Introduction to Complex Dynamical Systems

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Abstract

An introductory course of lectures is presented on the theory of complex dynamical systems. The main attention is paid to the case of holomorphic mappings of the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The course starts by presenting detailed information on the simplest holomorphic mappings $\Delta \to \mathbb{C}$, which includes the group of homographic maps. Then a description of the theory of holomorphic self-mappings of Δ based on the Schwarz Lemma is given. It is followed by a detailed discussion of the fixed points of such self-mappings, based on the Denjoy-Wolff theory. Thereafter, a presentation of the most important results in the theory of continuous-time semigroups of holomorphic self-mappings of Δ is given. It includes the question of the generation of such semigroups and a continuous version of the Denjoy-Wolff theory.

1 Introduction

The theory of complex dynamical systems has appeared as an interdisciplinary branch of mathematics at the edge of such classical areas as Complex Analysis, Iteration Theory, Probability Theory, Functional Analysis. Richness of its topics originated from the transparency of geometric ideas and depth of analytic methods forming a peculiarity of this branch. The applications of the results, as well as the sources of new ideas and problems, extend to such domains as:

- stochastic Markov processes based on Kolmogorov's forward and backward equations, including branching processes;
- stochastic differential equations, especially Loewner equations and SLE problem;
- geometric function theory and hyperbolic geometry;
- invariant subspaces of linear operators on Krein and Pontryagin spaces;
- composition operators on Hardy and Bergman spaces;

- Banach algebras and complete vector fields.

For a holomorphic self-mapping F of a domain $\mathcal{D} \subset \mathbb{C}$, iterates are defined as compositions, i.e., $F^n := F \circ F^{n-1}$. Each iteration procedure generates a discrete dynamical system, which can be considered as a semigroup with respect to a discrete time parameter. Studying the behavior of such semigroups in a domain of the complex plane $\mathbb{C}$ goes back to classical works of Julia, Fatou, Denjoy and Wolff. Although the theory of discrete semigroups, and hence discrete dynamical systems, was developed extensively, little was known about semigroups with respect to a continuous time parameter. At the same time, continuous semigroups appear in all domains of mathematics mentioned above, where they constitute an important part of methods applied therein. In 1978, Berkson and Porta showed that each continuous semigroup $\{F_t\}_{t\geq 0}$ of holomorphic mappings $F_t : \Delta \to \Delta$, $\Delta := \{z \in \mathbb{C} : |z| < 1\}$, is everywhere differentiable with respect to t. Hence the limit

$$f(z) := \lim_{t \downarrow 0} \frac{z - F_t(z)}{t} \tag{1.1}$$

exists and defines a holomorphic function f on Δ. This is the *infinitesimal* generator of the semigroup $\{F_t\}_{t\geq 0}$. In the same work, Berkson and Porta showed that a holomorphic function f on Δ is the generator of a continuous semigroup of holomorphic mappings $F_t : \Delta \to \Delta$ if and only if it admits the following representation

$$f(z) = (z - \tau)(1 - z\bar{\tau})p(z), \qquad z \in \Delta, \tag{1.2}$$

where τ is a point in the closed unit disc $\overline{\Delta}$, and p is a holomorphic function on Δ such that $\mathrm{Re}p(z) \geq 0$ for all $z \in \Delta$. The representation in (1.2) is unique. Since $f(\tau) = 0$, by the semigroup property it follows that $F_t(\tau) = \tau$ for all $t \geq 0$. That is τ is a *fixed point* of the semigroup. It can be attractive or repelling, which means that $F_t(z) \to \tau$ as $t =\to +\infty$ (attractive), or $F_t(z)$ never approaches τ (repelling). Such properties of fixed point of $\{F_t\}_{t\geq 0}$ are very important in applications.

The proposed lectures will give an introductory description of the development of the theory of such semigroups and its possible applications in the domains mentioned above.

2 Linear Fractional Transformations

In these lectures, by $\mathbb{C}$ we denote the field of complex numbers, and $\Delta \subset \mathbb{C}$ will always stand for the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. A linear fractional (homographic) transformation $T : \mathbb{C} \to \mathbb{C}$ is

$$T(z) = \frac{az + b}{cz + d}, \tag{2.1}$$

where $a, b, c, d \in \mathbb{C}$ are its parameters. It is naturally to assume that c and d cannot be zero simultaneously. If $ad - bc = 0$, then T is a constant function. Otherwise, it is meromorphic with pole at $z = -d/c$ for $c \neq 0$, and it is an affine function $a'z + b'$ for $c = 0$. The map (2.1) can be represented by a 2×2 matrix

$$A_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.2}$$

In the sequel, we suppose that

$$\det A_T = ad - bc \neq 0. \tag{2.3}$$

The representation as in (2.2) implies that each T obeying (2.3) is invertible, that is, the equation

$$w = T(z) \tag{2.4}$$

has a unique solution

$$z = T^{-1}(w), \qquad T^{-1}(w) = \frac{\alpha w + \beta}{\gamma w + \delta}, \tag{2.5}$$

where the matrix

$$A_{T^{-1}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2.6}$$

is inverse to the one in (2.2). Moreover, the composition $T_1 \circ T_2$ of two transformations (2.1) is again of this type with matrix

