

# Universal homogeneous structures (lecture notes)

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 December 31, 2012

## Abstract

We present category-theoretic framework for universal homogeneous structures, with selected applications.

Most of the material had been presented as a series of lectures for graduate students at Department of Mathematics, University of Silesia in Katowice, Poland (21 – 23 November 2012).

## Introduction

When studying a given class of structures, it is an interesting and important issue to find a “special” object that is universal for the class, namely, every other object “embeds” into this special one. A better property of this special object would be some sort of homogeneity, namely, that every isomorphism between two “small” substructures extends to an automorphism of the special object (as we shall see later, the exact meaning of “small” will depend on the class under consideration). Probably one of the earliest results in this spirit was Cantor’s theorem on the uniqueness of the set of rational numbers among all countable linear orders. More precisely,  $\mathbb{Q}$  is the only (up to isomorphism) countable linear order which contains all other countable linear orders and every isomorphism between two finite subsets extends to an automorphism of  $\mathbb{Q}$ . A typical statement of Cantor’s theorem is different: the extension property is equivalent to saying that  $\mathbb{Q}$  is order dense and has no endpoints. The argument used in the proof is now called the “back-and-forth” method: an automorphism extending a given finite isomorphism is constructed inductively, interchanging the domain and the co-domain at each step.

It was Urysohn who found an analogue of Cantor’s back-and-forth argument for metric spaces. Namely, he found a complete separable metric space  $\mathbb{U}$  which has very similar properties to the rationals, now considering isometries between finite subsets. Urysohn’s work [45] was actually forgotten for many years; it received some significant attention in the end of the 20th century due to problems in topological dynamics; perhaps the most representative work is [24].

One of the most important works on this subject, independent of Urysohn, was done by Roland Fraïssé [11] in 1954. This is a model-theoretic approach to Cantor’s back-and-forth argument, which can be partially applied also to the case of Urysohn’s metric space. Roughly speaking, Fraïssé considers a class  $\mathfrak{K}$  of finite (or, at least, finitely generated) models of some first order language. The class should have the amalgamation property, that is, each two embeddings of the same model  $Z \in \mathfrak{K}$  can be extended to a further embedding into a bigger model in  $\mathfrak{K}$ . In other words, given two embeddings  $f: Z \rightarrow X$ ,  $g: Z \rightarrow Y$ , there should exist a model  $W \in \mathfrak{K}$  and embeddings  $f': X \rightarrow W$  and  $g': Y \rightarrow W$  satisfying  $f' \circ f = g' \circ g$ . If such a class has only countably many isomorphic types and each two models embed into a common one (the joint embedding property), then there exists a countable model  $\mathbb{U}$  that can be represented as the union of a chain of models from the class  $\mathfrak{K}$ , contains isomorphic copies of all models in  $\mathfrak{K}$  and has the following strong homogeneity property: every isomorphism between submodels of  $\mathbb{U}$  which are in  $\mathfrak{K}$  extends to an automorphism of  $\mathbb{U}$ . Furthermore, the model  $\mathbb{U}$  is unique up to isomorphism. It is often called the *Fraïssé limit* of the class  $\mathfrak{K}$ .

This line of investigation was further continued by Jónsson [19] and Morley & Vaught [35] (see also Yasuhara [46]), where uncountable classes of models were studied. Besides the amalgamation property, typical cardinal-arithmetic assumption  $\kappa = 2^{<\kappa}$  is needed for the existence of the universal homogeneous structure of cardinality  $\kappa$ . One has to mention a curious independent work of Trnková [44] with metamathematical results on universal categories, in the setting of Bernays-Gödel set theory. One of the main tools is the amalgamation property for certain classes of categories, treated just as first-order structures.

All the authors cited above assume that the class of structures is closed under unions of chains of length less than the size of the universal homogeneous structure (in the last case, the union of any chain of small categories is a category). One of the objectives of this work is to relax this assumption and to make the theory general enough for capturing new cases and obtaining new examples of universal homogeneous objects.

We believe that category theory is the proper language for Fraïssé-Jónsson limits. In fact, this has already been confirmed by the works of Droste & Göbel [9, 10], where the authors consider some categories of first-order structures with special types of embeddings, obtaining new applications in algebra and theoretical computer science. In this context, one has to mention a recent work of Pech & Pech [39] where the authors, based on the results of Droste & Göbel, develop the theory of Fraïssé limits in comma categories, leading to universal homomorphisms and uni-

versal retractions. Finally, Irwin & Solecki [16] presented a version of Fraïssé theory with reversed arrows, i.e., epimorphisms of finite structures instead of embeddings. By this way, they obtained an interesting new characterization of the *pseudo-arc*, a certain connected compact metric space, never associated to Fraïssé limits before.

There is no doubt that universal structures with strong homogeneity properties can be discovered or identified in various areas of pure mathematics, theoretical computer science and even mathematical physics (see [7]). Model-theoretic Fraïssé limits are nowadays important objects of study in combinatorics, permutation group theory and topological dynamics, see Macpherson’s survey [33] for more information and further references. Category theory brings much more freedom for dealing with Fraïssé limits, offering the possibility of constructing new objects from old and eliminating superfluous assumptions. It has been demonstrated in [9, 10, 39] that category-theoretic approach brings new important examples of universal homogeneous objects and the work [16] shows that one of the simplest constructions in category theory, namely, passing to the opposite category, leads to new and somewhat surprising examples. Actually, one of the author’s construction of a universal pre-image for a certain class of compact linearly ordered spaces [26] turns out to be the Fraïssé limit of a category whose arrows are increasing quotient maps. Summarizing, category-theoretic approach allows constructing new universal homogeneous objects as well as identifying existing objects, discovering their homogeneity properties.

Here a notational issue has to be pointed out. Namely, in category theory the notion of a “universal object” is totally different from the notion of a “universal structure” in model theory and related areas. In order to avoid this confusion, we shall replace the adjective “universal” by “cofinal” in the latter case. Namely, an object  $U$  is defined to be *cofinal* for a class  $\mathfrak{K}$  if every object from  $\mathfrak{K}$  embeds into  $U$ , where “embedding” will be just an arrow of a category under consideration.



The purpose of this note is to present category-theoretic framework for universal homogeneous structures. We shall explain the Fraïssé theory using the language of category theory, emphasizing on new applications in topology and functional analysis. Part of the material is devoted to *continuous* Fraïssé limits, that is, objects like the universal homogeneous metric space of Urysohn. As it happens, these structures can be easily described and studied using the language of categories enriched over metric spaces.

The key point of our approach is dealing with sequences instead of (co-)limits, where a *sequence* is nothing but a functor from an ordinal into a fixed category. We shall concentrate on countable sequences, i.e., functors from the set of natural numbers. The crucial notion is that of a *Fraïssé sequence*, a sequence which is supposed to “converge” to a universal (cofinal, in our terminology) and homogeneous object in a bigger category, where homogeneity is meant with respect to the original category. In other words, we deal with the base category of “small” objects and we

use sequences for encoding the category of “large” objects. This approach is similar in spirit to the idea of forcing in set theory, where one deals with approximations of a generic object, working in the base model instead of its forcing extension.

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## 1 Preliminaries

☞ We recall relevant basic notions from category theory. (1 hour)

We start with the definition of a category. Namely, a *category* is a structure of the form  $\mathfrak{K} = \langle V, A, \text{dom}, \text{cod}, \circ \rangle$ , where  $V$  is the class of *objects*,  $A$  is the class of *arrows*,  $\text{dom}: A \rightarrow V$  and  $\text{cod}: A \rightarrow V$  are the *domain* and *codomain* functions, and  $\circ$  is a partial binary operation on arrows, such that the following conditions are satisfied:

- (1)  $f \circ g$  is defined iff  $\text{cod}(g) = \text{dom}(f)$  and  $\circ$  is associative, that is,  $(f \circ g) \circ h = f \circ (g \circ h)$  whenever  $\text{cod}(h) = \text{dom}(g)$  and  $\text{cod}(g) = \text{dom}(f)$ .
- (2) For each object  $a \in V$  there is an arrow  $\text{id}_a$  with  $\text{dom}(\text{id}_a) = a = \text{cod}(\text{id}_a)$  and  $\text{id}_a \circ g = g$ ,  $f \circ \text{id}_a = f$  holds for every  $f, g \in A$  such that  $\text{dom}(f) = a = \text{cod}(g)$ .

(3) For every objects  $a, b \in V$  the class

$$\mathfrak{K}(a, b) := \{f \in A : \text{dom}(f) = a \text{ and } \text{cod}(f) = b\}$$

is a set.

We omit the metamathematical discussion of classes, since it is not relevant for our subject. We shall actually consider mainly categories whose class of objects has a countable set of “representatives”.

Categories will usually be denoted by letters  $\mathfrak{K}, \mathfrak{L}, \mathfrak{M}$ , etc. The class of objects of  $\mathfrak{K}$  will be denoted by  $\text{Ob}(\mathfrak{K})$  and the class of arrows of  $\mathfrak{K}$  will be denoted by the same letter  $\mathfrak{K}$ . All functors considered here are assumed to be covariant, unless otherwise specified. Let  $\mathfrak{K}$  be a category. We shall write “ $a \in \mathfrak{K}$ ” for “ $a$  is an object of  $\mathfrak{K}$ ”. Given  $a, b \in \mathfrak{K}$ , we shall denote by  $\mathfrak{K}(a, b)$  the set of all  $\mathfrak{K}$ -morphisms from  $a$  to  $b$ . A *subcategory* of  $\mathfrak{K}$  is a category  $\mathfrak{L}$  such that each object of  $\mathfrak{L}$  is an object of  $\mathfrak{K}$  and each arrow of  $\mathfrak{L}$  is an arrow of  $\mathfrak{K}$  (with the same domain and co-domain). We write  $\mathfrak{L} \subseteq \mathfrak{K}$ . Recall that a subcategory  $\mathfrak{L}$  of  $\mathfrak{K}$  is *full* if  $\mathfrak{L}(a, b) = \mathfrak{K}(a, b)$  for every objects  $a, b \in \mathfrak{L}$ . We say that  $\mathfrak{L}$  is *cofinal* in  $\mathfrak{K}$  if for every object  $x \in \mathfrak{K}$  there exists an object  $y \in \mathfrak{L}$  such that  $\mathfrak{K}(x, y) \neq \emptyset$ . The opposite category to  $\mathfrak{K}$  will be denoted by  $\mathfrak{K}^{\text{op}}$ . That is, the objects of  $\mathfrak{K}^{\text{op}}$  are the objects of  $\mathfrak{K}$  and all arrows are reversed, i.e.  $\mathfrak{K}^{\text{op}}(a, b) = \mathfrak{K}(b, a)$  for every  $a, b \in \mathfrak{K}$ . Recall that a category  $\mathfrak{K}$  is *ordered* if  $|\mathfrak{K}(x, y)| \leq 1$  and  $\mathfrak{K}(x, y) \neq \emptyset \neq \mathfrak{K}(y, x)$  implies  $x = y$  for every  $\mathfrak{K}$ -objects  $x, y$ . Removing the last condition we get the notion of a *quasi-ordered category*. Every (not necessarily ordered) category induces a partial order  $\leq$  on the objects of  $\mathfrak{K}$  defined by the formula  $\mathfrak{K}(x, y) \neq \emptyset$  iff  $x \leq y$ . Every partially ordered set  $\langle P, \leq \rangle$  can be viewed as an ordered category  $\mathfrak{K}_P$  with  $P$  the class of objects and the class of arrows defined by  $\mathfrak{K}_P(x, y) = \{\langle x, y \rangle\}$  whenever  $x \leq y$  and  $\mathfrak{K}_P(x, y) = \emptyset$  otherwise. In particular, ordinals treated as well ordered sets are important examples of ordered categories.

Let  $\mathfrak{K}$  be a category. We say that  $\mathfrak{K}$  has the *amalgamation property* if for every  $a, b, c \in \mathfrak{K}$  and for every morphisms  $f \in \mathfrak{K}(a, b)$ ,  $g \in \mathfrak{K}(a, c)$  there exist  $d \in \mathfrak{K}$  and morphisms  $f' \in \mathfrak{K}(b, d)$  and  $g' \in \mathfrak{K}(c, d)$  such that  $f' \circ f = g' \circ g$ . If, additionally, for every arrows  $f'', g''$  such that  $f'' \circ f = g'' \circ g$  there exists a unique arrow  $h$  satisfying  $h \circ f' = f''$  and  $h \circ g' = g''$  then the pair  $\langle f', g' \rangle$  is a *pushout* of  $\langle f, g \rangle$ . Reversing the arrows, we define the *reversed amalgamation* and the *pullback*. We say that  $\mathfrak{K}$  is *directed* if for every  $a, b \in \mathfrak{K}$  there exists  $g \in \mathfrak{K}$  such that both sets  $\mathfrak{K}(a, g)$ ,  $\mathfrak{K}(b, g)$  are nonempty. In model theory, where the arrows are embeddings, this is usually called the *joint embedding property*.

Fix a category  $\mathfrak{K}$  and fix an ordinal  $\delta > 0$ . An *inductive  $\delta$ -sequence* in  $\mathfrak{K}$  is formally a covariant functor from  $\delta$  (treated as a poset category) into  $\mathfrak{K}$ . In other words, it could be described as a pair of the form  $\langle \{a_\xi\}_{\xi < \delta}, \{a_\xi^\eta\}_{\xi < \eta < \delta} \rangle$ , where  $\delta$  is an ordinal,  $\{a_\xi : \xi < \delta\} \subseteq \mathfrak{K}$  and  $a_\xi^\eta \in \mathfrak{K}(a_\xi, a_\eta)$  are such that  $a_\eta^\varrho \circ a_\xi^\eta = a_\xi^\varrho$  for every  $\xi < \eta < \varrho < \delta$ . We shall denote such a sequence shortly by  $\vec{a}$ . The ordinal  $\delta$  is the *length* of  $\vec{a}$ .

Let  $\kappa$  be an infinite cardinal. A category  $\mathfrak{K}$  is  $\kappa$ -complete if all inductive sequences of length  $< \kappa$  have co-limits in  $\mathfrak{K}$ . Every category is  $\aleph_0$ -complete, since the co-limit of a finite sequence is its last object. A category  $\mathfrak{K}$  is  $\kappa$ -bounded if for every inductive sequence  $\vec{x}$  in  $\mathfrak{K}$  of length  $\lambda < \kappa$  there exist  $y \in \mathfrak{K}$  and a co-cone of arrows  $\{y_\alpha\}_{\alpha < \lambda}$  such that  $y_\alpha: x_\alpha \rightarrow y$  and  $y_\beta \circ x_\alpha^\beta = y_\alpha$  for every  $\alpha < \beta < \lambda$ . Obviously, every  $\kappa$ -complete category is  $\kappa$ -bounded. We shall write “ $\sigma$ -complete” and “ $\sigma$ -bounded” for “ $\aleph_1$ -complete” and “ $\aleph_1$ -bounded” respectively.

We shall need the following notion concerning families of arrows. Fix a family of arrows  $\mathcal{F}$  in a given category  $\mathfrak{K}$ . We shall write  $\text{Dom}(\mathcal{F})$  for the set  $\{\text{dom}(f) : f \in \mathcal{F}\}$ . We say that  $\mathcal{F}$  is *dominating* in  $\mathfrak{K}$  if the family of objects  $\text{Dom}(\mathcal{F})$  is cofinal in  $\mathfrak{K}$  and moreover for every  $a \in \text{Dom}(\mathcal{F})$  and for every arrow  $f: a \rightarrow x$  in  $\mathfrak{K}$  there exists an arrow  $g$  in  $\mathcal{F}$  such that  $g \circ f \in \mathcal{F}$ .

For all undefined category-theoretic notions we refer to Mac Lane [32] or Johnstone [18].

