty(G)$.

This extend the concept of amenability for the discrete groups. Indeed, any discrete group G is locally compact and $L^\infty(G) = B(G)$. Each compact group is amenable with LIM equals the normalized Haar integral.

Let $C_0(G)$ denote the space of continuous functions f on G with $\lim_{g\to\infty} f(g) = 0$. By $M(G)$ denote the dual space $C_0(G)'$ of bounded regular measures on G. It is a Banach space with the norm equal to the full variation. The convolution is continuous on $M(G)$:

$$\langle \mu * \nu, f\rangle := \int_{G\times G} f(g_1 g_2)\, d\mu(g_1) d\nu(g_2).$$

We note that $L^1(G)$ is embedded isometrically into $M(G)$ via the one-to-one map $\phi \mapsto \mu_\phi$, $d\mu_\phi(g) := \phi(g) d\mu(g)$.

Exercise 3. $L^1(G)$ is an 2-sided ideal in the algebra $(M(G), *)$. More precisely, if

$$\mu * \phi(g) := \int_G \phi(x^{-1}g)\, d\mu(x),$$

$$\phi * \mu(g) := \int_G \phi(gx^{-1})\Delta(x^{-1})\, d\mu(x))$$

then $\mu * \mu_\phi = \mu_{\mu*\phi}$ and $\mu_\phi * \mu = \mu_{\phi*\mu}$. Moreover, $\|\mu * \phi\|_1 \le \|\mu\| \cdot \|\phi\|_1$ and $\|\phi * \mu\|_1 \le \|\mu\| \cdot \|\phi\|_1$. In particular, $\mu_\phi * \mu_\psi = \mu_{\phi*\psi}$, where

$$\phi * \psi(g) = \int_G \phi(x)\psi(x^{-1}g)\, d\lambda(g).$$

Since $L^\infty(G) = L^1(G,\mu)'$, we use this duality to define the following convolution.

Definition 4. Given $\phi \in L^1(G)$ and $f \in L^\infty(G)$ we let

$$\phi * f(g) := \int_G \phi(y) f(yg) d\lambda(y), \qquad f * \phi(g) := \int_g f(gy)\phi(y) d\lambda(y).$$

We see that $\|\phi * f\|_\infty \leq \|\phi\|_1 \cdot \|f\|_\infty$ and $\|f * \phi\|_\infty \leq \|\phi\|_1 \cdot \|f\|_\infty$. Moreover,

$$\langle \phi * \psi, f\rangle = \langle \psi, \phi * f\rangle = \langle \phi, f * \psi\rangle.$$

Definition 5. A linear functional m on $L^\infty(G)$ is called a topological LIM on $L^\infty(G)$ if it is positive, normalized and $m(\phi * f) = m(f)$ for all

$$\phi \in \mathcal{P}(G) := \{\phi \in L^1(G) \mid \phi \geq 0, \|\phi\|_1 = 1\}.$$

A linear functional m on $L^\infty(G)$ is called a topological RIM on $L^\infty(G)$ if it is positive, normalized and $m(f * \phi) = m(f)$ for all $\phi \in \mathcal{P}(G)$.

In a similar way one can define topological LIM and RIM on $BC(G), BUC_r(G), BUC(G)$. $\blacksquare$

Lemma 6 (On regularization). *Let $f \in L^\infty(G)$ and $\phi \in \mathcal{P}(G)$. Then $\phi * f \in BUC_l(G)$, $f * \phi \in BUC_r(G)$. If $f_1 \in BUC_r(G)$ then $\phi * f_1 \in BUC(G)$. If $f_2 \in BUC_l(G)$ then $f * \phi \in BUC(G)$.*

Proof. We check only the first claim.

$$\begin{aligned}
|\phi * f(x) - \phi * f(yx)| &= |\int \phi(t) f(tx) d\lambda(t) - \int \phi(t) f(tyx) d\lambda(t)| \\
&\leq \int |\phi(tx^{-1}) - \phi(tx^{-1}y^{-1})\Delta(y)||f(t)|\Delta(x) d\lambda(t) \\
&\leq \|f\|_\infty \int |\phi(t) - \phi(ty^{-1})\Delta(y)| d\lambda(t) \\
&= \|f\|_\infty \|\phi - R_{y^{-1}}(\phi)\|_1
\end{aligned}$$

□

Recall that L_g and R_g denote the following isometries in $L^1(G)$. $L_g f(x) := f(g^{-1}x)$, $R_g f(x) := f(xg)\Delta(g^{-1})$.

