Equivariant fixed point theory and related topics

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0 Introduction

In the 1920s, algebraic topology began to develop partially due to the need for new topological techniques in solving problems in fixed point theory. Indeed, the celebrated Lefschetz fixed point theorem is a far reaching generalization of the well-known Brouwer fixed point theorem. The field of topological fixed point theory continued to flourish through the 1940s during which much of the foundations was laid by the early pioneers such as S. Lefschetz, J. Nielsen, H. Hopf, K. Reidemeister, W. Franz, and F. Wecken among others.

The main purpose of these lectures is to present the analogous topological fixed point theory in the presence of a group action. Many topics in classical topological fixed point theory have their equivariant analogs.

This material will be presented in a series of four lectures as follows.

Lecture I: Equivariant Lefschetz fixed point theory
After reviewing the classical Lefschetz fixed point theorem, we introduce equivariant Lefschetz numbers for G-maps. We establish an equivariant analog of the classical Lefschetz fixed point theorem. Equivariant analogs of the classical Euler characteristic, topological degree and vector fields will be discussed.

Lecture II: Equivariant Nielsen fixed point theory
We first present elements of the classical Nielsen fixed point theory and a converse of the Lefschetz fixed point theorem. Equivariant Nielsen type numbers will be defined and a converse of the

equivariant Lefschetz fixed point theorem will be proven. The equivariant Reidemeister trace will also be discussed.

Lecture III: Equivariant Gottlieb groups, Rhodes groups, and torus homotopy groups
The first Gottlieb group, also known as the Jiang subgroup, plays an important role in Nielsen fixed point theory, while the higher Gottlieb groups are important objects of study in classical homotopy theory. We introduce equivariant Gottlieb groups and investigate the relationships among equivariant Gottlieb groups, Rhodes groups and Fox torus homotopy groups. Along this lines, we re-formulate some classical results in homotopy theory using generalized Fox torus homotopy groups.

Lecture IV: Equivariant Nielsen root theory
Fixed point theory can be generalized to coincidence theory of two maps between two spaces. When one of the maps is a constant map, we refer this as 'root theory'. In 1930, H. Hopf already introduced the notion of Nielsen root theory. Here, we generalize this theory to the equivariant setting. We relate Borsuk–Ulam type results with equivariant Nielsen root theory and illustrate how certain positive codimensional coincidence problem can be transformed into an equivariant Nielsen root problem.

1 Lecture I: Equivariant Lefschetz fixed point theory

This series of lectures intends to be a survey of certain topics in equivariant topological fixed point theory: we will look at the field of fixed point theory, but in the presence of a group action. This will be very diverse!

Notation. All maps are assumed to be continuous. Given a map $f\colon X \to X$, we will write $\operatorname{Fix} f$ for the fixed point set of f, i.e., $\operatorname{Fix} f = \{x \in X \mid f(x) = x\}$. Furthermore, D^n denotes the n-dimensional closed unit disc in $\mathbb{R}^n$; likewise, S^n denotes the n-dimensional unit sphere.

1.1 Lefschetz fixed point theorem

The story starts with the Brouwer fixed point theorem.

Theorem 1.1. *Any map $f\colon D^n \to D^n$ has a fixed point.*[1]

In 1921 Lefschetz generalized the above. Recall that given a topological space X, the q-th homology $H_q(X;\mathbb{Q})$ is a vector space, and a map $f\colon X \to X$ induces a linear transformation $f_{*_q}\colon H_q(X;\mathbb{Q}) \to H_q(X;\mathbb{Q})$ for any integer $q \geq 0$. Consequently, f_{*_q} is a matrix. If $H_q = 0$ for almost all $q \geq 0$ and the remaining H_q's are finite-dimensional, the *Lefschetz number* of f is defined to be $L(f) = \sum_{q=0}^{n}(-1)^q \operatorname{tr}(f_{*_q})$.

Theorem 1.2 (Lefschetz fixed point theorem)**.** *Let M be a compact, connected, triangulated manifold, $f\colon M \to M$ a map. If $L(f) \neq 0$, then* $\operatorname{Fix} f \neq \emptyset$.[2]

Remark 1.3. Clearly, if $f \sim g$, then $L(f) = L(g)$. In particular, if $L(f) \neq 0$, then every map homotopic to f has a fixed point.

How to prove such a theorem? The key idea is to use:

Theorem 1.4 (Hopf Trace Theorem)**.** *Let M be a compact, connected, triangulated manifold, $f\colon M \to M$ a map. Then $L(f) = \sum_{q=0}(-1)^q \operatorname{tr}(f_{\#_q})$, where $f_{\#q}\colon C_q(M;\mathbb{Q}) \to C_q(M;\mathbb{Q})$ is the transformation induced by f on the group of q-chains. In other words, $L(f)$ can be computed on the chain level.*

[1] Note that in dimension 1 this is equivalent to the intermediate value theorem.

