Version of 24.9.12

Lectures on a Stochastic Integral

D.H.Fremlin

University of Essex, Colchester, England

Lectures 1-3

Measure algebras Let (Ω, Σ, μ) be a probability space. Then we have an equivalence relation $\sim$ on Σ defined by saying that $E \sim F$ if $\mu(E \triangle F) = 0$. Let $\mathfrak{A}$ be the set of equivalence classes $a = E^\bullet$ for $E \in \Sigma$. Then we have operations $\cup$, $\cap$, $\setminus$ and $\triangle$ on $\mathfrak{A}$ defined by saying that

$$E^\bullet \cup F^\bullet = (E \cup F)^\bullet, \quad E^\bullet \cap F^\bullet = (E \cap F)^\bullet,$$

$$E^\bullet \setminus F^\bullet = (E \setminus F)^\bullet, \quad E^\bullet \triangle F^\bullet = (E \triangle F)^\bullet$$

for E, $F \in \Sigma$. These operations behave in the same way as the ordinary Boolean operations $\cup$, $\cap$, $\setminus$ and $\triangle$, so that, for instance,

$$a \cap (b \cup c) = (a \cup b) \triangle (a \cup c)$$

for all a, b, $c \in \mathfrak{A}$. We have a zero $0 = \emptyset^\bullet$ and a unit $1 = \Omega^\bullet$ in $\mathfrak{A}$, so that, for instance,

$$1 \setminus (a \cup b) = (1 \setminus a) \cap (1 \setminus b);$$

$\mathfrak{A}$ is a **Boolean algebra**. Next, we have a partial ordering $\subseteq$ on $\mathfrak{A}$ defined by saying that

$$a \subseteq b \iff a = a \cap b \iff b = a \cup b \iff a \setminus b = 0,$$

just like $\subseteq$; for this partial ordering, $a \cap b = \inf\{a, b\}$ and $a \cup b = \sup\{a, b\}$ for all a, $b \in \mathfrak{A}$, and $\mathfrak{A}$ is a distributive lattice. For *countable* infinitary operations, we get a simple correspondence with the corresponding operations in $\mathcal{P}\Omega$:

$$(\bigcup_{n \in \mathbb{N}} E_n)^\bullet = \sup_{n \in \mathbb{N}} E_n^\bullet, \quad (\bigcap_{n \in \mathbb{N}} E_n)^\bullet = \inf_{n \in \mathbb{N}} E_n^\bullet$$

for all sequences $\langle E_n \rangle_{n \in \mathbb{N}}$ in Σ. For *uncountable* infinitary operations, there is a difference. In the probability spaces of interest in this course, not all families $\mathcal{A} \subseteq \Sigma$ have unions in Σ; but in any measure algebra $\mathfrak{A}$ defined from a probability space, every subset of $\mathfrak{A}$ has a supremum and an infimum in $\mathfrak{A}$ (counting $\sup \emptyset$ as 0 and $\inf \emptyset$ as 1), that is, $\mathfrak{A}$ is **Dedekind complete** with greatest and least elements.

On $\mathfrak{A}$, we have a 'measure' $\bar{\mu}$ defined by saying that $\bar{\mu} E^\bullet = \mu E$ for every $E \in \Sigma$; we have the ordinary rules

$$\bar{\mu} 0 = 0, \quad \bar{\mu}(a \cup b) = \bar{\mu} a + \bar{\mu} b \text{ if } a \cap b = 0, \quad \bar{\mu} 1 = 1$$

of elementary probability theory. In addition we have

$$\bar{\mu}(\sup A) = \sup_{a \in A} \bar{\mu} a$$

whenever $A \subseteq \mathfrak{A}$ is non-empty and **upwards-directed**, that is, for any a, $b \in A$ there is a $c \in A$ such that $a \subseteq c$ and $b \subseteq c$.

L^0-spaces Let (Ω, Σ, μ) be a probability space, and write $\mathcal{L}^0$ for the set of all Σ-measurable real-valued functions defined on Ω. Then we have an equivalence relation $\sim$ on $\mathcal{L}^0$ defined by saying that

$$f \sim g \text{ if } f =_{\text{a.e.}} g, \text{ that is, } \mu\{\omega : f(\omega) \neq g(\omega)\} = 0.$$

Let L^0 be the set of equivalence classes $u = f^\bullet$ for $f \in \mathcal{L}^0$. Then we have operations $+$, $\times$ on L^0, and a scalar multiplication, defined by saying that

$$f^\bullet + g^\bullet = (f+g)^\bullet, \quad \alpha f^\bullet = (\alpha f)^\bullet, \quad f^\bullet \times g^\bullet = (f \times g)^\bullet,$$

for f, $g \in \mathcal{L}^0$ and $\alpha \in \mathbb{R}$. L^0 is a commutative algebra with additive identity $0 = (\chi\emptyset)^\bullet$ and multiplicative identity $(\chi\Omega)^\bullet$. Next, L^0 has a partial ordering defined by saying that

$$f^\bullet \leq g^\bullet \iff f \leq_{\text{a.e.}} g, \text{ that is, } \mu\{\omega : f(\omega) \leq g(\omega)\} = 1;$$

this makes L^0 a Dedekind complete distributive lattice with

$$f^\bullet \vee g^\bullet = \max(f,g)^\bullet, \quad f^\bullet \wedge g^\bullet = \min(f,g)^\bullet$$

for f, $g \in \mathcal{L}^0$. The familiar algebraic rules apply in L^0 as in $\mathcal{L}^0$, e.g.,

$$u \times (v + w) = u \times v + u \times w, \quad u \leq v \Longrightarrow u + w \leq v + w, \quad u \times v \geq 0 \text{ if } u,\, v \geq 0;$$

L^0 is an **f-algebra**. We have a map $\chi : \mathfrak{A} \to L^0$ defined by saying that $\chi(E^\bullet) = (\chi E)^\bullet$ for $E \in \Sigma$.

Mappings on L^0 If $h : \mathbb{R} \to \mathbb{R}$ is any Borel measurable function, we have a function $\bar{h} : L^0 \to L^0$ defined by saying that

$$\bar{h}(f^\bullet) = (hf)^\bullet \text{ for every } f \in \mathcal{L}^0,$$

where hf here is the composition $h \circ f : \Omega \to \mathbb{R}$. The most important special case is when $h(x) = |x|$ for $x \in \mathbb{R}$, so that $\bar{h}(u) = |u| = u \wedge (-u)$ for $u \in L^0$.

Regions in $\mathfrak{A}$ For u, $v \in L^0$, $\alpha \in \mathbb{R}$ and Borel sets $E \subseteq \mathbb{R}$ we can define 'regions' $[\![u > \alpha]\!]$, $[\![u \in E]\!]$, $[\![u = v]\!]$ in $\mathfrak{A}$ by saying that

$$[\![f^\bullet > \alpha]\!] = \{\omega : f(\omega) > \alpha\}^\bullet, \quad [\![f^\bullet \in E]\!] = (f^{-1}[E])^\bullet,$$

$$[\![f^\bullet = g^\bullet]\!] = \{\omega : f(\omega) = g(\omega)\}^\bullet$$

when $f^\bullet = u$ and $g^\bullet = v$. For some purposes it is helpful to think of a member u of L^0 as being defined by the family $\langle [\![u > \alpha]\!] \rangle_{\alpha \in \mathbb{R}}$.

L^1 spaces and integration If we write $\mathcal{L}^1_{\text{strict}}$ for the space of measurable integrable functions $f : \Omega \to \mathbb{R}$, and $L^1 = L^1_{\bar{\mu}}$ for $\{f^\bullet : f \in \mathcal{L}^1_{\text{strict}}\}$, we get a linear subspace of L^0 which is **solid**, that is, if $u \in L^1$ and $|v| \leq |u|$ then $v \in L^1$. I will write $\mathbb{E} = \mathbb{E}_{\bar{\mu}}$ for the corresponding notion of integration in L^1, so that

$$\mathbb{E}(f^\bullet) = \int_\Omega f(\omega)\mu(d\omega)$$

for $f \in \mathcal{L}^1_{\text{strict}}$.

Convergence in measure We have a functional $\theta = \theta_{\bar\mu} : L^0 \to [0,\infty[$ defined by setting

$$\theta(u) = \mathbb{E}(|u| \wedge \chi 1) \text{ for } u \in L^0,$$

$$\theta(f^\bullet) = \int \min(|f(\omega)|, 1)\mu(d\omega) \text{ for } f \in \mathcal{L}^0.$$

Now

$$\theta(u+v) \le \theta(u) + \theta(v), \quad \theta(\alpha u) \le \theta(u) \text{ if } |\alpha| \le 1, \quad \lim_{\alpha\to 0}\theta(\alpha u) = 0$$

for all u, $v \in L^0$, so we have a metric $(u,v) \mapsto \theta(u-v)$ on L^0 which defines a linear space topology on L^0, not normally locally convex; this is the **topology of convergence in measure** on L^0. Under this metric, L^0 is complete.

σ-subalgebras of Σ, closed subalgebras of $\mathfrak{A}$ Let (Ω, Σ, μ) be a probability space with measure algebra $\mathfrak{A}$. If T is a σ-subalgebra of Σ, that is,

$$\emptyset \in \mathrm{T}, \quad \Omega \setminus E \in \mathrm{T} \text{ whenever } E \in \mathrm{T},$$

$$\bigcup_{n\in\mathbb{N}} E_n \in \mathrm{T} \text{ whenever } \langle E_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } \mathrm{T},$$

then $\mathfrak{B} = \{E^\bullet : E \in \mathrm{T}\}$ is a closed subalgebra of $\mathfrak{A}$, that is,

$$0 \in \mathfrak{B}, \quad 1 \setminus a \in \mathfrak{B} \text{ whenever } a \in \mathfrak{B},$$

$$\sup A \in \mathfrak{B} \text{ for every } A \subseteq \mathfrak{B}.$$

In this case, $(\Omega, \mathrm{T}, \mu\restriction\mathrm{T})$ is a probability space, and its measure algebra can be identified with $(\mathfrak{B}, \bar\mu\restriction\mathfrak{B})$, while $L^0(\mathfrak{B})$ can be identified with

$$\{u : u \in L^0(\mathfrak{A}), [\![u > \alpha]\!] \in \mathfrak{B} \text{ for every } \alpha \in \mathbb{R}\},$$

and $L^1(\mathfrak{B}, \bar\mu\restriction\mathfrak{B})$ with $L^0(\mathfrak{B}) \cap L^1(\mathfrak{A}, \bar\mu)$.

Lecture 4

Filtrations Let (Ω,Σ,μ) be a probability space. A family $\langle\Sigma_t\rangle_{t\ge0}$ of σ-subalgebras of Σ is a **filtration** if $\Sigma_s\subseteq\Sigma_t$ whenever $s\le t$. Associated with this is the family $\langle\mathfrak{A}_t\rangle_{t\ge0}$ where $\mathfrak{A}_t=\{E^\bullet:E\in\Sigma_t\}$ for $t\ge0$; this is a 'filtration of closed subalgebras'.

A filtration $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is **right-continuous** if $\mathfrak{A}_t=\bigcap_{s>t}\mathfrak{A}_s$ for every t. If every Σ_t contains every set of measure 0, this will be the case iff $\Sigma_t=\bigcap_{s>t}\Sigma_s$ for every t.

Stopping times If $\langle\Sigma_t\rangle_{t\ge0}$ is a filtration of σ-subalgebras of Σ, a function $h:\Omega\to[0,\infty]$ is a **stopping time** if $\{\omega:h(\omega)>t\}$ belongs to Σ_t for every $t\ge0$. In this case, we have a corresponding 'stopping time' $\tau=\langle a_t\rangle_{t\ge0}$ where $a_t=\{\omega:h(\omega)>t\}^\bullet$ for every $t\ge0$. The family $\langle a_t\rangle_{t\ge0}$ will have the properties

$$a_t\in\mathfrak{A}_t,\quad a_t=\sup_{s>t}a_s \text{ for every } t\ge0.$$

In this context I will write $[\![\tau>t]\!]$ for a_t.

If $t\ge0$, we have a **constant stopping time** $\dot t$ defined by

$$[\![\dot t>s]\!]=1 \text{ if } s<t,\ 0 \text{ if } s\ge t.$$

The lattice of stopping times Let $\mathcal{T}\subseteq\prod_{t\ge0}\mathfrak{A}_t$ be the set of all stopping times associated with a filtration $\langle\mathfrak{A}_t\rangle_{t\ge0}$. We have a partial ordering on $\mathcal{T}$ defined by saying that

$$\sigma\le\tau \text{ if } [\![\sigma>t]\!]\subseteq[\![\tau>t]\!] \text{ for every } t\ge0.$$

Under this ordering, $\mathcal{T}$ is a Dedekind complete Boolean lattice with lattice operations defined by saying that

$$[\![\sigma\vee\tau>t]\!]=[\![\sigma>t]\!]\cup[\![\tau>t]\!],\quad [\![\sigma\wedge\tau>t]\!]=[\![\sigma>t]\!]\cap[\![\tau>t]\!]$$

for $t\ge0$, while if $A\subseteq\mathcal{T}$ is not empty,

$$[\![\sup A>t]\!]=\sup_{\tau\in A}[\![\tau>t]\!]$$

for all t. $\mathcal{T}$ has a least element $\min\mathcal{T}=\dot0$ such that $[\![\min\mathcal{T}>t]\!]=0$ for every t, and a greatest element $\max\mathcal{T}$ such that $[\![\max\mathcal{T}>t]\!]=1$ for every t.

I will write $\mathcal{T}_f$ for the set of **finite** stopping times τ such that $\inf_{t\ge0}[\![\tau>t]\!]=0$, and $\mathcal{T}_b$ for the set of **bounded** stopping times τ such that $[\![\tau>t]\!]=0$, that is, $\tau\le\dot t$, for some $t\ge0$.

The algebra associated with a stopping time If τ is a stopping time, write

$$\mathfrak{A}_\tau=\{a:a\setminus[\![\tau>t]\!]\in\mathfrak{A}_t \text{ for every } t\ge0\}.$$

Then $\mathfrak{A}_\tau$ is a closed subalgebra of $\mathfrak{A}$. We have $\mathfrak{A}_{\dot t}=\mathfrak{A}_t$ for every t, and $\mathfrak{A}_\sigma\subseteq\mathfrak{A}_\tau$ if $\sigma\le\tau$. Generally, $\mathfrak{A}_{\sigma\wedge\tau}=\mathfrak{A}_\sigma\cap\mathfrak{A}_\tau$ for all σ, $\tau\in\mathcal{T}$.

Regions associated with stopping times If σ, τ are stopping times, set

$$[\![\sigma < \tau]\!] = \sup_{t\ge 0} [\![\tau > t]\!] \setminus [\![\sigma > t]\!],$$

$$[\![\sigma = \tau]\!] = 1 \setminus ([\![\sigma < \tau]\!] \cup [\![\tau < \sigma]\!]).$$

We find that $[\![\sigma < \tau]\!]$, $[\![\sigma = \tau]\!]$, $[\![\tau < \sigma]\!]$ form a partition of unity and all belong to $\mathfrak{A}_{\sigma\wedge\tau}$.

Stopping-time intervals If σ, τ are stopping times, with $\sigma \le \tau$, write $c(\sigma,\tau)$ for the **stopping-time interval**

$$\langle [\![\tau > t]\!] \setminus [\![\sigma > t]\!] \rangle_{t\ge 0} \in \prod_{t\ge 0} \mathfrak{A}_t.$$

In this context, it is helpful to think of the product $\mathfrak{D} = \prod_{t\ge 0} \mathfrak{A}_t$ as a Boolean algebra (using coordinate-by-coordinate definitions of the Boolean operations). If we think of a stopping time τ as neither more nor less than the family $\langle [\![\tau > t]\!] \rangle_{t\ge 0}$, then τ actually becomes an element of $\mathfrak{D}$ (not arbitrary, because we demand the property

$$[\![\tau > t]\!] = \sup_{s>t} [\![\tau > s]\!]$$

for every $t \ge 0$), and $c(\sigma,\tau)$ is the Boolean difference $\tau \setminus \sigma$, interpreted in $\mathfrak{D}$. Note that $\sigma \le \tau$, as defined above, iff $\sigma \subseteq \tau$ when they are thought of as elements of $\mathfrak{D}$. Note also that the expression of a stopping-time interval e as $c(\sigma,\tau)$ is practically never unique. In fact we have

$$c(\sigma,\tau) = c(\sigma',\tau') \text{ iff } [\![\sigma < \tau]\!] = [\![\sigma' < \tau']\!] \subseteq [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!].$$

Now suppose that $I \subseteq \mathcal{T}$ is a finite sublattice of $\mathcal{T}$. If we interpret I as a subset of $\mathfrak{D}$, with greatest and least elements $\min I$ and $\max I$, it generates a finite subalgebra $\mathfrak{D}_0$ of $\mathfrak{D}$. $\mathfrak{D}_0$, being in itself a finite Boolean algebra, has (finitely) many 'atoms' (minimal non-zero elements), all disjoint, and each element of $\mathfrak{D}_0$ being the supremum of the atoms it includes. We can identify these atoms as being either $c(\min \mathcal{T}, \min I)$, $c(\max I, \max \mathcal{T})$ or of the form $c(\sigma,\tau)$ where σ, $\tau \in I$. The latter I will call ***I*-cells**.

Lecture 5

Fully adapted processes Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$. A **fully adapted process** with domain $\mathcal{S}$ is a family $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ such that

$$u_\sigma \in L^0(\mathfrak{A}_\sigma), \qquad [\![\sigma = \tau]\!] \subseteq [\![u_\sigma = u_\tau]\!]$$

for all σ, $\tau \in \mathcal{S}$.