$$A_{T_1 \circ T_2} = A_{T_1} \cdot A_{T_2}. \tag{2.7}$$

That is, all transformations (2.1) obeying (2.3) constitute a group $\mathfrak{T}$ such that the map

$$\mathfrak{M}_2(\mathbb{C}) \ni A \mapsto T \in \mathfrak{T}, \tag{2.8}$$

defined as in Eqs. (2.1) and (2.2), is a group homomorphism. Here $\mathfrak{M}_2(\mathbb{C})$ is the group of all nonsingular 2×2 matrices over the field $\mathbb{C}$. The unit element of $\mathfrak{T}$ is the identity map $T(z) = z$. Hence, the kernel of the homomorphism in (2.8) is the set $\mathfrak{I} := \{\alpha I : \alpha \in \mathbb{C}\}$, where I is the unit 2×2 matrix. The quotient group $\mathfrak{M}_2(\mathbb{C})/\mathfrak{I}$ is called the *projective linear group* and is usually denoted $\mathrm{PGL}(2, \mathbb{C})$.

As noted above, for $c \neq 0$, the transformation (2.1) is meromorphic. It can however be turned into a holomorphic function mapping the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ into itself by setting:

$$T(-d/c) = \infty, \qquad T(\infty) = a/c. \tag{2.9}$$

The multiplication in (2.7) can naturally be extended to such 'extended' mappings T, and hence the group $\mathfrak{T}$ can be considered as that comprising such 'extended' mappings $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. It is called the *Möbius group*. It turns out, that this is exactly the group of all *automorphisms* of $\widehat{\mathbb{C}}$, cf. Definition 3.3 below.

Example 2.1 *In (a) below $T(z)$ is a linear fractional transformation, whereas the one in (b) is not.*

$$(a) \quad T(z) = \frac{z^2+1}{z^2-3\mathrm{i}z-2} = \frac{z+\mathrm{i}}{z-2\mathrm{i}}, \qquad (b) \quad T(z) = \frac{2z^2+1}{3z^2+1}.$$

The following statement can be proven.

Theorem 2.2 *Let T be a linear fractional transformation. Then T maps circles or straight lines to circles or straight lines. Moreover, if D is a domain, then T maps its boundary ∂D onto $\partial T(D)$.*

So, in order to find the image $T(D)$ of a linear fractional transformation T one need to:

- find the image of ∂D;
- find from which side of $\partial T(D)$ lies the image $T(D)$.

By definition, a *fixed point* of T is a solution of the equation $T(z) = z$. By (2.1) it has the form

$$\frac{az+b}{cz+d} = z. \tag{2.10}$$

For $c \neq 0$, it is a quadratic equation, which has two, possibly coinciding, solutions

$$\xi_{1,2} = \frac{(a-d) \pm \sqrt{(a-d)^2+4bc}}{2c}. \tag{2.11}$$

Therefore, if there exist three different fixed points of a linear fractional transformation T then $T(z) \equiv z$.

3 Self-mappings of the open unit disk

3.1 Definitions and general properties

Definition 3.1 *Given a domain $\mathcal{D}$, a function $f : \mathcal{D} \to \mathbb{C}$ is said to be a self- mapping of $\mathcal{D}$ if $f(\mathcal{D}) \subseteq \mathcal{D}$.*

Let us consider the following example: $\mathcal{D} = \Delta$ and $f(z) = z/(2-z)$. Check that f is a self-mapping of Δ. Indeed, the inverse of f is $g(w) = 2w/(1+w)$. Hence, $|z| < 1$ readily implies

$$\left| w - \frac{1}{3} \right| < \frac{2}{3},$$

and therefore f maps Δ in $\{w : |w - 1/3| < 2/3\} \subset \Delta$. The fundamental role in the theory of self-mappings of Δ is played by the following statement.

Theorem 3.2 (The Schwarz Lemma) *Let f be a holomorphic mapping $f : \Delta \to \mathbb{C}$, which enjoys the following properties: (a) $f(0) = 0$; (b) $|f(z)| \leq 1$ for all $z \in \Delta$. Then: (i) $|f'(0)| \leq 1$; (ii) $|f(z)| \leq |z|$. Moreover, if the equality in (i) holds for at least one point $z_0 \in \Delta \setminus \{0\}$, or if the equality in (ii) holds, then $f(z) = \exp(\mathrm{i}\theta)z$ for some constant $\theta \in \theta$. That is, the equality in (i) holds for all $z \in \Delta$.*

A particular case of the homographic map (2.1) is called the *Möbius transformation.* It is

$$M_a(z) = \frac{a - z}{1 - \bar{a}z}, \qquad a \in \Delta. \tag{3.1}$$

For each $a \in \Delta$, the transformation in (3.1) has the following properties:

1. M_a is holomorphic on Δ;
2. M_a is a bijective self-mapping of Δ;
3. the inverse M_a^{-1} is also a Möbius transformation, with the same a, i.e., $M_a^{-1} = M_a$.

The latter property means that M_a is an involution, that is, $M_a \circ M_a = \mathrm{Id}$.