[2] Throughout these lectures, we often will not state a result in full generality, unless it is beneficial, or the generalization is straightforward; the Lefschetz fixed point theorem is known to hold in more general spaces, e.g., compact metric ANRs.

1.2 Fixed point index

Let $U \subseteq \mathbb{R}^n$ be an open subset and $f\colon U \to \mathbb{R}^n$ a map. Assume that $\operatorname{Fix} f$ is compact in U. Write $o_{\operatorname{Fix} f} \in H_n(U, U \setminus \operatorname{Fix} f)$ for the fundamental class around $\operatorname{Fix} f$ and $o_0 \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ for the fundamental class around $\{0\}$. The *fixed point index of* f is the integer $\operatorname{ind}(f, U)$ such that

$$(i - f)_*(o_{\operatorname{Fix} f}) = \operatorname{ind}(f, U) \cdot o_0,$$

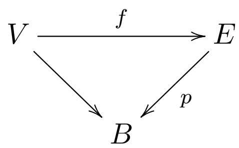
where $i\colon U \hookrightarrow \mathbb{R}^n$ is the inclusion.

Basic properties of $\operatorname{ind}(f, U)$**.** (1) If $\operatorname{ind}(f, U) \neq 0$, then f has a fixed point in U.

(2) (HOMOTOPY INVARIANCE) If $H\colon U \times I \to \mathbb{R}^n$ is a homotopy such that $\bigcup_{t\in[0,1]} \operatorname{Fix} H_t \subseteq U$ is a compact subset, then $\operatorname{ind}(H_0, U) = \operatorname{ind}(H_1, U)$.

(3) (NORMALIZATION) $\operatorname{ind}(f, X) = L(f)$.[3]

Remark 1.5. There is a more abstract approach to the notion of fixed point index: in 1974, Dold introduced the so called fixed point index over B.[4] It is concerned with maps $f\colon V \to E$ which factor through a fibration over B:



Here E is an ENR, $V \subseteq E$ is an open subspace, B is paracompact and $p\colon E \to B$ is a fibration.

Given two such maps $f_0\colon V_0 \to E_0$ and $f_1\colon V_1 \to E_1$ over B, we write $f_0 \sim f_1$ if there exists a map $g\colon W \to E$ over $B \times [0,1]$ such that $g_t\colon W_t \to E_t$ is over $B \times \{t\}$ for any $t \in [0,1]$ and $\bigcup_{t\in[0,1]} \operatorname{Fix} g_t$ is compact. Define $\mathrm{FIX}_B = \{[f]\}$; this is an additive monoid with the neutral element $[\emptyset]$ (no fixed points).

The *Dold fixed point index* is a map $I\colon \mathrm{FIX}_B \to \pi_s^0(B \oplus \{\mathrm{pt}\})$, where the target is the 0-th stable cohomotopy group of $B \oplus \{\mathrm{pt}\}$.

1.3 Basics of equivariant topology

Let G be a finite group. Recall that a G*-space* is a topological space X equipped with a (continuous) G-action.[5] The set $X^G = \{x \in X \mid gx = x \text{ for any } g \in G\}$ is called the *fixed point set* of the action.

Example 1.6. Consider the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The formula $gz = \bar{z}$ defines a $\mathbb{Z}_2$-action on S^1. One easily sees that $(S^1)^{\mathbb{Z}_2} = \{-1, 1\}$.

Given two G-spaces X, Y, we say that $f\colon X \to Y$ is a G*-map* if $f(gx) = gf(x)$ for any $g \in G$ and $x \in X$.

Example 1.7. Consider S^1 equipped with the $\mathbb{Z}_2$-action of Example 1.6. The map $f\colon S^1 \to S^1$ given by $f(x, y) = (-x, y)$ for any $(x, y) \in S^1$ is clearly a $\mathbb{Z}_2$-map.

In general, any G-map $f\colon X \to X$ has the following properties:

(1) For any subgroup $H \subseteq G$, we have $f^H = f_{|X^H}\colon X^H \to X^H$.

(2) $\operatorname{Fix} f$ is a G-subset.

[3] The index has other properties (additivity, commutativity, ...) which we will not talk about right now.

[4] B should be thought of as a parameter space.

[5] We will restrict attention to finite groups, but this definition clearly makes sense for any topological group G.

Remark 1.8. Recall that the ordinary fixed point index is a map $I\colon \mathrm{FIX}_{\{pt\}} \to \mathbb{Z}$. In the equivariant setting, we have $I_G\colon G - \mathrm{FIX}_{\{pt\}} \to A(G)$, where $A(G)$ is the Burnside ring. As an abelian group, $A(G)$ is generated by $[G/H]$. Multiplicatively, $A(G) = \prod_{(H)} \mathbb{Z}$, where (H) stands for the conjugacy class of a subgroup $H \subseteq G$.