Theorem 1 Suppose that $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of a complete probability space (Ω, Σ, μ), and that $\Sigma_t = \{E : E \in \Sigma,\ E^\bullet \in \mathfrak{A}_t\}$ for $t \geq 0$. Let $\langle X_t \rangle_{t \geq 0}$ be a family of real-valued random variables on Ω which is **progressively measurable**, that is, $(s, \omega) \mapsto X_s(\omega) : [0, t] \times \Omega \to \mathbb{R}$ is $\mathcal{B}([0,t]) \widehat{\otimes} \Sigma_t$-measurable for every $t \geq 0$, where $\mathcal{B}([0,t]) \widehat{\otimes} \Sigma_t$ is the σ-algebra of subsets of $[0,t] \times \Omega$ generated by $\{[a,b] \times E : 0 \leq a \leq b \leq t,$ $E \in \Sigma_t\}$.

(a) For any stopping time $h : \Omega \to [0, \infty[$, the function $X_h = \langle X_{h(\omega)}(\omega) \rangle_{\omega \in \Omega}$ is Σ_h-measurable, where $\Sigma_h = \{E : E \in \Sigma,\ E \setminus \{\omega : h(\omega) > t\} \in \Sigma_t \text{ for every } t \geq 0\}$.

(b) We have a fully adapted process $\boldsymbol{u} = \langle u_\tau \rangle_{\tau \in \mathcal{T}_f}$ defined by saying that $u_\tau = X_h^\bullet$ whenever $h : \Omega \to [0, \infty[$ is a stopping time and $\tau = h^\bullet$ is the corresponding stopping time in $\mathcal{T}_f$.

The class of fully adapted processes Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $M = M(\mathcal{S}) \subseteq (L^0)^{\mathcal{S}}$ the set of fully adapted processes with domain $\mathcal{S}$. Then M is an f-subalgebra of $(L^0)^{\mathcal{S}}$ (that is, a linear subspace closed under multiplication and the lattice operations), and $\bar{h}(\boldsymbol{u}) = \langle \bar{h}(u_\sigma) \rangle_{\sigma \in \mathcal{S}} \in M$ whenever $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ belongs to M and $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

Riemann sums Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ are fully adapted processes.

For a stopping-time interval e with endpoints in $\mathcal{S}$, we can define $\Delta_e(\boldsymbol{u}, d\boldsymbol{v})$ by saying that

$$\Delta_e(\boldsymbol{u}, d\boldsymbol{v}) = u_\sigma \times (v_\tau - v_\sigma)$$

whenever $e = c(\sigma, \tau)$ where $\sigma \leq \tau$ in $\mathcal{S}$.

For a finite sublattice I of $\mathcal{S}$, set

$$S_I(\boldsymbol{u}, d\boldsymbol{v}) = \textstyle\sum_{e \text{ is an } I\text{-cell}} \Delta_e(\boldsymbol{u}, d\boldsymbol{v}).$$

The stochastic integral Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ are fully adapted processes defined on $\mathcal{S}$. Write $\mathcal{I}(\mathcal{S})$ for the set of finite sublattices of $\mathcal{S}$. Then

$$\textstyle\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = \lim_{I \uparrow \mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\boldsymbol{v})$$

if this is defined in L^0 for the topology of convergence in measure; that is, $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v} = z$ iff

> for every $\epsilon > 0$ there is a $J \in \mathcal{I}(\mathcal{S})$ such that $\theta(z - S_I(\boldsymbol{u}, d\boldsymbol{v})) \leq \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$ and $J \subseteq I$.

Warning! This is not quite the standard stochastic integral. PROTTER 03 would call it $\int \boldsymbol{u}_{-}\, d\boldsymbol{v}$, because in the Riemann sums we always use the value u_σ at the lower end of the interval $c(\sigma,\tau)$.

The deterministic case Consider the case in which $\mathfrak{A}$ is the trivial Boolean algebra $\{0,1\}$ with two elements (corresponding to the case in which Ω has just one point). In this case, every $\mathfrak{A}_t$ has to be equal to $\mathfrak{A}$, the only possible values for a region $[\![\tau > t]\!]$ are 0 and 1 (so every stopping time is either a constant stopping time or $\max \mathcal{T}$), and every member of L^0 is of the form $\alpha\chi 1$ for some α. So we can identify $\mathcal{T}$ with $[0,\infty]$ and L^0 with $\mathbb{R}$. Every subset of $[0,\infty]$ is a sublattice, and if $I = \{t_0, \dots, t_n\}$ where $t_0 < \ldots \le t_n$, then the I-cells are the intervals $[t_i, t_{i+1}[$ for $i < n$. So to calculate $\int_{[0,\infty[} f\, dg$, where f, $g : [0,\infty[\to \mathbb{R}$ are real-valued functions, we look at sums of the form $\sum_{i=0}^{n-1} f(t_i)(g(t_{i+1}) - g(t_i))$.

This looks like a Stieltjes integral of some kind. But it is not the Lebesgue-Stieltjes integral, even if g is non-decreasing, so that we have an associated Radon measure on $\mathbb{R}$. Consider, for instance, the case in which

$$\begin{aligned} f(x) = g(x) &= 0 \text{ if } x < 1, \\ &= 1 \text{ if } x \ge 1. \end{aligned}$$

If we look at a term $\Delta_{[s,t[}(f, dg) = f(s)(g(t) - g(s))$, this will always be zero, because either $s < 1$ and $f(s) = 0$, or $s \ge 1$ and $g(s) = g(t)$. So we get $\int_{[0,\infty[} f\, dg = 0$. But if we look at a measure ν_g on $[0,\infty[$ to represent dg, the only candidate is the Dirac measure concentrated at 1, in which case $\int f\, d\nu_g = f(1) = 1$.

Lecture 6

Basic properties of the integral: Theorem 2 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) If $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u}'$, $\boldsymbol{v}'$ are fully adapted processes with domain $\mathcal{S}$, and $\alpha \in \mathbb{R}$, then

$$\int_{\mathcal{S}} \boldsymbol{u} + \boldsymbol{u}' \, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u} \, d\boldsymbol{v} + \int_{\mathcal{S}} \boldsymbol{u}' \, d\boldsymbol{v},$$

$$\int_{\mathcal{S}} \boldsymbol{u} \, d(\boldsymbol{v} + \boldsymbol{v}') = \int_{\mathcal{S}} \boldsymbol{u} \, d\boldsymbol{v} + \int_{\mathcal{S}} \boldsymbol{u} \, d\boldsymbol{v}',$$

$$\int_{\mathcal{S}} (\alpha \boldsymbol{u}) \, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{u} \, d(\alpha \boldsymbol{v}) = \alpha \int_{\mathcal{S}} \boldsymbol{u} \, d\boldsymbol{v}$$

whenever the right-hand sides are defined.

(b) Suppose that $\boldsymbol{u}$, $\boldsymbol{v}$ are fully adapted processes with domain $\mathcal{S}$, and $\tau \in \mathcal{S}$. Set $\mathcal{S} \wedge \tau = \{\sigma \wedge \tau : \sigma \in \mathcal{S}\} = \mathcal{S} \cap [\min \mathcal{T}, \tau]$, $\mathcal{S} \vee \tau = \{\sigma \vee \tau : \sigma \in \mathcal{S}\} = \mathcal{S} \cap [\tau, \max \mathcal{T}]$. Then

$$\int_{\mathcal{S}} \boldsymbol{u} \, d\boldsymbol{v} = \int_{\mathcal{S} \wedge \tau} \boldsymbol{u} \, d\boldsymbol{v} + \int_{\mathcal{S} \vee \tau} \boldsymbol{u} \, d\boldsymbol{v}$$

if either side is defined.

(c) Suppose that $\boldsymbol{u}$, $\boldsymbol{v}$ are fully adapted processes with domain $\mathcal{S}$, and that $\int_{\mathcal{S}} \boldsymbol{u} \, d\boldsymbol{v}$ is defined. Set $z_\tau = \int_{\mathcal{S} \wedge \tau} \boldsymbol{u} \, d\boldsymbol{v}$ for $\tau \in \mathcal{S}$.

(i) The **indefinite integral** $ii_{\boldsymbol{v}}(\boldsymbol{u}) = \langle z_\tau \rangle_{\tau \in \mathcal{S}}$ is a fully adapted process.

(ii) If $\mathcal{S} \neq \emptyset$, then $\lim_{\tau \downarrow \mathcal{S}} z_\tau = 0$ and $\lim_{\tau \uparrow \mathcal{S}} z_\tau = \int_{\mathcal{S}} \boldsymbol{u} \, d\boldsymbol{v}$.

Remark What I am calling $ii_{\boldsymbol{v}}(\boldsymbol{u})$ would (allowing for a different definition of the integral) be denoted $\boldsymbol{u} \cdot \boldsymbol{v}$ by most authors.

Simple processes Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. A fully adapted process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$ is **simple**, with **breakpoints** $\tau_0, \dots, \tau_n$ and **root value** $u_\downarrow$, if

$$\tau_i \in \mathcal{S} \text{ for every } i \le n, \quad \tau_0 \le \dots \le \tau_n,$$

$$[\![\sigma < \tau_0]\!] \subseteq [\![u_\sigma = u_\downarrow]\!], \quad [\![\tau_n \le \sigma]\!] \subseteq [\![u_\sigma = u_{\tau_n}]\!],$$

$$[\![\tau_i \le \sigma]\!] \cap [\![\sigma < \tau_{i+1}]\!] \subseteq [\![u_\sigma = u_{\tau_i}]\!] \text{ for every } i < n$$

for every $\sigma \in \mathcal{S}$.

In this case, if $\tau \in \mathcal{S}$ and we write $\mathcal{S} \wedge \tau = \{\sigma \wedge \tau : \sigma \in \mathcal{S}\}$, then $\boldsymbol{u} \restriction \mathcal{S} \wedge \tau$ is simple, with breakpoints $\tau_0 \wedge \tau, \dots, \tau_n \wedge \tau$ and root value $u_\downarrow$.

Integrating simple processes Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in \mathcal{S}}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ fully adapted processes, of which $\boldsymbol{u}$ is simple, with breakpoints $\tau_0, \dots, \tau_n$ and root value $u_\downarrow$. Suppose that $v_\downarrow = \lim_{\sigma \downarrow \mathcal{S}} v_\sigma$ and $v_\uparrow = \lim_{\sigma \uparrow \mathcal{S}} v_\sigma$ are defined in L^0. Then $\int_{\mathcal{S}} \boldsymbol{u} \, d\boldsymbol{v}$ is defined and equal to

$$u_\downarrow \times (v_{\tau_0} - v_\downarrow) + \sum_{i=0}^{n-1} u_{\tau_i} \times (v_{\tau_{i+1}} - v_{\tau_i}) + u_{\tau_n} \times (v_\uparrow - v_{\tau_n}).$$

Lecture 7

Near-simple processes A fully adapted process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is **order-bounded** if $\{u_\sigma : \sigma \in \mathcal{S}\}$ is bounded above and below in L^0. In this case, write $\sup|\boldsymbol{u}|$ for $\sup_{\sigma\in\mathcal{S}}|u_\sigma|$ (taking the supremum in $(L^0)^+$, so that $\sup|\boldsymbol{u}| = 0$ if $\mathcal{S}$ is empty).

An order-bounded fully adapted process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is **near-simple** if for every $\epsilon > 0$ there is a simple process $\boldsymbol{u}' = \langle u'_\sigma \rangle_{\sigma\in\mathcal{S}}$ such that $\theta(\sup|\boldsymbol{u} - \boldsymbol{u}'|) \le \epsilon$.

Integrators: Definitions (a) Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma\in\mathcal{S}}$ a fully adapted process. The **capped-stake variation set of $\boldsymbol{v}$ over $\mathcal{S}$** is the set $Q_{\mathcal{S}}(d\boldsymbol{v})$ of Riemann sums $S_I(\boldsymbol{u}, d\boldsymbol{v})$ where $I \in \mathcal{I}(\mathcal{S})$, $\boldsymbol{u}$ is a fully adapted process with domain I and $\sup|\boldsymbol{u}| \le \chi 1$.

(b) $\boldsymbol{v}$ is an **integrator** if

$Q_{\mathcal{S}}(d\boldsymbol{v})$ is **topologically bounded** in L^0, that is, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\theta(\delta z) \le \epsilon$ for every $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$,

either $\mathcal{S}$ is empty or $\lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ and $\lim_{\sigma\uparrow\mathcal{S}} v_\sigma$ are defined in L^0.

Remark Actually the second condition here, on the existence of limits at each end of $\mathcal{S}$, is redundant, being provable from the topological boundedness of Q. But this seems to be deep, and for the elementary theory it is much easier to carry the extra condition through the arguments.

Theorem 3 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$. If $\boldsymbol{u}$ is near-simple and $\boldsymbol{v}$ is an integrator, then $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v}$ is defined.

proof Let $\epsilon > 0$. Let $\delta > 0$ be such that $\theta(\delta z) \le \epsilon$ whenever $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$. Let $\boldsymbol{u}'$ be a simple process such that $\theta(\sup|\boldsymbol{u} - \boldsymbol{u}'|) \le \delta\epsilon$; then $a = [\![\sup|\boldsymbol{u} - \boldsymbol{u}'| \ge \delta]\!]$ has measure at most ϵ. It follows that $\theta(S_I(\boldsymbol{u}, d\boldsymbol{v}) - S_I(\boldsymbol{u}', d\boldsymbol{v})) \le 2\epsilon$ for every $I \in \mathcal{I}(\mathcal{S})$. **P** Set

$$\boldsymbol{w} = \operatorname{med}(-\mathbf{1}, \frac{1}{\delta}(\boldsymbol{u} - \boldsymbol{u}'), \mathbf{1}).$$

Then $\sup|\boldsymbol{w}| \le \chi 1$ so $S_I(\boldsymbol{w}, d\boldsymbol{v}) \in Q_{\mathcal{S}}(d\boldsymbol{v})$ and $\theta(\delta S_I(\boldsymbol{w}, d\boldsymbol{v})) \le \epsilon$. Now

$$\begin{aligned}[\![S_I(\boldsymbol{u} - \boldsymbol{u}' - \delta\boldsymbol{w}, d\boldsymbol{v}) \ne 0]\!] &\subseteq \sup_{\sigma\in I}[\![u_\sigma - u'_\sigma \ne \delta w_\sigma]\!] = \sup_{\sigma\in I}[\![|u_\sigma - u'_\sigma| > \delta]\!] \\ &\subseteq [\![\sup|\boldsymbol{u} - \boldsymbol{u}'| > \delta]\!] \subseteq a,\end{aligned}$$

so

$$\theta(S_I(\boldsymbol{u}, d\boldsymbol{v}) - S_I(\boldsymbol{u}', d\boldsymbol{v})) \le \theta(\delta S_I(\boldsymbol{w}, d\boldsymbol{v})) + \theta(S_I(\boldsymbol{u} - \boldsymbol{u}' - \delta\boldsymbol{w}, d\boldsymbol{v})) \le 2\epsilon. \textbf{ Q}$$

We know that $w = \int_{\mathcal{S}} \boldsymbol{u}'\,d\boldsymbol{v}$ is defined; let $J \in \mathcal{I}(\mathcal{S})$ be such that $\theta(w - S_I(\boldsymbol{u}', d\boldsymbol{v})) \le \epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$ and $I \supseteq J$. Then $\theta(w - S_I(\boldsymbol{u}, d\boldsymbol{v})) \le 3\epsilon$ whenever $I \in \mathcal{I}(\mathcal{S})$ and $I \supseteq J$. As ϵ is arbitrary, and L^0 is a complete linear topological space, $\int_{\mathcal{S}} \boldsymbol{u}\,d\boldsymbol{v} = \lim_{I\uparrow\mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\boldsymbol{v})$ is defined.

Càdlàg processes Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$ which is **order-convex**, that is, $\sigma \in \mathcal{S}$ whenever τ, $\tau' \in \mathcal{S}$ and $\tau \le \sigma \le \tau'$, and has a least element. I say that a fully additive process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma\in\mathcal{S}}$ is **càdlàg** if

$u_\tau = \lim_{\sigma\downarrow A} u_\sigma$ whenever $A \subseteq \mathcal{S}$ is non-empty and downwards-directed and has infimum τ,

$\lim_{\sigma\uparrow A} u_\sigma$ is defined in L^0 whenever $A \subseteq \mathcal{S}$ is non-empty and upwards-directed and has an upper bound in $\mathcal{S}$.

Theorem 4 Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least element, and $\boldsymbol{u}$ a fully adapted process with domain $\mathcal{S}$.

(a) If $\boldsymbol{u}$ is càdlàg, it is **locally near-simple**, that is, $\boldsymbol{u}\restriction\mathcal{S}\wedge\tau$ is near-simple for every $\tau \in \mathcal{S}$.

(b) Suppose that $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is right-continuous. If $\boldsymbol{u}$ is locally near-simple, it is càdlàg.

Theorem 5 Suppose that $(\mathfrak{A},\bar\mu)$ is the measure algebra of the probability space (Ω,Σ,μ), that $\Sigma_t = \{E : E^\bullet \in \mathfrak{A}_t\}$ for every $t \ge 0$, that $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is right-continuous, and that $\langle X_t\rangle_{t\ge0}$ is a progressively measurable stochastic process with corresponding fully adapted process $\boldsymbol{u}$ defined on $\mathcal{T}_f$. If almost every path $t \mapsto X_t(\omega) : [0,\infty[\to \mathbb{R}$ is a càdlàg real function, then $\boldsymbol{u}$ is càdlàg.

Remark The 'usual conditions' of most authors include the hypothesis that the filtration is right-continuous; integration is normally over order-convex sublattices with least elements; and processes are normally assumed to be càdlàg.