## 2 Categories of sequences

☞ We look at the class of sequences as a category. (1 hour)

Fix a category  $\mathfrak{K}$  and denote by  $\sigma\mathfrak{K}$  the class of all sequences in  $\mathfrak{K}$ . We would like to turn  $\sigma\mathfrak{K}$  into a category in such a way that an arrow from a sequence  $\vec{a}$  into a sequence  $\vec{b}$  induces an arrow from  $\lim \vec{a}$  into  $\lim \vec{b}$ , whenever  $\mathfrak{K}$  is embedded into a category in which sequences  $\vec{a}, \vec{b}$  have co-limits.

Fix two sequences  $\vec{a}$  and  $\vec{b}$  in a given category  $\mathfrak{K}$ . A *transformation* from  $\vec{a}$  to  $\vec{b}$  is, by definition, a natural transformation from  $\vec{a}$  into  $\vec{b} \circ \varphi$ , where  $\varphi: \omega \rightarrow \omega$  is a strictly order preserving map (i.e. a covariant functor from  $\omega$  to  $\omega$ , treated as an ordered category).

In order to define an arrow from  $\vec{a}$  to  $\vec{b}$  we need to identify some transformations. Fix two natural transformations  $F: \vec{a} \rightarrow \vec{b} \circ \varphi$  and  $G: \vec{a} \rightarrow \vec{b} \circ \psi$ . We shall say that  $F$  and  $G$  are *equivalent* if the following conditions hold:

- (1) For every  $k$  there exists  $\ell \geq k$  such that  $\varphi(k) \leq \psi(\ell)$  and

$$b_{\varphi(k)}^{\psi(\ell)} \circ F(k) = G(\ell) \circ a_k^\ell.$$

- (2) For every  $k$  there is  $\ell \geq k$  such that  $\psi(k) \leq \varphi(\ell)$  and

$$b_{\psi(k)}^{\varphi(\ell)} \circ G(k) = F(\ell) \circ a_k^\ell.$$

It is rather clear that this defines an equivalence relation, which is actually a congruence on the category of transformations. Every equivalence class of this relation will be called an *arrow* (or *morphism*) from  $\vec{a}$  to  $\vec{b}$ . It is easy to check that this indeed

defines a category structure on all sequences in  $\mathfrak{K}$ . Formally, this is the quotient category with respect to the equivalence relation described above. The identity arrow of  $\vec{a}$  is the equivalence class of the identity natural transformation  $\text{id}_{\vec{a}}: \vec{a} \rightarrow \vec{a}$ .

Categories of sequences are special cases (or rather “parts”) of more general categories called *Ind-completions*, see Chapter VI of Johnstone’s monograph [18].

We shall later need the following two facts.

**Lemma 2.1.** *Let  $\mathfrak{K}$  be a category and let  $\vec{x}$  be a sequence in  $\sigma\mathfrak{K}$ . Then  $\vec{x}$  has the co-limit in  $\sigma\mathfrak{K}$ .*

*Proof.* Refining inductively each  $\vec{x}_n$  ( $n \in \omega$ ) to a cofinal subsequence, we may assume that all bonding maps are natural transformations. Now look at  $\vec{x}$  as a functor from  $\omega \times \omega$  into  $\mathfrak{K}$  and let  $\vec{y}$  be the diagonal sequence. Then  $\vec{y}$  is easily seen to be the co-limit of  $\vec{x}$ .  $\square$

**Corollary 2.2.** *For every category  $\mathfrak{K}$ , the category of sequences  $\sigma\mathfrak{K}$  is  $\sigma$ -complete.*

### 3 Fraïssé sequences

☞ We define the crucial notion of a Fraïssé sequence, characterizing its existence and listing its basic properties. (1 hour)

Below we introduce the key notion of this course.

Let  $\mathfrak{K}$  be a fixed category. A *Fraïssé sequence* in  $\mathfrak{K}$  is an inductive sequence  $\vec{u}$  satisfying the following conditions:

- (U) For every  $x \in \mathfrak{K}$  there exists  $n \in \omega$  such that  $\mathfrak{K}(x, u_n) \neq \emptyset$ .
- (A) For every  $k \in \omega$  and for every arrow  $f \in \mathfrak{K}(u_k, y)$ , there exist  $\ell > k$  and  $g \in \mathfrak{K}(y, u_\ell)$  such that  $u_k^\ell = g \circ f$ .

A sequence satisfying (U) will be called  *$\mathfrak{K}$ -cofinal*. More generally, a collection  $\mathcal{U}$  of objects of  $\mathfrak{K}$  is  *$\mathfrak{K}$ -cofinal* if for every  $x \in \text{Ob}(\mathfrak{K})$  there is  $u \in \mathcal{U}$  such that  $\mathfrak{K}(x, u) \neq \emptyset$ . Condition (A) will be called *amalgamation property*.

Below is the existence result, proved by using the Baire Category Theorem. Recall that a family of arrows  $\mathcal{F}$  is *dominating* in  $\mathfrak{K}$  if it satisfies the following two conditions.

- (D1) The family  $\text{Dom}(\mathcal{F})$  is cofinal in  $\mathfrak{K}$ , i.e. for every  $x \in \mathfrak{K}$  there is  $a \in \text{Dom}(\mathcal{F})$  such that  $\mathfrak{K}(x, a) \neq \emptyset$ .
- (D2) Given  $a \in \text{Dom}(\mathcal{F})$  and  $f: a \rightarrow y$  in  $\mathfrak{K}$ , there exist  $g: y \rightarrow b$  in  $\mathfrak{K}$  such that  $g \circ f \in \mathcal{F}$ .

**Theorem 3.1.** *Let  $\mathfrak{K}$  be a directed category with the amalgamation property. Assume that  $\mathfrak{K}$  is dominated by a countable family of arrows. Then  $\mathfrak{K}$  has a Fraïssé sequence.*

*Proof.* Without loss of generality, we may assume that a countable dominating family  $\mathcal{F}$  is a subcategory of  $\mathfrak{K}$ . A sequence  $\vec{x}$  in  $\mathcal{F}$  may be regarded as a function from  $\Delta = \{\langle m, n \rangle : m \leq n\}$  into  $\mathcal{F}$  satisfying the obvious conditions. Thus, the set  $S$  of all sequences in  $\mathcal{F}$  is a closed subspace of the Polish space  $\mathcal{F}^\Delta$ , endowed with the product topology. Given an object  $x$  in  $\mathcal{F}$ , let  $U_x$  be the set of all  $\vec{x} \in S$  for which there exists an arrow  $x \rightarrow \vec{x}$ . Clearly,  $U_x$  is open and dense in  $S$ . Given an arrow  $f: a \rightarrow b$  in  $\mathcal{F}$  and  $n \in \omega$ , let

$$V_{f,n} = \{\vec{x} \in S : x_n = a \implies (\exists m > n)(\exists g) g \circ f = x_n^m\}.$$

Again,  $V_{f,n}$  is open and dense in  $S$  (for the density one needs to use amalgamations). Using the Baire Category Theorem, we can find a sequence  $\vec{u} \in S$  that belongs to all the sets defined above. It is easy to check, using the fact that  $\mathcal{F}$  is dominating in  $\mathfrak{K}$ , that  $\vec{u}$  is a Fraïssé sequence in  $\mathfrak{K}$ .  $\square$

Below we list some of the basic properties of Fraïssé sequences. The proofs are easy exercises.

Let  $\vec{v}$  be a sequence in a category  $\mathfrak{K}$ . We say that  $\vec{v}$  has the *extension property* if the following holds:

- (E) For every arrows  $f: a \rightarrow b$ ,  $g: a \rightarrow v_k$  in  $\mathfrak{K}$ , where  $k \in \omega$ , there exist  $\ell > k$  and an arrow  $h: b \rightarrow v_\ell$  such that  $v_k^\ell \circ g = h \circ f$ .

Clearly, this condition implies (A).

**Proposition 3.2.** *Let  $\vec{u}$  be a Fraïssé sequence in a category  $\mathfrak{K}$ . Then  $\mathfrak{K}$  is directed. Moreover, the following conditions are equivalent:*

- (a)  $\vec{u}$  has the extension property.  
(b)  $\mathfrak{K}$  has the amalgamation property.

**Proposition 3.3.** *Assume  $\mathfrak{K}$  is a directed category. Then every sequence in  $\mathfrak{K}$  satisfying condition (A) is Fraïssé.*

**Proposition 3.4.** *Let  $\mathfrak{K}$  be a category, let  $\vec{u}$  be a sequence in  $\mathfrak{K}$  and let  $S \subseteq \omega$  be an infinite set.*

- (a) *If  $\vec{u}$  is a Fraïssé sequence in  $\mathfrak{K}$  then  $\vec{u} \upharpoonright S$  is Fraïssé in  $\mathfrak{K}$ .*  
(b) *If  $\mathfrak{K}$  has the amalgamation property and  $\vec{u} \upharpoonright S$  is a Fraïssé sequence in  $\mathfrak{K}$  then so is  $\vec{u}$ .*

**Proposition 3.5.** *Let  $\mathfrak{K}$  be a category with the amalgamation property and let  $\vec{g}: \vec{u} \rightarrow \vec{v}$ ,  $\vec{h}: \vec{v} \rightarrow \vec{u}$  be transformations of sequences such that  $\vec{g} \circ \vec{h}$  is equivalent to the identity of  $\vec{v}$ . If  $\vec{u}$  is a Fraïssé sequence then so is  $\vec{v}$ .*

The last statement says that, assuming the amalgamation property, the definition of a Fraïssé sequence indeed does not depend on its representation with respect to the equivalence relation on sequences defined in Section 2.

## 4 Cofinality and homogeneity

☞ We show that a Fraïssé sequence is cofinal (universal, in model-theoretic terminology), homogeneous with respect to the base category and unique. (1 hour)

We start with the first important property of Fraïssé sequences.

**Theorem 4.1** (Cofinality). *Assume  $\vec{u}$  is a Fraïssé sequence in a category  $\mathfrak{K}$  with the amalgamation property. Then for every sequence  $\vec{x}$  in  $\mathfrak{K}$  there exists an arrow of sequences  $F: \vec{x} \rightarrow \vec{u}$ .*

*Proof.* We use the extension property (property (E)) of the sequence  $\vec{u}$ , which is equivalent to the amalgamation property of  $\mathfrak{K}$  (Proposition 3.2). Let  $\vec{x}$  be a sequence in  $\mathfrak{K}$ . Using (U), find an arrow  $f_0: x_0 \rightarrow u_{\alpha_0}$ , where  $\alpha_0 \in \omega$ . Now assume that arrows  $f_0, \dots, f_{n-1}$  have been defined so that  $f_m: x_m \rightarrow u_{\alpha_m}$  and the diagram

$$\begin{array}{ccc} x_\ell & \xrightarrow{f_\ell} & u_{\alpha_\ell} \\ x_k^\ell \uparrow & & \uparrow u_{\alpha_k}^{\alpha_\ell} \\ x_k & \xrightarrow{f_k} & u_{\alpha_k} \end{array}$$

commutes for every  $k < \ell < n$  (in particular  $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1}$  are natural numbers). Using (E), find  $\alpha_n > \alpha_{n-1}$  and an arrow  $f: x_n \rightarrow u_{\alpha_n}$  so that  $f \circ x_{n-1}^n = u_{\alpha_n}^{\alpha_{n-1}} \circ f_{n-1}$  and define  $f_n := f$ . Given  $m < n - 1$ , by the inductive hypothesis, we get

$$f_n \circ x_m^n = f_n \circ x_{n-1}^n \circ x_m^{n-1} = u_{\alpha_n}^{\alpha_{n-1}} \circ f_{n-1} \circ x_m^{n-1} = u_{\alpha_n}^{\alpha_{n-1}} \circ u_{\alpha_{n-1}}^{\alpha_m} \circ f_m = u_{\alpha_n}^{\alpha_m} \circ f_m.$$

Finally, setting  $F = \{f_n\}_{n \in \omega}$ , we obtain the required morphism  $F: \vec{x} \rightarrow \vec{u}$ .  $\square$

In model theory, when the sequences are interpreted as certain countable structures, the property described above is called *universality*.

We now turn to the question of homogeneity and uniqueness.

**Theorem 4.2.** *Assume that  $\vec{u}, \vec{v}$  are Fraïssé sequences in a given category  $\mathfrak{K}$ . Furthermore, assume that  $k, \ell \in \omega$  and  $f: u_k \rightarrow v_\ell$  is an arrow in  $\mathfrak{K}$ . Then there exists an isomorphism  $F: \vec{u} \rightarrow \vec{v}$  in  $\sigma\mathfrak{K}$  such that the diagram*

$$\begin{array}{ccc} \vec{u} & \xrightarrow{F} & \vec{v} \\ u_k^\infty \uparrow & & \uparrow v_\ell^\infty \\ u_k & \xrightarrow{f} & v_\ell \end{array}$$

*commutes. In particular  $\vec{u} \approx \vec{v}$ .*

*Proof.* We construct inductively arrows  $f_n: u_{k_n} \rightarrow v_{\ell_n}$ ,  $g_n: v_{\ell_n} \rightarrow u_{k_{n+1}}$ , where  $k_0 \leq \ell_0 < k_1 \leq \ell_1 < \dots$  and for each  $n \in \omega$  the diagram

$$\begin{array}{ccccc}
u_{k_n} & \xrightarrow[u_{k_n}]{k_{n+1}} & u_{k_{n+1}} & \xrightarrow[u_{k_{n+1}}]{k_{n+2}} & u_{k_{n+2}} \dashrightarrow \\
& \searrow f_n & \nearrow g_n & \searrow f_{n+1} & \nearrow g_{n+1} \\
& & v_{\ell_n} & \xrightarrow[v_{\ell_n}]{\ell_{n+1}} & v_{\ell_{n+1}} \dashrightarrow
\end{array}$$

commutes.

We start with  $f_0 := f$ ,  $k_0 := k$ ,  $\ell_0 := \ell$ , possibly replacing  $f$  by some arrow of the form  $j_\ell^m \circ f$  to ensure that  $k_0 \leq \ell_0$ . Using property (A) of the sequence  $\vec{u}$ , find  $k_1 > k_0$  and  $g_1: v_0 \rightarrow u_{k_1}$  such that  $g_1 \circ f = u_0^{k_1}$ . Assume that  $f_m, g_m$  have already been constructed for  $m \leq n$ . Using the amalgamation of  $\vec{v}$ , find  $\ell_{n+1} \geq k_{n+1}$  and an arrow  $f_{n+1}: u_{k_{n+1}} \rightarrow v_{\ell_{n+1}}$  such that  $f_{n+1} \circ g_n = v_{\ell_n}^{\ell_{n+1}}$ . Now, using the amalgamation of  $\vec{u}$ , we find  $k_{n+2} > \ell_{n+1}$  and an arrow  $g_{n+1}: v_{\ell_{n+1}} \rightarrow u_{k_{n+2}}$  such that  $g_{n+1} \circ f_{n+1} = u_{k_{n+1}}^{k_{n+2}}$ . By the induction hypothesis,  $g_n \circ f_n = u_{k_n}^{k_{n+1}}$ , therefore the above diagram commutes. This finishes the construction.