It is clear that if we want to study fixed points of equivariant maps, we need to take into account fixed points of the action. One way of defining an equivariant Lefschetz number is to consider $L_G(f) = \big(L(f^{H_1}), L(f^{H_2}), \ldots, L(f^{H_k})\big)$. In the semi-free[6] case, this reduces to $L_G(f) = \big(L(f), L(f^G)\big)$.

It turns out that

$$L_G(f) = \frac{L(f) - L(f^G)}{|G|}[G/1] + L(f^G)[G/G].$$

Since the coefficients are integers, it means that $L(f) \equiv L(f^G) \bmod |G|$.

We will now give one example of application of this formula.

Theorem 1.9. *Let $f\colon M \to M$ be a map, p a prime number. Then $L(f^n) \equiv L(f^p) \bmod p$.*

Proof. Let M be the p-fold product of X. Write $G = \langle \zeta \rangle$ for the cyclic group of order p. Consider the G-action on M given by $\zeta(x_1, x_2, \ldots, x_p) = (x_p, x_1, x_2, \ldots, x_{p-1})$.

Define $g_f\colon M \to M$ by $g_f(x_1, \ldots, x_p) = \big(f(x_p), f(x_1), f(x_2), \ldots, f(x_{p-1})\big)$. Clearly, g_f is a G-map. Furthermore, since $M^G = \big\{(x, x, \ldots, x) \mid x \in X\big\} \approx X$, g_f^G is basically f.

Note that if $x \in \mathrm{Fix}\, f^p$, then $\big(x, f(x), \ldots, f^{p-1}(x)\big) \in \mathrm{Fix}\, g_f$. Likewise, if $(x_1, x_2, \ldots, x_p) \in \mathrm{Fix}\, g_f$, then $x_1 \in \mathrm{Fix}\, f^p$. Consequently, $\mathrm{Fix}\, g_f \sim \mathrm{Fix}\, f^p$. Moreover, $L(g_f) = \mathrm{ind}(g_f) = L(f^p)$. By the preceding formula, $L(g_f) - L(g_f^G)$ is divisible by p and the conclusion follows. □

2 Lecture II: Equivariant Nielsen fixed point theory

Roughly speaking, Lefschetz fixed point theory uses algebraic (precisely, homological) methods to count fixed points. In 1927 Nielsen introduced a more geometric approach to this problem. We will see how it works; to begin with, let us take a look at an example.

Example 2.1. Let $X = S^1 \vee S^1$. Consider the common point x, and a point y antipodal to x on one of the circles. It is well-known that $\pi_1(X, x) = \langle \alpha, \beta \rangle$, the free group on two generators. Consequently, $H_1(X; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Write $\bar{\alpha}$, $\bar{\beta}$ for the images of α, β in H_1.

Let $f\colon X \to X$ be a map such that $\mathrm{Fix}\, f = \{x, y\}$ and $f_\#(\alpha) = \alpha^2$, $f_\#(\beta) = \beta^{-1}$. Since $f_{*_1}(\bar{\alpha}) = 2\bar{\alpha}$ and $f_{*_1}(\bar{\beta}) = -\bar{\beta}$, it follows that $L(f) = 0$.

On the other hand, every map homotopic to f must have at least two fixed points. To see this we will use the 'Nielsen way' of counting. The bottom line, however, is that the Lefschetz theory is good as long as the Lefschetz number is nonzero.

2.1 Nielsen fixed point theory

The idea is to partition $\mathrm{Fix}\, f$ into disjoint classes as follows. We say that x, $y \in \mathrm{Fix}\, f$ are *Nielsen equivalent* (as fixed points) if there exists a path $\sigma\colon [0,1] \to X$ such that $\sigma(0) = x$, $\sigma(1) = y$ and $f \circ \sigma \sim \sigma$ relative to endpoints. It is easy to see that this is an equivalence relation.

Now we have these classes F_i of fixed points. We always assume that the space in question is compact, so $\mathrm{Fix} f$ is also compact, and in this situation the number of F_i's is finite. From now on, we will use the name *fixed point classes* (or simply *fpc*s).

Note that every fpc F has an open neighbourhood U which does not contain any other fixed point, hence we can define the *fixed point index* of a class F to be $i(F) = \mathrm{ind}(f, U)$. The *Nielsen number* of f is then $N(f) = \#\{F \mid i(F) \neq 0\}$.

[6] An action $G \times X \to X$ is said to be *semi-free* if it is free outside the fixed point set, i.e., for any $x \in X$, the isotropy group G_x is either trivial or the whole group.