Lecture 8

Variations on integration: adapted local interval functions A general feature of 'gauge integrals', such as the stochastic integral described here, is that they suggest variations. We have a structure with elements

$$\mathcal{S}, \quad \mathcal{I}(\mathcal{S}), \quad \{e : e \text{ is an } I\text{-cell}\}$$

and a formula

$$\Delta_e(\boldsymbol{u}, d\boldsymbol{v}) = u_\sigma \times (v_\tau - v_\sigma)$$

leading naturally to Riemann sums $S_I(\boldsymbol{u}, d\boldsymbol{v})$ and integrals $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$. If we replace the difference $v_\tau - v_\sigma$ by a more general function $\psi(\sigma, \tau)$, we shall be able to proceed as before provided that we always have

$$u_\sigma \times \psi(\sigma, \tau) = u_{\sigma'} \times \psi(\sigma', \tau')$$

whenever $c(\sigma, \tau) = c(\sigma', \tau')$, that is, whenever $[\![\sigma < \tau]\!] = [\![\sigma' < \tau']\!] \subseteq [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!]$. For this we shall need, first,

$$\psi(\sigma, \tau) = \psi(\sigma', \tau') \text{ whenever } [\![\sigma < \tau]\!] = [\![\sigma' < \tau']\!] \subseteq [\![\sigma = \sigma']\!] \cap [\![\tau = \tau']\!],$$

and then

$$[\![\psi(\sigma, \tau) \neq 0]\!] \subseteq [\![\sigma < \tau]\!].$$

For a general theory which will be useful in the present context, we need also

$$\psi(\sigma, \tau) \in L^0(\mathfrak{A}_\tau) \text{ whenever } \sigma \leq \tau.$$

Such a function ψ, defined on $\{(\sigma, \tau) : \sigma \leq \tau \text{ in } \mathcal{S}\}$ for a sublattice $\mathcal{S}$ of $\mathcal{T}$, I will call an **adapted local integrable function**; the associated constructions are

$$\Delta_e(\boldsymbol{u}, d\psi) = u_\sigma \times \psi(\sigma, \tau), \qquad S_I(\boldsymbol{u}, d\psi) = \textstyle\sum_{e \text{ is an } I\text{-cell}} \Delta_e(\boldsymbol{u}, d\psi),$$

$$\textstyle\int_{\mathcal{S}} \boldsymbol{u}\, d\psi = \lim_{I \uparrow \mathcal{I}(\mathcal{S})} S_I(\boldsymbol{u}, d\psi)$$

when this is defined.

Examples (a) $\psi(\sigma, \tau) = v_\tau - v_\sigma$ where $\boldsymbol{v}$ is fully adapted.

(b) $\psi(\sigma, \tau) = \bar{h}(\phi(\sigma, \tau))$ for an adapted local interval function ϕ and a Borel measurable $h : \mathbb{R} \to \mathbb{R}$. When $\psi(\sigma, \tau) = |v_\tau - v_\sigma|$ for a fully adapted process $\boldsymbol{v}$, I will write $\Delta_e(\boldsymbol{u}, |d\boldsymbol{v}|)$, etc.

(c) Sums and products of adapted local interval functions.

Bounded variation Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process. Then $\boldsymbol{v}$ is of **bounded variation** if $\{S_I(\mathbf{1}, |d\boldsymbol{v}|) : I \in \mathcal{I}(\mathcal{S})\}$ is bounded above in L^0.

We find that $S_J(\mathbf{1}, |d\boldsymbol{v}|) \leq S_I(\mathbf{1}, |d\boldsymbol{v}|)$ whenever $J \subseteq I$ in $\mathcal{I}(\mathcal{S})$. So $\{S_I(\mathbf{1}, |d\boldsymbol{v}|) : I \in \mathcal{I}(\mathcal{S})\}$ is upwards-directed and has an upper bound iff $\int_{\mathcal{S}} \mathbf{1}\, |d\boldsymbol{v}| = \int_{\mathcal{S}} |d\boldsymbol{v}|$ is defined.

In this case, we can set $z_\tau = \int_{\mathcal{S}\wedge\tau} |d\boldsymbol{v}|$ for every $\tau \in \mathcal{S}$ and find that $\boldsymbol{z}$ and $\boldsymbol{z} - \boldsymbol{v}$ are both non-decreasing; it is also the case that $\lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ is defined, so $\boldsymbol{v}$ can be expressed as the difference of non-negative non-decreasing order-bounded processes.

Theorem 6 A process of bounded variation is an integrator.

Lecture 9

Brownian motion The most important of all continuous-time stochastic processes is 'Brownian motion'. There are many ways of describing this. For definiteness I will take the following formulation from FREMLIN 03[1]. Let Ω be the set of continuous functions $\omega : [0,\infty[\to \mathbb{R}$ such that $\omega(0) = 0$. For $t \geq 0$ set $X_t(\omega) = \omega_t$. Let Σ' be the σ-algebra of subsets of Ω generated by these coordinate functionals, and Σ'_t the σ-algebra generated by $\{X_s : s \leq t\}$ for $t \geq 0$. Then there is a unique probability measure μ' with domain Σ' such that

> whenever $0 \leq s < t$, $X_t - X_s$ is normally distributed with expectation 0 and variance $t - s$, and is independent of Σ'_s.

Let μ be the completion of μ' and set $\Sigma_t = \{E\triangle F : E \in \Sigma'_t,\ \mu F = 0\}$ for each t. Then the conditions of Theorem 1 are satisfied. Let $\boldsymbol{w}$ be the stochastic process defined by the construction there, based on the measure algebra of μ. I will call $\boldsymbol{w}$ **Brownian motion**.

The Poisson process The fact that Brownian motion has continuous sample paths gives it a large number of special properties. 'Modern' theories of stochastic calculus, from the 1960s on, have been developed to deal with discontinuous processes, of which the most important is the 'Poisson process'. Once again, I fix on a formulation based on material in FREMLIN 03. Let Ω be the set of infinite, locally finite subsets of $]0,\infty[$. For $t \geq 0$, set $X_t(\omega) = \#(\omega \cap [0,t])$. Once again, let Σ' be the σ-algebra of subsets of Ω generated by these functionals, and Σ'_t the σ-algebra generated by $\{X_s : s \leq t\}$ for $t \geq 0$. Then there is a unique probability measure μ' with domain Σ' such that

> whenever $0 \leq s < t$, $X_t - X_s$ has a Poisson distribution with expectation $t - s$, and is independent of Σ'_s.

Let μ be the completion of μ' and set $\Sigma_t = \{E\triangle F : E \in \Sigma'_t,\ \mu F = 0\}$ for each t. Then the conditions of Theorem 1 are satisfied. I will say that the stochastic process $\boldsymbol{v}$ defined by the construction there, based on the measure algebra of μ, is the **Poisson process**.

Remarks Let $\boldsymbol{v}$ be the Poisson process as just described. Note that the sample paths $t \mapsto X_t(\omega)$ are càdlàg, so Theorem 5 is applicable, and $\boldsymbol{v}$ is càdlàg, therefore locally near-simple. Next, $\boldsymbol{v}$ is non-decreasing, so is a **local integrator**, that is, $\boldsymbol{v}\restriction \operatorname{dom}\boldsymbol{v} \wedge \tau$ is an integrator for every $\tau \in \operatorname{dom}\boldsymbol{v}$. Consequently we shall have, for instance, an indefinite integral $ii_{\boldsymbol{v}}(\boldsymbol{v}) = \langle \int_{[\min \mathcal{T},\tau]} \boldsymbol{v}\, d\boldsymbol{v} \rangle_{\tau\in\mathcal{T}_f}$ defined everywhere on $\mathcal{T}_f = \operatorname{dom}\boldsymbol{v}$.

Even if you work through all the details of all the proofs of the theorems I have given so far, you will find yourselves singularly lacking in techniques for evaluating particular integrals. If I say that

$$ii_{\boldsymbol{v}}(\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{v}^2 - \boldsymbol{v}),$$

you have at least a chance of checking this by methods based on the definition I gave of the integral. The corresponding formula for Brownian motion

[1] Later editions only.

$$ii_{\boldsymbol{w}}(\boldsymbol{w}) = \frac{1}{2}(\boldsymbol{w}^2 - \boldsymbol{\iota})$$

where $\boldsymbol{\iota} = \langle \tau \rangle_{\tau \in \mathcal{T}_f}$ is the **identity process**, seems to be much harder. In fact while we can use Theorem 5 to see that $\boldsymbol{w}$ is locally near-simple, I have not even shown that it is a local integrator, so it is far from clear that $ii_{\boldsymbol{w}}(\boldsymbol{w})$ is defined. For this we need the first really hard theorem of the subject, which I will give in the next lecture.

Lecture 10

Revision and clarification.

Lecture 11

Conditional expectations If $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, $(\mathfrak{B},\bar\mu\restriction\mathfrak{B})$ is again a probability algebra. The Radon-Nikodým theorem tells us that for every $u\in L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)$ we have a unique $Pu\in L^1(\mathfrak{B},\bar\mu\restriction\mathfrak{B})=L^1_{\bar\mu}\cap L^0(\mathfrak{B})$ such that $\mathbb{E}(Pu\times\chi b)=\mathbb{E}(u\times\chi b)$ for every $b\in\mathfrak{B}$. The map $P:L^1_{\bar\mu}\to L^1_{\bar\mu}$ is linear, positive ($Pu\ge 0$ if $u\ge 0$), of norm 1 ($\|Pu\|_1=\mathbb{E}(|Pu|)\le\|u\|_1$ for every $u\in L^1_{\bar\mu}$)), a projection ($P^2=P$), and $\|Pu\|_\infty\le\|u\|_\infty$ for every $u\in L^\infty(\mathfrak{A})$ (that is, whenever $|u|\le\gamma\chi 1$ for some γ).

We shall need to know that

— if $u\in L^0(\mathfrak{B})$, $v\in L^1_{\bar\mu}$ and $u\times v\in L^1_{\bar\mu}$, then $P(u\times v)=u\times Pv$,

— if $v\in L^1_{\bar\mu}$, $h:\mathbb{R}\to\mathbb{R}$ is convex and $\bar h(v)\in L^1_{\bar\mu}$, then $\bar h(Pv)\le P(\bar h(v))$.

So, for instance, if $v\in L^2_{\bar\mu}$, that is, $v^2\in L^1_{\bar\mu}$, then $Pv\in L^2_{\bar\mu}$ and $\|Pv\|_2\le\|v\|_2$.

Finite martingales Now suppose that $\mathfrak{A}_0\subseteq\ldots\subseteq\mathfrak{A}_n$ are closed subalgebras of $\mathfrak{A}$. A finite sequence $\langle v_i\rangle_{i\le n}$ in $L^1_{\bar\mu}$ is a **martingale adapted to** $\langle\mathfrak{A}_i\rangle_{i\le n}$ if $v_i\in L^0(\mathfrak{A}_i)$ (that is, $v_i\in L^1(\mathfrak{A}_i,\bar\mu\restriction\mathfrak{A}_i)$) for each $i\le n$ and $P_iv_j=v_i$ whenever $i\le j$, where $P_i:L^1_{\bar\mu}\to L^1_{\bar\mu}$ is the conditional expectation associated with $\mathfrak{A}_i$.

Theorem 7 Let $\langle v_i\rangle_{i\le n}$ be a martingale adapted to $\langle\mathfrak{A}_i\rangle_{i\le n}$. Suppose that $\langle u_i\rangle_{i<n}$ is such that $u_i\in L^0(\mathfrak{A}_i)$ and $|u_i|\le\chi 1$ for $i<n$. Take M, $\delta>0$. Setting $z=\sum_{i=0}^{n-1}u_i\times(v_{i+1}-v_i)$,

$$\theta(\delta z)\le\delta M+\frac{1}{M}\|v_n\|_1.$$

Doob's maximal inequality Let $\langle v_i\rangle_{i\le n}$ be a martingale. Setting $\bar v=\sup_{i\le n}|v_i|$,

$$\bar\mu[\![\bar v>M]\!]\le\frac{1}{M}\|v_n\|_1$$

for every $M>0$.

Lemma 1 Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\langle v_i\rangle_{i\le n}$ be a martingale in $L^1_{\bar\mu}$ adapted to a sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$. Suppose that $M>0$. Then there are a probability algebra $(\mathfrak{B},\bar\nu)$ with closed subalgebras $\mathfrak{B}_0,\ldots,\mathfrak{B}_{2n}$, a martingale $\langle w_j\rangle_{j\le 2n}$ adapted to $\langle\mathfrak{C}_j\rangle_{j\le 2n}$, and an em embedding of $\mathfrak{A}$ as a closed subalgebra of $\mathfrak{B}$ such that

$$\mathfrak{A}_i=\mathfrak{A}\cap\mathfrak{B}_{2i}\text{ for }i\le n,$$

$$w_{2i}=v_i\text{ for }i\le n,$$

$$[\![|w_j|\ge M]\!]\subseteq[\![|v_0|\ge M]\!]\cup\sup\nolimits_{k\le j}[\![|w_k|=M]\!]\text{ for }j\le 2n.$$

Lemma 2 Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\langle v_i\rangle_{i\le n}$ be a martingale in $L^1_{\bar\mu}$ adapted to a sequence $\langle\mathfrak{A}_i\rangle_{i\le n}$ of closed subalgebras of $\mathfrak{A}$. Suppose that $M>0$. Then there are a probability algebra $(\mathfrak{B},\bar\nu)$ with closed subalgebras $\mathfrak{C}_0\subseteq\ldots\subseteq\mathfrak{C}_n$, a martingale $\langle\hat v_i\rangle_{i\le n}$ adapted to $\langle\mathfrak{C}_i\rangle_{i\le n}$, and an embedding of $\mathfrak{A}$ as a closed subalgebra of $\mathfrak{B}$ such that

$$\mathfrak{A}_i=\mathfrak{A}\cap\mathfrak{C}_i \text{ for } i\le n,$$

$$|\hat v_i|\le M\chi 1 \text{ for } i\le n,$$

$$\bar\nu(\sup_{i\le n}[\![v_i\ne\hat v_i]\!])\le\frac{1}{M}\|v_n\|_1.$$

proof of theorem from Lemma 2 Regarding the u_i as members of $L^0(\mathfrak{C}_i)\subseteq L^0(\mathfrak{B})$, set

$$\hat z=\sum_{i=0}^{n-1}u_i\times(\hat v_{i+1}-\hat v_i).$$

Then

$$[\![\delta z\ne\delta\hat z]\!]=[\![\hat z\ne z]\!]\subseteq\sup_{i\le n}[\![v_i\ne\hat v_i]\!]$$

has measure at most $\frac{1}{M}\|v_n\|_1$, and $\theta(\delta z-\delta\hat z)\le\frac{1}{M}\|v_n\|_1$. Next, setting $\hat z_i=\sum_{j=0}^{n-1}u_j\times(\hat v_{j+1}-\hat v_j)$ for $i\le n$ (starting from $\hat z_0=0$), we see that $\hat z_i\in L^0(\mathfrak{C}_i)$ while $\hat z_{i+1}-\hat z_i=u_i\times(\hat v_{i+1}-\hat v_i)$ so, taking P_i to be the conditional expectation associated with $\mathfrak{C}_i$,

$$\begin{aligned}\mathbb{E}(\hat z_i\times(\hat z_{i+1}-\hat z_i))&=\mathbb{E}(\hat z_i\times u_i\times(\hat v_{i+1}-\hat v_i))=\mathbb{E}(P_i(\hat z_i\times u_i\times(\hat v_{i+1}-\hat v_i)))\\&=\mathbb{E}(\hat z_i\times u_i\times P_i(\hat v_{i+1}-\hat v_i))=\mathbb{E}(\hat z_i\times u_i\times(P_i\hat v_{i+1}-\hat v_i))=0.\end{aligned}$$

Consequently

$$\begin{aligned}\|\hat z_{i+1}\|_2^2&=\mathbb{E}((\hat z_i+(\hat z_{i+1}-\hat z_i))^2)=\mathbb{E}(\hat z_i^2)+\mathbb{E}((\hat z_{i+1}-\hat z_i)^2)\\&=\mathbb{E}(\hat z_i^2)+\mathbb{E}(u_i^2\times(\hat v_{i+1}-\hat v_i)^2)\\&\le\mathbb{E}(\hat z_i^2)+\mathbb{E}((\hat v_{i+1}-\hat v_i)^2)=\mathbb{E}(\hat z_i^2)+\mathbb{E}(\hat v_{i+1}^2)-\mathbb{E}(\hat v_i^2)\end{aligned}$$

for $i\le n$. It follows that

$$\|\hat z_n\|_2^2\le\|\hat v_n\|_2^2-\|\hat v_0\|_2^2\le M^2$$

and

$$\theta(\delta\hat z)=\theta(\delta\hat z_n)\le\mathbb{E}(\delta|\hat z_n|)=\delta\|\hat z_n\|_1\le\delta\|\hat z_n\|_2\le\delta M.$$

Putting these together,

$$\theta(\delta z)\le\theta(\delta\hat z)+\theta(\delta z-\delta\hat z)\le\delta M+\frac{1}{M}\|v_n\|_1.$$

proof of Lemma 2 from Lemma 1 Set $\mathfrak{C}_i=\mathfrak{B}_{2i}$, so $\mathfrak{A}_i=\mathfrak{A}\cap\mathfrak{C}_i$ for each i. Start by taking $w'_j=w_j\times\chi[\![|w_0|<M]\!]$; then $\langle w'_j\rangle_{j\le 2n}$ is a martingale and $[\![|w'_j|\ge M]\!]\subseteq\sup_{k\le j}[\![|w'_k|=M]\!]$ for $j\le 2n$. Now let $\langle\hat w_j\rangle_{j\le 2n}$ be the stopped martingale which freezes $\langle w'_j\rangle_{j\le 2n}$ at the first time it takes the value $\pm M$, so that

$$|\hat{w}_j| \le M\chi 1 \text{ for every } j \le 2n,$$

$$\sup_{j\le 2n} [\![\hat{w}_j \ne w'_j]\!] \subseteq \sup_{k\le 2n} [\![|w'_k| \ge M]\!].$$

Set $\hat{v}_i = \hat{w}_{2i}$ for $i \le n$; then $\langle \hat{v}_i \rangle_{i\le n}$ is a martingale adapted to $\langle \mathfrak{C}_i \rangle_{i\le n}$ and

$$\begin{aligned}\sup_{i\le n} [\![v_i \ne \hat{v}_i]\!] &\subseteq \sup_{i\le n} [\![w_{2i} \ne w'_{2i}]\!] \cup \sup_{i\le n} [\![w'_{2i} \ne \hat{w}_{2i}]\!] \\ &\subseteq [\![|w_0| \ge M]\!] \cup \sup_{k\le 2n} [\![|w'_k| \ge M]\!] \subseteq \sup_{k\le 2n} [\![|w_k| \ge M]\!]\end{aligned}$$

has measure at most

$$\frac{1}{M}\|w_{2n}\|_1 = \frac{1}{M}\|v_n\|_1$$

by Doob's maximal inequality.