Finally, set  $F = \{f_n\}_{n \in \omega}$  and  $G = \{g_n\}_{n \in \omega}$ . Then  $F: \vec{u} \rightarrow \vec{v}$ ,  $G: \vec{v} \rightarrow \vec{u}$  are morphisms of sequences and by a simple induction we show that

$$(*) \quad g_n \circ v_{\ell_m}^{\ell_n} \circ f_m = u_{k_m}^{k_{n+1}} \quad \text{and} \quad f_n \circ u_{k_{m+1}}^{k_n} \circ g_m = v_{\ell_m}^{\ell_n}$$

holds for every  $m < n < \omega$ . This shows that  $F \circ G = \text{id}_{\vec{v}}$  and  $G \circ F = \text{id}_{\vec{u}}$ , therefore  $F$  is an isomorphism. The equality  $v_0^\infty \circ f = F \circ u_0^\infty$  means that  $v_0^{\ell_n} \circ f = f_n \circ u_0^{k_n}$  should hold for every  $n \in \omega$ . Fix  $n > 0$ . Applying (\*) twice (with  $m = 0$  and  $m = n - 1$  respectively), we get

$$f_n \circ u_0^{k_n} = f_n \circ g_{n-1} \circ v_0^{\ell_{n-1}} \circ f = v_{\ell_{n-1}}^{\ell_n} \circ v_0^{\ell_{n-1}} \circ f = v_0^{\ell_n} \circ f.$$

Thus  $v_0^\infty \circ f = F \circ u_0^\infty$ .

Finally, notice that, by property (U) of the sequence  $\vec{v}$ , for some  $\ell < \omega$  there exists an arrow  $f: u_0 \rightarrow v_\ell$ , so applying the first part we see that  $\vec{u} \approx \vec{v}$ .  $\square$

As a corollary we obtain the homogeneity of a Fraïssé sequence:

**Corollary 4.3.** *Let  $\mathfrak{K}$  be a directed category with the amalgamation property and let  $\vec{u}$  be a Fraïssé sequence in  $\mathfrak{K}$ . Then for every  $\mathfrak{K}$ -arrow  $f: a \rightarrow b$ , for every  $\sigma \mathfrak{K}$ -arrows  $i: a \rightarrow \vec{u}$ ,  $j: b \rightarrow \vec{u}$  there exists an automorphism  $H: \vec{u} \rightarrow \vec{u}$  for which the diagram*

$$\begin{array}{ccc}
\vec{u} & \xrightarrow{H} & \vec{u} \\
i \uparrow & & \uparrow j \\
a & \xrightarrow{f} & b
\end{array}$$

is commutative.

## 5 Metric categories and norms

☞ We say what a metric category is. We revise some notions and define a norm on a category. (2 hours)

Let  $\mathfrak{M}_k$  denote the category of metric spaces with non-expansive (i.e., 1-Lipschitz) mappings. A category  $\mathfrak{K}$  is *enriched over*  $\mathfrak{M}_k$  if for every  $\mathfrak{K}$ -objects  $a, b$  there is a metric  $\varrho$  on the set of  $\mathfrak{K}$ -arrows  $\mathfrak{K}(a, b)$  so that the composition operator is non-expansive from both sides. More precisely, we have

$$(M) \quad \varrho(f_0 \circ g, f_1 \circ g) \leq \varrho(f_0, f_1) \quad \text{and} \quad \varrho(h \circ f_0, h \circ f_1) \leq \varrho(f_0, f_1)$$

whenever the compositions make sense. This allows us to consider  $\varepsilon$ -commutative diagrams, with the obvious meaning. Formally, on each hom-set we have a different metric, but there is no ambiguity with using always the same letter  $\varrho$ . For the sake of convenience, we allow  $+\infty$  as a possible value of the metric.

Later on, we shall say that  $\mathfrak{K}$  is *metric-enriched* having in mind that  $\mathfrak{K}$  is enriched over  $\mathfrak{M}_k$ .

In order to present the approximate variant of Fraïssé theory, it is necessary to work with a pair of categories  $\langle \mathfrak{K}_0, \mathfrak{K} \rangle$ , where  $\mathfrak{K}_0$  is a subcategory of  $\mathfrak{K}$  with  $\text{Ob}(\mathfrak{K}_0) = \text{Ob}(\mathfrak{K})$  and we always assume that  $\mathfrak{K}$  is a metric-enriched category.

We now define the notion of a *norm* of  $\mathfrak{K}$ -arrows, with respect to the subcategory  $\mathfrak{K}_0$ . Namely, we define the *norm* of  $f \in \mathfrak{K}$  by the formula

$$\mu(f) = \inf\{\varrho(i, j \circ f) : i, j \in \mathfrak{K}_0\},$$

where only  $\mathfrak{K}_0$ -arrows  $i, j$  satisfying  $\text{dom}(i) = \text{dom}(f)$ ,  $\text{cod}(i) = \text{cod}(j)$  and  $\text{dom}(j) = \text{cod}(f)$  are taken into account. By definition,  $\mu(f) = 0$  whenever  $f \in \mathfrak{K}_0$ , although it may formally happen that  $\mu(f) = 0$  for some  $f \in \mathfrak{K} \setminus \mathfrak{K}_0$ .

A pair of categories  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  as above will be called a *normed category*.

Let  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  be a fixed normed category. We now re-define the category of sequences, taking into account the metric. A *sequence* in  $\mathfrak{K}$  is formally a covariant functor from the set of natural numbers  $\omega$  into  $\mathfrak{K}$ .

Denote by  $\sigma(\mathfrak{K}, \mathfrak{K}_0)$  the category of all sequences in  $\mathfrak{K}_0$  with arrows in  $\mathfrak{K}$ . This is indeed a category with arrows being equivalence classes of semi-natural transformations. A *semi-natural transformation* from  $\vec{x}$  to  $\vec{y}$  is, by definition, a natural transformation from  $\vec{x}$  to  $\vec{y} \circ \varphi$  for some increasing function  $\varphi: \omega \rightarrow \omega$ . Slightly abusing notation, we shall consider transformations (arrows) of sequences, having in mind their equivalence classes. Thus, an arrow from  $\vec{x}$  to  $\vec{y}$  is a sequence of arrows  $\vec{f} = \{f_n\}_{n \in \omega} \subseteq \mathfrak{K}$  together with an increasing map  $\varphi: \omega \rightarrow \omega$  such that for each  $n < m$  the diagram

$$\begin{array}{ccc} y_{\varphi(n)} & \xrightarrow{y_{\varphi(n)}^{\varphi(m)}} & y_{\varphi(m)} \\ f_n \uparrow & & \uparrow f_m \\ x_n & \xrightarrow{x_n^m} & x_m \end{array}$$

is commutative.

Since the category  $\mathfrak{K}$  is enriched over  $\mathfrak{Mk}$ , it is natural to allow more arrows in  $\sigma(\mathfrak{K}, \mathfrak{K}_0)$ . An *approximate arrow* from a sequence  $\vec{x}$  into a sequence  $\vec{y}$  is  $\vec{f} = \{f_n\}_{n \in \omega} \subseteq \mathfrak{K}$  together with an increasing map  $\varphi: \omega \rightarrow \omega$  satisfying the following condition:

( $\circlearrowleft$ ) For every  $\varepsilon > 0$  there exists  $n_0$  such that all diagrams of the form

$$\begin{array}{ccc} y_{\varphi(n)} & \xrightarrow{y_{\varphi(n)}^{\varphi(m)}} & y_{\varphi(m)} \\ f_n \uparrow & & \uparrow f_m \\ x_n & \xrightarrow{x_n^m} & x_m \end{array}$$

are  $\varepsilon$ -commutative, i.e.,  $\varrho(f_m \circ x_n^m, y_{\varphi(n)}^{\varphi(m)} \circ f_n) < \varepsilon$  whenever  $n_0 \leq n < m$ .

It is obvious that the composition of approximate arrows is an approximate arrow, therefore  $\sigma(\mathfrak{K}, \mathfrak{K}_0)$  is indeed a category. It turns out that  $\sigma(\mathfrak{K}, \mathfrak{K}_0)$  is naturally metric-enriched. Indeed, given approximate arrows  $\vec{f}: \vec{x} \rightarrow \vec{y}$ ,  $\vec{g}: \vec{x} \rightarrow \vec{y}$ , define

$$(\Rightarrow) \quad \varrho(\vec{f}, \vec{g}) := \lim_{n \rightarrow \infty} \lim_{m > n} \varrho(y_n^m \circ f_n, y_n^m \circ g_n),$$

assuming that  $\vec{f}$  and  $\vec{g}$  are natural transformations. Replacing  $\vec{y}$  by its cofinal subsequence, we can make such assumption, without loss of generality. We need to show that the limit above exists.

Given  $n < m < \ell$ , we have

$$\varrho(y_n^\ell \circ f_n, y_n^\ell \circ g_n) = \varrho(y_m^\ell \circ y_n^m \circ f_n, y_m^\ell \circ y_n^m \circ g_n) \leq \varrho(y_n^m \circ f_n, y_n^m \circ g_n),$$

therefore for each  $n \in \omega$  the sequence  $\{\varrho(y_n^m \circ f_n, y_n^m \circ g_n)\}_{m > n}$  is decreasing. On the other hand, given  $\varepsilon > 0$ , given  $n_0 \leq n < k < m$  such that ( $\circlearrowleft$ ) holds for both  $\{f_n\}_{n \geq n_0}$  and  $\{g_n\}_{n \geq n_0}$ , we have that

$$\begin{aligned} \varrho(y_n^m \circ f_n, y_n^m \circ g_n) &\leq \varrho(y_k^m \circ y_n^k \circ f_n, y_k^m \circ y_n^k \circ f_n) + \varrho(y_k^m \circ f_k \circ x_n^k, y_k^m \circ g_k \circ x_n^k) \\ &\quad + \varrho(y_k^m \circ g_k \circ x_n^k, y_k^m \circ y_n^k \circ g_n) \leq \varrho(y_k^m \circ f_k, y_k^m \circ g_k) \\ &\quad + \varrho(y_n^k \circ f_n, f_k \circ x_n^k) + \varrho(g_k \circ x_n^k, y_n^k \circ g_n) \\ &< \varrho(y_k^m \circ f_k, y_k^m \circ g_k) + 2\varepsilon. \end{aligned}$$

Passing to the limit as  $m \rightarrow \infty$ , we see that the sequence

$$\left\{ \lim_{m > n} \varrho(y_n^m \circ f_n, y_n^m \circ g_n) \right\}_{n \in \omega}$$

is increasing. This shows that the double limit in ( $\Rightarrow$ ) exists.

One should mention that a much more natural definition for  $\varrho$  would be

$$(\rightarrow) \quad \varrho(\vec{f}, \vec{g}) = \lim_{n \rightarrow \infty} \varrho(f_n, g_n).$$

The problem is that this limit may not exist in general. In practice however, we shall always have

$$\varrho(i \circ f, i \circ g) = \varrho(f, g)$$

whenever  $i \in \mathfrak{K}_0$ . To be more precise, when  $\mathfrak{K}$  is a metric-enriched category, an arrow  $i \in \mathfrak{K}$  is called a *monic* (or a *monomorphism*) if the above equation holds for arbitrary compatible arrows  $f, g \in \mathfrak{K}$ . This is an obvious generalization of the notion of a monic in category theory. In fact, every category is metric-enriched over the 0-1 metric:  $\varrho(f, g) = 0$  iff  $f = g$ .

In all natural examples of normed categories  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  the subcategory  $\mathfrak{K}_0$  consists of arrows that are monics in  $\mathfrak{K}$ . Thus, we can use  $(\rightarrow)$  instead of the less natural  $(\rightrightarrows)$  as the definition of  $\varrho(\vec{f}, \vec{g})$ . It is an easy exercise to check that  $\varrho$  is indeed a metric on each hom-set of  $\sigma(\mathfrak{K}, \mathfrak{K}_0)$  and that all composition operators are non-expansive.

The important fact is that every normed category  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  naturally embeds into  $\sigma(\mathfrak{K}, \mathfrak{K}_0)$ , identifying a  $\mathfrak{K}_0$ -object  $x$  with the sequence of identities

$$x \longrightarrow x \longrightarrow x \longrightarrow \dots$$

and every  $\mathfrak{K}$ -arrow becomes a natural transformation between sequences of identities. Actually, it may happen that  $\mathfrak{K}$  is not a full subcategory of  $\sigma(\mathfrak{K}, \mathfrak{K}_0)$ . Namely, every Cauchy sequence of  $\mathfrak{K}$ -arrows  $f_n: x \rightarrow y$  is an approximate arrow from  $x$  to  $y$ , regarded as sequences. Now, if  $\mathfrak{K}(x, y)$  is not complete, there may be no  $f: x \rightarrow y$  satisfying  $f = \lim_{n \rightarrow \infty} f_n$ . The problem disappears if all hom-sets of  $\mathfrak{K}$  are complete with respect to the metric  $\varrho$ .

In practice, we can partially ignore the construction described above, because usually there is a canonical faithful functor from  $\sigma(\mathfrak{K}, \mathfrak{K}_0)$  into a natural category containing  $\mathfrak{K}$  and in which all countable sequences in  $\mathfrak{K}_0$  have co-limits. Two relevant examples are described below.

**Example 5.1.** Let  $\mathfrak{K}$  be the category of all finite metric spaces with non-expansive mappings and let  $\mathfrak{K}_0$  be the subcategory of isometric embeddings. Let  $\mathfrak{C}$  be the category of all separable complete metric spaces. Given a  $\mathfrak{K}_0$ -sequence  $\vec{x}$ , we can identify it with a chain of finite metric spaces, therefore it is natural to consider  $\lim \vec{x}$  to be the completion of the union of this chain. This is in fact the co-limit of  $\vec{x}$  in the category  $\mathfrak{M}\mathfrak{S}$ . In particular,  $\lim \vec{x} = \lim \vec{y}$ , whenever  $\vec{x}$  and  $\vec{y}$  are equivalent. Furthermore, every approximate arrow  $\vec{f}: \vec{x} \rightarrow \vec{y}$  “converges” to a non-expansive map  $F: \lim \vec{x} \rightarrow \lim \vec{y}$  and again it is defined up to an equivalence of approximate arrows. By this way we have defined a canonical faithful functor  $\lim: \sigma(\mathfrak{K}, \mathfrak{K}_0) \rightarrow \mathfrak{C}$ . Unfortunately, since we are restricted to 1-Lipschitz mappings, the functor  $F$  is not onto. The simplest example is as follows.

Let  $X = \{0, 1\} \cup \{\pm 1/n : n \in \mathbb{N}\}$  with the metric induced from the real line. Let  $X_n = \{\pm 1/k : k < n\}$  and let  $Y = \{0, 1\}$ . Then  $X = \lim_{n \in \omega} X_n$  although the canonical embedding  $e: Y \rightarrow X$  is not the co-limit of any approximate arrow from the sequence  $\{X_n\}_{n \in \omega}$  into  $Y$  (the space  $Y$  can be treated as the infinite constant sequence with identities).

Note that if we change  $\mathfrak{C}$  to the category of all countable metric spaces then the canonical co-limit is just the union of the sequence, however some approximate arrows of sequences would not have co-limits.

Concerning applications, the situation where the canonical “co-limiting” functor is not surjective does not cause any problems, since our main results say about the existence of certain arrows (or isomorphisms) of sequences only. In the next example we have a better situation.

**Example 5.2.** Let  $\mathfrak{B}$  be the category of all finite-dimensional Banach spaces with linear operators of norm  $\leq 1$  and let  $\mathfrak{B}_0$  be the subcategory of all isometric embeddings. It is clear that  $\mathfrak{B}$  is a metric-enriched category, where  $\varrho(f, g) = \|f - g\|$  for  $f, g: X \rightarrow Y$  in  $\mathfrak{B}$ .

Now assume  $f: X \rightarrow Y$  is a linear operator satisfying

$$(*) \quad (1 - \varepsilon)\|x\| \leq \|f(x)\| \leq \|x\|.$$

We claim that  $\mu(f) \leq \varepsilon$ . In fact, consider  $Z = X \oplus Y$  with the norm defined by the following formula:

$$\|\langle x, y \rangle\| = \inf\{\|u\|_X + \|v\|_Y + \varepsilon\|w\|_X : \langle x, y \rangle = \langle u, v \rangle + \langle w, -f(w) \rangle, u, w \in X, v \in Y\}.$$

Let  $i: X \rightarrow Z, j: Y \rightarrow Z$  be the canonical injections. Note that  $\|i - j \circ f\| \leq \varepsilon$ , just by definition. It remains to check that  $i, j$  are isometric embeddings, that is, they belong to the category  $\mathfrak{B}_0$ .