Definition 3.3 *For a domain $\mathcal{D}$, a function $F : \mathcal{D} \to \mathbb{C}$ is said to be an automorphism of $\mathcal{D}$ if F is a holomorphic bijection $F : \mathcal{D} \to \mathcal{D}$.*

Clearly, if F is an automorphism of a domain $\mathcal{D}$, then its inverse F^{-1} also has this property; if F and G are automorphisms of $\mathcal{D}$, then their composition $F \circ G$ is also an automorphism. That is, all automorphisms of a domain constitute a group with respect to composition. We shall denote it $\mathrm{Aut}(\mathcal{D})$. In principle, one can also consider $\widehat{\mathbb{C}}$ as a domain $\mathcal{D}$. In this case, as was mentioned above the group $\mathrm{Aut}(\widehat{\mathbb{C}})$ is exactly the group $\mathfrak{T}$ of homographic maps obeying (2.3) and extended as in (2.9).

As mentioned above, $M_a \in \mathrm{Aut}(\Delta)$. The following statement gives a general form of the automorphisms of the unit disk Δ.

Theorem 3.4 *For each $F \in \mathrm{Aut}(\Delta)$, there exist $a \in \Delta$ and $\theta \in \mathbb{R}$ such that*

$$F(z) = e^{\mathrm{i}\theta} M_a(z). \tag{3.2}$$

That is, each $F \in \mathrm{Aut}(\Delta)$ is the composition of a Möbius transformation and a rotation.

Classification 3.5 *The following classification of the elements of $\mathrm{Aut}(\Delta)$ is used. Let ξ_1 and ξ_2 be fixed points of a given $F \in \mathrm{Aut}(\Delta)$. Then F is*

- **elliptic** *if $|\xi_1| < 1$ and $\xi_2| > 1$;*
- **parabolic** *if $\xi_1 = \xi_2$ and $|\xi_1| = 1$;*

- **hyperbolic** *if* $\xi_1 \neq \xi_2$ *and* $|\xi_1| = |\xi_2| = 1$.

Let us give some examples.

Example 3.6 *An elliptic automorphism of* Δ*:*

$$F(z) = e^{\mathrm{i}\pi/2} \frac{z - \frac{\mathrm{i}}{2}}{1 + \frac{z\mathrm{i}}{2}};$$

A parabolic automorphism of Δ*:*

$$F(z) = \mathrm{i} \frac{z - \frac{\sqrt{2}}{2}}{1 - z\frac{\sqrt{2}}{2}};$$

A hyperbolic automorphism of Δ*:*

$$F(z) = \frac{z - \frac{1}{2}}{1 - z\frac{1}{2}}.$$

The Möbius transformation (3.1) can be used to describe the set of all holomorphic self-mappings of Δ which we denote by $\mathrm{Hol}(\Delta)$.

Theorem 3.7 (Schwarz-Pick Lemma) *For every* $F \in \mathrm{Hol}(\Delta)$ *and each pair of points* $z, w \in \Delta$*, the following inequalities holds*

$$(a) \quad \left| \frac{F(w) - F(z)}{1 - F(z)\overline{F(w)}} \right| \leq \left| \frac{z - w}{1 - z\overline{w}} \right| \tag{3.3}$$

$$(b) \quad \frac{|F'(z)|}{1 - |F(z)|^2} \leq \frac{1}{1 - |z|^2}, \qquad z \in \Delta.$$

Moreover, equalities in (a) and (b) hold if and only if $F \in \mathrm{Aut}(\Delta)$.

The following important fact follows directly from (b) in (3.3).

Corollary 3.8 *Let* $F \neq \mathrm{const}$ *be holomorphic on* Δ *with values not necessarily in* Δ*. Then Then* $F \in \mathrm{Hol}(\Delta)$ *if and only if, for each* $z \in \Delta$*,*

$$|F'(z)|^2 \leq \frac{1 - |F(z)|^2}{1 - |z|^2}.$$

An important property of $\mathrm{Hol}(\Delta)$ is established by the following classical statement.

Theorem 3.9 (Julia Lemma) *Let F be in* $\mathrm{Hol}(\Delta)$ *and a unimodular, i.e.,* $|a| = 1$*. Suppose that a sequence* $a_n \in \Delta$ *be such that* $a_n \to a$*,* $n \to \infty$*, and the following limits exist*

$$\alpha := \lim_{n\to\infty} \frac{1 - |F(a_n)|}{1 - |a_n|}, \qquad b := \lim_{n\to\infty} F(a_n). \tag{3.4}$$

Then, for each $z \in \Delta$*, the following holds:*

$$\frac{|1 - \bar{b}F(z)|^2}{1 - |F(z)|^2} \leq \alpha \frac{|1 - \bar{a}z|^2}{1 - |z|^2}. \tag{3.5}$$

3.2 Fixed points

First we study the fixed points of the members of $\mathrm{Aut}(\Delta) \subset \mathrm{Hol}(\Delta)$.

3.2.1 Fixed points of $F \in \mathrm{Aut}(\Delta)$

As follows from Theorem 3.4, each $F \in \mathrm{Aut}(\Delta)$ is a composition of a rotation r_θ and a Möbius transformation (3.1). By this representation we readily conclude that the equation

$$F(z) = z$$

can have the following solutions, cf. (2.11):

1. a single solution $\xi \in \Delta$;
2. no solution in Δ and a single solution $\xi \in \partial\Delta$;
3. two solutions $\xi_1 \neq \xi_2$, both in $\partial\Delta$.