Lecture 12

Martingale processes Returning to the context developed in Lectures 1-8, let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a fully adapted process. Then $\boldsymbol{v}$ is a **martingale** if

$$v_\sigma \in L^1_{\bar{\mu}} \text{ for every } \sigma \in \mathcal{S},$$

$$v_\sigma = P_\sigma v_\tau \text{ whenever } \sigma \le \tau \text{ in } \mathcal{S},$$

where $P_\sigma : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}}$ is the conditional expectation operator corresponding to the closed subalgebra $\mathfrak{A}_\sigma$.

Proposition Suppose that a $\boldsymbol{v}$ is a càdlàg fully adapted process with domain $\mathcal{T}_b$ and that $\boldsymbol{v} \restriction \dot{\mathcal{T}}$ is a martingale, where $\dot{\mathcal{T}}$ is the lattice of constant stopping times. Then $\boldsymbol{v}$ is a martingale.

Corollary The restriction $\boldsymbol{w} \restriction \mathcal{T}_b$ of Brownian motion to the bounded stopping times is a martingale.

Lemma Suppose that I is a non-empty finite sublattice of $\mathcal{T}$.
(a) There are $\sigma_0 \le \ldots \le \sigma_n$ in I such that $\{c(\sigma_i, \sigma_{i+1}) : i < n\}$ is the set of I-cells.
(b) If $\boldsymbol{u}$, $\boldsymbol{v}$ are fully adapted processes with domains including I,

$$S_I(\boldsymbol{u}, d\boldsymbol{v}) = \sum_{i=0}^{n-1} u_{\sigma_i} \times (v_{\sigma_{i+1}} - v_{\sigma_i}).$$

Theorem 8 If $\mathcal{S}$ is a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ a martingale, then $\boldsymbol{v}$ is a local integrator.

proof (a) Suppose that to begin with that $\mathcal{S}$ has a greatest member. Of course $\lim_{\sigma \uparrow \mathcal{S}} v_\sigma = v_{\max \mathcal{S}}$ is defined. If $\langle \sigma_n \rangle_{n \in \mathbb{N}}$ is any non-increasing sequence in $\mathcal{S}$ then $\lim_{n \to \infty} v_{\sigma_n}$ is defined by the reverse martingale theorem; it follows that $v_\downarrow = \lim_{\sigma \downarrow \mathcal{S}} v_\sigma$ is defined (because L^0 is complete).

(b) Now consider $Q_{\mathcal{S}}(d\boldsymbol{v})$. Let $\epsilon > 0$. Let M, $\delta > 0$ be such that $\delta M + \frac{1}{M}\|v_{\max \mathcal{S}}\|_1 \le \epsilon$. Suppose that $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$. Then there are a finite sublattice I of $\mathcal{S}$ and a fully adapted process $\boldsymbol{u} = \langle u_\sigma \rangle_{\sigma \in I}$ with domain I such that $|u_\sigma| \le \chi 1$ for every $\sigma \in I$ and and $z = S_I(\boldsymbol{u}, d\boldsymbol{v})$. By the lemma, there are $\sigma_0 \le \ldots \sigma_n$ in I such that $z = \sum_{i=0}^{n-1} u_{\sigma_i} \times (v_{\sigma_{i+1}} - v_{\sigma_i})$.
Applying Theorem 7 to $\langle A_{\sigma_i} \rangle_{i \le n}$, $\langle v_{\sigma_i} \rangle_{i \le n}$ and $\langle u_{\sigma_i} \rangle_{i \le n}$, we see that

$$\theta(\delta z) \le \delta M + \frac{1}{M}\|v_{\sigma_n}\|_1 = \delta M + \frac{1}{M}\|P_{\sigma_n} v_\tau\|_1 \le \delta M + \frac{1}{M}\|v_\tau\|_1 \le \epsilon,$$

and this is true for every $z \in Q_{\mathcal{S}}(d\boldsymbol{v})$.

(c) As ϵ is arbitrary, $Q_{\mathcal{S}}(d\boldsymbol{v})$ is topologically bounded and $\boldsymbol{v}$ is an integrator. This was on the assumption that $\mathcal{S}$ had a greatest member. For general lattices $\mathcal{S}$ and martingales $\boldsymbol{v}$ with domain $\mathcal{S}$, apply this to $\mathcal{S} \wedge \tau$ and $\boldsymbol{v} \restriction \mathcal{S} \wedge \tau$ to see that $\boldsymbol{v}$ is a local integrator.

Corollary Brownian motion is a local integrator.

Lecture 13

We are collecting a classification of stochastic processes: so far, I have talked about simple processes, order-bounded processes, near-simple processes, integrators, càdlàg processes, processes of bounded variation and martingales, with 'local' versions of many of these. Associated with every class is a string of natural questions: is it closed under addition/scalar multiplication/multiplication/lattice operations/operations $\boldsymbol{u} \mapsto \bar{h}(\boldsymbol{u})$ (and for which functions h), restriction to sublattices, restriction to initial segments $\mathcal{S} \wedge \tau$? And then we have the operation of indefinite integration: when can we deduce properties of $ii_{\boldsymbol{v}}(\boldsymbol{u})$ from properties of $\boldsymbol{v}$ and $\boldsymbol{u}$? Some of these questions are easy, some are hard, some depend on whether the filtration is right-continuous. I can testify that there are months of innocent enjoyment to be had from them. Here I can mention only a handful.

The class of simple processes on a given sublattice $\mathcal{S}$ is closed under all the operations described, including $(\boldsymbol{u}, \boldsymbol{v}) \mapsto ii_{\boldsymbol{v}}(\boldsymbol{u})$; and also under restriction to initial segments, but not restriction to arbitrary sublattices. Integration you would probably have to think about, the rest are straightforward. After this, things get trickier. Near-simple processes are closed under the operation $\boldsymbol{u} \mapsto \bar{h}(\boldsymbol{u})$ for continuous h, but not for general Borel functions h. If $\boldsymbol{u}$ is near-simple and $\boldsymbol{v}$ is a near-simple integrator, then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is near-simple; the ideas of the proof of Theorem 3 are essentially sufficient for this. We have a couple of further results of this kind; the first straightforward and useful, the second really important.

Theorem 9 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$ such that $\boldsymbol{u}$ is near-simple and $\boldsymbol{v}$ is of bounded variation. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is of bounded variation.

proof If $\bar{u} = \sup |\boldsymbol{u}|$ and $\bar{v} = \int_{\mathcal{S}} |d\boldsymbol{v}|$, and $e = c(\sigma, \tau)$ is a stopping-time interval with endpoints in $\mathcal{S}$, then

$$|\Delta_e(\boldsymbol{u}, d\boldsymbol{v})| \le \bar{u} \times \Delta_e(\mathbf{1}, |d\boldsymbol{v}|),$$

$$|S_I(\boldsymbol{u}, d\boldsymbol{v})| \le \bar{u} \times S_I(\mathbf{1}, |d\boldsymbol{v}|) \le \bar{u} \times \int_{\mathcal{S} \cap [\min I, \max I]} |d\boldsymbol{v}|,$$

$$\Delta_e(\mathbf{1}, |d\, ii_{\boldsymbol{v}}(\boldsymbol{u})|) = |\int_{\mathcal{S} \cap [\sigma, \tau]} \boldsymbol{u}\, d\boldsymbol{v}| \le \bar{u} \times \int_{\mathcal{S} \cap [\sigma, \tau]} |d\boldsymbol{v}|,$$

$$S_I(\mathbf{1}, |d\, ii_{\boldsymbol{v}}(\boldsymbol{u})|) \le \bar{u} \times \int_{\mathcal{S} \cap [\min I, \max I]} |d\boldsymbol{v}| \le \bar{u} \times \bar{v},$$

$$\int_{\mathcal{S}} |d\, ii_{\boldsymbol{v}}(\boldsymbol{u})| \le \bar{u} \times \bar{v}.$$

Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) A **covering ideal** of $\mathcal{S}$ is a sublattice $\mathcal{S}'$ of $\mathcal{S}$ such that
$\sigma \wedge \tau \in \mathcal{S}'$ whenever $\sigma \in \mathcal{S}'$ and $\tau \in \mathcal{S}$,
$\sup_{\sigma \in \mathcal{S}'} [\![\sigma = \tau]\!] = 1$ for every $\tau \in \mathcal{S}$.

Remarks In this case, any fully adapted process with domain $\mathcal{S}'$ has a unique extension to a fully adapted process with domain $\mathcal{S}$. A process $\boldsymbol{u}$ with domain $\mathcal{S}$ is a (local) integrator iff $\boldsymbol{u} \restriction \mathcal{S}'$ is a (local) integrator. Note that $\mathcal{T}_b$ is a covering ideal of $\mathcal{T}_f$.

(b) A fully adapted process $\boldsymbol{u}$ with domain $\mathcal{S}$ is a **local martingale** if there is a covering ideal $\mathcal{S}'$ of $\mathcal{S}$ such that $\boldsymbol{u}{\restriction}\mathcal{S}'$ is a martingale.

Remarks Note that I am *not* talking about restrictions $\boldsymbol{u}{\restriction}\mathcal{S} \wedge \tau$! Observe that Brownian motion, regarded as defined on $\mathcal{T}_b$, is a local martingale. Local martingales are local integrators.

Lemma Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v} = \langle v_\sigma \rangle_{\sigma \in \mathcal{S}}$ fully adapted processes such that $\boldsymbol{u}$ is near-simple, $\sup |\boldsymbol{u}| \le \chi 1$, $\boldsymbol{v}$ is a martingale, and $v_\sigma \in L^2_{\bar{\mu}}$ for every $\sigma \in \mathcal{S}$. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a martingale.

Theorem 10 Suppose that the filtration $\langle \mathfrak{A}_t \rangle_{t \ge 0}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least element and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes with domain $\mathcal{S}$ such that $\boldsymbol{u}$ is càdlàg and $\boldsymbol{v}$ is a local martingale. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is a local martingale.

Lecture 14

It is easy to check that sums and scalar multiples of integrators are integrators; the key fact is that

$$\begin{aligned}Q_{\mathcal{S}}(d(\boldsymbol{v}+\boldsymbol{w})) &= \{S_I(\boldsymbol{u}, d(\boldsymbol{v}+\boldsymbol{w})) : I \in \mathcal{I}(\mathcal{S}),\ \sup|u| \le \chi 1\}\\ &= \{S_I(\boldsymbol{u}, d\boldsymbol{v}) + S_(\boldsymbol{u}, d\boldsymbol{w}) : I \in \mathcal{I}(\mathcal{S}),\ \sup|u| \le \chi 1\} \subseteq Q_{\mathcal{S}}(d\boldsymbol{v}) + Q_{\mathcal{S}}(d\boldsymbol{w})\end{aligned}$$

and the linear sum of topologically bounded sets (in any linear topological space) is topologically bounded. But products $\boldsymbol{v} \times \boldsymbol{w}$ are much harder. We can get at these through the following theorem.

Theorem 11 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\sigma\rangle_{\sigma\in\mathcal{S}}$ an integrator, and $h : \mathbb{R} \to \mathbb{R}$ a convex function. Then $\bar{h}(\boldsymbol{v})$ is an integrator.

proof (a) First note that, because h is continuous,

$$\lim_{\sigma\downarrow\mathcal{S}} \bar{h}(v_\sigma) = \bar{h}(\lim_{\sigma\downarrow\mathcal{S}} v_\sigma)$$

is defined, and similarly for $\sigma \uparrow \mathcal{S}$.

(b) Write Q for $Q_{\mathcal{S}}(d\boldsymbol{v})$, Q^* for $Q_{\mathcal{S}}(d\bar{h}(\boldsymbol{v}))$. Let g be the right derivative of h, that is, $g(x) = \lim_{y\downarrow x} \frac{h(y)-h(x)}{y-x}$ for $x \in \mathbb{R}$, so that g is non-decreasing and $(y-x)g(x) \le h(y)-h(x)$ for all x, $y \in \mathbb{R}$. Consequently

$$\bar{g}(v_\sigma) \times (v_\tau - v_\sigma) \le \bar{h}(v_\tau) - \bar{h}(v_\sigma)$$

whenever $\sigma \le \tau$ in $\mathcal{S}$. Suppose for the time being that $|g(x)| \le M$ for every $x \in \mathbb{R}$.

(c) Check that integrators are always order-bounded (this is not quite trivial), so $\boldsymbol{v}$ is order-bounded, and consequently (because h is bounded on bounded intervals) $\bar{h}(\boldsymbol{v})$ is order-bounded. Set $\bar{w} = \sup|\bar{h}(\boldsymbol{v})|$.

(d) $A_0 = [-2\bar{w}, \bar{w}] + MQ$, its solid hull $A_1 = \{u : |u| \le |v| \text{ for some } v \in A_0\}$ and $A = A_1 + MQ$ are topologically bounded. Now $Q^* \subseteq A$. **P** Suppose that $I \in \mathcal{I}(\mathcal{S})$, $\sup|\boldsymbol{u}| \le \chi 1$ and $z = S_I(\boldsymbol{u}, d\bar{h}(\boldsymbol{v}))$. Let $\sigma_0 \le \ldots \le \sigma_n \in I$ be such that $\{c(\sigma_i, \sigma_{i+1}) : i < n\}$ is the set of I-cells (see the Lemma in Lecture 12). For $i \le n$ set

$$w_i = \bar{h}(v_{\sigma_{i+1}}) - \bar{h}(v_{\sigma_0}) - \textstyle\sum_{j=0}^{i-1} \bar{g}(v_{\sigma_j}) \times (v_{\sigma_{j+1}} - v_{\sigma_j}) \in [-2\bar{w}, 2\bar{w}] + MQ = A_0.$$

We have

$$w_{i+1} - w_i = \bar{h}(v_{\sigma_{i+1}}) - \bar{h}(v_{\sigma_i}) - \bar{g}(v_{\sigma_i}) \times (v_{\sigma_{i+1}} - v_{\sigma_i}) \ge 0$$

for each i. Now

$$\begin{aligned}z &= \sum_{i=1}^{n-1} u_{\sigma_i} \times (\bar{h}(v_{\sigma_{i+1}}) - \bar{h}(v_{\sigma_i}))\\ &= \sum_{i=1}^{n-1} u_{\sigma_i} \times (w_{i+1} - w_i) + \sum_{i=1}^{n-1} u_{\sigma_i} \times \bar{g}(v_{\tau_i}) \times (v_{\tau_{i+1}} - v_{\tau_i}).\end{aligned}$$

But

$$|\sum_{i=1}^{n-1} u_{\sigma_i} \times (w_{i+1} - w_i)| \leq \sum_{i=1}^{n-1} w_{i+1} - w_i = w_n \in A_0,$$

$$\sum_{i=1}^{n-1} u_{\sigma_i} \times \bar{g}(v_{\tau_i}) \times (v_{\tau_{i+1}} - v_{\tau_i}) \in MQ,$$

so $z \in A_1 + MQ = A$. **Q**

(e) Thus $Q_{\mathcal{S}}(d\bar{h}(\boldsymbol{v}))$ is topologically bounded and $\bar{h}(\boldsymbol{v})$ is an integrator, at least when its right derivative is bounded. In general, look at

$$\begin{aligned} h_K(x) &= h(x) \text{ if } |x| \leq K, \\ &= h(K) + (K - x)g(K) \text{ if } x \geq K, \\ &= h(-K) + (-K - x)g(-K) \text{ if } x \leq -K; \end{aligned}$$

use (b)-(d) to see that $Q_{\mathcal{S}}(d\bar{h}_K(\boldsymbol{v}))$ is always topologically bounded; and show that $Q_{\mathcal{S}}(d\bar{h}(\boldsymbol{v}))$ is approximated in the right way by $Q_{\mathcal{S}}(d\bar{h}_K(\boldsymbol{v}))$, for large K, to be topologically bounded.

Corollary If $\boldsymbol{v}$, $\boldsymbol{w}$ are integrators, then $\boldsymbol{v}^2$, $\boldsymbol{v} \times \boldsymbol{w} = \frac{1}{2}((\boldsymbol{v} + \boldsymbol{w})^2 - \boldsymbol{v}^2 - \boldsymbol{w}^2)$ and $|\boldsymbol{v}|$ are integrators. Moreover, if $h : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, so that it is the difference of two convex functions, then $\bar{h}(\boldsymbol{v})$ is an integrator.

Lecture 15

Integrating interval functions In Lecture 8 I briefly mentioned the possibility of integrating with respect to an ‘adapted local interval function’. I now return to this idea.

Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. An **integrating interval function** on $\mathcal{S}$ is an adapted local interval function ψ on $\mathcal{S}$ such that

$\int_{\mathcal{S}} d\psi = \int_{\mathcal{S}} \mathbf{1}\, d\psi$ is defined,

$Q_{\mathcal{S}}(d\psi) = \{S_I(\boldsymbol{u}, d\psi) : I \in \mathcal{I}(\mathcal{S}),\ \sup|\boldsymbol{u}| \le \chi 1\}$ is topologically bounded.

Theorem 12 If $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ is a near-simple process with domain $\mathcal{S}$, and ψ is an integrating interval function on $\mathcal{S}$, then $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ is defined.

proof As Theorem 3.

Theorem 13 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ a near-simple process with domain $\mathcal{S}$, and ψ an integrating interval function on $\mathcal{S}$.

(a) Set $\boldsymbol{v} = ii_\psi(\boldsymbol{u})$, that is, $v_\tau = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\psi$ for $\tau \in \mathcal{S}$. Then $\boldsymbol{v}$ is an integrator.

(b) Let $\boldsymbol{u}\psi$ be the interval function defined by saying that $(\boldsymbol{u}\psi)(\sigma,\tau) = u_\sigma \times \psi(\sigma,\tau)$ for $\sigma \le \tau$ in $\mathcal{S}$. Then $\boldsymbol{u}\psi$ is an integrating interval function.