It is clear that  $\|i\| \leq 1$  and  $\|j\| \leq 1$ . On the other hand, if  $x = u + w$  and  $0 = v - f(w)$  then

$$\|u\|_X + \|v\|_Y + \varepsilon\|w\|_X \geq \|u\|_X + (1 - \varepsilon)\|w\|_X + \varepsilon\|w\|_X \geq \|u + w\|_X = \|x\|_X.$$

This shows that  $\|i(x)\| = \|x\|_X$ . A similar calculation shows that  $\|j(y)\| = \|y\|_Y$ .

Thus, if  $f$  satisfies  $(*)$  then  $\mu(f) \leq \varepsilon$ . On the other hand, if  $\|f(x)\| = 1 - \varepsilon$  for some  $x$  with  $\|x\| = 1$ , then given isometric embeddings  $i, j$ , we have

$$\|i(x) - j(f(x))\| \geq |\|i(x)\| - \|j(f(x))\|| = |1 - \|f(x)\|| = \varepsilon.$$

Finally, we conclude that  $\mu(f) = \varepsilon$ , where  $\varepsilon \geq 0$  is minimal for which the inequality  $(*)$  holds.

It follows that  $\langle \mathfrak{B}, \mathfrak{B}_0 \rangle$  is a normed category.

Again, we have a canonical functor  $\lim: \sigma(\mathfrak{B}, \mathfrak{B}_0) \rightarrow \mathfrak{C}$ , where  $\mathfrak{C}$  is the category of all separable Banach spaces with non-expansive linear operators. It turns out

that this functor is surjective. Namely, fix two Banach spaces  $X = \overline{\bigcup_{n \in \omega} X_n}$  and  $Y = \overline{\bigcup_{n \in \omega} Y_n}$ , where  $\{X_n\}_{n \in \omega}$  and  $\{Y_n\}_{n \in \omega}$  are chains of finite-dimensional spaces. Fix a linear operator  $T: X \rightarrow Y$  such that  $\|T\| \leq 1$  and let  $T_n = T \upharpoonright X_n$ . By an easy induction, we define a sequence of linear operators  $T'_n: X_n \rightarrow Y_n$  so that  $T'_{n+1}$  extends  $T'_n$  and  $\|T_n - T'_n\| < 1/n$  for each  $n \in \omega$ . Note that  $\|T'_n\| \leq 1 + 1/n$ . Define  $T''_n = \frac{n}{n+1} T'_n$ . Now  $\vec{t} = \{T''_n\}_{n \in \omega}$  is a sequence of  $\mathfrak{B}$ -arrows and standard calculation shows that  $\|T''_n - T_n\| < 2/n$  for every  $n \in \omega$ . Thus  $\vec{t}$  is an approximate arrow from  $\{X_n\}_{n \in \omega}$  to  $\{Y_n\}_{n \in \omega}$  with  $\lim \vec{t} = T$ .

It is natural to extend the norm  $\mu$  to the category of sequences. More precisely, given an approximate arrow  $\vec{f}: \vec{x} \rightarrow \vec{y}$ , we define

$$\mu(\vec{f}) = \lim_{n \rightarrow \infty} \mu(f_n).$$

This is indeed well defined, because given  $\varepsilon > 0$  and taking  $n < m$  as in  $(\odot)$ , we have that  $\mu(f_n) \leq \mu(f_m) + \varepsilon$ . The function  $\mu$  obviously extends the norm of  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ , although it is formally not a norm, because it is defined in a different way, without referring to any subcategory of  $\sigma(\mathfrak{K}, \mathfrak{K}_0)$ . An approximate arrow  $\vec{f}$  is a  $\theta$ -arrow if  $\mu(\vec{f}) = 0$ . We shall be interested in 0-arrows only.

We say that a normed category  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  has the *almost amalgamation property* if for every  $\mathfrak{K}_0$ -arrows  $f: c \rightarrow a$ ,  $g: c \rightarrow b$ , for every  $\varepsilon > 0$ , there exist  $\mathfrak{K}_0$ -arrows  $f': a \rightarrow w$ ,  $g': b \rightarrow w$  such that the diagram

$$\begin{array}{ccc} b & \xrightarrow{g'} & w \\ g \uparrow & & \uparrow f' \\ c & \xrightarrow{f} & a \end{array}$$

is  $\varepsilon$ -commutative, i.e.  $\varrho(f' \circ f, g' \circ g) < \varepsilon$ . We say that  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  has the *strict amalgamation property* if for each  $f, g$  the diagram above is commutative (i.e. no  $\varepsilon$  is needed).

It turns out that almost amalgamations can be moved to the bigger category  $\mathfrak{K}$ . Namely:

**Proposition 5.3.** *Let  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  have the almost amalgamation property. Then for every  $\varepsilon, \delta > 0$ , for every  $\mathfrak{K}$ -arrows  $f: c \rightarrow a$ ,  $g: c \rightarrow b$  with  $\mu(f) < \varepsilon$ ,  $\mu(g) < \delta$ , there exist  $\mathfrak{K}_0$ -arrows  $f': a \rightarrow w$  and  $g': b \rightarrow w$  such that the diagram*

$$\begin{array}{ccc} b & \xrightarrow{g'} & w \\ g \uparrow & & \uparrow f' \\ c & \xrightarrow{f} & a \end{array}$$

*is  $(\varepsilon + \delta)$ -commutative.*

*Proof.* Fix  $\eta > 0$ . Find  $\mathfrak{K}_0$ -arrows  $i, j$  such that  $\varrho(j \circ f, i) < \mu(f) + \eta$ . Find  $\mathfrak{K}_0$ -arrows  $k, \ell$  such that  $\varrho(\ell \circ g, k) < \mu(g) + \eta$ . Using the almost amalgamation property, find  $\mathfrak{K}_0$ -arrows  $j', \ell'$  such that  $\varrho(j' \circ i, \ell' \circ k) < \eta$ . Define  $f' := j' \circ j, g' := \ell' \circ \ell$ . Then

$$\begin{aligned} \varrho(f' \circ f, g' \circ g) &\leq \varrho(j' \circ j \circ f, j' \circ i) + \varrho(j' \circ i, \ell' \circ k) + \varrho(\ell' \circ k, \ell' \circ \ell \circ g) \\ &< \varrho(j \circ f, i) + \eta + \varrho(k, \ell \circ g) < \mu(f) + \mu(g) + 3\eta. \end{aligned}$$

Thus, it is clear that if  $\eta$  is small enough then  $\varrho(f' \circ f, g' \circ g) < \varepsilon + \delta$ .  $\square$

The next simple proposition shows that the norm  $\mu$  satisfies natural conditions that look like triangle inequalities. In general, it may happen that  $\mu(f) \not\leq \mu(f \circ g) + \mu(g)$ .

**Proposition 5.4.** *Assume  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  is a normed category with the almost amalgamation property. Then for every compatible  $\mathfrak{K}$ -arrows  $f, g$  the following inequalities hold:*

$$(N_1) \quad \mu(f \circ g) \leq \mu(f) + \mu(g).$$

$$(N_2) \quad \mu(g) \leq \mu(f) + \mu(f \circ g).$$

*Proof.* Fix  $\varepsilon > 0$  and fix  $i, j, k, \ell \in \mathfrak{K}_0$  such that

$$\varrho(j \circ f, i) < \mu(f) + \varepsilon/3 \quad \text{and} \quad \varrho(\ell \circ g, k) < \mu(g) + \varepsilon/3.$$

Using the almost amalgamation property, we can find  $i', \ell' \in \mathfrak{K}$  such that

$$\varrho(i' \circ i, \ell' \circ i) < \varepsilon/3.$$

Combining these inequalities we obtain that  $\varrho(\ell' \circ k, i' \circ j) < \mu(f) + \mu(g) + \varepsilon$ , which shows  $(N_1)$ .

A similar argument shows  $(N_2)$ .  $\square$

## 6 Approximate Fraïssé sequences

$\clubsuit$  We re-define the notion of a Fraïssé sequence and discuss its basic properties and existence. (1 hour)

As usual, we assume that  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  is a normed category. A sequence  $\vec{u}: \omega \rightarrow \mathfrak{K}_0$  is *Fraïssé* in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  if it satisfies the following two conditions:

(U) For every  $\mathfrak{K}$ -object  $x$ , for every  $\varepsilon > 0$ , there exist  $n \in \omega$  and a  $\mathfrak{K}$ -arrow  $f: x \rightarrow u_n$  such that  $\mu(f) < \varepsilon$ .

(A) Given  $\varepsilon > 0$ , given a  $\mathfrak{K}_0$ -arrow  $f: u_n \rightarrow y$ , there exist  $m > n$  and a  $\mathfrak{K}$ -arrow  $g: y \rightarrow u_m$  such that  $\mu(g) < \varepsilon$  and  $\varrho(u_n^m, g \circ f) < \varepsilon$ .

Recall that we identify a sequence with all of its cofinal subsequences. It turns out that the definition above is “correct” because of the almost amalgamation property:

**Proposition 6.1.** *Assume  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  is a normed category with the almost amalgamation property. Let  $\vec{u}$  be a sequence in  $\mathfrak{K}_0$ . The following conditions are equivalent.*

- (a)  $\vec{u}$  is Fraïssé in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ .
- (b)  $\vec{u}$  has a cofinal subsequence that is Fraïssé in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ .
- (c) Every cofinal subsequence of  $\vec{u}$  is Fraïssé in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ .

*Proof.* Implications (a)  $\implies$  (c) and (c)  $\implies$  (b) are obvious. In fact, the almost amalgamation property is used only for showing that (b)  $\implies$  (a).

Suppose  $M \subseteq \omega$  is infinite and such that  $\vec{u} \upharpoonright M$  is Fraïssé in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ . Fix  $n \in \omega \setminus M$  and fix a  $\mathfrak{K}_0$ -arrow  $f: u_n \rightarrow y$ . Fix  $\varepsilon > 0$ . Using the almost amalgamation property, we can find  $\mathfrak{K}_0$ -arrows  $f': u_m \rightarrow w$  and  $j: y \rightarrow w$  such that  $m \in M$ ,  $m > n$  and the diagram

$$\begin{array}{ccc} u_n & \xrightarrow{u_n^m} & u_m \\ & \searrow f & \searrow f' \\ & y & \xrightarrow{j} w \end{array}$$

is  $\varepsilon/2$ -commutative. Since  $\vec{u} \upharpoonright M$  is Fraïssé, there is a  $\mathfrak{K}$ -arrow  $g: w \rightarrow u_\ell$  with  $\ell > m$ ,  $\mu(g) < \varepsilon$ , and such that the triangle

$$\begin{array}{ccc} u_m & \xrightarrow{u_m^\ell} & u_\ell \\ & \searrow f' & \nearrow g \\ & w & \end{array}$$

is  $\varepsilon/2$ -commutative. Finally,  $\mu(g \circ j) \leq \mu(g) < \varepsilon$  and  $\varrho(g \circ j \circ f, u_n^\ell) < \varepsilon$ , which shows that  $\vec{u}$  satisfies (A).  $\square$

The following characterization of a Fraïssé sequence turns out to be useful:

**Proposition 6.2.** *Let  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  be a normed category and let  $\vec{u}$  be a sequence in  $\mathfrak{K}_0$  satisfying (U). Then  $\vec{u}$  is Fraïssé in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  if and only if it satisfies the following condition:*

- (B) *Given  $\varepsilon > 0$ , given  $n \in \omega$ , given a  $\mathfrak{K}$ -arrow  $f: u_n \rightarrow y$  with  $\mu(f) < +\infty$ , there exist  $m > n$  and a  $\mathfrak{K}$ -arrow  $g: y \rightarrow u_m$  such that*

$$\mu(g) < \varepsilon \quad \text{and} \quad \varrho(g \circ f, u_n^m) < \mu(f) + \varepsilon.$$

*Proof.* It is obvious that (B) implies (A). Suppose  $\vec{u}$  is Fraïssé and choose  $\mathfrak{K}_0$ -arrows  $i: u_n \rightarrow w$  and  $j: y \rightarrow w$  such that  $\varrho(j \circ f, i) < \mu(f) + \varepsilon/2$ . Using (A), find  $m > n$  and a  $\mathfrak{K}$ -arrow  $k: y \rightarrow u_m$  such that  $\mu(k) < \varepsilon$  and  $\varrho(k \circ i, u_n^m) < \varepsilon/2$ . Define  $g = k \circ j$ . Then  $\mu(g) \leq \mu(k) + \mu(j) = \mu(k) < \varepsilon$  and

$$\varrho(g \circ f, u_n^m) \leq \varrho(k \circ j \circ f, k \circ i) + \varrho(k \circ i, u_n^m) < \mu(f) + \varepsilon/2 + \varepsilon/2 = \mu(f) + \varepsilon,$$

which shows that  $\vec{u}$  satisfies (A).  $\square$

In order to characterize the existence of a Fraïssé sequence, we shall introduce the concept of *separability*. A subcategory  $\mathcal{F}$  of  $\mathfrak{K}_0$  is *dominating* in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  if

(D<sub>1</sub>) Every  $\mathfrak{K}$ -object has  $\mathfrak{K}$ -arrows into  $\mathcal{F}$ -objects of arbitrarily small norm. More precisely, given  $x \in \text{Ob}(\mathfrak{K})$ , given  $\varepsilon > 0$ , there exists  $f: x \rightarrow y$  such that  $y \in \text{Ob}(\mathcal{F})$  and  $\mu(f) < \varepsilon$ .

(D<sub>2</sub>) Given  $\varepsilon > 0$ , a  $\mathfrak{K}_0$ -arrow  $f: a \rightarrow y$  such that  $a \in \text{Ob}(\mathcal{F})$ , there exist a  $\mathfrak{K}$ -arrow  $g: y \rightarrow b$  and an  $\mathcal{F}$ -arrow  $u: a \rightarrow b$  such that  $\mu(g) < \varepsilon$  and  $\varrho(g \circ f, u) < \varepsilon$ .

A normed category  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  is *separable* if there exists a countable  $\mathcal{F} \subseteq \mathfrak{K}_0$  that is dominating in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ .

Finally, we need to adapt the notion of directedness. Namely, we say that  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  is *directed* if for every  $\mathfrak{K}_0$ -objects  $a, b$ , for every  $\varepsilon > 0$  there exist  $\mathfrak{K}$ -arrows  $f: a \rightarrow d$  and  $g: b \rightarrow d$  such that  $\mu(f) < \varepsilon$  and  $\mu(g) < \varepsilon$ .

**Theorem 6.3.** *Let  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  be a directed normed category with the almost amalgamation property. The following conditions are equivalent:*

- (a)  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  is separable.
- (b)  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  has a Fraïssé sequence.

*Furthermore, if  $\mathcal{F}$  is a countable directed dominating subcategory of  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  with the almost amalgamation property, then there exists a sequence in  $\mathcal{F}$  that is Fraïssé in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ .*

*Proof.* Implication (b)  $\implies$  (a) is obvious: the image of a Fraïssé sequence is a countable dominating subcategory of  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ . It remains to show that (a)  $\implies$  (b). We shall use the simple folklore fact, known as the Rasiowa-Sikorski Lemma: given a directed partially ordered set  $\mathbb{P}$ , given a countable family  $\{D_n\}_{n \in \omega}$  of cofinal subsets of  $\mathbb{P}$ , there exists an increasing sequence  $\{p_n\}_{n \in \omega} \subseteq \mathbb{P}$  such that  $p_n \in D_n$  for every  $n \in \omega$ .