According to Classification 3.5, case 1 corresponds to an elliptic F, case 2 to a parabolic F, and case 3 to a hyperbolic $F \in \mathrm{Aut}(\Delta)$. By the Schwarz Lemma, see Theorem 3.2, an elliptic $F \in \mathrm{Aut}(\Delta)$ can be written

$$F = M_a \circ r_\theta \circ M_{-a}, \tag{3.6}$$

for some $a \in \Delta$ and $\theta \in [0, 2\pi)$. Here r_θ is the rotation around zero, which acts as $r_\theta(z) = e^{\mathrm{i}\theta} z$. Thus, F can be interpreted as the 'rotation' around a. The remaining two cases can be described by the following alternative: *if $F \in \mathrm{Aut}(\Delta)$ has a fixed point at the boundary of Δ, then either it is unique or the second fixed point lies also at $\partial\Delta$.* Assume now that $F \in \mathrm{Aut}(\Delta)$ has two distinct fixed points $\xi_1, \xi_2 \in \partial\Delta$, and let a be the same as in (3.6). Set

$$\lambda = \frac{1 - \bar{a} \cdot \xi_2}{1 - \bar{a} \cdot \xi_1} = \frac{\overline{\xi_2} - \bar{a}}{\overline{\xi_1} - \bar{a}} \cdot \frac{\xi_2}{\xi_1}. \tag{3.7}$$

Since $F(\xi_i) = \xi_i$, $i = 1, 2$, we have that

$$1 - \bar{a} \cdot \xi_i = e^{\mathrm{i}\theta} (\xi_i - a)/\xi_i, \qquad i = 1, 2,$$

and hence

$$\lambda = \frac{\overline{\xi_2} - \bar{a}}{\overline{\xi_1} - \bar{a}} \cdot \frac{\xi_2}{\xi_1} = \frac{\xi_2 - a}{\xi_1 - a} \cdot \frac{\overline{\xi_2}}{\overline{\xi_1}} = \bar{\lambda},$$

that is, λ is real. It can be shown that $\lambda > 0$ and $\lambda \neq 1$. The latter holds since $a \neq 0$ and $\xi_1 \neq \xi_2$. Consider

$$L(z) = \frac{z - \xi_1}{z - \xi_2}, \tag{3.8}$$

which immediately yields

$$(L \circ F)(z) = \lambda L(z),$$

and hence

$$F(z) = (L^{-1} \circ \lambda \cdot L)(z). \tag{3.9}$$

Let F^n, $n \in \mathbb{N}$ stand for the corresponding iterate of F, i.e., $F^n := F \circ F^{n-1}$. Then by (3.9) we get

$$F^n(z) = (L^{-1} \circ \lambda^n \cdot L)(z). \tag{3.10}$$

Thus, by (3.10) we have that, as $n \to +\infty$, the following holds

$$\text{if} \quad \lambda < 1, \qquad \text{then} \quad F^n(z) \to L^{-1}(0) = \xi_1,$$

$$\text{if} \quad \lambda > 1, \qquad \text{then} \quad F^n(z) \to L^{-1}(0) = \xi_2.$$

Thereby we have come to the following

Conclusion 3.10 *If $F \in \text{Aut}(\Delta)$ is hyperbolic, and thus has two distinct fixed points $\xi_1, \xi_2 \in \partial\Delta$, then the sequence of iterates $\{F^n\}_{n\geq 1}$ converges uniformly on compact subsets of Δ to either ξ_1 or ξ_2.*

Moreover, the rate of the mentioned convergence can be estimated. Set $\zeta = \xi_1$ and $\varepsilon = \lambda$ if $\lambda < 1$, and $\zeta = \xi_2$ and $\varepsilon = \lambda^{-1}$ if $\lambda > 1$. Then

$$|F^n(z) - \zeta| \leq \frac{4\varepsilon^n}{1 - |z|}. \tag{3.11}$$

Let us consider now the parabolic case, where F has exactly one fixed point $\xi \in \partial\Delta$. In this case ξ is a double root of the equation $F(z) = z$, and

$$2\xi = (1 - e^{\mathrm{i}\theta})/\overline{a}.$$

Since $|\xi| = 1$ and $|a| < 1$, it follows that $e^{\mathrm{i}\theta} \neq -1$, and hence one can divide by $(1 + e^{\mathrm{i}\theta})$. By direct calculation, one verifies that

$$\frac{\xi}{F(z) - \xi} = \frac{e^{\mathrm{i}\theta} - 1}{e^{\mathrm{i}\theta} + 1} + \frac{\xi}{z - \xi},$$

and hence

$$\frac{\xi}{F^n(z) - \xi} = \frac{e^{\mathrm{i}\theta} - 1}{e^{\mathrm{i}\theta} + 1} + \frac{\xi}{F^{n-1}(z) - \xi} = \cdots = n\frac{e^{\mathrm{i}\theta} - 1}{e^{\mathrm{i}\theta} + 1} + \frac{\xi}{z - \xi}, \tag{3.12}$$

which holds for all $n \in \mathbb{N}$ and $z \in \Delta$. From (3.12) we then get

$$\frac{1}{|F^n(z) - \xi|} \geq n\left|\frac{e^{\mathrm{i}\theta} - 1}{e^{\mathrm{i}\theta} + 1}\right| - \frac{1}{|z - \zeta|},$$

which yields

$$F^n(z) \to \zeta, \qquad n \to +\infty. \tag{3.13}$$

The above arguments lead to the following

Conclusion 3.11 *If $F \in \mathrm{Aut}(\Delta)$ is not an elliptic automorphism, then the sequence $\{F^n\}_{n\geq 1}$ is convergent uniformly on compact subsets of Δ. If F is not the identity map, then the limit is a $\xi \in \partial\Delta$, which is a fixed point of F.*