Remark 2.2. We immediately have that $N(f) = N(f')$ whenever $f \sim f'$. Furthermore, $N(f) \leq \#\mathrm{Fix}\, f$. Putting these two together yields $N(f) \leq \min\{\#\mathrm{Fix}\, f' \mid f' \sim f\}$. This shows that the Nielsen number gives more information than the Lefschetz number.

Coming back to Example 2.1, we will sketch that $N(f) = 2$. We either have one class, containing both x and y, or two classes, each consisting of a single point. We will show that the latter holds.

Suppose x and y are Nielsen equivalent, i.e., there exists a path σ from x to y such that $\sigma \sim f \circ \sigma$. Write ω for $f \circ \sigma$. Then $\sigma\omega^{-1}$ is a loop based at x. Let $\gamma = [\sigma\omega^{-1}] \in \pi_1(X, x)$. Consider $\gamma\beta f_\#(\gamma)^{-1} = [\sigma\omega^{-1}]\beta[f \circ \omega][f \circ \sigma^{-1}]$. It is not hard to see that $\beta[f \circ \omega] = [\omega]$. Consequently, $\gamma\beta f_\#(\gamma)^{-1} = [\sigma(f \circ \sigma^{-1})] = [\sigma\sigma^{-1}] = 1$.

Observe that in H_1, $\gamma\beta f_\#(\gamma)^{-1} = 1$ becomes $\bar{\gamma} + \bar{\beta} = f_{*_1}(\bar{\gamma})$. This cannot happen. Indeed, since $H_1 = \mathbb{Z} \oplus \mathbb{Z} = \langle \bar{\alpha}, \bar{\beta} \rangle$, we can write $\bar{\gamma} = a\bar{\alpha} + b\bar{\beta}$ for some $a, b \in \mathbb{Z}$. Hence in matrix notation

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

This is impossible, and so such a γ cannot possibly exist.

We know that $L(f) = \mathrm{ind}(f, x) + \mathrm{ind}(f, y)$. It turns out that neither of the indices is zero, hence one must be equal to 1, the other -1. It follows that $N(f) = 2$.

Theorem 2.3 (Wecken). *Let M be a compact, connected, triangulated manifold, $\dim M \geq 3$. For any map $f \colon X \to X$, there exists a map f' homotopic to f such that $\#\mathrm{Fix}\, f' = N(f)$.*

In other words, not only $N(f)$ is the best lower bound, but it can always be achieved. It simply means that main question now is: how do we compute the Nielsen number?

Remark 2.4. Wecken's theorem is false in dimension 2; a counterexample was provided by Jiang in 1984. In fact, counterexamples exist for any surface of negative Euler characteristic.

Observe that Nielsen theory gives a sort of a converse to the Lefschetz fixed point theorem: in many cases, if $N(f) = 0$, then there exists a map homotopic f which does not have a fixed point. To obtain a 'true' converse of the Lefschetz fixed point theorem, however, we need to have: $L(f) = 0$ implies $N(f) = 0$. This happens, for example, when the space in question is simply connected, since in this case every loop is contractible. As a consequence, $N(f)$ is either 0 or 1, because there exists only one fpc. But $\mathrm{ind}(f, X) = L(f)$.

Theorem 2.3 also has its equivariant version. Let $G = \mathbb{Z}_p$, so that the action is semi-free.

Theorem 2.5 (Wilczyński, Vidal). *Let M be a compact, simply connected G-smooth manifold with M^G simply connected and $\dim M^G \geq 3$. Let $f \colon M \to M$ be a G-map. If $L(f) = 0$ and $L(f^G) = 0$, then $f \sim_G f'$ with $\mathrm{Fix}\, f' = \emptyset$.*

In 1988 Fadell and Wong proved the same thing for the Nielsen number, except they did not use the assumption of simple connectedness. Instead, they assumed that $\mathrm{codim}\, M^G \geq 2$.[7]

Example 2.6. Consider the $\mathbb{Z}_2$-action on S^2 given by the reflection across the equator, so that $(S^2)^{\mathbb{Z}_2} = S^1$. Let $f \colon S^2 \to S^2$ be any rotation about the axis through N/S poles. It is clear that f is a $\mathbb{Z}_2$-map with exactly two fixed points, namely the poles. Since M is simply connected, they belong to the same equivalence class.

It turns out that $N(f) = 1$ and $N(f^G) = 0$. Since $f \sim \mathrm{id}$, we have

$$L(f) = L(\mathrm{id}) = \chi(S^2) = 2 \neq 0.$$

Even though the minimal number of fixed points for any $f' \sim_G f$ appears to be 2, it actually is 1.

[7] The assumption on codimension is necessary: there exist relevant counterexamples.