Assume  $\mathcal{F} \subseteq \mathfrak{K}_0$  is countable and dominating in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ . Enlarging  $\mathcal{F}$  if necessary, we may assume that it is directed and has the almost amalgamation property. We are going to find a Fraïssé sequence in  $\mathcal{F}$ , which by (D<sub>1</sub>) and (D<sub>2</sub>) must also be Fraïssé in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ . This will also show the “furthermore” statement.

Define the following partially ordered set  $\mathbb{P}$ : Elements of  $\mathbb{P}$  are finite sequences in  $\mathcal{F}$  (i.e. covariant functors from  $n < \omega$  into  $\mathcal{F}$ ). The order is end-extension, that is,  $\vec{x} \leq \vec{y}$  if  $\vec{y} \upharpoonright n = \vec{x}$ , where  $n = \text{dom}(\vec{x})$ .

Fix  $n, k \in \omega$  and fix an  $\mathcal{F}$ -arrow  $f: a \rightarrow b$ . We define  $D_{f,n,k} \subseteq \mathbb{P}$  to be the set of all  $\vec{x} \in \mathbb{P}$  such that  $\text{dom}(\vec{x}) > n$  and the following two conditions are satisfied:

- (1) There exists  $\ell < \text{dom}(\vec{x})$  such that  $\mathcal{F}(a, x_\ell) \neq \emptyset$ .
- (2) If  $a = x_n$  then there exists an  $\mathcal{F}$ -arrow  $g: b \rightarrow x_m$  such that  $n \leq m < \text{dom}(\vec{x})$  and  $\varrho(g \circ f, x_n^m) < 1/k$ .

Since  $\mathcal{F}$  is directed and has the almost amalgamation property, it is clear that all sets of the form  $D_{f,n,k}$  are cofinal in  $\mathbb{P}$ . It is important that there are only countably many such sets. Thus, by the Rasiowa-Sikorski Lemma, there exists a sequence

$$\vec{u}_0 < \vec{u}_1 < \vec{u}_2 < \dots$$

such that for each suitable triple  $f, n, k$  there is  $r \in \omega$  satisfying  $\vec{u}_r \in D_{f,n,k}$ . It is now rather clear that  $\vec{u} := \bigcup_{n \in \omega} \vec{u}_n$  is a Fraïssé sequence in  $\mathcal{F}$  which, by the remarks above, is also a Fraïssé sequence in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ .  $\square$

## 7 Approximate back-and-forth argument and cofinality

*☞ We present an approximate variant of the back-and-forth argument, which gives uniqueness and almost homogeneity. (1 hour)*

We now show that a Fraïssé sequence is “almost homogeneous” in the sense described below.

**Lemma 7.1.** *Let  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  be a normed category and let  $\vec{u}, \vec{v}$  be Fraïssé sequences in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$ . Furthermore, let  $\varepsilon > 0$  and let  $h: u_0 \rightarrow v_0$  be a  $\mathfrak{K}$ -arrow with  $\mu(f) < \varepsilon$ . Then there exists an approximate isomorphism  $H: \vec{u} \rightarrow \vec{v}$  such that  $\mu(H) = 0$  and the diagram*

$$\begin{array}{ccc} \vec{u} & \xrightarrow{H} & \vec{v} \\ u_0^\infty \uparrow & & \uparrow v_0^\infty \\ u_0 & \xrightarrow{h} & v_0 \end{array}$$

*is  $\varepsilon$ -commutative.*

*Proof.* Fix a decreasing sequence of positive reals  $\{\varepsilon_n\}_{n \in \omega}$  such that

$$\mu(h) < \varepsilon_0 < \varepsilon \quad \text{and} \quad 2 \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon - \varepsilon_0.$$

We define inductively sequences of  $\mathfrak{K}$ -arrows  $f_n: u_{\varphi(n)} \rightarrow v_{\psi(n)}$ ,  $g_n: v_{\psi(n)} \rightarrow u_{\varphi(n+1)}$  such that

- (1)  $\varphi(n) \leq \psi(n) < \varphi(n+1)$ ;
- (2)  $\varrho(g_n \circ f_n, u_{\varphi(n)}^{\varphi(n+1)}) < \varepsilon_n$ ;
- (3)  $\varrho(f_n \circ g_{n-1}, v_{\psi(n-1)}^{\psi(n)}) < \varepsilon_n$ ;
- (4)  $\mu(f_n) < \varepsilon_n$  and  $\mu(g_n) < \varepsilon_{n+1}$ ;

We start by setting  $\varphi(0) = \psi(0) = 0$  and  $f_0 = h$ . We find  $g_0$  and  $\varphi(1)$  by using condition (B) of Proposition 6.2.

We continue, using condition (B) for both sequences repeatedly. More precisely, having defined  $f_{n-1}$  and  $g_{n-1}$ , we first use property (B) of the sequence  $\vec{v}$  is Fraïssé, constructing  $f_n$  satisfying (3) and with  $\mu(f_n) < \varepsilon_n$ ; next we use the fact that  $\vec{u}$  satisfies (B) in order to find  $g_n$  satisfying (2) and with  $\mu(g_n) < \varepsilon_{n+1}$ .

We now check that  $\vec{f} = \{f_n\}_{n \in \omega}$  and  $\vec{g} = \{g_n\}_{n \in \omega}$  are approximate arrows. Fix  $n \in \omega$  and observe that

$$\begin{aligned} \varrho\left(v_{\psi(n)}^{\psi(n+1)} \circ f_n, f_{n+1} \circ u_{\varphi(n)}^{\varphi(n+1)}\right) \\ \leq \varrho\left(v_{\psi(n)}^{\psi(n+1)} \circ f_n, f_{n+1} \circ g_n \circ f_n\right) + \varrho\left(f_{n+1} \circ g_n \circ f_n, f_{n+1} \circ u_{\varphi(n)}^{\varphi(n+1)}\right) \\ \leq \varrho\left(v_{\psi(n)}^{\psi(n+1)}, f_{n+1} \circ g_n\right) + \varrho\left(g_n \circ f_n, u_{\varphi(n)}^{\varphi(n+1)}\right) < \varepsilon_{n+1} + \varepsilon_n. \end{aligned}$$

Since the series  $\sum_{n \in \omega} \varepsilon_n$  is convergent, we conclude that  $\{f_n\}_{n \in \omega}$  is an approximate arrow from  $\vec{u}$  to  $\vec{v}$ .

By symmetry, we deduce that  $\vec{g}$  is an approximate arrow from  $\vec{v}$  to  $\vec{u}$ . Conditions (2) and (3) tell us that the compositions  $\vec{f} \circ \vec{g}$  and  $\vec{g} \circ \vec{f}$  are equivalent to the identities, which shows that  $H := \vec{f}$  is an isomorphism. Condition (4) ensures us that  $\mu(H) = 0$ . Finally, recalling that  $h = f_0$ , we obtain

$$\begin{aligned} \varrho(v_0^\infty \circ h, H \circ u_0^\infty) &\leq \sum_{n=0}^{\infty} \varrho\left(v_{\psi(n)}^{\psi(n+1)} \circ f_n, f_{n+1} \circ u_{\varphi(n)}^{\varphi(n+1)}\right) \\ &< \sum_{n=0}^{\infty} (\varepsilon_n + \varepsilon_{n+1}) = \varepsilon_0 + 2 \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon. \end{aligned}$$

This completes the proof. □

The lemma above has two interesting corollaries. Recall that a  $\theta$ -isomorphism is an isomorphism  $H$  with  $\mu(H) = 0$ . In such a case also  $\mu(H^{-1}) = 0$ .

**Theorem 7.2** (Uniqueness). *A normed category may have at most one Fraïssé sequence, up to an approximate  $\theta$ -isomorphism.*



where the first triangle is commutative, the second one is  $\delta$ -commutative, and the internal square is  $(\varepsilon - 4\delta)$ -commutative. Applying (1), (2), (3), the triangle inequality and inequalities (M), we obtain

$$\begin{aligned}
\varrho(j \circ f, H \circ i) &\leq \varrho(j \circ f, u_k^\infty \circ j_k \circ f) + \varrho(u_k^\infty \circ f_1, u_\ell^\infty \circ g_2 \circ g_1 \circ f_1) \\
&\quad + \varrho(u_\ell^\infty \circ g_2 \circ g_1 \circ f_1, u_\ell^\infty \circ g_2 \circ f_2 \circ i_k) + \varrho(u_\ell^\infty \circ g \circ i_k, H \circ u_k^\infty \circ i_k) \\
&\quad + \varrho(H \circ u_k^\infty \circ i_k, H \circ i) \\
&\leq \varrho(j, u_k^\infty \circ j_k) + \varrho(u_k^\ell, g_2 \circ g_1) + \varrho(g_1 \circ f_1, f_2 \circ i_k) + \varrho(u_\ell^\infty \circ g, H \circ u_k^\infty) \\
&\quad + \varrho(u_k^\infty \circ i_k, i) \\
&< \delta + \delta + (\varepsilon - 4\delta) + \delta + \delta = \varepsilon.
\end{aligned}$$

This completes the proof.  $\square$

We finish with showing that a Fraïssé sequence is cofinal in the category of sequences.

**Theorem 7.4.** *Assume  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  is a normed category with the almost amalgamation property and with a Fraïssé sequence  $\vec{u}$ . Then for every sequence  $\vec{x}$  in  $\mathfrak{K}_0$  there exists an approximate arrow*

$$\vec{f}: \vec{x} \rightarrow \vec{u}$$

such that  $\mu(\vec{f}) = 0$ .

*Proof.* We construct inductively a strictly increasing sequence of natural numbers  $\{k_n\}_{n \in \omega}$  and a sequence of  $\mathfrak{K}$ -arrows  $f_n: x_n \rightarrow u_{k_n}$  satisfying for each  $n \in \omega$  the condition

$$(*) \quad \varrho(u_{k_n}^{k_{n+1}} \circ f_n, f_{n+1} \circ x_n^{n+1}) < 3 \cdot 2^{-n} \text{ and } \mu(f_n) < 2^{-n}.$$

We start by finding  $f_0$  using condition (U) of a Fraïssé sequence. Fix  $n \in \omega$  and suppose  $f_n$  and  $k_n$  have been defined already.

Let  $\varepsilon = 2^{-n}$ . Since  $\mu(f_n) < 2^{-n}$ , there exist  $\mathfrak{K}_0$ -arrows  $i: x_n \rightarrow v$ ,  $j: u_{k_n} \rightarrow v$  such that

$$\varrho(i, j \circ f_n) < 2^{-n}.$$

Next, using the almost amalgamation property, we find  $\mathfrak{K}_0$ -arrows  $k: v \rightarrow w$  and  $\ell: x_{n+1} \rightarrow w$  such that

$$\varrho(k \circ i, \ell \circ x_n^{n+1}) < 2^{-n}.$$

Finally, using the fact that  $\vec{u}$  is Fraïssé, find  $k_{n+1} > k_n$  and a  $\mathfrak{K}$ -arrow  $g: w \rightarrow u_{k_{n+1}}$  such that  $\mu(g) < 2^{-(n+1)}$  and

$$\varrho(g \circ k \circ j, u_{k_n}^{k_{n+1}}) < 2^{-n}.$$

The situation is described in the following diagram, where both internal squares and the triangle are  $2^{-n}$ -commutative:

$$\begin{array}{ccccc}
 u_{k_n} & \longrightarrow & u_{k_{n+1}} & \longrightarrow & \dots \\
 \uparrow f_n & \searrow j & & \uparrow g & \\
 & & v & \xrightarrow{k} & w \\
 & \nearrow i & & \uparrow \ell & \\
 x_n & \longrightarrow & x_{n+1} & \longrightarrow & \dots
 \end{array}$$

Define  $f_{n+1} := g \circ \ell$ . Then  $\mu(f_{n+1}) \leq \mu(g) + \mu(\ell) = \mu(g) < 2^{-(n+1)}$ . The diagram above shows that condition (\*) is satisfied. This completes the inductive construction.

Finally,  $\vec{f} = \{f_n\}_{n \in \omega}$  is an approximate arrow from  $\vec{x}$  to  $\vec{u}$  satisfying  $\mu(\vec{f}) = \lim_{n \rightarrow \infty} \mu(f_n) = 0$ .  $\square$

## 8 Selected applications

*☞ We present some examples and general constructions, including embedding-projection pairs. (2 hours)*

First of all, properties of the classical Fraïssé limits can be easily described using the results of Sections 3 and 4.

Recall that a *Fraïssé class* is a class  $\mathfrak{K}$  of finitely generated models of a fixed first-order language, satisfying the following conditions:

1. Given  $A, B \in \mathfrak{K}$  there exists  $C \in \mathfrak{K}$  such that both  $A$  and  $B$  embed into  $C$ .
2. Given embeddings  $f: C \rightarrow A, g: C \rightarrow B$  with  $A, B, C \in \mathfrak{K}$  there exist  $D \in \mathfrak{K}$  and embeddings  $f': A \rightarrow D, g': B \rightarrow D$  such that  $f' \circ f = g' \circ g$ .
3.  $\mathfrak{K}$  is hereditary, i.e., given  $A \in \mathfrak{K}$ , the class  $\{X: X \text{ embeds into } A\}$  is contained in  $\mathfrak{K}$ .
4.  $\mathfrak{K}$  has countably many isomorphic types.

Looking at  $\mathfrak{K}$  as a category (with embeddings as arrows) it is clear that  $\sigma\mathfrak{K}$  can be identified with the class of all countable models whose finitely generated substructures are in  $\mathfrak{K}$ . Thus we obtain:

**Theorem 8.1** (Fraïssé [11]). *Given a Fraïssé class  $\mathfrak{K}$  there exists a unique countable model  $U = \text{Flim } \mathfrak{K}$  (called the Fraïssé limit of  $\mathfrak{K}$ ) satisfying the following conditions:*

1. *Every countable model whose all finitely generated substructures are in  $\mathfrak{K}$  is embeddable into  $U$ .*

2. Up to isomorphism,  $\mathfrak{K} = \{X \leq U : X \text{ is finitely generated}\}$ .
3. Every isomorphism between finitely generated substructures of  $U$  extends to an automorphism of  $U$ .

## 8.1 Reversed Fraïssé limits

Now we describe briefly the theory of inverse limits of models, the so-called *projective Fraïssé theory*, developed by Irwin & Solecki [16]. Fix a first-order language  $L$  and consider some class  $\mathfrak{M}$  of  $L$ -models. The classical Fraïssé theory deals with the category of all embeddings of  $L$ -models. In the reversed Fraïssé theory one deals with *quotient maps*  $f: M \rightarrow N$  that are, by definition, surjective homomorphisms satisfying the following formula

$$R^N(y_0, \dots, y_{n-1}) \implies (\exists x_0, \dots, x_{n-1} \in M) R^M(x_0, \dots, x_{n-1}) \wedge (\forall i < n) f(x_i) = y_i$$

for each  $n$ -ary relation symbol  $R \in L$  and for each  $y_0, \dots, y_{n-1} \in N$ . More precisely, we consider the category  $\mathfrak{K}$  whose class of objects consists of finite  $L$ -models and an arrow from  $A \in \text{Ob}(\mathfrak{K})$  to  $B \in \text{Ob}(\mathfrak{K})$  is a quotient map of models  $f: B \rightarrow A$ .

The following result from [16] is a direct application of the results from Sections 3 and 4. It can also be derived from the results of [10], however our approach is more direct and explains why the topology is needed.