By results of Denjoy and Wolff, it is possible to show that the above result extends also to all $F \in \mathrm{Hol}(\Delta)$.

3.2.2 Fixed points of $F \in \mathrm{Hol}(\Delta)$

For $a \in \Delta$ and $r \in (0,1)$, we set

$$s = \frac{1-r^2}{1-r^2|a|^2}, \qquad t = \frac{1-|a|^2}{1-r^2|a|^2}, \tag{3.14}$$

and then

$$\Omega_r(a) := \{z \in \Delta : |z - sa| < rt\}. \tag{3.15}$$

By the Schwarz-Pick Lemma, cf. Theorem 3.7, one can prove the following

Proposition 3.12 *For each $F \in \mathrm{Hol}(\Delta)$, $a \in \Delta$, $r \in (0,1)$ and s and t as in (3.14), the following holds*

$$F(\Omega_r(a)) \subset \Omega_r(F(a)), \tag{3.16}$$

that is, F maps the disk $\Omega_r(a)$ into itself.

The latter result immediately yields

$$F^n(\Omega_r(a)) \subset \Omega_r(F(a)), \qquad n \in \mathbb{N},$$

which poses a natural question as to the convergence of $F^n(z)$, as $n \to +\infty$, to the fixed point of F. The answer is affirmative, but not for elliptic $F \in \mathrm{Hol}(\Delta)$.

Theorem 3.13 *Suppose that $F \in \mathrm{Hol}(\Delta)$ is not elliptic, and $a \in \Delta$ be its fixed point. The the sequence $\{F^n\}_{n\geq 1}$ converges uniformly on compact subsets of Δ to a $f \in \mathrm{Hol}(\Delta)$. If F is not the identity $F(z) = z$, $z \in \Delta$, then $f(z) \equiv a$ for all $z \in \Delta$.*

Next, we adopt the following

Definition 3.14 *We will say that $F \in \mathrm{Hol}(\Delta)$ is power convergent if the sequence $\{F^n\}_{n\geq 1}$ converges uniformly on any disk strictly inside Δ. If the limit f of $\{F^n\}_{n\geq 1}$is constant $f(z) = a$, $z \in \Delta$, then clearly a is a unique fixed point of F in Δ, which will be called a locally uniformly attractive fixed point.*

In terms of this definition, we can rephrase Theorem 3.13 as follows.

Corollary 3.15 *Suppose that $F \in \mathrm{Hol}(\Delta)$ has a fixed point $a \in \Delta$. Then a is a locally uniformly attractive if and only if*

$$|F'(a)| < 1.$$

For $a \in \partial\Delta$ and $K > 0$, the set

$$D(a, K) = \left\{ z \in \Delta : \frac{|1 - \overline{a}z|^2}{1 - |z|^2} < K \right\} \tag{3.17}$$

is called a *horocycle*. It is *internally tangent* to $\partial\Delta$, as it is internal to the disk Δ and has a single common point with its boundary. The following statement is a classical result in this area.

Theorem 3.16 (Wolff Lemma) *Let $F \in \mathrm{Hol}(\Delta)$ have no fixed point in Δ. Then there exists a unique $a \in \partial\Delta$, such that, for each $K > 0$ such that the horocycle (3.17) is internally tangent to $\partial\Delta$, it follows that*

$$F(D(a, K)) \subset D(a, K).$$

That is, $D(a, K)$ is F-invariant, and hence, for all $n \in \mathbb{N}$,

$$F^n(D(a, K)) \subset D(a, K). \tag{3.18}$$

The next question which appears here is whether the point in Theorem 3.16 is attractive, cf. Definition 3.14. An affirmative answer to this question is given by the following statement.

Theorem 3.17 (Denjoy-Wolff Theorem) *Let $F \in \mathrm{Hol}(\Delta)$ have no fixed point in Δ. Then there exists a unique $a \in \partial\Delta$, such that the sequence of iterates $\{F^n\}_{n\in\mathbb{N}}$ converges to a constant map $f(z) \equiv a$. Thus, $F \in \mathrm{Hol}(\Delta)$ is power-convergent on Δ to a unique $a \in \partial\Delta$ if and only if F is nor elliptic.*

The point in the latter theorem is called the *Dejoy-Wolff* point of $F \in \mathrm{Hol}(\Delta)$. The following result of Shields is a corollary of the Denjoy-Wolff Theorem. Let $\mathfrak{F} \subset \mathrm{Hol}(\Delta)$ have the property: for each distinct $F, G \in \mathfrak{F}$, it follows that $(F \circ G)(z) = (G \circ F)(z)$ for all $z \in \Delta$. Then we say that $\mathfrak{F}$ is a *commuting family.*