Lemma 7 (On approximation in $L^1(G)$). *Let $f \in L^1(G)$.*

(1) *The maps $G \ni g \mapsto L_g(f) \in L^1(G)$ and $G \ni g \mapsto R_g(f) \in L^1(G)$ are continuous.*
(2) *For each $\epsilon > 0$, there is a neighborhood U of e such that for each non-negative $\psi \in L^1(G)$ vanishing out of U and such that $\int \psi(g) d\lambda(g) = 1$,*

$$\|\psi * f - f\|_1 < \epsilon \text{ and } \|f * \psi - f\|_1 < \epsilon.$$

Proof. (1) Approximate f with a function from $C_{00}(G)$ in $\|.\|_1$ norm.

(2)

$$\begin{aligned}
\|\psi * f - f\|_1 &\leq \int \psi(y) \cdot \int |f(yx) - f(x)| d\lambda(x) d\lambda(y) \\
&= \int \psi(y) \|L_{y^{-1}}(f) - f\|_1 d\lambda(y)
\end{aligned}$$

It remains to apply (1). In a similar way,

$$\begin{aligned}\|f * \psi - f\|_1 &\le \int \psi(y) \int |f(xy) - f(x)| d\lambda(x) d\lambda(y) \\ &= \int \psi(y) \int |R_y f(x)\Delta(y) - f(x)| d\lambda(x) d\lambda(y)\end{aligned}$$

□

Corollary 8. *There is an approximate unit $(e_\alpha)_{\alpha\in\mathcal{A}}$ in $L^1(G)$, i.e. $e_\alpha \in L^1(G)$ and $\lim_\alpha e_\alpha * f = \lim_\alpha f * e_\alpha = f$, where $\mathcal{A}$ is a directed set. Moreover, we will additionally assume that $e_\alpha \in \mathcal{P}$ and e_α is compactly supported.*

We need an analogue of Lemma 7(2) for $BUC(G)$.

Lemma 9. *Let $f \in BUC(G)$. Given $\epsilon > 0$, there is a neighborhood U of e with compact support such that for each $\psi \in L^1_+(G)$ vanishing out of U and such that $\int \psi\, d\lambda = 1$, $\|\psi * f - f\|_\infty < \epsilon$.*

Proof.

$$\begin{aligned}|\psi * f(x) - f(x)| &\le \int \psi(y) \cdot |f(yx) - f(x)| d\lambda(y) \\ &= \int \psi(y) \|L_{y^{-1}}(f) - f\|_\infty \, d\lambda(y) \\ &\le \sup_{y\in U} \|L_{y^{-1}}(f) - f\|_\infty\end{aligned}$$

□

Theorem 10. *If there is a LIM on $B(G)$ then there is a LIM on $BC(G)$. The converse is not true. The following are equivalent:*

(1) *there is a topological LIM on $L^\infty(G)$,*
(2) *there is a LIM on $L^\infty(G)$,*
(3) *there is a LIM on $BC(G)$,*
(4) *there is a LIM on $BUC_l(G)$,*
(5) *there is a LIM on $BUC(G)$.*

Proof. The first claim is obvious.

To show the second one, consider $O_3(\mathbb{R})$. It is amenable as a compact group but it is non-amenable as a discrete group since it contains F_2.

(1)⇒(2) Let m be a topological LIM on $L^\infty(G)$. Let us show that m is a LIM on $L^\infty(G)$. Take $g \in G$ and $\phi \in \mathcal{P}(G)$. For each $f \in L^\infty(G)$,

$$\begin{aligned}(\phi * L_g f)(x) &= \int \phi(t) f(gtx)\, d\lambda(t) \\ &= \int \phi(g^{-1}t) f(tx)\, d\lambda(u) = L_g(\phi) * f.\end{aligned}$$

Hence

$$m(L_g f) = m(\phi * L_g f) = m(L_g(\phi) * f) = m(f).$$

(2)⇒(3)⇒(4)⇒(5) are obvious.