(c) $\int_{\mathcal{S}} \boldsymbol{w}\, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{w}\, d(\boldsymbol{u}\psi) = \int_{\mathcal{S}} \boldsymbol{w} \times \boldsymbol{u}\, d\psi$ for any near-simple process $\boldsymbol{w}$ with domain $\mathcal{S}$.

Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v} = \langle v_\tau\rangle_{\tau\in\mathcal{S}}$ an integrator. Then $Q_{\mathcal{S}}((d\boldsymbol{v})^2)$ is topologically bounded.

proof Set $\psi(\sigma,\tau) = v_\tau - v_\sigma$ for $\sigma \le \tau$ in $\mathcal{S}$. If $I \in \mathcal{I}(\mathcal{S})$ and $\boldsymbol{u} = \langle u_\sigma\rangle_{\sigma\in I}$ is fully adapted,

$$u_\sigma \times (v_\tau - v_\sigma)^2 = u_\sigma \times (v_\tau^2 - v_\sigma^2) - 2u_\sigma \times v_\sigma \times (v_\tau - v_\sigma).$$

Hence

$$Q_{\mathcal{S}}((d\boldsymbol{v})^2) \subseteq Q_{\mathcal{S}}(d(\boldsymbol{v}^2)) - 2Q_{\mathcal{S}}(\boldsymbol{v}d\boldsymbol{v})$$

is topologically bounded.

Quadratic variation Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a near-simple integrator with domain $\mathcal{S}$. The **quadratic variation** $\boldsymbol{v}^*$ of $\boldsymbol{v}$ is $\boldsymbol{v}^2 - v_\downarrow^2\mathbf{1} - 2ii_{\boldsymbol{v}}(\boldsymbol{v})$, where $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$.

Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a fully adapted process defined on $\mathcal{S}$. Then we have an adapted local interval function $\psi(\sigma,\tau) = (v_\tau - v_\sigma)^2$ for $\sigma \le \tau$ in $\mathcal{S}$. I will write $\int_{\mathcal{S}} \boldsymbol{u}\,(d\boldsymbol{v})^2$ for $\int_{\mathcal{S}} \boldsymbol{u}\, d\psi$ when this is defined.

Theorem 14 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ a near-simple integrator with domain $\mathcal{S}$. Then $\boldsymbol{v}^*$ is an integrator and

$$\int_{\mathcal{S}} \boldsymbol{z}\,(d\boldsymbol{v})^2 = \int_{\mathcal{S}} \boldsymbol{z}\, d(\boldsymbol{v}^2) - 2\int_{\mathcal{S}} \boldsymbol{z} \times \boldsymbol{v}\, d\boldsymbol{v} = \int_{\mathcal{S}} \boldsymbol{z}\, d\boldsymbol{v}^*$$

for every near-simple process $\boldsymbol{z}$ with domain $\mathcal{S}$.

proof For $\sigma \le \tau$ in $\mathcal{S}$,

$$(v_\tau - v_\sigma)^2 = (v_\tau^2 - v_\sigma^2) - 2v_\sigma \times (v_\tau - v_\sigma),$$

so

$$\begin{aligned}\int_{\mathcal{S}} \boldsymbol{z}\,(d\boldsymbol{v})^2 &= \int_{\mathcal{S}} \boldsymbol{z}\,d(\boldsymbol{v}^2) - 2\int_{\mathcal{S}} \boldsymbol{z} \times \boldsymbol{v}\,d\boldsymbol{v} \\ &= \int_{\mathcal{S}} \boldsymbol{z}\,d(\boldsymbol{v}^2) - 2\int_{\mathcal{S}} \boldsymbol{z}\,d(ii_{\boldsymbol{v}}(\boldsymbol{v})) = \int_{\mathcal{S}} \boldsymbol{z}\,d\boldsymbol{v}^*.\end{aligned}$$

Corollary $\boldsymbol{v}^*$ is non-negative and non-decreasing, for any near-simple integrator $\boldsymbol{v}$.

Lecture 16

The original development of stochastic calculus was based on 'continuous' processes, that is, processes with continuous sample paths. In the language I am using here the following concept seems to be a useful way of focusing on these.

Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$ an order-bounded fully adapted process with domain $\mathcal{S}$.

(a) For $I \in \mathcal{I}(\mathcal{S})$, set

$$\mathrm{Oscll n}^*_I(\boldsymbol{u}) = \sup_{J \in \mathcal{I}(\mathcal{S}), J \supseteq I} \sup_{e \text{ is a } J\text{-cell}} \Delta_e(\mathbf{1}, |d\boldsymbol{u}|).$$

(b) $\boldsymbol{u}$ is **jump-free** if $\inf_{I \in \mathcal{I}(\mathcal{S})} \theta(\mathrm{Oscll n}^*_I(\boldsymbol{u})) = 0$.

(c) $\boldsymbol{u}$ is **locally jump-free** if $\boldsymbol{u} \restriction \mathcal{S} \wedge \tau$ is jump-free for every $\tau \in \mathcal{S}$.

Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{u}$, $\boldsymbol{v}$ jump-free order-bounded processes with domain $\mathcal{S}$. If $\alpha \in \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $\boldsymbol{u} + \boldsymbol{v}$, $\alpha\boldsymbol{u}$, $\boldsymbol{u} \times \boldsymbol{v}$, $\bar{h}(\boldsymbol{u})$ are jump-free.

Proposition A jump-free process is locally jump-free and near-simple.

Theorem 15 Suppose that $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of the probability space (Ω, Σ, μ), that $\Sigma_t = \{E : E^\bullet \in \mathfrak{A}_t\}$ for every $t \geq 0$, that $\langle \mathfrak{A}_t \rangle_{t \geq 0}$ is right-continuous, and that $\langle X_t \rangle_{t \geq 0}$ is a progressively measurable stochastic process with corresponding fully adapted process $\boldsymbol{u}$ defined on $\mathcal{T}_f$. If almost every path $t \mapsto X_t(\omega) : [0, \infty[\to \mathbb{R}$ is a continuous real function, then $\boldsymbol{u}$ is locally jump-free.

proof (a) Reduce to the case in which every path is continuous. Take $\tau \in \mathcal{T}_f$ and $\epsilon > 0$. For $\omega \in \Omega$ set $h_0(\omega) = 0$ and

$$\begin{aligned} h_{n+1}(\omega) &= \inf\{t : t > h_n(\omega),\ |X_t(\omega) - h_n(\omega)| \geq \epsilon\} \text{ if } h_n(\omega) \text{ is finite} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{counting } \inf \emptyset \text{ as } \infty) \\ &= \infty \text{ if } h_n(\omega) = \infty. \end{aligned}$$

Then every $h_n : \Omega \to [0, \infty]$ is a stopping time. If $n \in \mathbb{N}$ and $h_{n+1}(\omega) < \infty$ then

$$|X_{h_{n+1}}(\omega) - X_{h_n}(\omega)| = \epsilon,$$

$$|X_t(\omega) - X_s(\omega)| \leq 2\epsilon \text{ whenever } h_n(\omega) \leq s \leq t \leq h_{n+1}(\omega).$$

Note that $\lim_{n\to\infty} h_n(\omega) = \infty$ for every ω.

(b) For $n \in \mathbb{N}$ let $\tau_n \in \mathcal{T}$ be the stopping time represented by h_n. Then $\sup_{n\in\mathbb{N}} \tau_n = \max \mathcal{T}$. Set $a_n = [\![\tau > \tau_n]\!]$. Then $\inf_{n\in\mathbb{N}} a_n = 0$ so $\lim_{n\to\infty} \bar{\mu} a_n = 0$ and there is an n such that $\bar{\mu} a_n \leq \epsilon$.

Note that $\boldsymbol{u}$ is càdlàg, therefore locally near-simple, and $\boldsymbol{u} \restriction [\min \mathcal{T}, \tau]$ is order-bounded; set $\bar{u} = \sup_{\sigma \leq \tau} |u_\sigma|$.

Set $I = \{\tau_i \wedge \tau : i \le n\}$. If J is a finite sublattice of $[\min\mathcal{T}, \tau]$ including I and e is a J-cell, we can express e as $c(\sigma, \sigma')$ where either $\tau_i \le \sigma \le \sigma' \le \tau_{i+1}$ for some $i < n$, or $\tau_n \wedge \tau \le \sigma \le \sigma' \le \tau$. In the former case, $\Delta_e(\mathbf{1}, |d\boldsymbol{v}|) \le 2\epsilon\chi 1$; in the latter, $\Delta_e(\mathbf{1}, |d\boldsymbol{v}|) \le 2\bar{u} \times \chi a_n$. So

$$\mathrm{Osclln}^*_I(\boldsymbol{u}) \le 2\epsilon\chi 1 + 2\bar{u} \times \chi a_n, \quad \theta(\mathrm{Osclln}^*_I(\boldsymbol{u})) \le 2\epsilon + 2\bar{\mu} a_n \le 4\epsilon.$$

As ϵ and τ are arbitrary, $\boldsymbol{u}$ is locally jump-free.

Examples The identity process and Brownian motion are jump-free, but the Poisson process is not.

Lecture 17

Itô's Formula, first form: Theorem 16 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{v} = \langle v_\tau \rangle_{\tau \in \mathcal{S}}$ a jump-free integrator, and $\boldsymbol{v}^*$ its quadratic variation. If $h : \mathbb{R} \to \mathbb{R}$ is a twice-differentiable function with continuous second derivative, then

$$\int_{\mathcal{S}} \bar{h}'(\boldsymbol{v})\, d\boldsymbol{v} + \frac{1}{2} \int_{\mathcal{S}} \bar{h}''(\boldsymbol{v})\, d\boldsymbol{v}^*$$

is defined and equal to $\bar{h}(v_\uparrow) - \bar{h}(v_\downarrow)$, where

$$v_\uparrow = \lim_{\sigma \uparrow \mathcal{S}} v_\sigma, \quad v_\downarrow = \lim_{\sigma \downarrow \mathcal{S}} v_\sigma.$$

proof (a) Being jump-free, $\boldsymbol{v}$ is near-simple, so $\bar{h}'(\boldsymbol{v})$ and $\bar{h}'(\boldsymbol{v})$ are near-simple, and $\boldsymbol{v}^*$ is defined; being non-negative and non-decreasing, $\boldsymbol{v}^*$ is an integrator; so both integrals are defined. Moreover, $\bar{h}(\boldsymbol{v})$ is an integrator, so

$$\bar{h}(v_\uparrow) - \bar{h}(v_\downarrow) = \lim_{\sigma \uparrow \mathcal{S}} \bar{h}(v_\sigma) - \lim_{\sigma \downarrow \mathcal{S}} \bar{h}(v_\sigma) = \int_{\mathcal{S}} d\bar{h}(\boldsymbol{v}).$$

Finally, $Q_{\mathcal{S}}((d\boldsymbol{v})^2)$ is topologically bounded, by the Corollary in Lecture 15.

(b) Consider first the case in which h'' is uniformly continuous. Let $\epsilon > 0$. Let $\eta > 0$ be such that $\theta(\eta w) \le \epsilon$ for every $w \in Q_{\mathcal{S}}((d\boldsymbol{v})^2)$. Then there is a $\delta > 0$ such that $|h''(\alpha) - h''(\beta)| \le \eta$ whenever $|\alpha - \beta| \le 2\delta$. Now take any such α and β. By Taylor's theorem with remainder, there is a γ lying between α and β such that

$$h(\beta) = h(\alpha) + (\beta - \alpha) h'(\alpha) + \frac{1}{2}(\beta - \alpha)^2 h''(\gamma),$$

so that

$$|h(\beta) - h(\alpha) - (\beta - \alpha) h'(\alpha) - \frac{1}{2}(\beta - \alpha)^2 h''(\alpha)| \le \eta(\beta - \alpha)^2.$$

It follows that if w, $w' \in L^0$ then $[\![|w' - w| \le \delta]\!]$ is included in

$$[\![|\bar{h}(w') - \bar{h}(w) - \bar{h}'(w) \times (w' - w) - \frac{1}{2}\bar{h}''(w) \times (w' - w)^2| \le \eta(w' - w)^2]\!].$$

Let $J \in \mathcal{I}(\mathcal{S})$ be such that $\theta(z) \le \delta\epsilon$, where $z = \mathrm{Osclln}^*_J(\boldsymbol{v})$. Then $a = [\![z \le \delta]\!]$ has measure at least $1 - \epsilon$. Take any $I \in \mathcal{I}(\mathcal{S})$ such that $I \supseteq J$. Now if $e = c(\sigma, \sigma')$ is any I-cell, and we set

$$y_e = \Delta_e(\mathbf{1}, d(\bar{h}(\boldsymbol{v}))) - \Delta_e(\bar{h}(\boldsymbol{v}), d\boldsymbol{v}) - \frac{1}{2}\Delta_e(\bar{h}''(\boldsymbol{v}), (d\boldsymbol{v})^2)$$

$$= \bar{h}(v_\tau) - \bar{h}(v_\sigma) - \bar{h}'(v_\sigma) \times (v_\tau - v_\sigma) - \frac{1}{2}\bar{h}''(v_\sigma) \times (v_\tau - v_\sigma)^2,$$

we have

$$|v_\tau - v_\sigma| = \Delta_e(\mathbf{1}, |d\boldsymbol{v}|) \le \mathrm{Osclln}_I(\boldsymbol{v}) \le z$$

and

$$a \subseteq [\![|v_\tau - v_\sigma| \le \delta]\!] \subseteq [\![|y_e| \le \eta(v_\tau - v_\sigma)^2]\!] = [\![|y_e| \le \eta\Delta_e(\mathbf{1}, (d\boldsymbol{v})^2]\!].$$

Summing over e,

$$a \subseteq [\![|S_I(\mathbf{1}, d\bar{h}(\boldsymbol{v})) - S_I(\bar{h}'(\boldsymbol{v}), d\boldsymbol{v}) - S_I(\bar{h}''(\boldsymbol{v}), (d\boldsymbol{v})^2| \leq \eta S_I(\mathbf{1}, (d\boldsymbol{v})^2)]\!]$$

and

$$\theta\big(S_I(\mathbf{1}, d\bar{h}(\boldsymbol{v})) - S_I(\bar{h}'(\boldsymbol{v}), d\boldsymbol{v}) - S_I(\bar{h}''(\boldsymbol{v}), (d\boldsymbol{v})^2)\big) \leq \bar{\mu}(1 \setminus a) + \bar{\theta}(\eta S_I(\mathbf{1}, (d\boldsymbol{v})^2) \leq 2\epsilon$$

by the choice of η. And this is true whenever $I \supseteq J$. Accordingly, taking the limit as $I \uparrow \mathcal{I}(\mathcal{S})$, we have

$$\begin{aligned} &\bar{h}(v_\uparrow) - \bar{h}(v_\downarrow) - \int_{\mathcal{S}} \bar{h}'(\boldsymbol{v})\, d\boldsymbol{v} - \frac{1}{2}\int_{\mathcal{S}} \bar{h}''(\boldsymbol{v})\, d\boldsymbol{v}^* \\ &\qquad = \bar{h}(v_\uparrow) - \bar{h}(v_\downarrow) - \int_{\mathcal{S}} \bar{h}'(\boldsymbol{v})\, d\boldsymbol{v} - \frac{1}{2}\int_{\mathcal{S}} \bar{h}''(\boldsymbol{v})\, (d\boldsymbol{v})^2 = 0 \end{aligned}$$

as required.

Itô's Formula, second form: Theorem 17 Let $k \geq 1$ be an integer, and $h : \mathbb{R}^k \to \mathbb{R}$ a twice-differentiable function with continuous second derivative. Denote its first partial derivatives by $h_1, \ldots, h_k$ and its second partial derivatives by $h_{11}, \ldots, h_{kk}$. Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ jump-free integrators with domain $\mathcal{S}$; let $\boldsymbol{u}$ be a near-simple fully adapted process with domain $\mathcal{S}$. Write $\boldsymbol{V} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$. Then

$$\textstyle\int_{\mathcal{S}} \boldsymbol{u}\, d\bar{h}(\boldsymbol{V}) = \sum_{i=1}^k \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}_i(\boldsymbol{V})\, d\boldsymbol{v}_i + \frac{1}{2}\sum_{i=1}^k \sum_{j=1}^k \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}_{ij}(\boldsymbol{V})\, d[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j].$$

where $[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j]$ is the **covariation**

$$\begin{aligned} \frac{1}{2}((\boldsymbol{v}_i + \boldsymbol{v}_j)^* - \boldsymbol{v}_i^* - \boldsymbol{v}_j^*) &= \boldsymbol{v}_i \times \boldsymbol{v}_j - (v_{i\downarrow} \times v_{j\downarrow})\mathbf{1} - ii_{\boldsymbol{v}_i}(\boldsymbol{v}_j) - ii_{\boldsymbol{v}_j}(\boldsymbol{v}_i) \\ &= \langle \int_{\mathcal{S}\wedge\tau} d\boldsymbol{v}_i d\boldsymbol{v}_j \rangle_{\tau \in \mathcal{S}}. \end{aligned}$$

Oops! We don't know what the quadratic variation of Brownian motion is.

Lecture 18

Definition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. I will say that a sublattice $\mathcal{S}'$ of $\mathcal{S}$ is **separating in** $\mathcal{S}$ if whenever τ, $\tau' \in \mathcal{S}$ and $[\![\tau < \tau']\!] \neq 0$, there is a $\sigma \in \mathcal{S}'$ such that $[\![\tau \le \sigma]\!] \cap [\![\sigma < \tau']\!] \neq 0$.

Example If $t \ge 0$, then $\{\dot{s} : s \le t\}$ is separating in $[\dot{0}, \dot{t}] \subseteq \mathcal{T}$.

Theorem 18 Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least element, and $\boldsymbol{u}$, $\boldsymbol{v}$ fully adapted processes such that $z = \int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}$ is defined. Suppose that $\boldsymbol{v}$ is càdlàg and $\boldsymbol{u}$ is order-bounded. Let $\mathcal{S}'$ be a sublattice of $\mathcal{S}$, cofinal with $\mathcal{S}$, which is separating in $\mathcal{S}$. Then $\int_{\mathcal{S}'} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to z.