Theorem 3.18 (Shields) *Let $\mathfrak{F}$ be a commuting family such that each $F \in \mathfrak{F}$ is continuous on the closure $\overline{\Delta}$. Then there exists a common fixed point $a \in \overline{\Delta}$, i.e., $F(a) = a$ for all $F \in \mathfrak{F}$.*

4 Hyperbolic Metric on the Unit Disk

Another look at the Denjoy–Wolff theory is using the so-called hyperbolic metric of a domain. It is a natural approach for a class of self-mappings of a domain to introduce a metric, such that each mapping of this class becomes nonexpansive with respect to such a metric. For the class of holomorphic mappings there are different approaches to define such metrics – for example, by using the local uniform Lipschitz property.

4.1 The Poincaré metric on Δ

Here we consider some elements of the classical hyperbolic geometry on the unit disk Δ and show that the Denjoy–Wolff theory can be extended to a wider class of self-mappings as compared to $\mathrm{Hol}(\Delta)$. Let z and w be arbitrary points in Δ. Let also $\gamma \subset \mathbb{C}$ be a circle, orthogonal to $\partial\Delta$ and containing both z and w. Let $\bar{\gamma}$ be its closure. Then we pick two intersection points $z^* \in \bar{\gamma} \cap \partial\Delta$ and $w^* \in \bar{\gamma} \cap \partial\Delta$ such that z^* and z are at the same side. Similarly for w^* and w. By definition, the anharmonic relation for these points is

$$(z^*, z, w, w^*) = \frac{z - z^*}{z - w^*} \cdot \frac{w - w^*}{w - z^*}. \tag{4.1}$$

Let T be a homographic map as in (2.1). Set $u = T(z)$, $v = T(w)$, $u^* = T(z^*)$, and $v^* = T(w^*)$. It is an exercise to show the following fact.

Proposition 4.1 *For each homographic map T, it follows that*

$$(z^*, z, w, w^*) = (u^*, u, v, v^*). \tag{4.2}$$

In particular, by setting

$$T(\zeta) = \frac{\zeta - z}{1 - \bar{z}\zeta},$$

we get $T(z) = 0$. Denote $r = |T(w)|$ and $\theta = \arg T(w)$. Observe that $r < 1$ as $w \in \Delta$. After some calculations from (4.2) we obtain

$$(z^*, z, w, w^*) = \frac{1 - r}{1 + r} = \frac{1 - |T(w)|}{1 + |T(w)|}. \tag{4.3}$$

The following facts shows the importance of (4.1)

Proposition 4.2 *Let $\rho : \Delta \times \Delta \to \mathbb{R}$ be a map with the following properties:*

(a) *for each $F \in \mathrm{Aut}(\Delta)$ and arbitrary $z, w \in \Delta$, it follows that*

$$\rho(F(z), F(w)) = \rho(z, w);$$

(b) *for each $s, t \in (0, 1)$ such that $s < t$, it follows that*

$$\rho(0, t) = \rho(0, s) + \rho(s, t);$$

(c) *the following holds*

$$\lim_{t \downarrow 0} \rho(0, t)/t = 1.$$

Then

$$\rho(z, w) = -\frac{1}{2} \log(z^*, z, w, w^*). \tag{4.4}$$

Proposition 4.3 *The function defined in (4.4) is a metric on Δ.*

It is clear that $\rho(z, w) \geq 0$ for all $z, w \in \Delta$. I(t is clearly symmetric, and $\rho(z, w) = 0$ if and only if $z = w$, see (4.1). The triangle inequality can be shown by some calculations.

The importance of this metric ρ can be seen from the following fact.

Theorem 4.4 *For each $F \in \mathrm{Hol}(\Delta)$, it follows that, for arbitrary $z, w \in \Delta$,*

$$\rho(F(z), F(w)) \leq \rho(z, w),$$

that is, F is non-expansive with respect to ρ. Moreover, the equality holds if and only if $F \in \mathrm{Aut}(\Delta)$.

The metric established in Proposition 4.3 is called the *Poincaré metric.*

Proposition 4.5 *The pair (Δ, ρ) is a complete unbounded metric space. The metric topology defined by ρ Δ is equivalent to the original topology on thereon.*

4.2 Compatibility of the Poincaré metric with convexity

As is well-known, convexity is an extremely important property of the unit disk Δ, which it possesses as a subset of the complex plane. On the other hand, the property established Theorem 4.4 shows the importance of the Poincaré metric in the analysis of the properties of $\mathrm{Hol}(\Delta)$. Then it would be much desirable to combine both these properties. The present subsection deals with this task.