(5)⇒(1) Let m be a LIM on $BUC(G)$. First of all we show that m is a topological LIM on $BUC(G)$. Take $f \in BUC(G)$, $\phi \in L^1(G)$ and $g \in G$. We set $J(\phi)(g) := \phi(g^{-1})\Delta_G(g)$. Of course, $J(\phi) \in L^1(G)$. We claim that $J(L_g(\phi)) * f = L_g(J(\phi) * f)$.

$$\begin{aligned} J(L_g(\phi)) * f(x) &= \int_G \phi(g^{-1}t^{-1}) f(tx) \Delta_G(t) d\lambda(t) \\ &= \int_G \phi(t_1^{-1}) f(t_1 g^{-1} x) \Delta_G(g^{-1}t_1) \Delta_G(g) d\lambda(t_1) \\ &= \int_G J(\phi)(t_1) f(t_1 g^{-1} x) d\lambda(t) \\ &= J(\phi) * f(g^{-1}x). \end{aligned}$$

Then $J(L_g(\phi)) * f$ and $J(\phi) * f$ are in $BUC(G)$ by Lemma 6 and

$$m(J(L_g(\phi)) * f) = m(J(\phi) * f).$$

Thus the map $\phi \mapsto m(J(\phi) * f)$ is a bounded linear right-invariant functional on $L^1(G)$. Hence (uniqueness of Haar measure) there is a constant c_f such that

$$m(J(\phi) * f) = c_f \int \phi \, d\lambda.$$

We note that $J(\phi) \in \mathcal{P}(G)$ if $\phi \in \mathcal{P}(G)$ and $\phi \mapsto J(\phi)$ is an affine isomorphism of $\mathcal{P}(G)$. Hence

$$m(\phi * f) = c_f$$

for each $\phi \in \mathcal{P}(G)$. Take an approximate unit $(e_\alpha)_{\alpha \in \mathcal{A}}$ in $L^1(G)$. Take $\phi \in \mathcal{P}(G)$. We have $e_\alpha * f \to f$ in $\|.\|_\infty$ by Lemma 9 and

$$m(\phi * f) = c_f = m(e_\alpha * f) \to m(f),$$

as desired.

Take a symmetric neighborhood U of e with compact closure. Let $\psi := 1_U/\lambda(U)$. Then $\psi \in \mathcal{P}(G)$ and $J(\psi) = \psi$. By Lemma 6, $\psi * f * \psi \in BUC(G)$ for each $f \in L^\infty(G)$. Hence

$$m'(f) := m(\psi * f * \psi)$$

is a well defined mean on $L^\infty(G)$, i.e. it is positive, normalized linear functional. We want to have for each $\phi \in \mathcal{P}(G)$ and $f \in L^\infty(G)$,

$$m(\psi * \phi * f * \psi) = m'(\phi * f) = m'(f) = m(\psi * f * \psi).$$

It will follow from the more general assertion (in view of Lemma 6):

$$m(\tau_1 * f') = m(\tau_2 * f')$$

for all $\tau_1, \tau_2 \in \mathcal{P}(G)$ and $f' \in BUC_r(G)$. To prove the latter, we note that $\tau_j * e_\alpha \to \tau_j$ in $L^1(G)$ and hence $\|\tau_j * e_\alpha * f' - \tau_j * f'\|_\infty \le \|\tau_j * e_\alpha - \tau_j\|_1 \|f'\|_\infty \to 0$.

$$m(\tau_j * f') = \lim_\alpha m(\tau_j * e_\alpha * f') = \lim_\alpha m(e_\alpha * f').$$

□

Theorem 11. *Let G be a locally compact group and N a closed subgroup of G.*

(1) *Let G and H be a locally compact groups and let $\pi : G \to H$ be a continuous onto homomorphism. If G is amenable then H is amenable.*
(2) *If H is a closed subgroup of a locally compact amenable group G then H is amenable.*
(3) *Let N be an amenable closed normal subgroup of G and let the quotient group G/N is also amenable. Then G is amenable.*
(4) *Let $G = \operatorname{inj\,lim}_{\alpha\in\mathcal{A}} G_\alpha$ for a directed set $\mathcal{A}$, every G_α is an amenable closed subgroup of G. Then G is also amenable.*

Proof. (1) Just note that $BC(H)$ is embedded into $BC(G)$.

(3) Let m_N be a LIM on $BC(N)$ and let $m_{G/N}$ be a LIM on $BC(G/N)$. Given $f \in BC(G)$, we let $f'(g) := m_N(L_g(f))$. Since $g \mapsto L_g(f)$ is a continuous map from G to $BC(G)$, $f' \in C(G)$. Of course, f' is bounded. Since $L_n(f') = f'$ for all $n \in N$, we conclude that f' is a continuous function on G/N. We now set $m(f) := m_{G/N}(f')$. Then m is a LIM on G.