3 Lecture III: Equivariant Gottlieb groups, Rhodes groups, and torus homotopy groups

Let f be a G-map and $x, y \in \operatorname{Fix} f$. We say that x, y are *G-Nielsen equivalent* if either $y = gx$ for some $g \in G$ or there exists a path $\sigma\colon [0,1] \to X$ such that $\sigma(0) = x$, $\sigma(1) = gy$ for some $g \in G$, and $f \circ \sigma \sim \sigma$ relative to the endpoints. Note that such a G-fixed point class is a disjoint union of ordinary fixed point classes. Therefore we can define the fixed point index as before and set $N_G(f) = \#\{G\text{-fpc } F \mid \operatorname{ind}(f, F) \neq 0\}$.

We will make a little detour before further investigating N_G.

3.1 Reidemeister trace

Let $\eta\colon \tilde{X} \to X$ be the universal covering of X. It is well-known that given a map $f\colon X \to X$, we have a lift (in fact, many lifts) $\tilde{f}\colon \tilde{X} \to \tilde{X}$; choose one. Clearly, if $\tilde{x} \in \operatorname{Fix} \tilde{f}$, then $\eta(\tilde{x}) \in \operatorname{Fix} f$. Even better than that, $\eta(\operatorname{Fix} \tilde{f}) \subseteq \operatorname{Fix} f$, if nonempty, is a single fixed point class. Indeed, that all fixed points in $\eta(\operatorname{Fix} \tilde{f})$ belong to a single class follows immediately from the fact that $\tilde{X}$ is simply connected: take any path between fixed points in the covering and project it 'downstairs'. It remains to show that any two fixed points in the base come from fixed points of the universal covering, but this also is not too difficult.

Sometimes two distinct lifts, say $\alpha\tilde{f}$, $\beta\tilde{f}$, may produce the same fixed point class. (Here α and β are deck transformations.) It follows from the preceding discussion that $\eta(\operatorname{Fix} \alpha\tilde{f}) \cap \eta(\operatorname{Fix} \beta\tilde{f})$ is either empty or $\eta(\operatorname{Fix} \alpha\tilde{f}) = \eta(\operatorname{Fix} \beta\tilde{f})$. The second situation happens exactly when $\beta = \sigma\alpha f_{\#}(\sigma)^{-1}$ for some $\sigma \in \pi_1(X)$. This brings us to the so called *Reidemeister classes*, which are the orbits of the action of π_1 on π_1 given by $\sigma\alpha \mapsto \sigma\alpha f_{\#}(\sigma)^{-1}$.

Reidemeister has been able to produce something very similar to the Lefschetz number, but using the fact that fixed points come from the fixed points of lifts to the universal cover.

Recall that π_1 acts freely on $\tilde{X}$. That makes $C_q(\tilde{X}, \mathbb{Z})$, the q-th cellular chains of $\tilde{X}$, a free $\mathbb{Z}\pi$-module. Consider the transformation $\tilde{f}_{\#_q}\colon C_q(\tilde{X}, \mathbb{Z}) \to C_q(\tilde{X}, \mathbb{Z})$ induced by $\tilde{f}$; this is a matrix over $\mathbb{Z}\pi$, say P_q.

We now need to take into account that different lifts may produce the same classes. A remedy to this is looking not at π, but rather at π modulo the action – precisely the Reidemeister classes. Set $R(\varphi, \pi)$ to be the set of Reidemeister classes. Clearly, we have a surjection $\varphi\colon \mathbb{Z}\pi \to \mathbb{Z}R(\varphi, \pi)$.

Define the *Reidemeister trace* to be

$$RT(\varphi, \pi) = \sum (-1)^q \varphi(\operatorname{tr} P_q) \in \mathbb{Z}R(\varphi, \pi).$$

Since RT lives in an abelian group, $RT(\varphi, \pi) = \sum_{\alpha \in R(\varphi,\pi)} = i_\alpha \alpha$, where $i_\alpha \in \mathbb{Z}$. It turns out that $i_\alpha = 0$ if the corresponding fpc is empty and $i_\alpha = \operatorname{ind}(\alpha, F_\alpha)$, where F_α is the nonempty fpc corresponding to α. This contains information about both Lefschetz number and Nielsen number: $L(f) = \sum i_\alpha$ and $N(f) = \#\{\alpha \mid i_\alpha \neq 0\}$.

A question that we will be interested in is: what is the covering space approach to the equivariant Nielsen fixed point theory? We will come back to this.

3.2 Jiang subgroup

Example 3.1. Consider the lens space $L_p = S^3/\mathbb{Z}_p$. Its universal covering is S^3; this is a p-fold cover. This means that the degree of $\tilde{f}$ is the same as the degree of $\alpha\tilde{f}$, regardless of the lift that we are using. Consequently, all the lifts have the same degree. Hopf's theorem now says that $\alpha\tilde{f}$ is homotopic to $\beta\tilde{f}$ for all α, $\beta \in \pi_1$. So if we look at the fixed points of the covering, we have $L(\alpha\tilde{f}) = 1 - \deg \alpha\tilde{f}$.