Corollary Suppose that $t \ge 0$, that $\boldsymbol{u}$, $\boldsymbol{v}$ are càdlàg fully adapted processes defined on $[\dot{0}, \dot{t}]$ and that $z = \int_{[\dot{0},\dot{t}]} \boldsymbol{u}\, d\boldsymbol{v}$ is defined. Set $\mathcal{S}' = \{\dot{s} : s \le t\}$. Then $\int_{\mathcal{S}'} \boldsymbol{u}\, d\boldsymbol{v}$ is defined and equal to z.

Remarks Thus we can (in the most important cases) calculate an integral $\int \boldsymbol{u}\, d\boldsymbol{v}$ directly from the values $u_{\dot{s}}$, $v_{\dot{s}}$.

Note however that it is possible for $\int_{\{\dot{s}:s\le t\}} \boldsymbol{u}\, d\boldsymbol{v}$ to be defined when $\int_{[\dot{0},\dot{t}]} \boldsymbol{u}\, d\boldsymbol{v}$ is not.

Corollary If $\boldsymbol{w}$ is Brownian motion, its quadratic variation $\boldsymbol{w}^*$ is equal to the identity process $\boldsymbol{\iota}$ on $\mathcal{T}_f$.

proof I show first that $w^*_{\dot{t}} = t\chi 1$ for $t \ge 0$. **P** Set $\mathcal{S}' = \{\dot{s} : s \le t\}$. Then $\mathcal{S}'$ is separating in $[\dot{0}, \dot{t}]$. We have

$$w^*_{\dot{t}} = w^2_{\dot{t}} - 2\int_{[\dot{0},\dot{t}]} \boldsymbol{w}\, d\boldsymbol{w} = \int_{[\dot{0},\dot{t}]} d(\boldsymbol{w}^2) - 2\int_{[\dot{0},\dot{t}]} \boldsymbol{w}\, d\boldsymbol{w}$$
$$= \int_{\mathcal{S}'} d(\boldsymbol{w}^2) - 2\int_{\mathcal{S}'} \boldsymbol{w}\, d\boldsymbol{w} = \int_{\mathcal{S}'} (d\boldsymbol{w})^2 \doteq S_I(\mathbf{1}, (d\boldsymbol{w})^2)$$

for all sufficiently large $I \in \mathcal{I}(\mathcal{S}')$. Expressing I as $\{\dot{s}_0, \dots, \dot{s}_n\}$ where $0 = s_0 \le \dots \le s_n = t$,

$$S_I(\mathbf{1}, (d\boldsymbol{w})^2) = \textstyle\sum_{i=0}^{n-1} (w_{\dot{s}_{i+1}} - w_{\dot{s}_i})^2 = \sum_{i=0}^{n-1} (s_{i+1} - s_i) z_i$$

where $z_0, \dots, z_{n-1}$ are independent and identically distributed and have the $\chi^2(1)$-distribution with mean 1 and variance 2. But this means that $S_I(\mathbf{1}, (d\boldsymbol{w})^2)$ has mean t and variance $2\sum_{i=0}^{n-1} (s_{i+1} - s_i)^2 \doteq 0$ and is close to $t\chi 1$ for the topology of convergence in measure. **Q**

Because $\boldsymbol{w}^*$ is non-decreasing, $w^*_\tau = \tau$ for every $\tau \in \mathcal{T}_f$.

Brownian processes Brownian motion, as I have described it, is attached explicitly to a particular probability space and filtration. This is inadequate for investigating more complex evolving worlds (e.g., two-dimensional Brownian motion) in which a simple Brownian process is only part of the structure. Probabilists since Kolmogorov have generally approached such models in terms of probabilities on product spaces. In the measure-algebra context, this leads us to closed subalgebras. If we have a probability algebra $(\mathfrak{A}, \bar{\mu})$ with a filtration $\langle \mathfrak{A}_t \rangle_{t \geq 0}$ and a closed subalgebra $\mathfrak{B}$, we can look at the filtration $\langle \mathfrak{B}_t \rangle_{t \geq 0}$, where $\mathfrak{B}_t = \mathfrak{A}_t \cap \mathfrak{B}$ for $t \geq 0$, and the corresponding lattice $\mathcal{T}_{\mathfrak{B}} \subseteq \mathcal{T} = \mathcal{T}_{\mathfrak{A}}$ of stopping times. We find that if $\tau \leq \tau'$ in $\mathcal{T}_{\mathfrak{B}}$ then $\mathcal{T}_{\mathfrak{B}} \cap [\tau, \tau']$ is separating in $[\tau, \tau'] \subseteq \mathcal{T}_{\mathfrak{A}}$, so that if $\boldsymbol{u}$ is a càdlàg process and $\boldsymbol{v}$ is a càdlàg integrator, both defined on $[\tau, \tau']$, $\int_{\mathcal{T}_{\mathfrak{B}} \cap [\tau, \tau']} \boldsymbol{u}\, d\boldsymbol{v}$ will be defined and equal to $\int_{[\tau, \tau']} \boldsymbol{u}\, d\boldsymbol{v}$.

It does not quite follow that $\boldsymbol{v} \restriction \mathcal{T}_{\mathfrak{B}} \cap [\tau, \tau']$ will be a martingale whenever $\boldsymbol{v}$ is a martingale on $[\tau, \tau']$; for this we need $\mathfrak{B}$ and $\mathfrak{A}_t$ to be 'relatively independent' over $\mathfrak{B}_t$ for every t, that is, the conditional expectation on $\mathfrak{A}_t$ of any $z \in L^1_{\bar{\mu}} \cap L^0(\mathfrak{B})$ must belong to $L^0(\mathfrak{B}_t)$.

I will say that a **Brownian process** is a càdlàg process $\boldsymbol{v}$, defined on $\mathcal{T}_f$, such that there is a closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, with $\mathfrak{B}$ and $\mathfrak{A}_t$ relatively independent over $\mathfrak{B}_t = \mathfrak{B} \cap \mathfrak{A}_t$ for every t, and

$$(\mathfrak{B}, \bar{\mu} \restriction \mathfrak{B}, \langle \mathfrak{B}_t \rangle_{t \geq 0}, \mathcal{T}_{\mathfrak{B}}, \boldsymbol{v} \restriction \mathcal{T}_{\mathfrak{B}})$$

isomorphic to Brownian motion as described in Lecture 9. In this case, $\boldsymbol{v}$ will be jump-free, $\boldsymbol{v} \restriction \mathcal{T}_b$ will be a martingale, and $\boldsymbol{v}^*$ will be $\boldsymbol{\iota}$, all these being interpreted in $\mathcal{T}_{\mathfrak{A}}$.

Lecture 19

Change of law I have been speaking so far as though the 'law', or measure, $\bar{\mu}$ was immutable. But if we look at the structures discussed in the lectures so far, we find that it does not enter directly into most of the formulae. In setting up the probability algebra $(\mathfrak{A}, \bar{\mu})$ itself from a probability space (Ω, Σ, μ), the most important thing was not μ, but the ideal $\mathcal{N}$ of negligible sets. The same is true of the space L^0. 'Filtrations' (and 'right-continuity' of filtrations) don't refer to the measure at all, and so $\mathcal{T}$ is describable in terms of the triple $(\Omega, \Sigma, \mathcal{N})$, as are fully adapted processes and the Riemann sums of Lecture 5.

When we come to integration, of course, we do need to involve the measure. The notions of $L^1 = L^1_{\bar{\mu}}$, $\mathbb{E} = \mathbb{E}_{\bar{\mu}}$ and $\theta = \theta_{\bar{\mu}}$ from Lecture 2 directly involve μ and $\bar{\mu}$. But at this point we have a striking fact. We can easily find probability measures ν on Ω with domain Σ and null ideal $\mathcal{N}$; for instance, take $\nu E = \int_E f(\omega)\mu(d\omega)$ where f is any strictly positive function such that $\int_\Omega f(\omega)\mu(d\omega)$ is defined and equal to 1. But if we do this we find that although $\bar{\nu}$ and $\mathbb{E}_{\bar{\nu}}$ (derived from integration with respect to ν) may seem very different from $\bar{\mu}$ and $\mathbb{E}_{\bar{\mu}}$, the functionals $\theta_{\bar{\mu}}$ and $\theta_{\bar{\nu}}$ are mutually absolutely continuous; that is,

for every $\epsilon > 0$ there is a $\delta > 0$ such that $\theta_{\bar{\nu}}(u) \le \epsilon$ whenever $u \in L^0$ and $\theta_{\bar{\mu}}(u) \le \delta$,

for every $\epsilon > 0$ there is a $\delta > 0$ such that $\theta_{\bar{\mu}}(u) \le \epsilon$ whenever $u \in L^0$ and $\theta_{\bar{\nu}}(u) \le \delta$.

But this means that the metrics on L^0 defined by $\theta_{\bar{\mu}}$ and $\theta_{\bar{\nu}}$ are uniformly equivalent, and in particular $\theta_{\bar{\mu}}$ and $\theta_{\bar{\nu}}$ give the same topology of convergence in measure. Since integration is defined in Lecture 5 in terms of limits for this topology, we get exactly the same notion of stochastic integration in the structures $(\mathfrak{A}, \bar{\mu}, \langle \mathfrak{A}_t \rangle_{t \ge 0}, L^0, \theta_{\bar{\mu}}, \mathcal{T})$ and $(\mathfrak{A}, \bar{\nu}, \langle \mathfrak{A}_t \rangle_{t \ge 0}, L^0, \theta_{\bar{\nu}}, \mathcal{T})$.

Working through the definitions, we find that 'simple' processes (Lecture 6), 'order-bounded' processes (Lecture 7), 'bounded variation' (Lecture 8) and 'separating sublattice' (Lecture 18) are defined from $(\mathfrak{A}, \langle \mathfrak{A}_t \rangle_{t \ge 0}, L^0, \mathcal{T})$ alone, and that 'near-simple' processes, 'integrators' and 'càdlàg' processes (Lecture 7), 'integrating interval functions' (Lecture 13), 'jump-free processes' (Lecture 16) and Itô's formula (Lecture 17) depend on the topology but not on the measure (because changing θ to a mutually absolutely continous functional has no effect on the truth of a statement of the form

'for every $\epsilon > 0$ there is a $\ldots$ such that $\theta(\ldots) \le \epsilon$').

Similarly, of course, the operation $(\boldsymbol{u}, \boldsymbol{v}) \mapsto ii_{\boldsymbol{v}}(\boldsymbol{u})$ (Lecture 6) is unaffected by change of law, so 'quadratic variation' (Lecture 13) also is.

Where the measure does intervene essentially is in the notions of 'conditional expectation' and 'martingale' (Lecture 11). If we change the law, we are surprised if there is a single non-constant martingale which is still a martingale. But of course many processes cannot possibly be martingales under any law. So the following result (a version of the Bichteler-Dellacherie theorem) is very striking.

Theorem 19 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ an integrator with domain $\mathcal{S}$. Then there is a functional $\bar{\nu} : \mathfrak{A} \to [0, 1]$ such that $(\mathfrak{A}, \bar{\nu})$ is a probability algebra and whenever $\epsilon > 0$ there are fully adapted processes $\hat{\boldsymbol{v}}$ and $\boldsymbol{v}'$, both with domain $\mathcal{S}$, such that $\bar{\nu}[\![\boldsymbol{v} \ne \hat{\boldsymbol{v}} + \boldsymbol{v}']\!] \le \epsilon$, $\hat{\boldsymbol{v}}$ is a $\bar{\nu}$-uniformly integrable $\bar{\nu}$-martingale, and $\boldsymbol{v}'$ is of bounded variation.

Remarks Recall that a set $A \subseteq L^1$ is **uniformly integrable** if for every $\epsilon > 0$ there is an $M \geq 0$ such that $\sup_{u\in A} \mathbb{E}((|u| - M\chi 1)^+) \leq \epsilon$; equivalently, if A is relatively weakly compact.

By $[\![\boldsymbol{v} \neq \boldsymbol{u}]\!]$ I mean $[\![\sup|\boldsymbol{v} - \boldsymbol{u}| \neq 0]\!]$.

The ucp topology: Definitions Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$.

(a) Write $M_{\text{n-s}} = M_{\text{n-s}}(\mathcal{S})$ for the set of near-simple fully adapted processes with domain $\mathcal{S}$. Then $M_{\text{n-s}}$ is an f-subalgebra of $(L^0)^{\mathcal{S}}$ closed under the action of continuous functions $h : \mathbb{R} \to \mathbb{R}$.

(b) For $\boldsymbol{u} \in M_{\text{n-s}}$ set $\widehat{\theta}_{\mathcal{S}}(\boldsymbol{u}) = \theta(\sup|\boldsymbol{u}|)$. Then $\widehat{\theta}_{\mathcal{S}}$ defines a linear space topology on $M_{\text{n-s}}$ for which $\boldsymbol{u} \mapsto \bar{h}(\boldsymbol{u})$ is continuous whenever $h : \mathbb{R} \to \mathbb{R}$ is continuous.

I will call this the **ucp topology**. Note that a change of law does not change $M_{\text{n-s}}$ and changes $\widehat{\theta}_{\mathcal{S}}$ into an equivalent functional, so does not change the ucp topology.

Theorem 20 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a near-simple integrator with domain $\mathcal{S}$. Then $ii_{\boldsymbol{v}}(\boldsymbol{u}) \in M_{\text{n-s}}$ whenever $\boldsymbol{u} \in M_{\text{n-s}} = M_{\text{n-s}}(\mathcal{S})$, and $ii_{\boldsymbol{v}} : M_{\text{n-s}} \to M_{\text{n-s}}$ is continuous for the ucp topology.

Remark This theorem seems to be difficult; my method is, in effect, to prove it for $\boldsymbol{v}$ of bounded variation (which is easy) and for martingales $\boldsymbol{v}$ (which involves an elaboration of Theorem 7), and then to quote the Bichteler-Dellacherie theorem. The result that $ii_{\boldsymbol{v}} : M_{\text{n-s}} \to (L^0)^{\mathcal{S}}$ is continuous, where $(L^0)^{\mathcal{S}}$ is given its product topology, is much easier.

Theorem 21 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a jump-free integrator with domain $\mathcal{S}$. Then $ii_{\boldsymbol{v}}(\boldsymbol{u})$ is jump-free for every $\boldsymbol{u} \in M_{\text{n-s}} = M_{\text{n-s}}(\mathcal{S})$.

proof Deal with simple $\boldsymbol{u}$, using the formula in Lecture 6, and then extend by continuity, using Theorem 20.

Remark Recall Theorems 8 and 9; indefinite integration with respect to a process of bounded variation yields a process of bounded variation, and (under rather more restricted conditions) indefinite integration with respect to a martingale yields a local martingale.

Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, and $\boldsymbol{v}$ a near-simple jump-free integrator with domain $\mathcal{S}$; then $\boldsymbol{v}^*$ is jump-free.

proof $\boldsymbol{v}^* = \boldsymbol{v}^2 - v_{\downarrow}^2\mathbf{1} - 2ii_{\boldsymbol{v}}(\boldsymbol{v})$.

Theorem 22 Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Suppose that $\boldsymbol{v}$, $\boldsymbol{v}'$, $\boldsymbol{u}$, $\boldsymbol{z}$ are near-simple fully adapted processes, all with domain $\mathcal{S}$, of which $\boldsymbol{v}$ and $\boldsymbol{v}'$ are integrators. Set $\boldsymbol{w} = ii_{\boldsymbol{v}}(\boldsymbol{u})$. Then $\int_{\mathcal{S}} \boldsymbol{z}\, d\boldsymbol{w} d\boldsymbol{v}' = \int_{\mathcal{S}} \boldsymbol{z} \times \boldsymbol{u}\, d\boldsymbol{v} d\boldsymbol{v}'$.

Remarks Compare Theorem 13: this gives $\int_{\mathcal{S}} \boldsymbol{z}\, d\boldsymbol{w} = \int_{\mathcal{S}} \boldsymbol{z} \times \boldsymbol{u}\, d\boldsymbol{v}$. I do not think it is safe just to tack the $d\boldsymbol{v}'$ on the ends of these formulae, because it is *not* normally the case that $z_\sigma \times (w_\tau - w_\sigma) = z_\sigma \times u_\sigma \times (v_\tau - v_\sigma)$. I think in fact that Theorem 22 demands a deeper look, and my own proof depends on the continuity of the operators $ii_{\boldsymbol{v}}$, $ii_{\boldsymbol{v}'}$, which in turn depends on the Bichteler-Dellacherie theorem.

Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Suppose that $\boldsymbol{v}$, $\boldsymbol{v}'$, $\boldsymbol{u}$, $\boldsymbol{u}'$ and $\boldsymbol{z}$ are near-simple processes, all with domain $\mathcal{S}$, of which $\boldsymbol{v}$ and $\boldsymbol{v}'$ are integrators. Set $\boldsymbol{w} = ii_{\boldsymbol{v}}(\boldsymbol{u})$ and $\boldsymbol{w}' = ii_{\boldsymbol{v}'}(\boldsymbol{u}')$. Then $\int_{\mathcal{S}} \boldsymbol{z}\, d\boldsymbol{w} d\boldsymbol{w}' = \int_{\mathcal{S}} \boldsymbol{z} \times \boldsymbol{u} \times \boldsymbol{u}'\, d\boldsymbol{v} d\boldsymbol{v}'$.

proof $\int_{\mathcal{S}} \boldsymbol{z}\, d\boldsymbol{w} d\boldsymbol{w}' = \int_{\mathcal{S}} \boldsymbol{z} \times \boldsymbol{u}\, d\boldsymbol{v} d\boldsymbol{w}' = \int_{\mathcal{S}} \boldsymbol{z} \times \boldsymbol{u} \times \boldsymbol{u}'\, d\boldsymbol{v} d\boldsymbol{w}'$.