(4) Let m_α be a LIM on $BC(G_\alpha)$. Then the m_α can be viewed as a map defined on $BC(G)$. It is a mean and it is G_α-invariant. Denote by M_α the set of all G_α-invariant LIMs on G. Then M_α is *-weakly compact subset of the unit ball in $BC(G)'$. It is non-empty. Given $\alpha, \beta \in \mathcal{A}$, there is $\gamma \in \mathcal{A}$ with $\gamma \succ \alpha$ and $\gamma \succ \beta$. Then $\emptyset \neq M_\gamma \subset M_\alpha \cap M_\beta$. Hence $\bigcap_{\gamma\in\mathcal{A}} M_\gamma \neq \emptyset$.

(2) Suppose that G is Polish. Then there is a Borel map $s : G/H \to G$ such that $Hs(Hg) = Hg$. Every g uniquely decomposes into $g = hs(Hg)$, where $h = gs(Hg)^{-1}$. Given $f \in CB(H)$, let $f'(g) := f(h)$. Then $f' \in B(G)$. Moreover, it is Borel. Hence $f' \in L^\infty(G)$. Moreover, the map $f \mapsto f'$ is linear and $\|f'\|_\infty \leq \|f\|_\infty$. It remains to put $m_H(f) := m(f')$. If m is LIM on $L^\infty(G)$ then m_H is a LIM on $BC(H)$. $\square$

Literature.

1. Greenleaf, Invariant means on topological groups and their applications

Let $M(G) \subset L^\infty(G)'$ denote the space of means on $L^\infty(G)$. Then $\mathcal{P}(G) \subset M(G)$.

Lemma 12. *$\mathcal{P}(G)$ is *-weakly dense in $M(G)$.*

Proof. We first note that $M(G)$ is a *-weakly compact subset of $L^\infty(G)'$. Let K be the closure of $\mathcal{P}(G)$ in $L^\infty(G)'$. Then K is compact. If there is $k_0 \in M(G) \setminus K$ then by Hahn-Banach separation theorem, there is $f \in L^\infty(G)$ and $\delta > 0$ such that $k_0(f) > \langle \phi, f\rangle + \delta$ for all $\phi \in \mathcal{P}(G)$. There is a compact set $C \subset G$ such that $\lambda(C) > 0$ and $f(g) \geq \|f\|_\infty - \delta/2$ for all $g \in C$. Then $\phi_0 := \lambda(C)^{-1} 1_C \in \mathcal{P}(G)$ and $k_0(f) > \int_C f\, d\lambda/\lambda(C) + \delta \geq \|f\|_\infty + \delta/2$, a contradiction. □

Theorem 13.

(1) *G is amenable if and only if there is a net $\phi_\alpha \in \mathcal{P}(G)$, $\alpha \in \mathcal{A}$, such that for each $g \in G$, $L_g(\phi_\alpha) - \phi_\alpha \to 0$ *-weakly in $L^\infty(G)'$.*

(2) *G is amenable if and only if there is a net $\phi_\alpha \in \mathcal{P}(G)$, $\alpha \in \mathcal{A}$, such that for each $\phi \in \mathcal{P}(G)$, $\phi * \phi_\alpha - \phi_\alpha \to 0$ *-weakly in $L^\infty(G)'$.*

Proof. We prove only (2).

($\Leftarrow$) There is a subnet ϕ_α converging to some $m \in M(G)$. Then

$$m(\phi * f) - m(f) = \lim_\alpha (\langle \phi_\alpha, \phi * f)\rangle - \langle \phi_\alpha, f\rangle) = \lim_\alpha \langle \phi * \phi_\alpha - \phi_\alpha, f\rangle = 0.$$

($\Rightarrow$) Take a topological LIM m on $L^\infty(G)$. By Lemma 12, there is a net $\phi_\alpha \in \mathcal{P}(G)$ with $\lim_\alpha \phi_\alpha = m$. Then

$$\langle \phi * \phi_\alpha - \phi_\alpha, f\rangle = \langle \phi_\alpha, \phi * f\rangle - \langle \phi_\alpha, f\rangle \to m(\phi * f) - m(f) = 0.$$

□

Theorem 14.