Proposition 4.6 *The Poincaré metric ρ on the unit disk Δ has the following properties:*

(a) *for distinct $z, w \in \Delta$ and each $k \in (0, 1)$, it follows that*

$$\rho(kz, kw) \leq k\rho(z, w);$$

(b) *for each triple of distinct points $z, w, u \in \Delta$, and for each $k \in (0, 1)$, it follows that*

$$\rho((1 - k)z + ku, (1 - k)w + ku) \leq q\rho(z, w),$$

where $q = (1 - k) + k|u| < 1$;

(c) *for distinct $z, w, u, v \in \Delta$, and each $k \in [0, 1]$, it follows that*

$$\rho((1 - k)zv + ku, (1 - k)w + kv) \leq q \max\{\rho(z, w); \rho(u, v)\}.$$

Now we are ready to formulate our main result in this subsection. For four points $z, w, u, v \in \Delta$, let

$$\phi(t) := \rho((1-t)zv + tu, (1-t)w + tv). \tag{4.5}$$

Proposition 4.7 *For arbitrary four points $z, w, u, v \in \Delta$, the following statements are equivalent:*

(a) *the function $\phi : [0,1] \to [0, +\infty)$ defined in (4.5) is nondecreasing;*

(b) $\phi'(0^+) \geq 0$*;*

(c) *the following holds*

$$\mathrm{Re}\left[\frac{\overline{z}(z-u)}{1-|z|^2} + \frac{\overline{w}(w-v)}{1-|w|^2}\right] \leq \mathrm{Re}\left[\frac{\overline{z}(w-v) + w(\overline{z}-\overline{u})}{1-\overline{z}w}\right]. \tag{4.6}$$

5 Generation theory on the unit disk

As mentioned in Introduction, we pay some attention to the theory of holomorphic semigroups of mappings on Δ. The generator of such a semigroup is defined as in (1.1).

5.1 One-parameter continuous semigroup of holomorphic self-mappings

Let $\mathcal{A}$ be a topological abelian (additive) semigroup with zero. Suppose there exists a natural ordering of $\mathcal{A}$, i.e., $\tau \geq t$ if and only if there is $s \in \mathcal{A}$ such that $\tau = t + s$. By the analytic action of $\mathcal{A}$ on Δ we mean a map $S : \mathcal{A} \to \mathrm{Hol}(\Delta)$, which preserves the additive structure of $\mathcal{A}$ with respect to composition operation on $\mathrm{Hol}(\Delta)$. That is:

(a) if $s, t, s+t \in \mathcal{A}$, then $S(t+s) = S(t) \circ S(s)$;

(b) $S(0) = I$ – the identity map.

If $\mathcal{A} = \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then $S = \{S_0, S_1, S_2, \dots\}$ is called a one-parameter discrete semigroup. Actually such a semigroup consists of iterates of a holomorphic self-mapping of Δ, i.e., $S_n = F^n$, $F \in \mathrm{Hol}(\Delta)$ because of conditions (a) and (b). If $\mathcal{A}$ is a sub-interval of $\mathbb{R}$, containing zero and action $S : \mathcal{A} \to \mathrm{Hol}(\Delta)$ is continuous with respect to the topology of the point-wise convergence on Δ, we say that S is a one-parameter continuous semigroup of holomorphic self-mapping of Δ. In other words, a one-parameter continuous semigroup of holomorphic self-mappings of Δ is a family $\{F_t\}_{t \in \mathcal{A}} \subset \mathrm{Hol}(\Delta)$ such that:

(a) if $t, s, t+s \in \mathcal{A}$, then $F_{t+s}(z) = (F_t \circ F_s)(z)$, for each $z \in \Delta$;

(b) $F_0(z) = z$ for all $z \in \Delta$, and

$$\lim_{t \to s} F_t(z) = F_s(z). \tag{5.1}$$

The following cases are of intere st: $\mathcal{A} = [0, T)$; $\mathcal{A} = (-T, T)$, for $0 < T \leq +\infty$. For $\mathcal{A} = \mathbb{R}$, we have $F_t \circ F_{-t} = I$, and hence $F_t \in \text{Aut}(\Delta)$ for all $t \in \mathbb{R}$. The converse is also true in the following form.

Proposition 5.1 *If $\mathcal{A} = [0, T)$ and $F_t \in \text{Aut}(\Delta)$ for at least one $t \in \mathcal{A}$, then so does each F_t, $t \in \mathcal{A}$, and the semigroup $\{F_t\}_{t \in \mathcal{A}}$ can be extended to a one-parameter group $\{F_t\}_{t \in (-T,T)} \subset \text{Aut}(\Delta)$.*

Example 5.2 *For $a \in \mathbb{C}$ with $\text{Re} a \geq 0$ and $t \in [0, T)$, define*

$$F_t(z) = e^{-at} z, \qquad z \in \Delta.$$

Clearly, $F_0(z) = z$ and $(F_t \circ F_s)(z) = F_{t+s} z$. Since T is arbitrary, we can extend the semigroup to a flow $\{F_t\}_{t \geq 0}$. As a matter of tact we will show later that each one-parameter semigroup can be extended to a holomorphic flow on Δ.

Example 5.3 *Define*

$$F_t(z) = \frac{z}{e^t - z(e^t - 1)}, \qquad z \in \Delta, \quad t \geq 0.$$

It is clear that $F_t \in \text{Hol}(\Delta)$ for all $t \geq 0$.