**Theorem 8.2** (Irwin & Solecki [16]). *Let  $\mathfrak{M}$  be a countable class of finite models of a fixed first-order language  $L$ . Suppose  $\mathfrak{M}$  satisfies the following conditions:*

- (J) For every  $a, b \in \mathfrak{M}$  there exist  $c \in \mathfrak{M}$  and quotient maps  $f: c \rightarrow a$  and  $g: c \rightarrow b$ .
- (A) Given quotient maps  $f: c \rightarrow a$  and  $g: c \rightarrow b$  with  $a, b, c \in \mathfrak{M}$ , there exist  $w \in \mathfrak{M}$  and quotient maps  $f', g'$  for which the diagram

$$\begin{array}{ccc} b & \xleftarrow{g'} & w \\ g \downarrow & & \downarrow f' \\ c & \xleftarrow{f} & a \end{array}$$

*commutes.*

*Then there exists a unique (up to a topological isomorphism) topological  $L$ -model  $\mathbb{M}$  satisfying the following conditions:*

- (1)  $\mathbb{M}$  is the inverse limit of a sequence of models from  $\mathfrak{M}$  with quotient maps.
- (2) Every model from  $\mathfrak{M}$  is a continuous quotient of  $\mathbb{M}$ .

- (3) Given continuous quotients  $p: \mathbb{M} \rightarrow a$  and  $q: \mathbb{M} \rightarrow b$  with  $a, b \in \mathfrak{M}$ , given a quotient map  $f: b \rightarrow a$ , there exists a topological isomorphism  $h: \mathbb{M} \rightarrow \mathbb{M}$  for which the diagram

$$\begin{array}{ccc} \mathbb{M} & \xleftarrow{h} & \mathbb{M} \\ p \downarrow & & \downarrow q \\ a & \xleftarrow{f} & b \end{array}$$

is commutative.

Furthermore, every  $L$ -model which is the inverse limit of a sequence of models from  $\mathfrak{M}$  with quotient maps is a continuous quotient of  $\mathbb{M}$ .

*Proof.* Let  $\mathfrak{K}$  be the category whose objects are elements of  $\mathfrak{M}$  and an arrow from  $a \in \mathfrak{M}$  into  $b \in \mathfrak{M}$  is a quotient map  $f: b \rightarrow a$ . It is obvious that conditions (J) and (A) translate to directedness and the amalgamation property. Since  $\mathfrak{M}$  is countable, the category  $\mathfrak{K}$  is countable. Now observe that the category  $\sigma\mathfrak{K}$  can be naturally identified with the category of compact  $L$ -models that are inverse limits of sequences of models from  $\mathfrak{M}$ . In fact, the topology becomes natural here, because given two sequences  $\vec{x}$  and  $\vec{y}$  in  $\mathfrak{K}$  and taking  $X$  and  $Y$  to be their inverse limits in the category of sets, one can easily check that precisely the continuous quotient maps  $f: Y \rightarrow X$  correspond to  $\sigma\mathfrak{K}$ -arrows from  $\vec{x}$  to  $\vec{y}$ . Summarizing: the existence and properties of  $\mathbb{M}$  follow directly from Theorems 3.1, 4.1 and 4.2.  $\square$

## 8.2 Universal diagrams

We shall describe a quite general procedure of building a category with the amalgamation property. First of all, let us note that typical categories with monics do not admit pushouts. For instance, this is the case with the category of finite sets with one-to-one maps. In order to capture these situations, we define the following natural notion.

Let  $\mathfrak{K} \subseteq \mathfrak{L}$  be two categories. We say that  $\mathfrak{K}$  has pushouts in  $\mathfrak{L}$  (or, that  $\langle \mathfrak{K}, \mathfrak{L} \rangle$  has pushouts) if for every  $\mathfrak{K}$ -arrows  $f: c \rightarrow a$ ,  $g: c \rightarrow b$  there exist  $\mathfrak{K}$ -arrows  $f': a \rightarrow w$  and  $g': b \rightarrow w$  such that

$$\begin{array}{ccc} b & \xrightarrow{g'} & w \\ g \uparrow & & \uparrow f' \\ c & \xrightarrow{f} & a \end{array}$$

is a pushout in  $\mathfrak{L}$ .

Now, fix a small category  $\mathbb{S}$  and let  $\mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$  be the category whose objects are covariant functors from  $\mathbb{S}$  to  $\mathfrak{L}$  taking objects from  $\mathfrak{K}$  and the arrows are natural transformations into  $\mathfrak{K}$ . More precisely,  $x$  is an object of iff  $x$  is a functor from  $\mathbb{S}$  into  $\mathfrak{L}$  such that  $x(s)$  is an object of  $\mathfrak{K}$  for every object  $s$  of  $\mathbb{S}$ . Furthermore,  $f: x \rightarrow y$  is an  $\mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$ -arrow iff it is a natural transformation from  $x$  to  $y$  whose all components

(arrows) are in  $\mathfrak{K}$ . In other words, for every object  $s$  of  $\mathbb{S}$ , the arrow  $f(s)$  belongs to  $\mathfrak{K}$ .

**Lemma 8.3.** *Let  $\mathfrak{K} \subseteq \mathfrak{L}$  be a pair of categories such that  $\mathfrak{K}$  has pushouts in  $\mathfrak{L}$  and let  $\mathbb{S}$  be a small category. Then  $\mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$  has pushouts in  $\mathfrak{f}(\mathbb{S}, \mathfrak{L}, \mathfrak{L})$ .*

*Proof.* Fix  $a, b, c \in \mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$  and fix natural transformations  $f: c \rightarrow a$ ,  $g: c \rightarrow b$ . We are going to define  $w \in \mathfrak{f}(\mathbb{S}, \mathfrak{K}, \mathfrak{L})$ . Fix  $p \in \mathbb{S}$ . Let  $\langle f'(p), g'(p) \rangle$  be the pushout of  $\langle f(p), g(p) \rangle$  in  $\mathfrak{L}$ , and let  $w(p)$  be the common co-domain of  $f'(p)$  and  $g'(p)$ . Now fix an arrow  $j: p \rightarrow q$  in  $\mathbb{S}$ . Using the property of a pushout, there is a unique arrow  $w(j)$  making the following diagram commutative.

$$\begin{array}{ccccc}
 & & c(q) & \xrightarrow{g(q)} & b(q) \\
 & & \searrow f(q) & & \searrow g'(q) \\
 & & a(q) & \xrightarrow{f'(q)} & w(q) \\
 c(j) \uparrow & & \uparrow a(j) & & \uparrow b(j) \\
 c(p) & \xrightarrow{g(p)} & b(p) & & \\
 \searrow f(p) & & \searrow g'(p) & & \\
 a(p) & \xrightarrow{f'(p)} & w(p) & & \\
 & & \uparrow w(j) & & 
 \end{array}$$

Indeed,  $\langle f'(q) \circ a(j), g'(q) \circ b(j) \rangle$  is an amalgamation of  $\langle f(p), g(p) \rangle$ . By uniqueness, we have that  $w(j \circ k) = w(j) \circ w(k)$ , whenever  $j, k$  are compatible arrows in  $\mathbb{S}$ .

Thus we have defined a functor  $w: \mathbb{S} \rightarrow \mathfrak{K}$ . Further,  $p \mapsto f'(p)$  and  $p \mapsto g'(p)$  are natural transformations from  $a$  to  $w$  and from  $b$  to  $w$  respectively.

It is clear that  $f' \circ f = g' \circ g$ .

It remains to check that  $\langle f', g' \rangle$  is a pushout of  $\langle f, g \rangle$  in  $\mathfrak{f}(\mathbb{S}, \mathfrak{L}, \mathfrak{L})$ . For this aim, fix  $v \in \mathfrak{f}(\mathbb{S}, \mathfrak{L}, \mathfrak{L})$  and natural transformations  $f'': a \rightarrow v$ ,  $g'': b \rightarrow v$  such that  $f'' \circ f = g'' \circ g$ . By the fact that  $\langle f', g' \rangle$  is a pushout of  $\langle f, g \rangle$  in  $\mathfrak{L}$ , for each  $p \in \mathbb{S}$  there is a unique  $\mathfrak{L}$ -arrow  $h(p): w(p) \rightarrow v(p)$  satisfying  $h(p) \circ f'(p) = f''(p)$  and  $h(p) \circ g'(p) = g''(p)$ . This defines uniquely a map  $h: \mathbb{S} \rightarrow \text{Arr } \mathfrak{L}$  that satisfies  $h \circ f' = f''$  and  $h \circ g' = g''$ . It remains to check that  $h$  is indeed a natural transformation.

Fix an arrow  $j: p \rightarrow q$  in  $\mathbb{S}$  and let  $k = h(q) \circ w(j)$ ,  $\ell = v(j) \circ h(p)$ . We need to show that  $k = \ell$ . Notice that  $k \circ f'(p) = f''(q) \circ a(j)$  and  $k \circ g'(p) = g''(q) \circ b(j)$ . Also,  $\ell \circ f'(p) = f''(q) \circ a(j)$  and  $\ell \circ g'(p) = g''(q) \circ b(j)$ . It follows that  $k = \ell$ , because  $\langle f'(p), g'(p) \rangle$  is a pushout of  $\langle f(p), g(p) \rangle$  in  $\mathfrak{L}$ .  $\square$

We now demonstrate a possible use of Lemma 8.3 in the context of Fraïssé model-theoretic structures.

Given categories  $\mathfrak{K} \subseteq \mathfrak{L}$ , we say that  $\langle \mathfrak{K}, \mathfrak{L} \rangle$  has the *mixed pushout property* if for every  $\mathfrak{K}$ -arrow  $i: c \rightarrow a$ , for every  $\mathfrak{L}$ -arrow  $f: c \rightarrow b$ , there exist  $j \in \mathfrak{K}$  and  $g \in \mathfrak{L}$  such that

$$\begin{array}{ccc} b & \xrightarrow{j} & w \\ f \uparrow & & \uparrow g \\ c & \xrightarrow{i} & a \end{array}$$

is a pushout in  $\mathfrak{L}$ . In case  $\mathfrak{M}$  is a class of models of some first-order language, it is natural to consider  $\mathfrak{K}(\mathfrak{M})$  to be the category of all embeddings between models of  $\mathfrak{M}$  and  $\mathfrak{L}(\mathfrak{M})$  to be the category of all homomorphisms between these models. We then say that  $\mathfrak{M}$  has the *mixed pushout property* if so does  $\langle \mathfrak{K}(\mathfrak{M}), \mathfrak{L}(\mathfrak{M}) \rangle$ . We denote by  $\overline{\mathfrak{M}}$  the class of all (countable) models that are unions of  $\omega$ -chains of models from  $\mathfrak{M}$ .

**Theorem 8.4.** *Let  $\mathfrak{M}$  be a countable Fraïssé class of finitely generated models, with the mixed pushout property. Let  $W$  denote the Fraïssé limit of  $\mathfrak{M}$ . Then there exists a unique (up to isomorphism) homomorphism  $L: W \rightarrow W$  satisfying the following conditions.*

- (a) *For every  $X, Y \in \overline{\mathfrak{M}}$ , for every homomorphism  $F: X \rightarrow Y$  there exist embeddings  $I_X: X \rightarrow W$  and  $I_Y: Y \rightarrow W$  such that the square*

$$\begin{array}{ccc} W & \xrightarrow{L} & W \\ I_X \uparrow & & \uparrow I_Y \\ X & \xrightarrow{F} & Y \end{array}$$

*is commutative.*

- (b) *Given finitely generated substructures  $x_0, x_1, y_0, y_1$  of  $W$  such that  $L[x_i] \subseteq y_i$  for  $i < 2$ , given isomorphisms  $h_i: x_i \rightarrow y_i$  for  $i < 2$  such that  $L \circ h_0 = h_1 \circ L$ , there exist automorphisms  $H_i: W \rightarrow W$  extending  $h_i$  for  $i < 2$ , and such that  $L \circ H_0 = H_1 \circ L$ .*

*Proof.* Consider the category  $\mathfrak{C} = \mathfrak{f}(2, \mathfrak{K}(\mathfrak{M}), \mathfrak{L}(\mathfrak{M}))$ , where 2 denotes the two-element poset category. The assumptions above, combined with Lemma 8.3, give a Fraïssé sequence  $\{\ell_n\}_{n \in \omega}$  which translated back to  $\mathfrak{L}(\mathfrak{M})$  looks as follows.

$$\begin{array}{ccccccc} u_0 & \longrightarrow & u_1 & \longrightarrow & \cdots & \longrightarrow & u_n & \longrightarrow & \cdots \\ \ell_0 \downarrow & & \ell_1 \downarrow & & & & \ell_n \downarrow & & \\ v_0 & \longrightarrow & v_1 & \longrightarrow & \cdots & \longrightarrow & v_n & \longrightarrow & \cdots \end{array}$$

The horizontal arrows are embeddings and the vertical arrows are homomorphisms of models. Without loss of generality, we may assume that the horizontal arrows are inclusions. Now let  $U = \bigcup_{n \in \omega} u_n$ ,  $V = \bigcup_{n \in \omega} v_n$  and define  $L = \bigcup_{n \in \omega} \ell_n$ . It is clear that  $L$  satisfies conditions (a) and (b) above. It remains to check that both  $U$  and

$V$  are isomorphic to the  $\mathfrak{M}$ -Fraïssé limit  $W$ . For this aim, it suffices to show that both  $\{u_n\}_{n \in \omega}$  and  $\{v_n\}_{n \in \omega}$  are Fraïssé sequences in  $\mathfrak{K}(\mathfrak{M})$ .

It is clear that both sequences satisfy (U), because identity arrows can be viewed as objects of  $\mathfrak{C}$ .

Fix  $n \in \omega$  and fix an embedding  $j: u_n \rightarrow x$ , where  $x \in \mathfrak{M}$ . Using the mixed pushout property, we can find an embedding  $k: v_n \rightarrow y$  and a homomorphism  $f: x \rightarrow y$  such that  $k \circ \ell_n = f \circ j$ . Now  $f$  can be regarded as an object of  $\mathfrak{C}$  and the pair  $\langle j, k \rangle$  is an arrow of  $\mathfrak{C}$ . Using the Fraïssé property of the sequence  $\{\ell_n\}_{n \in \omega}$ , we find  $m \geq n$  and embeddings  $j': x \rightarrow u_m$ ,  $k': y \rightarrow v_m$  such that  $\ell_m \circ j' = k' \circ f$  and  $j' \circ j$ ,  $k' \circ k$  are inclusions. In particular, this shows that  $\{u_n\}_{n \in \omega}$  is Fraïssé and therefore  $U = W$ .

Now fix an embedding  $i: v_n \rightarrow z$ . Let  $g = i \circ \ell_n$ . The pair  $\langle \text{id}_{u_n}, i \rangle$  can be viewed as a  $\mathfrak{C}$ -arrow into  $g$ , therefore by the Fraïssé property of  $\{\ell_n\}_{n \in \omega}$  we again find  $m \geq n$  and embeddings  $e: u_n \rightarrow u_m$ ,  $i': z \rightarrow u_m$  such that, among other conditions, the composition  $i' \circ i$  is the inclusion  $v_n \subseteq v_m$ . This shows that  $\{v_n\}_{n \in \omega}$  is Fraïssé and consequently  $V = W$ .  $\square$

Note that the *mixed* pushout property is needed for concluding that the domain of the universal homogeneous homomorphism is the Fraïssé limit of the class in question. Relaxing this by assuming only that embeddings have pushouts among all homomorphisms, one still gets a homomorphism with properties (a) and (b) above, whose co-domain is the Fraïssé limit of the considered class of models. It is clear how to formulate a similar statement for reversed Fraïssé limits, where pushouts are replaced by pullbacks.

### 8.3 Embedding-projection pairs

We describe a general construction on a given category, involving retractions. This construction had been first used by Dana Scott in order to get faithful models of untyped  $\lambda$ -calculus. It also appears in Droste & Göbel [10] in the context of Scott domains.