Corollary Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$. Suppose that $\boldsymbol{v}$ and $\boldsymbol{u}$ are near-simple processes, both with domain $\mathcal{S}$, of which $\boldsymbol{v}$ is an integrator. Set $\boldsymbol{w} = ii_{\boldsymbol{v}}(\boldsymbol{u})$. Then $\boldsymbol{w}^* = ii_{\boldsymbol{v}^*}(\boldsymbol{u}^2)$.

proof Taking $\boldsymbol{u}' = \boldsymbol{u}$, $\boldsymbol{v}' = \boldsymbol{v}$ and $\boldsymbol{z} = \mathbf{1}$ in the last corollary,

$$w^*_\tau = \int_{\mathcal{S}\wedge\tau} (d\boldsymbol{w})^2 = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u}^2\, (d\boldsymbol{v})^2.$$

Lecture 20

Revision and clarification.

Lecture 21

In Lecture 18, I showed how to calculate $\boldsymbol{w}^* = \boldsymbol{\iota}$ for Brownian motion $\boldsymbol{w}$. In effect, this shows that we cannot ignore terms of the form $\int \ldots (d\boldsymbol{v})^2$; in the multidimensional Itô's formula, we have terms $\int \ldots d\boldsymbol{v}_i d\boldsymbol{v}_j = \int \ldots d[\boldsymbol{v}_i \overset{*}{|} \boldsymbol{v}_j]$. The question arises, whether we must expect to have to deal with terms $\int \ldots (d\boldsymbol{v})^3$ or $\int \ldots d\boldsymbol{v}_i d\boldsymbol{v}_j d\boldsymbol{v}_k$. In Itô's formula, these seem not to have appeared; is there a reason for this?

Examples (a) Let $\boldsymbol{\iota}$ be the identity process. Then $\boldsymbol{\iota}^* = 0$. **P** Use the method of Lecture 18. If $t > 0$ and $\mathcal{S}' = \{\dot{s} : s \le t\}$ then

$$\iota^*_{\dot t} = \int_{[\dot 0,\dot t]} (d\boldsymbol{\iota})^2 = \int_{\mathcal{S}'} (d\boldsymbol{\iota})^2 \doteq S_I(\mathbf{1}, (d\boldsymbol{\iota})^2) = \sum_{i=0}^{n} (s_{i+1} - s_i)^2 \chi 1 \le \epsilon \chi 1$$

if $I = \{\dot s_0, \ldots, \dot s_n\}$ where $0 = s_0 < \ldots < s_n = t$ and $\max_{i<n} s_{i+1} - s_i \le \frac{\epsilon}{t}$. So $\iota^*_{\dot t} =$ for every t and $\boldsymbol{\iota}^* = 0$.

(b) Let $\boldsymbol{v}$ be the Poisson process. Then $\boldsymbol{v}^* = \boldsymbol{v}$. **P** Take $\Omega \subseteq \mathcal{P}([0, \infty[)$, Σ and μ as in Lecture 9. Set $h_0(\omega) = 0$ and for $n \ge 1$ let $h_n(\omega)$ be the nth point of ω, so that $\omega = \{h_1(\omega), h_2(\omega), \ldots\}$ and $\langle h_n(\omega)\rangle_{n\in\mathbb{N}}$ is strictly increasing and unbounded. The set $\{\omega : h_n(\omega) > t\} = \{\omega : \#(\omega \cap [0, t]) < n\}$ belongs to Σ_t for every t, so we have a corresponding stopping time $\tau_n = h_n^\bullet \in \mathcal{T}_f$; we have $\tau_0 = \dot 0$, $[\![\tau_n < \tau_{n+1}]\!] = 1$ for every n, and $\sup_{n\in\mathbb{N}} \tau_n = \max \mathcal{T}$. Now $[\![v_\sigma = n]\!] = [\![\tau_n \le \sigma]\!] \cap [\![\sigma < \tau_{n+1}]\!]$ whenever $\sigma \in \mathcal{T}_f$ and $n \in \mathbb{N}$.

If $\tau_n \le \sigma \le \sigma' \le \tau_{n+1}$, then

$$(v_{\sigma'} - v_\sigma)^2 = \chi(\sigma' = \tau_{n+1}) = v_{\sigma'} - v_\sigma.$$

So $S_I(\mathbf{1}, (d\boldsymbol{v})^2) = S_I(\mathbf{1}, d\boldsymbol{v})$ whenever $n \in \mathbb{N}$, $\tau \le \tau_n$ and I is a finite sublattice of $[\dot 0, \tau]$ containing $\tau \wedge \tau_i$ for $i \le n$. Accordingly

$$v^*_\tau = \int_{[\dot 0,\tau]} (d\boldsymbol{v})^2 = \int_{[\dot 0,\tau]} d\boldsymbol{v} = v_\tau$$

whenever $\tau \le \tau_n$. Because $\sup_{n\in\mathbb{N}} [\![\tau \le tau_n]\!] = 1$, $v^*_\tau = v_\tau$ whenever $\tau \in \mathcal{T}_f$.

Remarks Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$ and $\boldsymbol{v}$ is a near-simple integrator.

(a) By the definition of $\boldsymbol{v}^*$,

$$v_\uparrow^2 - v_\downarrow^2 = \lim_{\tau\uparrow\mathcal{S}} v_\tau^2 - v_\downarrow^2 = \lim_{\tau\uparrow\mathcal{S}} 2\int_{\mathcal{S}\wedge\tau} \boldsymbol{v}\, d\boldsymbol{v} - u^*_\tau$$

$$= \lim_{\tau\uparrow\mathcal{S}} 2\int_{\mathcal{S}\wedge\tau} \boldsymbol{v}\, d\boldsymbol{v} - \int_{\mathcal{S}\wedge\tau} d\boldsymbol{v}^* = 2\int_{\mathcal{S}} \boldsymbol{v}\, d\boldsymbol{v} - \int_{\mathcal{S}} d\boldsymbol{v}^*$$

(see (c-ii) of Theorem 3)

$$= 2\int_{\mathcal{S}} \boldsymbol{v}\, d\boldsymbol{v} - \frac{1}{2}\int_{\mathcal{S}} (d\boldsymbol{v})^2$$

exactly as declared by Itô's formula with $h(x) = x^2$.

(b) If $\sigma \le \tau$ in $\mathcal{T}$ then

$$(v_\tau - v_\sigma)^3 = v_\tau^3 - 3v_\sigma^2 \times (v_\tau - v_\sigma) - 3v_\sigma(v_\tau - v_\sigma)^2 - v_\sigma^3,$$

so, writing $(d\boldsymbol{v})^3$ to represent the adapted local interval function $(\sigma, \tau) \mapsto (v_\tau - v_\sigma)^3$,

$$\int_{\mathcal{S}} (d\boldsymbol{v})^3 = v_\uparrow^3 - v_\downarrow^3 - 3\int_{\mathcal{S}} \boldsymbol{v}^2\, d\boldsymbol{v} - 3\int_{\mathcal{S}} \boldsymbol{v}\,(d\boldsymbol{v})^2;$$

as with Itô's formula, we can make the step to

$$\int_{\mathcal{S}} \boldsymbol{u}\, d(\boldsymbol{v}^3) = 3\int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{v}^2\, d\boldsymbol{v} + 3\int_{\mathcal{S}} \boldsymbol{u} \times \boldsymbol{v}\,(d\boldsymbol{v})^2 + \int_{\mathcal{S}} \boldsymbol{u}\,(d\boldsymbol{v})^3$$

or, setting $h(x) = x^3$,

$$\int_{\mathcal{S}} \boldsymbol{u}\, d\bar{h}(\boldsymbol{v}) = \int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}'(\boldsymbol{v})\, d\boldsymbol{v} + \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}''(\boldsymbol{v})\,(d\boldsymbol{v})^2 + \frac{1}{6}\int_{\mathcal{S}} \boldsymbol{u} \times \bar{h}'''(\boldsymbol{v})\,(d\boldsymbol{v})^3$$

whenever $\boldsymbol{u}$ is a near-simple process with domain $\mathcal{S}$. The reason this works so generally is of course that the Taylor series

$$h(y) = h(x) + (y-x)h'(x) + \frac{1}{2}(y-x)^2 h''(x) + \frac{1}{6}(y-x)^3 h'''(x)$$

is exactly valid for all x and y.

We have already seen that with jump-free integrators we can expect to be able to ignore cubic terms $(d\boldsymbol{v})^3$ (and therefore, we can hope, terms $d\boldsymbol{v}_i d\boldsymbol{v}_j d\boldsymbol{v}_k$). There are important cases in which we can ignore quadratic terms.

Proposition Let $\mathcal{S}$ be a sublattice of $\mathcal{T}$, $\boldsymbol{u}$ an order-bounded process, $\boldsymbol{v}$ a jump-free process and $\boldsymbol{w}$ a process of bounded variation, all with domain $\mathcal{S}$. Then $\int_{\mathcal{S}} \boldsymbol{u}\, d\boldsymbol{v}d\boldsymbol{w} = 0$.

proof Take $\bar{u} = \sup|\boldsymbol{u}|$, $\bar{w} = \int_{\mathcal{S}} |d\boldsymbol{w}|$ and $\epsilon > 0$. Then there is a $\delta > 0$ such that $\theta(\bar{u} \times \bar{w} \times z) \le \epsilon$ whenever $\theta(z) \le \delta$. Let $I \in \mathcal{I}(\mathcal{S})$ be such that $\theta(\mathrm{Osclln}_I^*(\boldsymbol{u})) \le \delta$. Suppose that $J \in \mathcal{I}(\mathcal{S})$ and $J \supseteq I$. If $e = c(\sigma, \tau)$ is a J-cell, then

$$|\Delta_e(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w})| = |u_\sigma \times (v_\tau - v_\sigma) \times (w_\tau - w_\sigma)| \le \bar{u} \times \bar{v} \times \Delta_e(\mathbf{1}, |d\boldsymbol{w}|).$$

So

$$|S_J(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w})| \le \bar{u} \times \bar{v} \times S_J(\mathbf{1}, |d\boldsymbol{w}|) \le \bar{u} \times \mathrm{Osclln}_I^*(\boldsymbol{v}) \times \bar{w}$$

and

$$\theta(S_J(\boldsymbol{u}, d\boldsymbol{v}d\boldsymbol{w})) \le \theta(\bar{u} \times \bar{w} \times \mathrm{Osclln}_I^*(\boldsymbol{v})) \le \epsilon.$$

Corollary Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{T}$, $\boldsymbol{v}$ is a jump-free integrator and $\boldsymbol{w}$ is a near-simple process of bounded variation, both with domain $\mathcal{S}$.

(a) $[\boldsymbol{v} \overset{*}{|} \boldsymbol{w}] = 0$.

(b) $(\boldsymbol{v} + \boldsymbol{w})^* = \boldsymbol{v}^* + \boldsymbol{w}^*$.

Lecture 22

A stochastic differential equation I suppose we can all solve the equation

$$\frac{dz}{dv} = z, \quad z(0) = z_0$$

for a real function z of a real variable v. Can we make sense of this for stochastic processes? A naive view would start from

$$\frac{z_\tau - z_\sigma}{v_\tau - v_\sigma} \doteq z_\sigma$$

for $\sigma < \tau$ and $\tau \doteq \sigma$. The division might be problematic, so perhaps we are better off with

$$z_\tau - z_\sigma \doteq z_\sigma \times (v_\tau - v_\sigma), \qquad d\boldsymbol{z} = \boldsymbol{z} d\boldsymbol{v}.$$

Again, it is unclear what sort of approximation we should look for, but if we integrate both sides we get an equation

$$z_\tau - z_{\dot 0} = \int_{\dot 0}^{\tau} \boldsymbol{z}\, d\boldsymbol{v}, \qquad \boldsymbol{z} = z_{\dot 0}\mathbf{1} + ii_{\boldsymbol{v}}(\boldsymbol{z})$$

which looks much more manageable.

In the differential form, it is plain that jumps in $\boldsymbol{v}$ will give special problems. These are not so evidently disastrous in the integral form, but Itô's formula gives us an effective tool for handling jump-free integrators, so let us start with these.

Theorem 23 Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$. Suppose that $\boldsymbol{v}$ is a locally jump-free local integrator, and $\boldsymbol{u}$ a locally near-simple fully adapted process with domain $\mathcal{S}$. Set $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ and $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_\downarrow\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$. Then $\boldsymbol{z}$ is a locally jump-free local integrator, $\boldsymbol{z} = \mathbf{1} + ii_{\boldsymbol{v}}(\boldsymbol{z})$ and $ii_{\boldsymbol{z}}(\boldsymbol{u}) = ii_{\boldsymbol{v}}(\boldsymbol{u} \times \boldsymbol{z})$.

proof Set $\boldsymbol{w} = \boldsymbol{v} - v_\downarrow\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*$. Then $\boldsymbol{w}$ is a locally jump-free local integrator, so $\boldsymbol{z} = \overline{\exp}(\boldsymbol{w})$ also is. We have $w_\downarrow = v_\downarrow - v_\downarrow - \frac{1}{2}v^*_\downarrow = 0$ (using (c-ii) of Theorem 3). So $z_\downarrow = \chi 1$.

For any $\tau \in \mathcal{S}$,

$$z_\tau - \chi 1 = \int_{\mathcal{S}\wedge\tau} \overline{\exp}(\boldsymbol{w})\, d\boldsymbol{w} + \frac{1}{2}\int_{\mathcal{S}\wedge\tau} \overline{\exp}(\boldsymbol{w})\, d\boldsymbol{w}^*$$

by Itô's formula. But $\boldsymbol{w}^* = \boldsymbol{v}^*$ (see Lecture 21). So

$$z_\tau - \chi 1 = \int_{\mathcal{S}\wedge\tau} \boldsymbol{z}\, d\boldsymbol{v} - \frac{1}{2}\int_{\mathcal{S}\wedge\tau} \boldsymbol{z}\, d\boldsymbol{v}^* + \frac{1}{2}\int_{\mathcal{S}\wedge\tau} \boldsymbol{z}\, d\boldsymbol{v}^* = \int_{\mathcal{S}\wedge\tau} \boldsymbol{v}\, d\boldsymbol{z}$$

for every $\tau \in \mathcal{S}$, and $\boldsymbol{z} = \mathbf{1} + ii_{\boldsymbol{v}}(\boldsymbol{z})$.

The extension to $ii_{\boldsymbol{z}}(\boldsymbol{u}) = ii_{\boldsymbol{v}}(\boldsymbol{u} \times \boldsymbol{z})$ comes from the version

$$\int_{\mathcal{S}\wedge\tau} \boldsymbol{u}\, d\overline{\exp}(\boldsymbol{w}) = \int_{\mathcal{S}\wedge\tau} \boldsymbol{u} \times \overline{\exp}(\boldsymbol{w})\, d\boldsymbol{w} + \frac{1}{2}\int_{\mathcal{S}\wedge\tau} \boldsymbol{u} \times \overline{\exp}(\boldsymbol{w})\, d\boldsymbol{w}^*$$

of Itô's formula.

Corollary Let $\mathcal{S}$ be a non-empty sublattice of $\mathcal{T}$. Suppose that $\boldsymbol{v}$ is a locally jump-free local integrator, and $z \in L^0(\bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$. Set $v_\downarrow = \lim_{\sigma\downarrow\mathcal{S}} v_\sigma$ and $\boldsymbol{z}' = z \times \overline{\exp}(\boldsymbol{v} - v_\downarrow\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$. Then $\boldsymbol{z}' = z\mathbf{1} + ii_{\boldsymbol{v}}(\boldsymbol{z}')$.

proof Taking $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_\downarrow\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$, we have

$$z'_\sigma \times (v_\tau - v_\sigma) = z \times z_\sigma \times (v_\tau - v_\sigma),$$

whenever $\sigma \le \tau$ in $\mathcal{S}$,

$$S_I(\boldsymbol{z}', d\boldsymbol{v}) = z \times S_I(\boldsymbol{z}, d\boldsymbol{v})$$

whenever $I \in \mathcal{I}(\mathcal{S})$, and

$$z'_\tau = z \times z_\tau = z \times (\chi 1 + \int_{\mathcal{S}\wedge\tau} \boldsymbol{z}\, d\boldsymbol{v}) = z + \int_{\mathcal{S}\wedge\tau} \boldsymbol{z}'\, d\boldsymbol{v}$$

for every $\tau \in \mathcal{S}$.

We need to assume that $z \in L^0(\bigcap_{\sigma\in\mathcal{S}} \mathfrak{A}_\sigma)$ to be sure that $\boldsymbol{z}'$ is a fully adapted process.

Lecture 23

Theorem 24 Suppose that $\langle\mathfrak{A}_t\rangle_{t\ge 0}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least element, and $\boldsymbol{v}$ a jump-free local martingale with domain $\mathcal{S}$. Set $\boldsymbol{z} = \overline{\exp}(\boldsymbol{v} - v_{\min\mathcal{S}}\mathbf{1} - \frac{1}{2}\boldsymbol{v}^*)$. Then $\boldsymbol{z}$ is a local martingale.

proof $\boldsymbol{z} = \mathbf{1} + ii_{\boldsymbol{v}}(\boldsymbol{z})$, and $ii_{\boldsymbol{v}}(\boldsymbol{z})$ is a local martingale by Theorem 10.

Corollary If $\boldsymbol{w}$ is Brownian motion, $\overline{\exp}(\boldsymbol{w} - \frac{1}{2}\boldsymbol{\iota})$ is a local martingale.

Theoremx25 In Theorem 24, if $\sup_{\sigma\in\mathcal{S}}\mathbb{E}(\overline{\exp}(\frac{1}{2}(v_\sigma - v_{\min\mathcal{S}})))$ is finite, then $\boldsymbol{z}$ is a uniformly integrable martingale.

Corollary If $\boldsymbol{w}$ is Brownian motion, $\overline{\exp}(\boldsymbol{w} - \frac{1}{2}\boldsymbol{\iota})\restriction\mathcal{T}_b$ is a martingale.