Franz proved that for any map $f\colon L_p \to L_p$, all fixed point classes of f have the same index, say m. Let us take a look at the Lefschetz number:

$$L(f) = \sum_{\text{fpc } F} \operatorname{ind}(f, F) = \sum_{\text{fpc } F} m.$$

Now, if $L(f) = 0$, then $m = 0$ and $N(f) = 0$. On the other hand, if $L(f) \neq 0$, then $L(f) = mN(f)$. This looks very promising!

The key feature which allows this to work is that $f \sim f$ via a homotopy starts at fpc_α and finishes at fpc_β. (Recall that a homotopy from a map to itself is called a *cyclic homotopy*.)

It is exactly this idea that led Jiang to finding conditions under which all fixed point classes have the same index. Consider the group

$J(X, x_0) = \{\alpha \in \pi_1(X, x_0) \mid \text{there exists a cyclic homotopy } H_t \text{ of } \mathrm{id}_X \text{ such that } H_t(x_0) = \alpha(t)\}$.

Theorem 3.2. *If $J(X) = \pi_1(X)$, then for any map $f\colon X \to X$ the fpcs of f have the same index.*

Remark 3.3. Gottlieb studied the same group, though he named it $G_1(X, x_0)$ and used it to study different problems in algebraic topology. His approach was as follows.

Let $x_0 \in X$ be a basepoint. Consider the evaluation map $ev\colon \operatorname{Map}(X, X) \to X$, $ev(f) = f(x_0)$. Set $G_1(X, x_0) = ev_\#\big(\pi_1(\operatorname{Map}(X, X), 1_X)\big)$.

Using higher homotopy groups, one can define $G_n \subseteq \pi_n(X)$.

One especially beautiful application of Theorem 3.2 is:

Theorem 3.4. *Let M be a compact aspherical manifold. If $\chi(M) \neq 0$, then the center of $\pi_1(M)$ is trivial.*

Let us get back to the question of promoting this to the situation with group actions. Let X be G-space. Every $g \in G$ acts on X as a homemorphism; the identitity is lifted to a deck transformation. The homemorphism g can also be lifted to some homeomorphism of the universal cover $\tilde{X}$. This hints that instead of looking at deck transfotmations, we should look at all those homeomorphisms. Consider the group

$$\tilde{G} = \{\sigma \in \operatorname{Homeo}(\tilde{X}) \mid \eta\sigma = g_\sigma \eta \text{ for some } g_\sigma \in G\}.$$

Elements that cover the identity are deck transformations, so we have a short exact sequence

$$1 \to \pi \to \tilde{G} \to G \to 1.$$

Recall the G-fpcs. They are disjoint union of ordinary fpcs. In the non-equivariant case, we had a projection $\eta \operatorname{Fix} \alpha\tilde{f}$ which turned out to be one of the fpcs of the base space. We want to do exactly the same thing now, only that this time we will use $\tilde{G}$. Note that $G\eta(\operatorname{Fix} \alpha\tilde{f}) = G\eta(\operatorname{Fix} \beta\tilde{f})$ if and only if $\beta = \tilde{\gamma}\alpha\varphi_G(\tilde{\gamma})^{-1}$ for some $\tilde{\gamma} \in \tilde{G}$, $\varphi_G\colon \tilde{G} \to \tilde{G}$ induced by f.

Remark 3.5. While trying to introduce the equivariant Jiang subgroup ca. 2002, Wong realized that Rhodes discovered and studied $\tilde{G}$ already in 1966. He defined what he called the *fundamental group of a transformation group* (X, G).

Let $x_0 \in X$ be a basepoint, $g \in G$, and $\alpha\colon [0,1] \to X$ be a path such that $\alpha(0) = x_0$ and $\alpha(1) = gx_0$. If $g = e$, α is a loop. Write $[\alpha; g]$ for the relevant homotopy classes and define $[\alpha_1; g_1][\alpha_2; g_2] = [\alpha_1 * g\alpha_2; g_1 g_2]$, where $*$ stands for concatenation of paths. This operation turns $\sigma_1(X, x_0, G) = \{[\alpha; g]\}$ into a group. Furthermore, there is a short exact sequence

$$1 \to \pi_1(X, x_0) \to \sigma_1(X, x_0, G) \to G \to 1.$$

Wong realised that σ_1 is the same as previously defined $\tilde{G}$.

In 1969 Rhodes defined the higher Rhodes groups $\sigma_n(X, x_0, G)$. He proved that for any $n \geq 1$ there is a short exact sequence

$$1 \to \tau_n(X, x_0) \to \sigma_n(X, x_0, G) \to G \to 1,$$

where τ_n is the torus homotopy group.