(1) *If there is a net $\phi_\alpha \in \mathcal{P}(G)$, $\alpha \in \mathcal{A}$ such that for each $g \in G$, $L_g(\phi_\alpha) - \phi_\alpha \to 0$ *-weakly in $L^\infty(G)'$ then there is a net $\psi_\beta \in \mathcal{P}(G)$, $\beta \in \mathcal{B}$, such that for each $g \in G$, $\|L_g(\phi_\beta) - \phi_\beta\|_1 \to 0$.*

(2) *If there is a net $\phi_\alpha \in \mathcal{P}(G)$, $\alpha \in \mathcal{A}$ such that for each $\phi \in \mathcal{P}(G)$, $\phi * \phi_\alpha - \phi_\alpha \to 0$ *-weakly in $L^\infty(G)'$ then there is a net $\psi_\beta \in \mathcal{P}(G)$, $\beta \in \mathcal{B}$, such that for each $\phi \in \mathcal{P}(G)$, $\|\phi * \psi_\beta - \psi_\beta\|_1 \to 0$.*

Proof. We prove only (2).

Let $E = \prod_{\phi \in \mathcal{P}(G)} L^1(G)$ with the product topology, each L^1 endowed with $\|.\|_1$ norm. Then E is a locally convex space and $E' = \bigoplus_{\phi \in \mathcal{P}(G)} L^\infty(G)$. Let $T : L^1(G) \to E$ be defined by $T\psi(\phi) = \phi * \psi - \psi$. Then T is a linear map and hence $T(\mathcal{P}(G))$ is a convex subset of E. For each $l \in E'$, finite subset $A \subset \mathcal{P}(G)$ and $l_\phi \in L^\infty(G)$, $\phi \in A$,

$$\langle l, T\phi_\alpha\rangle = \sum_{\phi \in A} \langle l_\phi, \phi * \phi_\alpha - \phi_\alpha\rangle \to 0.$$

Thus $T\phi_\alpha \to 0$ $*$-weakly. Hence 0 belongs to the $*$-weal closure of $T(\mathcal{P}(G))$. Since $T(\mathcal{P}(G))$ is convex, the weak closure equals the strong closure. Hence there is a net $\psi_\beta \in \mathcal{P}(G)$ such that $T\psi_\beta \to 0$ strongly in E. □

Definition 15 (Reiter condition). A locally compact group G satisfies Reiter condition if for each $\epsilon > 0$ and compact subset $K \subset G$, there is $\phi \in \mathcal{P}(G)$ such that

$$\sup_{g\in K} \|L_g(\phi) - \phi\|_1 < \epsilon. \tag{R}$$

Theorem 16. *G satisfies the Reiter condition if and only if G is amenable.*

Proof. ($\Rightarrow$) Let $J = \{(\epsilon, K)\}$ and select ϕ_j, $j \in J$, satisfying (R). Order J in a usual way. By Theorem 13(1), G is amenable.

($\Leftarrow$) By Theorems 13(2) + 14(2), there is a net $(\phi_\alpha)_{\alpha\in\mathcal{A}}$ in $\mathcal{P}(G)$ such that $\|\phi * \phi_\alpha - \phi_\alpha\|_1 \to 0$ for each $\phi \in \mathcal{P}(G)$. Fix $\epsilon > 0$, a compact K in G and $\beta \in \mathcal{P}(G)$. Take a small neighborhood U of e with compact closure such that

$$\|\phi_U * \beta - \beta\|_1 < \epsilon, \quad \text{and} \quad \sup_{g\in U} \|L_g(\beta) - \beta\|_1 < \epsilon,$$

where $\phi_U = 1_U/\lambda(U) \in \mathcal{P}(G)$. There are $g_1, \dots, g_N \in G$ with $\bigcup_{j=1}^N g_j U \supset K$. We may assume that $g_1 = e$. Select ϕ_α such that

$$\max_{1\leq j\leq N} \|L_{g_j}(\phi_U) * \phi_\alpha - \phi_\alpha\|_1 < \epsilon, \quad \|\beta * \phi_\alpha - \phi_\alpha\|_1 < \epsilon.$$