Distributions Given $u \in L^0$, its **distribution** is the Radon probability measure ν_u on $\mathbb{R}$ such that $\nu_u(]0,\alpha]) = \bar\mu[\![u \le \alpha]\!]$ for every $\alpha \in \mathbb{R}$; in this case $\nu_u(E) = \bar\mu[\![u \in E]\!]$ for every Borel set $E \subseteq \mathbb{R}$. Similarly, if $u_1, \dots, u_k \in L^0$, we have the notion of 'joint distribution' ν_U of $U = (u_1, \dots, u_k)$ defined by saying that $\nu_U(E) = \bar\mu[\![U \in E]\!]$ for Borel sets $E \subseteq \mathbb{R}^k$. (If $u_i = f_i^\bullet$, where $f_i : \Omega \to \mathbb{R}$ is measurable for each i, then $[\![U \in E]\!] = \{\omega : (f_1(\omega), \dots, f_k(\omega)) \in E\}^\bullet$.) If $h : \mathbb{R}^k \to \mathbb{R}$ is a bounded Borel measurable function, then $\mathbb{E}(\bar h(U)) = \int_{\mathbb{R}^k} h(x)\nu_U(dx)$. The characteristic function ϕ_{ν_U} of ν_U or U is now given by

$$\begin{aligned}\phi_{\nu_U}(y) &= \int_{\mathbb{R}^k} e^{iy\,.\,x}\nu_U(dx) = \int_{\mathbb{R}^k}\cos(y\,.\,x)\nu_U(dx) + i\int_{\mathbb{R}^k}\sin(y\,.\,x)\nu_U(dx)\\ &= \mathbb{E}(\overline{\cos}(\eta_1 u_1 + \ldots + \eta_k u_k)) + i\mathbb{E}(\overline{\sin}(\eta_1 u_1 + \ldots + \eta_k u_k))\end{aligned}$$

for $y = (\eta_1, \dots, \eta_k) \in \mathbb{R}^k$.

Lecture 24

Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\ge 0}$ is right-continuous. Let $\mathcal{S}$ be an order-convex subset of $\mathcal{T}$ with a least member, and $\boldsymbol{v}$ a jump-free local martingale such that $v_{\min\mathcal{S}}=0$ and $\boldsymbol{v}^*$ is an L^∞-process. Then

$$\overline{\sin}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*),\qquad \overline{\cos}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*)$$

are martingales.

proof Apply Itô's formula with $h(x,y)=\sin x\,\exp(\frac{1}{2}y)$ to see that

$$\overline{\sin}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*)=ii_{\boldsymbol{v}}(\overline{\cos}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*))$$

is a local martingale. If $\tau\in\mathcal{S}$, the local martingale $\overline{\sin}(\boldsymbol{v})\times\overline{\exp}(\frac{1}{2}\boldsymbol{v}^*)\restriction\mathcal{S}\wedge\tau$ is uniformly bounded (by $\exp(\frac{1}{2}\|v^*_\tau\|_\infty)$) therefore uniformly integrable, and is a martingale; so $\overline{\sin}(\boldsymbol{v})\times\overline{\exp}(\frac{1}{2}\boldsymbol{v}^*)$ itself is a martingale.

Similarly,

$$\overline{\cos}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*)=\mathbf{1}-ii_{\boldsymbol{v}}(\overline{\sin}(\boldsymbol{v})\times\overline{\exp}(\tfrac{1}{2}\boldsymbol{v}^*))$$

is a martingale.

Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\ge 0}$ is right-continuous, and $\sigma\le\tau$ in $\mathcal{T}$. Let $\boldsymbol{v}$ be a jump-free martingale with domain $[\sigma,\tau]$ and quadratic variation $\boldsymbol{v}^*$. If $v_\sigma=0$ and $v^*_\tau=\gamma\chi 1$ for some $\gamma>0$, then v_τ has a normal distribution with mean 0 and variance γ and is independent of $\mathfrak{A}_\sigma$.

proof

$$\begin{aligned}\mathbb{E}(\overline{\sin}(v_\tau))&=\mathbb{E}(P_\sigma(\overline{\sin}(v_\tau)))=e^{-\gamma/2}\mathbb{E}(P_\sigma(\overline{\sin}(v_\tau)\times\overline{\exp}(\tfrac{1}{2}v^*_\tau)))\\&=e^{-\gamma/2}\mathbb{E}(\overline{\sin}(v_\sigma)\times\overline{\exp}(\tfrac{1}{2}v^*_\sigma))=0,\end{aligned}$$

and similarly

$$\mathbb{E}(\overline{\cos}(v_\tau))=e^{-\gamma/2}\mathbb{E}(\overline{\cos}(v_\sigma)\times\overline{\exp}(\tfrac{1}{2}v^*_\sigma))=e^{-\gamma/2}.$$

Applying the same argument to the martingale $\alpha\boldsymbol{v}$ with quadratic variation $\alpha^2\boldsymbol{v}^*$, we see that

$$\mathbb{E}(\overline{\sin}(\alpha v_\tau))=0,\qquad \mathbb{E}(\overline{\cos}(\alpha v_\tau))=e^{-\gamma\alpha^2/2}$$

for any $\alpha\in\mathbb{R}$. So v_τ has the same characteristic function $\alpha\mapsto e^{-\gamma\alpha^2/2}$ as the normal distribution with mean 0 and variance γ, and must have that distribution.

A refinement of the argument shows that v_τ is independent of $\mathfrak{A}_\sigma$.

Lemma Suppose that $\langle\mathfrak{A}_t\rangle_{t\ge 0}$ is right-continuous. Suppose that $\tau_0\le\ldots\le\tau_k$ in T and that $\boldsymbol{v}$ is a jump-free martingale with domain $[\tau_0,\tau_k]$ and quadratic variation $\boldsymbol{v}^*$. If $v_{\tau_0}=0$ and $v^*_{\tau_j}=\gamma_j\chi 1$ for $j\le k$, where $0=\gamma_0\le\gamma_1\le\ldots\le\gamma_k$, then $(v_{\tau_0},\ldots,v_{\tau_k})$ has a centered Gaussian distribution with covariance matrix $\mathbb{E}(v_{\tau_j}\times v_{\tau_l})=\gamma_{\min(j,l)}$ for $j,\,l\le k$.

Lévy's characterisation of Brownian motion: Theorem 26 Suppose that $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is right-continuous. Let $\boldsymbol{v}$ be a jump-free martingale defined on $\mathcal{T}_b$ such that $v_{\check 0}=0$ and $v^*_{\check t}=t\chi1$ for every $t\ge0$. Then $\boldsymbol{v}^*=\boldsymbol{\iota}$ and $\boldsymbol{v}$ is a Brownian process as described in Lecture 18.

Time change: Theorem 27 Suppose $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is right-continuous. Let family $\langle\bar\tau_t\rangle_{t\ge0}$ be a non-decreasing family in $\mathcal{T}$. Then $\langle\mathfrak{B}_t\rangle_{t\ge0}=\langle\mathfrak{A}_{\bar\tau_t}\rangle_{t\ge0}$ is a filtration. Write $\mathcal{Q}$ for the lattice of $\langle\mathfrak{B}_t\rangle_{t\ge0}$-stopping times.

(a) We have a lattice homomorphism $\rho\mapsto\sigma_\rho:\mathcal{Q}\to\mathcal{T}$ such that

$\sigma_{\check t}=\bar\tau_t$ for $t\ge0$, $\quad\sigma_{\max\mathcal{Q}}=\max\mathcal{T}$, $\quad\mathfrak{B}_\rho=\mathfrak{A}_{\sigma_\rho}$ for $\rho\in\mathcal{Q}$,

if $\mathcal{S}\subseteq\mathcal{T}$ is a sublattice and $\langle u_\sigma\rangle_{\sigma\in\mathcal{S}}$ is a process fully adapted to $\langle\mathfrak{A}_t\rangle_{t\ge0}$, then $\mathcal{S}'=\{\rho:\sigma_\rho\in\mathcal{S}\}$ is a sublattice of $\mathcal{Q}$ and $\langle u_{\sigma_\rho}\rangle_{\rho\in\mathcal{S}'}$ is fully adapted to $\langle\mathfrak{B}_t\rangle_{t\ge0}$.

(b) If $\bar\tau_t=\inf_{s>t}\bar\tau_s$ for every $t\ge0$, then $\langle\mathfrak{B}_t\rangle_{t\ge0}$ is right-continuous and $\sigma_{\inf D}=\inf_{\rho\in D}\sigma_\rho$ for every $D\subseteq\mathcal{Q}$.

Theorem 28 Suppose that $\langle\mathfrak{A}_t\rangle_{t\ge0}$ is right-continuous. Let $\mathcal{S}$ be an order-convex sublattice of $\mathcal{T}$ with a least element, and $\boldsymbol{v}$ a jump-free martingale defined on $\mathcal{S}$ such that $v_{\min\mathcal{S}}=0$ and for every $t\ge0$ there is a $\tau\in\mathcal{S}$ such that $v^*_\tau\ge t\chi1$. Then there is a right-continuous time change $\langle\bar\tau_t\rangle_{t\ge0}$ such that the corresponding family $\langle v_{\sigma_\rho}\rangle_{\rho\in\mathcal{S}'}$ is a Brownian process.

Lecture 25

The Black-Scholes model (`http://en.wikipedia.org/wiki/Black-Scholes`)

Start from a differential equation

$$d\boldsymbol{u} = \alpha\boldsymbol{u}d\boldsymbol{\iota} + \beta\boldsymbol{u}d\boldsymbol{w}, \qquad u_{\dot{0}} = u$$

($\boldsymbol{\iota}$ being the identity process and $\boldsymbol{w}$ a Brownian process), with (unique) solution

$$\begin{aligned}\boldsymbol{u} &= u\mathbf{1} + \alpha ii_{\boldsymbol{\iota}}(\boldsymbol{u}) + \beta ii_{\boldsymbol{w}}(\boldsymbol{u}) = u\mathbf{1} + ii_{\tilde{\boldsymbol{w}}}(\boldsymbol{u}) \\ &= u\mathbf{1} + u \times \overline{\exp}(\tilde{\boldsymbol{w}} - \tfrac{1}{2}\beta^2\boldsymbol{\iota})\end{aligned}$$

where $\tilde{\boldsymbol{w}} = \alpha\boldsymbol{\iota} + \beta\boldsymbol{w}$, so that $\tilde{\boldsymbol{w}}^* = \beta^2\boldsymbol{w}^* = \beta^2\boldsymbol{\iota}$ (see Lecture 22). We suppose that we have an 'option' $\boldsymbol{v}$ in the 'stock' $\boldsymbol{u}$ whose value at time t is $h(x,t)$ if $\boldsymbol{u}$ has value x at that time; so that $\boldsymbol{v} = \bar{h}(\boldsymbol{u},\boldsymbol{\iota})$. (If $\boldsymbol{v}$ corresponds to a process $\langle V_t\rangle_{t\ge 0}$ and $\boldsymbol{u}$ to a process $\langle U_t\rangle_{t\ge 0}$, we are supposing that $V_t(\omega) = h(U_t(\omega),t)$ for most pairs (ω,t).) Suppose also that h is twice continuously differentiable. Then we have

$$\boldsymbol{v} = \bar{h}(\boldsymbol{u},\boldsymbol{\iota}),$$

$$v_\tau - v_{\dot{0}} = \int_{[\dot{0},\tau]} \bar{h}_1(\boldsymbol{u},\boldsymbol{\iota})\,d\boldsymbol{u} + \int_{[\dot{0},\tau]} \bar{h}_2(\boldsymbol{u},\boldsymbol{\iota})\,d\boldsymbol{\iota} + \frac{1}{2}\int_{[\dot{0},\tau]} \bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota})\,d\boldsymbol{u}^*$$

by Itô's formula for two variables (because $\boldsymbol{u}$ and $\boldsymbol{\iota}$ are jump-free local integrators, and $\boldsymbol{\iota}^* = [\boldsymbol{\iota}\,{}^*_|\,\boldsymbol{u}] = 0$). To get an expression for the integral $\int \ldots d\boldsymbol{u}^*$, we use the second corollary to Theorem 22 to see that it is

$$\int \ldots \boldsymbol{u}^2\,d\tilde{\boldsymbol{w}}^* = \beta^2\int \ldots \boldsymbol{u}^2\,d\boldsymbol{w}^* = \beta^2\int \ldots \boldsymbol{u}^2\,d\boldsymbol{\iota}.$$

So we have

$$v_\tau - v_{\dot{0}} = \int_{[\dot{0},\tau]} \bar{h}_1(\boldsymbol{u},\boldsymbol{\iota})\,d\boldsymbol{u} + \int_{[\dot{0},\tau]} (\bar{h}_2(\boldsymbol{u},\boldsymbol{\iota})\,d\boldsymbol{\iota} + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota}))\,d\boldsymbol{\iota}.$$

Now consider a hedged version of $\boldsymbol{v}$; holding $\boldsymbol{v}$, we hedge by a quantity $\bar{h}_1(\boldsymbol{u},\boldsymbol{\iota})$ in the stock $\boldsymbol{u}$ to give us a portfolio $\tilde{\boldsymbol{v}} = \boldsymbol{v} - ii_{\boldsymbol{u}}(\bar{h}_1(\boldsymbol{u},\boldsymbol{\iota}))$. This is feasible if we can adjust the hedge at stopping time intervals small compared with the evolution of $\boldsymbol{u}$, so that

$$\tilde{v}_{\sigma'} - \tilde{v}_\sigma \doteq v_{\sigma'} - v_\sigma - \bar{h}_1(u_\sigma,\sigma) \times (u_{\sigma'} - u_\sigma)$$

with an approximation sufficiently close to ensure that

$$\begin{aligned}\tilde{v}_\tau - \tilde{v}_{\dot{0}} &\doteq v_\tau - v_{\dot{0}} - \int_{[\dot{0},\tau]} \bar{h}_1(\boldsymbol{u},\boldsymbol{\iota})\,d\boldsymbol{u} \\ &= \int_{[\dot{0},\tau]} \bar{h}_2(\boldsymbol{u},\boldsymbol{\iota}) + \frac{1}{2}\beta^2\boldsymbol{u}^2 \times \bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota})\,d\boldsymbol{\iota}.\end{aligned}$$

But now approximations of the same kind tell us that

$$\tilde{v}_{\sigma'} - \tilde{v}_\sigma \doteq (\bar{h}_2(u_\sigma,\sigma) + \frac{1}{2}\beta^2u_\sigma^2 \times \bar{h}_{11}(u_\sigma,\sigma)) \times (\sigma' - \sigma).$$

Since you are choosing the stopping time intervals, the step

$$(\bar{h}_2(u_\sigma,\sigma)+\frac{1}{2}\beta^2u_\sigma^2\times\bar{h}_{11}(u_\sigma,\sigma))\times(\sigma'-\sigma)$$

is risk-free; at time σ you know just how your gain or loss depend on the time σ'. And so does everyone else, so (in a perfect market with no hysteresis or arbitrage)

$$\tilde{v}_{\sigma'}-\tilde{v}_\sigma\doteqdot\rho(v_\sigma-\bar{h}_1(u_\sigma,\sigma)\times u_\sigma)\times(\sigma'-\sigma)$$

where ρ is the interest rate on risk-free investments. (Note that your current investment at time σ is the value v_σ of the option you hold, less what you have just spent to buy stock forward that day, which is $\bar{h}_1(u_\sigma,\sigma)\times u_\sigma$.) Taking the limit in the usual way,

$$\begin{aligned}\tilde{v}_\tau-\tilde{v}_{\dot{0}}&=\rho\int_{[\dot{0},\tau]}\boldsymbol{v}-\bar{h}_1(\boldsymbol{u},\boldsymbol{\iota})\times\boldsymbol{u}\;d\boldsymbol{\iota}\\&=\int_{[\dot{0},\tau]}\bar{h}_2(\boldsymbol{u},\boldsymbol{\iota})+\frac{1}{2}\beta^2\boldsymbol{u}^2\times\bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota})\;d\boldsymbol{\iota},\end{aligned}$$

so that

$$\int_{[\dot{0},\tau]}\bar{h}_2(\boldsymbol{u},\boldsymbol{\iota})+\frac{1}{2}\beta^2\boldsymbol{u}^2\times\bar{h}_{11}(\boldsymbol{u},\boldsymbol{\iota})+\rho\boldsymbol{u}\times\bar{h}_1(\boldsymbol{u},\boldsymbol{\iota})-\rho\bar{h}(\boldsymbol{u},\boldsymbol{\iota})\;d\boldsymbol{\iota}=0.$$

If this is to be true for every τ, or at least for every τ in an order-convex sublattice $\mathcal{S}$ containing $\dot{0}$, we must have

$$\bar{h}_2(u_\sigma,\sigma)+\frac{1}{2}\beta^2u_\sigma^2\times\bar{h}_{11}(u_\sigma,\sigma)+\rho u_\sigma\times\bar{h}_1(u_\sigma,\sigma)-\rho\bar{h}(u_\sigma,\sigma)=0$$

for every $\sigma\in\mathcal{S}$, so that for relevant x, y we shall need

$$h_2(x,y)+\frac{1}{2}\beta^2x^2h_{11}(x,y)+\rho xh_1(x,y)-\rho h(x,y)=0$$

or, if you prefer,

$$\frac{\partial h}{\partial y}+\frac{1}{2}\beta^2x^2\frac{\partial^2h}{\partial x^2}+\rho x\frac{\partial h}{\partial x}-\rho h=0$$

which is the **Black-Scholes equation**.

References

Fremlin D.H. [01] *Measure Theory, Vol. 2: Broad Foundations.* Torres Fremlin, 2001 (http://www.lulu.com/content/8005793).

Fremlin D.H. [02] *Measure Theory, Vol. 3: Measure Algebras.* Torres Fremlin, 2002 (http://www.essex.ac.uk/maths/people/fremlin/mtcont.htm).

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Spaces.* Torres Fremlin, 2003 (http://www.essex.ac.uk/maths/people/fremlin/mtcont.htm).

Fremlin D.H. [??] *Stochastic Calculus* (http://www.essex.ac.uk/maths/people/fremlin/sc.htm).

Protter P.E. [03] *Stochastic Integration and Differential Equations.* Springer-Verlag, 2003.