4 Lecture IV: Equivariant Nielsen root theory

Recall that given a polynomial $p(x)$, a solution of the equation $p(x) = 0$ is often called a *root*. We will now be interested in how topology can be applied to the problem of finding roots.

In 1927 Nielsen developed the notion of fixed point classes. Few years later, Hopf introduced what is now called the Hopf degree. But Hopf also studied preimages in the same fashion as Nielsen, i.e., using the notion of root classes. What does that mean?

4.1 Nielsen root theory

Let X, Y be orientable manifolds, $\varphi\colon X \to Y$ a map. Since we are not in the Euclidean situation, we pick a point $a \in Y$ and want to study solutions of the equation $\varphi(x) = a$.

Let $\Gamma_\varphi = \varphi^{-1}(a)$ be the set of roots. We say that x_1, $x_2 \in \Gamma_\varphi$ are *Nielsen equivalent* (as roots) if there exists a path $\sigma\colon [0,1] \to X$ such that $\sigma(0) = x_1$, $\sigma(1) = x_1$ and $\varphi \circ \sigma \sim \bar{a}$, where $\bar{a}\colon [0,1] \to Y$ is the constant path at a. Arising equivalence classes are now called *root classes.*

Let α be a root class. We want to define an index similar to the fixed point index. Choose an open set $U \subseteq X$ containing α but no other roots. Consider the following diagram:

$$X \overset{i}{\hookrightarrow} (X, X \setminus \alpha) \overset{j}{\hookleftarrow} (U, U \setminus \alpha) \overset{\varphi}{\to} (Y, Y \setminus \{a\}).$$

Note that j gives an excision. The *root index* of α is defined to be $\omega(\varphi, \alpha) = \varphi_{*n} \circ j_{*n}^{-1} \circ i_{*n}[X]$, where $[X]$ is the fundamental class of X.[8]

Now set $N(\varphi, a) = \#\{\alpha \mid \omega(\varphi, \alpha) \neq 0\}$; this is the *Nielsen root number* of φ.

Theorem 4.1. *Let X, Y be compact, connected, orientable manifolds, $a \in Y$. Let $\varphi\colon X \to Y$ be a map. For any two root classes α, β, we have $\omega(\varphi, \alpha) = \omega(\varphi, \beta)$, so that either $N(\varphi, a) = 0$ or $N(\varphi, a) = R(\varphi, a) = [\pi_1(Y) : \varphi_\#(\pi_1(X))] < \infty$.*

This makes the Nielsen root theory much more computable then the Nielsen fixed point theory.

Since root classes are disjoint, we have $\deg \varphi = \sum \omega(\varphi, \alpha)$. Hence if $\deg \varphi \neq 0$, then $\deg \varphi = mN(\varphi, \alpha)$, where $m = \omega(\varphi, \alpha)$. On the other hand, if $\deg \varphi = 0$, then $N(\varphi, \alpha) = 0$. These resemble the consequences of the Jiang condition. And it is exactly that!

We will show that given any two root classes α, β, there is a cylic homotopy of the identity relating α to β. Indeed, this is automatic if we work with a manifold, since in this case we have the so called homogeneity property. It implies that for any path σ between two points $p \neq q$ of the manifold, there exists a homotopy H_t of the identity to a map f such that $f(p) = q$ and $H_t(p) = \sigma(t)$.

Example 4.2. Suppose we want to study fixed point theory on the n-torus $T^n = S^1 \times \cdots \times S^1$. Let $f\colon T^n \to T^n$ be a map. Since T^n is a group, we can define a map $\varphi_f\colon T^n \to T^n$ by setting $\varphi_f(x) = x^{-1}f(x)$. Then $\varphi_f(x) = e$ if and only if $f(x) = x$, which means that the roots of φ_f are precisely the fixed points of f. With a little work, one sees that the root classes of φ_f

[8] In the case of nonorientable manifolds, we do not evaluate the homomorphism on the fundamental class: the index is just the homomorphism itself.

correspond to the fixed point classes of f, the root index corresponds to the fixed point index, and $(-1)^n \deg \varphi_f = L(f)$. Finding the degree is easier than finding the Lefschetz number, as the former depends only on the top homology.

The bottom line is that sometimes a fixed point problem can be translated into a root problem.

Of course, Example 4.2 is somewhat non-satisfactory, as T^n is a group. We can, however, improve on that easily.

Let G be a compact Lie group, $K \subseteq G$ a closed subgroup. Write M for the coset space $G/K = \{gK \mid g \in K\}$. Given a map $f\colon M \to M$, define $\varphi_f\colon G \to M$ by setting $\varphi_f(g) = g^{-1}f(gK)$. Similarly as before, if $f(gK) = gK$, then $\varphi_f(g) = eK$.