We fix a category  $\mathfrak{K}$ . Define  $\ddagger\mathfrak{K}$  to be the category whose objects are the objects of  $\mathfrak{K}$  and a morphism  $f: X \rightarrow Y$  is a pair  $\langle e, r \rangle$  of arrows in  $\mathfrak{K}$  such that  $e: X \rightarrow Y$ ,  $r: Y \rightarrow X$  and  $r \circ e = \text{id}_X$ . We set  $e(f) := e$  and  $r(f) := r$ , so  $f = \langle e(f), r(f) \rangle$ . Given morphisms  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  in  $\ddagger\mathfrak{K}$ , we define their composition in the obvious way:

$$g \circ f := \langle e(g) \circ e(f), r(f) \circ r(g) \rangle.$$

It is clear that this defines an associative operation on compatible arrows. Further, given an object  $a \in \mathfrak{K}$ , pair of the form  $\langle \text{id}_a, \text{id}_a \rangle$  is the identity morphism in  $\ddagger\mathfrak{K}$ . Thus,  $\ddagger\mathfrak{K}$  is indeed a category. Note that  $f \mapsto e(f)$  defines a covariant functor  $e: \ddagger\mathfrak{K} \rightarrow \mathfrak{K}$  and  $f \mapsto r(f)$  defines a contravariant functor  $r: \ddagger\mathfrak{K} \rightarrow \mathfrak{K}$ . Following [10], we shall call  $\ddagger\mathfrak{K}$  the *category of embedding-projection pairs* or briefly *EP-pairs*. The idea of considering EP-pairs is to obtain more special Fraïssé sequences, namely we would

like to obtain a unique cofinal object  $u$  in the category of sequences such that every other  $x$  can be both “embedded” into  $u$  and “projected” from  $u$ . In other words, we would like to have arrows  $j: x \rightarrow u$  and  $r: u \rightarrow x$  satisfying  $r \circ j = \text{id}_x$ . We shall demonstrate this in Example 8.7 below, dealing with the category of finite nonempty sets.

One can consider a stronger version of arrows between sequences (see Example 8.7 below for more explanations). This leads to the following concept.

Let  $f: Z \rightarrow X$  and  $g: Z \rightarrow Y$  be arrows in  $\ddagger\mathfrak{K}$ . We say that arrows  $h: X \rightarrow W$ ,  $k: Y \rightarrow W$  provide a *proper amalgamation* of  $f, g$  if  $h \circ f = k \circ g$  and moreover  $e(g) \circ r(f) = r(k) \circ e(h)$ ,  $e(f) \circ r(g) = r(h) \circ e(k)$  hold. Translating it back to the original category  $\mathfrak{K}$ , this means that the following four diagrams commute:

$$\begin{array}{cccc}
 \begin{array}{ccc} W & \xleftarrow{e(k)} & Y \\ \uparrow e(h) & & \uparrow e(g) \\ X & \xleftarrow{e(f)} & Z \end{array} & 
 \begin{array}{ccc} W & \xrightarrow{r(k)} & Y \\ \downarrow r(h) & & \downarrow r(g) \\ X & \xrightarrow{r(f)} & Z \end{array} & 
 \begin{array}{ccc} W & \xrightarrow{r(k)} & Y \\ \uparrow e(h) & & \uparrow e(g) \\ X & \xrightarrow{r(f)} & Z \end{array} & 
 \begin{array}{ccc} W & \xleftarrow{e(k)} & Y \\ \downarrow r(h) & & \downarrow r(g) \\ X & \xleftarrow{e(f)} & Z \end{array}
 \end{array}$$

We draw arrows  $\xrightarrow{\quad}$  and  $\xrightarrow{\quad}$  in order to indicate mono- and epimorphisms respectively. We shall say that  $\ddagger\mathfrak{K}$  has *proper amalgamations* if every pair of arrows in  $\ddagger\mathfrak{K}$  with common domain can be properly amalgamated in  $\ddagger\mathfrak{K}$ .

Below is a useful criterion for the existence of proper amalgamations.

**Lemma 8.5.** *Let  $\mathfrak{K}$  be a category and let  $f, g$  be arrows in  $\ddagger\mathfrak{K}$  with the same domain. If  $e(f), e(g)$  have a pushout in  $\mathfrak{K}$  then  $f, g$  can be properly amalgamated in  $\ddagger\mathfrak{K}$ .*

*Proof.* Let  $h: X \rightarrow W$  and  $k: Y \rightarrow W$  form a pushout of  $e(f), e(g)$ . Consider the following diagram:

$$\begin{array}{ccccc}
 & & Z & \xrightarrow{e(f)} & X & \xrightarrow{r(f)} & Z \\
 & & \uparrow r(g) & & \uparrow \ell & & \uparrow r(g) \\
 & & Y & \xrightarrow{k} & W & & Y \\
 & & \uparrow e(g) & & \uparrow h & & \uparrow e(g) \\
 Z & \xrightarrow{e(f)} & X & \xrightarrow{r(f)} & Z & & Z
 \end{array}$$

$\ell$  (dotted arrow from  $W$  to  $X$ ),  $j$  (dotted arrow from  $W$  to  $Z$ ),  $\text{id}_Y$  (solid arrow from  $Y$  to  $Y$ ),  $\text{id}_X$  (solid arrow from  $X$  to  $X$ ).

The dotted arrows indicate unique morphisms completing appropriate diagrams, i.e.  $j$  is the unique arrow satisfying equations  $j \circ h = e(g) \circ r(f)$ ,  $j \circ k = \text{id}_Y$  and  $\ell$  is the unique arrow satisfying equations  $\ell \circ k = e(f) \circ r(g)$ ,  $\ell \circ h = \text{id}_X$ . Consequently,  $\langle k, j \rangle$  and  $\langle h, \ell \rangle$  are morphisms in  $\ddagger\mathfrak{K}$ . Set  $s = r(f) \circ \ell$ . Then

$$(1) \quad s \circ k = r(f) \circ \ell \circ k = r(f) \circ e(f) \circ r(g) = r(g) \quad \text{and} \quad s \circ h = r(f) \circ \ell \circ h = r(f).$$

Recall that  $r(f) \circ e(f) = \text{id}_Z = r(g) \circ e(g)$ . Since  $k, h$  is a pushout of  $e(f), e(g)$ , we deduce that  $s$  must be the unique arrow satisfying (1). Now let  $t = r(g) \circ j$ . Similar

computations show that  $t \circ k = r(g)$  and  $t \circ h = r(f)$ , therefore by uniqueness we deduce that  $s = t$  or, in other words,  $r(f) \circ \ell = r(g) \circ j$ . This shows that the full diagram is commutative and hence  $\langle k, j \rangle$  and  $\langle h, \ell \rangle$  provide a proper amalgamation of  $f, g$  in the category  $\ddagger\mathfrak{K}$ .  $\square$

As an example, if  $\mathfrak{K}$  is the category of nonempty sets, then Lemma 8.5 says that category  $\ddagger\mathfrak{K}$  has proper amalgamations. We show below that not all amalgamations in  $\ddagger\mathfrak{K}$  are proper.

**Example 8.6.** Consider the category of nonempty finite sets  $\mathfrak{Set}^+$ . Let  $a, b, c, d$  be pairwise distinct elements and set  $Z = \{a\}$ ,  $X = \{a, b\}$ ,  $Y = \{a, c\}$  and  $W = \{a, b, c\}$ . We are going to define arrows  $f: Z \rightarrow X$ ,  $g: Z \rightarrow Y$ ,  $h: X \rightarrow W$  and  $k: Y \rightarrow W$  in the category  $\ddagger\mathfrak{Set}^+$ . Let  $e(f)$ ,  $e(g)$ ,  $e(h)$  and  $e(k)$  be the inclusion maps and let  $r(f)$  and  $r(g)$  be the obvious constant maps. Finally, let  $r(h)(c) = a$  and  $r(k)(b) = c$ . This already defines  $r(h)$  and  $r(k)$ , since these maps must be identity on the ranges of  $e(h)$  and  $e(k)$  respectively. It is clear that  $h \circ f = k \circ g$ , i.e.  $h, k$  amalgamate  $f, g$  in the category  $\ddagger\mathfrak{Set}^+$ . On the other hand,  $e(g) \circ r(f)(b) = a$  and  $r(k) \circ e(h)(b) = c$ , therefore  $e(g) \circ r(f) \neq r(k) \circ e(h)$ . Note that actually  $e(f) \circ r(g) = r(h) \circ e(k)$  holds, although redefining  $r(h)(c)$  to  $b$  we can even get  $e(f) \circ r(g) \neq r(h) \circ e(k)$ .

Now assume that  $\mathfrak{K}$  is a countable category with pushouts. Then  $\ddagger\mathfrak{K}$  has proper amalgamations and, assuming it is directed, it has a unique Fraïssé sequence  $\vec{u}$ . Every other sequence in  $\ddagger\mathfrak{K}$  has a proper arrow into  $\vec{u}$ . This turns out to be non-trivial even in one of the simplest cases, as we describe below.

**Example 8.7.** Let  $\mathfrak{Set}^+$  denote the category of nonempty finite sets. It is clear that monomorphisms have pushouts in  $\mathfrak{Set}^+$ , therefore it has proper amalgamations. We claim that  $\sigma(\ddagger\mathfrak{Set}^+)$  is isomorphic to the following category  $\mathfrak{L}$ . The objects of  $\mathfrak{L}$  are pairs of the form  $\langle K, D \rangle$ , where  $K$  is a totally disconnected compact metric space and  $D \subseteq K$  is a countable dense set. An arrow from  $\langle K, D \rangle$  to  $\langle L, E \rangle$  is a pair of functions  $\langle j, f \rangle$ , where  $j: D \rightarrow E$ ,  $f: L \rightarrow K$ ,  $f \circ j = \text{id}_D$  and  $f$  is continuous.

Given a sequence  $\vec{x}$  in  $\ddagger\mathfrak{Set}^+$ , we take  $D$  to be the co-limit of  $e[\vec{x}]$  in the category of sets. Clearly,  $D$  is a countable set. Now take  $K$  to be the inverse limit of  $r[\vec{x}]$  in the category of topological spaces. Why do we mention topology here? As we shall see in a moment, arrows correspond to continuous quotients. Clearly,  $K$  is compact metrizable and totally disconnected. Observe that there is a canonical embedding of  $D$  into  $K$ . Thus, we may think of  $D$  as a subset of  $K$ . Notice that every projection from  $K$  to an element of the sequence  $r[\vec{x}]$  is a continuous quotient; in other words, it corresponds to a partition into clopen sets.

Fix two sequences  $\vec{x}, \vec{y}$  in  $\ddagger\mathfrak{Set}^+$  and fix an arrow  $\vec{f}: \vec{x} \rightarrow \vec{y}$ . Let  $\langle K, D \rangle$  and  $\langle L, E \rangle$  be the  $\mathfrak{L}$ -objects corresponding to  $\vec{x}$  and  $\vec{y}$ , respectively. Notice that the sequence  $e(f_n)$  “converges to a one-to-one map  $j: D \rightarrow E$ , and the sequence  $r(f_n)$  “converges” to a continuous quotient  $f: L \rightarrow K$ . Clearly,  $f \circ j = \text{id}_D$ . By this way we have described a functor from  $\sigma(\ddagger\mathfrak{Set}^+)$  into  $\mathfrak{L}$ .

Now fix an  $\mathfrak{L}$ -object  $\langle K, D \rangle$  and write  $D = \{d_n\}_{n \in \omega}$ . Assume  $D$  is infinite and its enumeration is one-to-one. Construct inductively partitions  $\mathcal{U}_n$  of  $K$  into clopen sets, in such a way that for each  $U \in \mathcal{U}_n$  there is a unique index  $i \leq n$  such that  $d_i \in U$ . Let  $D_n = \{d_0, \dots, d_n\}$  and define  $r_n: K \rightarrow D_n$  so that  $r_n^{-1}(d_i) \in \mathcal{U}_n$  and  $r_n(d_i) = d_i$  for  $i \leq n$ . Let  $e_n: D_n \rightarrow D$  be the inclusion. Set  $x_n^\infty = \langle e_n, r_n \rangle$ . Then  $x_n^\infty$  is a  $\ddagger\mathfrak{Set}^+$  arrow and for each  $n < m$  there is a unique  $\ddagger\mathfrak{Set}^+$ -arrow  $x_n^m$  satisfying  $x_n^\infty = x_n^m \circ x_m^\infty$ . It is clear that this defines a sequence  $\vec{x}$  in  $\ddagger\mathfrak{Set}^+$  which induces the  $\mathfrak{L}$ -object  $\langle K, D \rangle$ . Call an  $\mathfrak{L}$ -arrow *proper* if it corresponds to a proper arrow of sequences in  $\ddagger\mathfrak{Set}^+$ .

**Claim 8.8.** *An  $\mathfrak{L}$ -arrow  $f: \langle K, D \rangle \rightarrow \langle L, E \rangle$  is proper if and only if  $f = \langle j, r \rangle$ , where  $j: K \rightarrow L$ ,  $r: L \rightarrow K$  are continuous maps such that  $j[D] \subseteq E$ ,  $r[E] \subseteq D$ , and  $r \circ j = \text{id}_K$ .*

The last three conditions actually imply that  $r[E] = D$ .

*Proof.* It is clear that a proper arrow of sequences induces a pair  $\langle j, r \rangle$  satisfying the conditions above. Fix continuous maps  $j: K \rightarrow L$ ,  $r: L \rightarrow K$  satisfying  $r \circ j = \text{id}_K$  and  $j[D] \subseteq E$ ,  $r[E] = D$ . Without loss of generality, we may assume that  $L \subseteq K$ ,  $j$  is the inclusion, and  $D = E \cap L$ . Choose a chain  $\{D_n\}_{n \in \omega}$  of finite subsets of  $D$  such that  $\bigcup_{n \in \omega} D_n = D$ . Choose inductively finite sets  $E_n \subseteq E$  so that  $D_n \subseteq E_n$ ,  $r[E_n] = D_n$ , and  $E_n \subseteq E_{n+1}$  for  $n \in \omega$ . We can do it in such a way that  $\bigcup_{n \in \omega} E_n = E$ , because  $r^{-1}[D] \supseteq E$ . Now observe that  $f_n := \langle e_n, r_n \rangle$  is a  $\ddagger\mathfrak{Set}^+$ -arrow, where  $e_n$  is the inclusion  $D_n \subseteq E_n$  and  $r_n = r \upharpoonright E_n$ . Finally,  $\{f_n\}_{n \in \omega}$  is a proper arrow from the sequence  $\{D_n\}_{n \in \omega}$  to the sequence  $\{E_n\}_{n \in \omega}$ .  $\square$

The category  $\ddagger\mathfrak{Set}^+$  is certainly countable, therefore it has a Fraïssé sequence  $\vec{u}$ . Let  $C$  be the inverse limit of  $r[\vec{u}]$  in the category of sets and let  $D$  be the co-limit of  $e[\vec{u}]$ , also in the category of sets. Then  $C$  can be regarded as the Cantor set and  $D$  is its countable dense subset. Note that a proper automorphism  $h$  of  $\langle C, D \rangle$  has the property that  $h[D] = D$ .

By this way we obtain in particular the well-known fact that every totally disconnected metric compact space  $K$  is a topological retract of the Cantor set. Moreover, the retraction maps the dense set  $D \subseteq C$  onto a given countable dense subset of  $K$ .