Let us show that $\phi := \beta * \phi_\alpha$ satisfies (R). Indeed,

$$\begin{aligned} \|\phi_U * \phi - L_g(\phi)\|_1 &\leq \|\phi_U * \phi - \phi\|_1 + \|\phi - L_g(\phi)\|_1 \\ &= \|(\phi_U * \beta - \beta) * \phi_\alpha\|_1 + \|(\beta - L_g(\beta)) * \phi_\alpha\|_1 < 2\epsilon \end{aligned}$$

for each $g \in U$. Hence

$$2\epsilon > \|L_{g_j}(\phi_U * \phi) - L_{g_j}(L_g(\phi))\|_1 = \|L_{g_j}(\phi_U) * \phi - L_{g_j g}(\phi)\|_1.$$

Now

$$\begin{aligned} \|L_{g_j g}(\phi) - \phi\|_1 \leq 2\epsilon + \|L_{g_j}(\phi_U) * \phi - \phi\|_1 &= 2\epsilon + \|L_{g_j}(\phi_U) * \beta * \phi_\alpha - \beta * \phi_\alpha\|_1 \\ &\leq 4\epsilon + \|L_{g_j}(\phi_U) * \phi_\alpha - \phi_\alpha\|_1 < 5\epsilon. \end{aligned}$$

□

Theorem 17. *A locally compact group G is amenable if and only if for each $\epsilon > 0$ and a compact set $K \subset G$, there is a Borel set F such that $0 \leq \lambda(F) < \infty$ and*

$$\frac{\lambda(gF \triangle F)}{\lambda(F)} < \epsilon.$$

Proof. ($\Leftarrow$) If $\phi := 1_F/\lambda(F)$ then $\phi \in \mathcal{P}(G)$ and $\|L_g(\phi) - \phi\|_1 = \frac{\lambda(gF\triangle F)}{\lambda(F)}$.

($\Rightarrow$) We first prove a weaker assertion.

(F) *Given $\epsilon > 0$ and $\delta > 0$ and compact set $K \subset G$, there are Borel subsets $F \subset G$ and $N \subset F$ such that $0 < \lambda(F) < \infty$, $\lambda(N) < \delta$ and $\frac{\lambda(gF\triangle F)}{\lambda(F)} < \epsilon$ for all $g \in K \setminus N$.*

From the Reiter condition, there is $\phi \in \mathcal{P}(G)$ with $\|L_g(\phi) - \phi\|_1 < \epsilon\delta/\lambda(K)$ for all $k \in K$. Without loss of generality we may assume that ϕ is a simple function. Moreover, we can represent ϕ as $\phi = \sum_{j=1}^N \lambda_j 1_{A_j}/\lambda(A_j)$ with $A_1 \subset A_2 \subset \cdots \subset A_N$ and $\lambda_j > 0$. Hence $\sum_j \lambda_j = 1$. Since $L_g(\phi) - \phi = \sum_j \lambda_j 1_{gA_j \setminus A_j}/\lambda(A_j) - \sum_j \lambda_j 1_{A_j \setminus gA_j}/\lambda(A_j)$ and $(gA_j \setminus A_j) \cap (A_i \setminus gA_i) = \emptyset$ then

$$\|L_g(\phi) - \phi\|_1 = \sum_{j=1}^N \lambda_j \frac{\lambda(gA_j \triangle A_j)}{\lambda(A_j)} < \epsilon\delta/\lambda(K).$$

Hence

$$\sum_{j=1}^N \lambda_j \int_K \frac{\lambda(gA_j \triangle A_j)}{\lambda(A_j)} d\lambda(g) < \epsilon\delta.$$

Hence there is j with

$$\int_K \frac{\lambda(gA_j \triangle A_j)}{\lambda(A_j)} d\lambda(g) < \epsilon\delta.$$

Hence $\frac{\lambda(gA_j \triangle A_j)}{\lambda(A_j)} > \epsilon$ on a subset N whose measure $\lambda(N) < \delta$. Thus (F) is proved.