Observe that K acts on M, and K also acts (freely) on G by $k \cdot g = gk^{-1}$. Clearly, φ_f is a K-map:

$$\varphi_f(k \cdot g) = \varphi_f(gk^{-1}) = (gk^{-1})^{-1}f(gk^{-1}K) = kg^{-1}f(gK) = k\varphi_f(g).$$

Example 4.3. Let $G = S^3$, $K = S^1$. Then $S^3/S^1 = S^2$, and given a map $f\colon S^2 \to S^2$, we have $\varphi_f\colon S^3 \to S^2$. (This is not very exciting since S^2 is simply connected, but it nevertheless gives a clue about how the whole thing works.)

Suppose that $\varphi_f\colon S^3 \to S^2$ is the constant map at eS^1. Then φ_f is homotopic to a map $\psi\colon S^3 \to S^2$ such that $\psi^{-1}(eS^1) = \emptyset$. However, every K-map which is K-homotopic to φ_f must have roots. This means that the homotopy $\varphi_f \sim \psi$ cannot be equivariant. (Here is another reason why this cannot happen: $eS^1 = \varphi_f(g) = g^{-1}f(gS^1)$, which implies $f = \mathrm{id}$.)

4.2 Equivariant Nielsen root theory

Let X, Y be G-spaces and $\varphi\colon X \to Y$ a G-map. Suppose $a \in Y^G$.

Clearly, $\Gamma_\varphi = \varphi^{-1}(a)$ is a G-set: if $x \in \Gamma_\varphi$, then $gx \in \Gamma_\varphi$. We say that x, $y \in \Gamma_\varphi$ are *G-Nielsen equivalent* as roots if (1) $y = gx$ for some $g \in G$ or there exists a path $\sigma\colon [0,1] \to X$ such that $\sigma(0) = x$, $\sigma(1) = gy$ for some $\sigma \in G$, and $\varphi \circ \sigma \sim \bar{a}$.

Define the *equivariant root index* and the *equivariant Nielsen root number* accordingly.

Using all of this as background, one can prove the following theorem.

Theorem 4.4. *Let G be a compact, connected Lie group, $K \subseteq G$ a closed subgroup. Assume that $M = G/K$ is an orientable manifold. Given a map $f\colon M \to M$, either $L(f) = 0$, and then $N(f) = 0$, or $L(f) \neq 0$ and $N(f) = R(f)$.*[9]

Roughly speaking, a theorem like this works because all the fixed point classes have fixed point indices of the same sign.

4.3 Relationship with the Borsuk–Ulam theorem and coincidence theory

Recall the celebrated Borsuk–Ulam theorem.

Theorem 4.5. *For any map $f\colon S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ such that $f(x) = f(-x)$.*[10]

Consider the usual antipodal action on S^n and the antipodal action on $\mathbb{R}^n$, so that $(\mathbb{R}^n)^{\mathbb{Z}_2} = \{0\}$. Phrased this way, the Borsuk–Ulam theorem says that for every $\mathbb{Z}_2$-map $\psi\colon S^n \to \mathbb{R}^n$, we have $\psi^{-1}(0) \neq \emptyset$. This is a perfect setting for applying the equivariant Nielsen root theory.

We conclude the lectures with the following remark. The equivariant root theory can be used even further: it can be helpful in solving problems in coincidence theory. Recall that concidence

[9] Spaces like G/K usually do not fulfill the Jiang condition.

[10] We note that there are hundreds of papers devoted to generalizing or re-interpreting this theorem.

theory is concerned with the set $C(f, h) = \{x \in X \mid f(x) = h(x)\}$, where f, $h\colon X \to Y$ are arbitrary maps.

Assume for simplicity that $Y = G/K$, where G is a compact, connected Lie group and $K \subseteq G$ is a closed subgroup. Recall that $p\colon G \to G/K$ is a fiber bundle. Pullback p along f; write $\hat{X}$ for the pullback. It is well-known that $\hat{X} \to X$ is also a fiber bundle.

$$\begin{CD} \hat{X} @>{\hat{f}}>> G \\ @VVV @VV{p}V \\ X @>{f}>> Y \end{CD}$$

As before, K acts on both G/K and G. Furthermore, K acts on $X \times G$ diagonally (with the trivial action on X), so that $\hat{X}$ is a K-space. Define $\varphi\colon \hat{X} \to G/K$ by $\varphi(x, g) = \hat{f}(x, g)^{-1}h(x)$. This is a K-map and, more importantly, $\varphi(x, g) = eK$ if and only if $f(x) = h(x)$.

This is not necessarily an easier approach, but looking at the same problem from a different point of view is always useful.