5.2 Infinitesimal generator of a one-parameter continuous semigroup

As was mentioned above, each one-parameter continuous semigroup of holomorphic self-mappings of Δ defined on an interval $[0, T)$ can be extended to the flow on Δ, i.e. we may always think that $T = +\infty$). Moreover, only the right continuity at zero of a semigroup implies continuity (right and left) at all $t \in [0, +\infty)$. These facts can be shown by different approaches, but here we will establish them by using a very strong property of a continuous one-parameter semigroup of holomorphic self-mappings of Δ to be differentiable by a parameter at each point $t \in [0, +\infty)$. This nice result for the one-dimensional case is due to E. Berkson and H. Porta.

Proposition 5.4 *Let $\{F_t\}_{t \in [0,T)}$ be a semigroup of holomorphic self-mappings of Δ such that, for each $z \in \Delta$,*

$$\lim_{t \downarrow 0} F_t(z) = z. \tag{5.2}$$

Then, for each $z \in \Delta$, there exists the limit

$$\lim_{t \downarrow 0} (z - F_t(z))/t =: f(z), \tag{5.3}$$

which defines a holomorphic map $f : \Delta \to \mathbb{C}$. The convergence in (5.3) is uniform on each subset strictly inside Δ. That is, a semigroup with the properties assumed is also right-differentiable at zero.

The function defined in (5.3) is called the *infinitesimal generator* of the semigroup $\{F_t\}_{t\in[0,T)}$. Usually, 'infinitesimal' is omitted, so we shall do in the sequel. The meaning of the notion just introduced can be better understood from the following fact.

Proposition 5.5 *Let $\{F_t\}_{t\in[0,T)}$ be a semigroup of holomorphic self-mappings of Δ satisfying the conditions of Proposition 5.4, and let $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ be its generator. Then the function $u : [0,T)\times\Delta \to \mathbb{C}$ defined by $u(t,z) = F_t(z)$ is a classical solution to the following Cauchy problem:*

$$\frac{\partial u(t,z)}{\partial t} + f(u(t,z)) = 0, \qquad u(0,z) = z, \quad t \in [0,T), \quad z \in \Delta. \tag{5.4}$$

The proof is immediate. Note that by a classical solution we mean the function $u : [0,T) \times \Delta \to \mathbb{C}$ which is: (a) continuously differentiable in t on $(0,T)$ such that the derivative is a holomorphic self-mapping of Δ; (b) continuous in t on $[0,T)$; (c) a point-wise solution of the Cauchy problem. From this we can also conclude that the solution is unique since f is locally Lipschitz (as a holomorphic function).

Example 5.6 *Let $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ be given. Consider the following Cauchy problem*

$$\frac{\partial u(t,z)}{\partial t} + \frac{\partial u(t,z)}{\partial z} f(z), \qquad u(0,z) = z.$$

Since every semigroup of holomorphic self-mappings of the unit disk is differentiable, it is natural to describe the asymptotic behavior of such a semigroup in terms of its generator. This becomes more desirable when a semigroup is not given explicitly, but it is known that such a semi-group is the solution of the Cauchy problem in (5.4).

5.3 Continuous versions of the Denjoy–Wolff theorem

Definition 5.7 *A point $a \in \overline{\Delta}$ is said to be a stationary point of a flow $\{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$ if, for all $t \geq 0$,*

$$\lim_{r\uparrow 1} F_t(ta) = a.$$

In other words, a is a stationary point of the flow if it is a common fixed point of all F_t.

Since $F_t \circ F_s = F_s \circ F_t = F_{s+t}$ for all $s,t \geq 0$, the flow is a commuting family. Then, if each F_t admits a continuous extension to the boundary $\partial\Delta$, by Shields' Theorem, cf. Theorem 3.18, then the stationary point set of the flow is nonempty. In addition, if for at least one $t > 0$, the map F_t has an interior fixed point $a \in \Delta$, then it is a unique fixed point for F_t, and we have for each $s > 0$

$$F_s(a) = (F_s \circ F_t)(a) = (F_t \circ F_s)(a) = a,$$

as a is the unique fixed point of F_t. Then a is also a fixed point of F_s, for each s, and also unique. A fundamental question here is whether and when this point is attractive, i.e. when the $F_t \to a$ as $t \to +\infty$? The answer can be found in the following assertion.

Proposition 5.8 *Let* $\{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$ *be a flow on* Δ. *Then the net* $\{F_t\}_{t\geq 0}$ *converges uniformly on compact subsets of* Δ *to a mapping* $F \in \mathrm{Hol}(\Delta, \overline{\Delta})$ *if and only if, for at least one* $t_0 > 0$, *the sequence* $\{F_{t_0}^n\}_{n\in\mathbb{N}}$ *converges uniformly on compact subsets of* Δ. *If this is the case, then, for each* $z \in \Delta$,

$$\lim_{t\to+\infty} F_t(z) = \lim_{n\to+\infty} F_{t_0}^n(z) = F(z). \tag{5.5}$$

Remark 5.9 *If the generator* f *is holomorphic in a neighborhood of the point* a, *then it follows by the uniqueness of the solution of the Cauchy problem in (5.4) that* $f(a) = 0$ *if and only if* a *is a stationary point of the semigroup* $\{F_t\}_{t\geq 0}$, *generated by* f. *Here we set* $F_t(z) = u(t, z)$.

6 Further Reading

6.1 Monographs

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