Now we note that if $\frac{\lambda(gF \triangle F)}{\lambda(F)} < \epsilon$ for each $g \in K \setminus N$ then $\frac{\lambda(gF \triangle F)}{\lambda(F)} < 2\epsilon$ for each $g \in (K \setminus N)(K \setminus N)^{-1}$. Indeed,

$$\begin{aligned}\frac{\lambda(g_1 g_2^{-1} F \triangle F)}{\lambda(F)} = \frac{\lambda(g_2^{-1} F \triangle g_1^{-1} F)}{\lambda(F)} &\leq \frac{\lambda(g_1^{-1} F \triangle F)}{\lambda(F)} + \frac{\lambda(g_2^{-1} F \triangle F)}{\lambda(F)} \\ &= \frac{\lambda(F \triangle g_1 F)}{\lambda(F)} + \frac{\lambda(F \triangle g_2 F)}{\lambda(F)}.\end{aligned}$$

It remains to show

Lemma 18. *If K is a compact in G then for each Borel subset $N \subset K \cup KK$, if $\lambda(N) < \lambda(K)/2$ then $(K \setminus N)(K \setminus N)^{-1} \supset K$.*

Proof. Let $K_1 := K \cup KK$ then

$$kK \subset kK_1 \cap K_1 \subset (k(K_1 \setminus N) \cap (K_1 \setminus N)) \cup kN \cup N.$$

Hence

$$\lambda(K) \leq \lambda(kK_1 \cap K_1) \leq \lambda(k(K_1 \setminus N) \cap (K_1 \setminus N)) + 2\lambda(N).$$

It follows that $\lambda(k(K_1 \setminus N) \cap (K_1 \setminus N)) > 0$. In particular, $k(K_1 \setminus N) \cap (K_1 \setminus N) \neq \emptyset$, i.e. $k \in (K \setminus N)(K \setminus N)^{-1}$. □

Thus to prove the theorem, first given K, consider K_1 and put $\delta := \lambda(K)/2$. Apply (F). Then apply Lemma 18. □

Definition 19. A net of Borel subsets $(F_\alpha)_{\alpha \in \mathcal{A}}$ of finite measure in a locally compact group is called a (left) Følner net if

$$\lim_{\alpha \in \mathcal{A}} \frac{\lambda(gF_\alpha \triangle F_\alpha)}{\lambda(F_\alpha)} = 0$$

for each $g \in G$.

Corollary 20. *Let G be a locally compact group. Then G is amenable if and only if there is a Følner net in it. If G is σ-compact then it is amenable if and only if tree is a Følner sequence it.*

Proof. Let $\mathcal{A}$ be the directed set $\{(K, \epsilon) \mid K \text{ is a compact in } G,\ \epsilon > 0\}$. For each $\alpha \in \mathcal{A}$, let F_α be the corresponding set in G such that $\frac{\lambda(gF_\alpha \triangle F_\alpha)}{\lambda(F_\alpha)} < \epsilon$ for each $g \in K$ (by Theorem 17). Then $(F_\alpha)_{\alpha \in \mathcal{A}}$ is a Følner net in G. Conversely, if $(F_\alpha)_{\alpha \in \mathcal{A}}$ is a Følner net in G then $\|L_g(1_{F_\alpha}/\lambda(F_\alpha)) - 1_{F_\alpha}/\lambda(F_\alpha)\|_1 \to 0$. It remains to apply Theorem 13(1) because the norm convergence implies the *-weak convergence. □

It follows from Theorem 17 that if G is amenable and σ-compact then there is a Følner sequence in it (the converse is also true). For example, if $G = \mathbb{Z}$ then $F_n := \{1, \ldots, n\}$ is a Følner sequence. If $G = \mathbb{R}$ then $F_n := [0, n]$ is a Følner sequence.

Corollary 21 (The invariant mean on the almost periodic functions on amenable groups). *Let $(F_\alpha)_\alpha$ be a Følner net in G. Let M be the (unique!) invariant mean on $AP(G)$. Then*

$$M(f) = \lim_\alpha \frac{1}{\lambda(F_\alpha)} \int_{F_\alpha} f(gx)\, d\lambda(g).$$

Proof. Consider a net $1_{F_\alpha}/\lambda(F_\alpha) \in \mathcal{P}(G)$. Take a limit point m of this net. Then m ia a LIM m on $L^\infty(G)$. We note that m is a LIM on $AP(G) \subset BUC(G) \subset L^\infty(G)$. Since the LIM on $AP(G)$ is unique, we conclude that $m \restriction AP(G) = M$. Thus, though there can be many limit points m, there restrictions to the $AP(G)$ is unique. Therefore there is $\lim_\alpha \langle 1_{F_n}/\lambda(F_n), f\rangle = M(f)$ if $f \in AP(G)$. □

Let V be a locally convex space and K a compact convex subset of V.

Definition 22. A continuous map $T : G \times K \ni (g, v) \mapsto T_g v \in K$ is called an affine action of G on K if T_g is an affine map of K and $T_g T_h = T_{gh}$ for all $g, h \in G$.

Theorem 23 (Markov-Kakutani fixed point theorem). *Let G be a locally compact group. It is amenable if and only if each affine action of G has a fixed point.*

Proof. ($\Leftarrow$) Let $\mathcal{M}$ be the set of means on $BUC_l(G)$. Then $\mathcal{M}$ is $*$-compact convex subset of $BUC_l(G)'$. We let $\langle T_g m, f\rangle := \langle m, L_{g^{-1}}(f)\rangle$. Then $(T_g)_{g \in G}$ is an affine action of G on $\mathcal{M}$. If $g_j \to g$ in G then $\|L_g(f) - L_{g_j}(f)\|_\infty \to 0$ because $f \in BUC_l(G)$. It is easy to check that T is continuous in 2 variables. Hence there is a fixed point for T, which is a LIM on $BUC(G)$.

($\Rightarrow$) Let T be an affine action of G on a compact convex set K. Since G is amenable, there is a Følner net $(F_\alpha)_{\alpha \in \mathcal{A}}$ in it. Take $v \in K$ and set $v_\alpha := \lambda(F_\alpha)^{-1} \int_{F_\alpha} T_g v\, d\lambda(g)$. Then $v_\alpha \in K$. Take a limit point $w \in K$ of $(v_\alpha)_\alpha$. Take a seminorm p on V. Then

$$p(T_h x_\alpha - x_\alpha) \le \frac{1}{\lambda(F_\alpha)} \int_{F_\alpha \triangle hF_\alpha} p(T_g v)\, d\lambda(g) \le \frac{\lambda(F_\alpha \triangle hF_\alpha) \max_K p(k)}{\lambda(F_\alpha)}.$$

Thus $p(T_h w - w) = 0$ for each $h \in G$ and each seminorm p defining the topology on V. □

Corollary 24 (Generalized Bogolubov-Krylov theorem). *Let G be a locally compact group. It is amenable if and only if there is an invariant probability Radon measure for each continuous action of G on a compact (Hausdorff) space.*

Proof. Let G be amenable. Let T be a continuous action of G on a compact space K. Define an action $T^*$of G on the space $C(K)'$ (endowed with the $*$-weak topology) by setting $\langle T_g^* F, f\rangle = \langle F, f \circ T_g\rangle$. It is continuous. Restrict T^* to the subset $\mathcal{P} \subset C(K)'$ of probability measures. We note that $\mathcal{P}$ is $*$-weakly compact and convex. Hence there is a fixed point in it.

Conversely, consider the space $\mathcal{M} \subset L^\infty(G)'$ of all mean on $L^\infty(G)$. It is $*$-weakly compact and convex. Of course, $(g, m) \mapsto m \circ L_g$ is a continuous action of G on $\mathcal{M}$. Hence there is a fixed point, which is an LIM on $L^\infty(G)$. □

Theorem 25 (Von Neumann mean ergodic theorem). *Let G be an amenable group and $(F_\alpha))\alpha \in \mathcal{A}$ a Føliner net in it. Let $G \ni g \mapsto U_g \in \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation of G in a Hilbert space $\mathcal{H}$. Then*

$$\lim_\alpha \frac{1}{\lambda(F_\alpha)} \int_{F_\alpha} U_g \, d\lambda(g) = P$$

strongly, where P is the orthogonal projection to the space of $(U_g)_{g \in G}$-invariant vectors.

Proof. Let $\mathcal{W}$ be the linear span of the space $\{U_h v - v \mid h \in G, v \in \mathcal{H}\}$. Let $\mathcal{I}$ be the subspace of U-invariant vectors. Put $A_\alpha := \frac{1}{\lambda(F_\alpha)} \int_{F_\alpha} U_g d\lambda(g)$.

— $A_\alpha w \to 0$ for each $w \in \mathcal{W}$. Hence $A_\alpha w \to 0$ for each $w \in \overline{\mathcal{W}}$.
— $A_\alpha v = v$ for each $v \in \mathcal{I}$.
— $\mathcal{W} \perp \mathcal{I}$.
— $\mathcal{W}^\perp \subset \mathcal{I}$.

Hence $\mathcal{H} = \overline{\mathcal{W}} \oplus \mathcal{I}$. Therefore verify the von Neumann theorem separately on vectors from $\overline{\mathcal{W}}$ and from $\mathcal{I}$. □

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