We finally give examples of improper arrows in  $\sigma(\ddagger\mathfrak{Set}^+)$ . Namely, let  $x_n = \{0, \dots, n\}$ ,  $y_n = \{0, \dots, n, \infty\}$ . Then  $x_0 \subseteq x_1 \subseteq \dots$  and  $y_0 \subseteq y_1 \subseteq \dots$ . Let  $K = L = \omega \cup \{\infty\}$ , where  $\omega$  is discrete and  $\infty = \lim_{n \rightarrow \infty} n$ . Given  $n < m$ , let  $x_n^m \in (\ddagger\mathfrak{Set}^+)$  be such that  $e(x_n^m)$  is inclusion and  $r(x_n^m)$  is the retraction mapping all elements of the set  $\{n+1, \dots, m\}$  to  $n$ . Let  $y_n^m \in (\ddagger\mathfrak{Set}^+)$  be such that  $e(y_n^m)$  is again inclusion, while  $r(y_n^m)$  maps the set  $\{n+1, \dots, m, \infty\}$  to  $\infty$ . This defines  $\omega$ -sequences  $\vec{x}$  and  $\vec{y}$  in  $\ddagger\mathfrak{Set}^+$ . Now let  $f_n: x_n \rightarrow y_n$  be defined by  $f_n = \langle e_n, r_n \rangle$ , where  $e_n$  is inclusion and  $r_n$  maps  $\infty$  to  $n$ . Notice that, given  $n < m$ , the equation  $y_n^m \circ f_n = f_m \circ x_n^m$  holds in  $\ddagger\mathfrak{Set}^+$ , although this amalgamation is not proper.

The sequences  $\vec{x}$  and  $\vec{y}$  correspond to pairs  $\langle K, \omega \rangle$  and  $\langle L, L \rangle$  in  $\mathfrak{L}$ , respectively. The sequence  $\vec{f} = \{f_n\}_{n \in \omega}$  induces maps  $e: \omega \rightarrow L$  and  $r: L \rightarrow K$ , where  $e$  is the

inclusion of  $\omega$  into  $L$  and  $r$  is the identity. Note that  $r[L] \not\subseteq \omega$ , therefore  $\langle e, r \rangle$  is not proper.

Another example of an improper arrow can be obtained as follows. Consider the compact space  $L = \mathbb{Z} \cup \{-\infty, \infty\}$ , where  $\mathbb{Z}$  is the (discrete!) set of the integers and  $\lim_{n \rightarrow \infty} \infty$  and  $\lim_{n \rightarrow \infty} -n = -\infty$ . Let  $K = \mathbb{Z} \cup \{\infty\}$  be a quotient of  $L$  obtained by identifying  $-\infty$  with  $\infty$ . Let  $r: L \rightarrow K$  be the quotient map. Furthermore, let  $D = K$ ,  $E = L$ , and let  $e: D \rightarrow E$  be the inclusion. We claim that  $\langle e, r \rangle$  is an arrow from  $\langle K, D \rangle$  into  $\langle L, E \rangle$ . Of course, it cannot be proper, because  $e$  is even not continuous.

Let  $x_n = \{i \in \mathbb{Z}: |i| \leq n\} \cup \{\infty\}$ ,  $y_n = \{i \in \mathbb{Z}: |i| \leq n\} \cup \{-\infty, \infty\}$ . Let  $x_n^m$  be such that  $e(x_n^m)$  is the inclusion  $x_n \subseteq x_m$  and  $r(x_n^m)$  is identity on  $x_n$  and maps all  $j \in x_m \setminus x_n$  to  $\infty$ . Define  $y_n^m$  in a similar manner, with the difference that  $r(y_n^m)(j) = -\infty$  whenever  $j < -n$  and  $r(y_n^m)(j) = \infty$  whenever  $j > n$ . We have defined sequences  $\vec{x}$  and  $\vec{y}$  corresponding to  $\langle K, K \rangle$  and  $\langle L, L \rangle$ , respectively. Finally, let  $f_n = \langle e_n, r_n \rangle$ , where  $e_n$  is the inclusion  $x_n \subseteq y_n$  and  $r_n$  is a quotient map of  $y_n$  onto  $x_n$  that maps  $-\infty$  onto  $\infty$  and satisfies  $r_n \circ e_n = \text{id}_{x_n}$ . Then  $\vec{f} = \{f_n\}_{n \in \omega}$  is an improper arrow of sequences inducing  $\langle e, r \rangle$ .

## 8.4 The Gurariĭ space

The Gurariĭ space is the unique separable Banach space  $\mathbb{G}$  satisfying the following condition:

- (G) Given  $\varepsilon > 0$ , given finite-dimensional Banach spaces  $X \subseteq Y$ , given an isometric embedding  $f: X \rightarrow \mathbb{G}$ , there exists an  $\varepsilon$ -isometric embedding  $g: Y \rightarrow \mathbb{G}$  such that  $g \upharpoonright X = f$ .

Recall that a linear operator  $T: E \rightarrow F$  is an  $\varepsilon$ -isometric embedding if  $(1+\varepsilon)^{-1}\|x\| < \|Tx\| < (1+\varepsilon)\|x\|$  holds for every  $x \in E$ .

The Gurariĭ space was constructed by Gurariĭ [13] in 1966, where it is shown that it is almost homogeneous in the sense that every linear isometry between finite-dimensional subspaces of  $\mathbb{G}$  extends to a bijective  $\varepsilon$ -isometry of  $\mathbb{G}$ . Furthermore, an easy back-and-forth argument shows that the Gurariĭ space is unique up to an  $\varepsilon$ -isometry for every  $\varepsilon > 0$ . The question of uniqueness of  $\mathbb{G}$  up to a linear isometry was open for some time, solved by Lusky [31] in 1976, using rather advanced methods. The first completely elementary proof of the isometric uniqueness of  $\mathbb{G}$  has been found very recently by Solecki and the author [28].

It turns out that our framework explains both the existence of  $\mathbb{G}$ , its isometric uniqueness and almost homogeneity with respect to isometries (already shown in [28]). Actually, a better understanding of the Gurariĭ space was one of the main motivations for our study.

Namely, as in Example 5.2, let  $\mathfrak{B}_0$  and  $\mathfrak{B}$  be the category of finite-dimensional Banach spaces with linear operators of norm  $\leq 1$  and with isometric embeddings,

respectively. For simplicity, we consider real Banach spaces, although the same arguments work for the complex ones. As mentioned before,  $\langle \mathfrak{B}, \mathfrak{B}_0 \rangle$  is a normed category.

It is well known that  $\mathfrak{B}$  has strict amalgamations. Obviously, it is directed. Let  $\mathcal{F}$  be the subcategory of  $\mathfrak{B}$  whose objects are Banach spaces of the form  $\langle \mathbb{R}^n, \|\cdot\| \rangle$ , where the norm is given by the formula

$$\|x\| = \max_{i < k} |\varphi_i(x)|$$

in which each  $\varphi_i$  is a functional satisfying  $\varphi_i[\mathbb{Q}^n] = \mathbb{Q}$ . Call such a space *rational*. An arrow of  $\mathcal{F}$  is a linear isometry  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  satisfying  $\varphi[\mathbb{Q}^n] \subseteq \mathbb{Q}^m$ . It is clear that  $\mathcal{F}$  is countable.

**Lemma 8.9.**  *$\mathcal{F}$  is dominating in  $\langle \mathfrak{B}, \mathfrak{B}_0 \rangle$ .*

*Proof.* It is rather clear that  $\mathcal{F}$  satisfies  $(D_1)$ . In order to see  $(D_2)$ , fix an isometric embedding  $f: X \rightarrow Y$ , where  $X$  is a rational Banach space. We may assume that  $X = \mathbb{R}^n$  and  $Y = X \oplus \mathbb{R}^k$  and  $f(x) = \langle x, 0 \rangle$  for  $x \in X$ . Given  $\varepsilon > 0$ , there exist functionals  $\varphi_0, \dots, \varphi_m$  on  $Y$  such that  $\|y\|_Y$  is  $\varepsilon/2$ -close to

$$\|y\|' = \max_{i < m} |\varphi_i(y)|.$$

We may assume that  $\|\varphi_i\| \leq 1$  for each  $i < m$  and that some of the  $\varphi_i$ s are extensions of the rational functionals defining the norm on  $X$ . We can now “correct” each non-rational  $\varphi_i$  so that it becomes rational and the respective norm is  $\varepsilon/2$ -close to  $\|\cdot\|'$ . Finally, the same map  $f$  becomes an isometric embedding  $\varepsilon$ -close to the original one, when  $Y$  is endowed with the new norm. This shows  $(D_2)$ .  $\square$

Thus,  $\langle \mathfrak{B}, \mathfrak{B}_0 \rangle$  is separable, therefore it has a Fraïssé sequence. Example 5.2 shows that  $\sigma(\mathfrak{B}, \mathfrak{B}_0)$  has a canonical co-limiting functor onto the category of separable Banach spaces. We still need to translate condition (G). By a chain of Banach spaces we mean a chain  $\{X_n\}_{n \in \omega}$ , where each  $X_n$  is a Banach space and the norm of  $X_{n+1}$  extends the norm of  $X_n$  for every  $n \in \omega$ .

**Lemma 8.10.** *Let  $\{G_n\}_{n \in \omega}$  be a chain of finite-dimensional Banach spaces with  $G_\infty = \overline{\bigcup_{n \in \omega} G_n}$ . If  $G_\infty$  satisfies (G) then  $\{G_n\}_{n \in \omega}$  is a Fraïssé sequence in  $\langle \mathfrak{B}, \mathfrak{B}_0 \rangle$ .*

*Proof.* We check that  $\vec{G} = \{G_n\}_{n \in \omega}$  satisfies (A). Fix  $\varepsilon > 0$  and an isometric embedding  $f: G_n \rightarrow Y$ , where  $Y$  is a finite-dimensional Banach space. Using condition (G) for the map  $f^{-1}: f[G_n] \rightarrow G_\infty$ , we find an  $\varepsilon$ -isometric embedding  $h: Y \rightarrow G_\infty$  such that  $h(f(x)) = x$  for every  $x \in G_n$ . Using the fact that  $Y$  is finite-dimensional, we can find  $m > n$  and an  $\varepsilon$ -isometric embedding  $h_1: Y \rightarrow G_m$  that is  $\varepsilon$ -close to  $h$ . Finally, define  $g := (1 + \varepsilon)^{-1}h_1$ . Then  $g: Y \rightarrow G_m$  is a  $\mathfrak{B}_0$ -arrow, because  $\|g\| \leq 1$ . Clearly,  $g$  is  $\varepsilon$ -close to  $h_1$ . Finally,  $g$  is  $2\varepsilon$ -close to  $h$ , therefore  $\|g(f(x)) - x\| = \|g(f(x)) - h(f(x))\| \leq 2\varepsilon\|x\|$  for  $x \in G_n$ . This shows (A).

Since  $\mathfrak{B}_0$  has the initial object  $\{0\}$ , condition (U) follows from (A).  $\square$

It turns out that the converse to the lemma above is also true, because of the uniqueness of the Fraïssé sequence, up to a linear isometry of its co-limit:

**Lemma 8.11.** *Let  $\vec{x}, \vec{y}$  be sequences in  $\mathfrak{B}$  and let  $\vec{f}: \vec{x} \rightarrow \vec{y}$  be an approximate 0-arrow in  $\sigma(\mathfrak{B}, \mathfrak{B}_0)$ . Let  $X$  and  $Y$  be the co-limits of  $\vec{x}, \vec{y}$  in the category of Banach spaces. Then  $\lim \vec{f}$  is an isometric embedding of  $X$  into  $Y$ .*

*Proof.* We may assume that  $\vec{x} = \{X_n\}_{n \in \omega}$ ,  $\vec{y} = \{Y_n\}_{n \in \omega}$  are chains of (finite-dimensional) Banach spaces and that  $f_n: X_n \rightarrow Y_n$  for each  $n \in \omega$ . Since  $\mu(f_n) \rightarrow 0$ , given  $\varepsilon > 0$  there is  $n_0$  such that

$$(1 - \varepsilon)\|x\| \leq \|f_n(x)\| \leq \|x\|$$

holds for every  $n \geq n_0$  and for every  $x \in X_n$ . Since  $\{f_n\}_{n \in \omega}$  is an approximate arrow, for each  $x \in X_k$ , the limit

$$f(x) = \lim_{n > k} f_n(x)$$

exists, because  $\{f_n(x)\}_{n > k}$  is a Cauchy sequence. Thus  $f(x)$  is defined on  $\bigcup_{n \in \omega} X_n$ , therefore it has a unique extension to a continuous linear operator from  $X$  to  $Y$  which is an  $\varepsilon$ -isometric embedding for every  $\varepsilon > 0$ . Thus, the extension of  $f$  is an isometric embedding.  $\square$

Thus, a Fraïssé sequence in  $\langle \mathfrak{B}, \mathfrak{B}_0 \rangle$  yields an isometrically unique separable Banach space  $\mathbb{G}$ , which must be the Gurariï space by Lemma 8.10. As we have mentioned before, an elementary proof of its isometric uniqueness [28] was one of the main inspirations for studying Fraïssé sequences in the context of metric-enriched categories.

*Remark 1.* The work [12] contains a construction of a universal linear operator on the Gurariï space. It is possible to describe it in the language of normed categories. Namely, the objects of the category  $\mathfrak{K}_0$  are linear operators  $T: X_0 \rightarrow X_1$ , where  $X_0, X_1$  are finite-dimensional Banach spaces and  $\|T\| \leq 1$ . An arrow from  $T$  to  $S$  is a pair  $\langle f_0, f_1 \rangle$  of linear operators of norm  $\leq 1$  satisfying  $S \circ f_0 = f_1 \circ T$ . Obviously,  $\mathfrak{K}$  should be the subcategory of all pairs of isometric embeddings. The main lemma in [12] says that  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  has the strict amalgamation property. The remaining issues are easily solved and the Fraïssé sequence in  $\langle \mathfrak{K}, \mathfrak{K}_0 \rangle$  leads to the universal (almost homogeneous) linear operator whose domain and range turn out to be isometric to the Gurariï space. The details can be found in [12], actually without referring to normed categories.

## 8.5 The pseudo-arc

We describe the universal chainable continuum, known under the name *pseudo-arc*, as the limit of a Fraïssé sequence in a suitable metric category. Actually, we shall work in a normed category of the form  $\langle \mathfrak{K}, \mathfrak{K} \rangle$ , so in particular  $\mu = 0$  and only the metric  $\varrho$  is relevant.

Recall that a *continuum* is a compact connected metrizable space. A continuum is *chainable* (also called *snake-like*) if it is homeomorphic to the limit of an inverse sequence of quotient maps of the unit interval. In particular, a chainable continuum maps onto the unit interval and therefore cannot be degenerate.

It turns out that we can restrict attention to piece-wise linear maps:

**Proposition 8.12.** *Every inverse sequence of quotient maps of the unit interval is equivalent to an inverse sequence of piece-wise linear quotient maps of the unit interval.*

*Proof.* Let  $\vec{f} = \{f_n^m\}_{n < m < \omega}$  be an inverse sequence of quotient maps of  $\mathbb{I}$ . We construct inductively piece-wise linear quotient maps  $g_n^{n+1}: \mathbb{I} \rightarrow \mathbb{I}$  such that

$$\varrho(f_n^{n+1}, g_n^{n+1}) < 2^{-n}/k_n$$

where

$$k_n = \text{Lip}(g_0^1) \cdot \text{Lip}(g_1^2) \cdot \dots \cdot \text{Lip}(g_{n-1}^n).$$

Note that every piece-wise linear map is Lipschitz, therefore  $k_n$  is well defined. We set  $g_i^j = g_i^{i+1} \circ \dots \circ g_{j-1}^j$ . Finally,  $\vec{g} = \{g_n^m\}_{n < m < \omega}$  is an inverse sequence, easily seen to be equivalent to  $\vec{f}$ .  $\square$

The following fact can be proved easily, using the linear structure of the unit interval:

**Lemma 8.13.** *Let  $\vec{q} = \{q_n^m\}_{n < m < \omega}$  be an inverse sequence of quotient maps of the unit interval with  $K = \varprojlim \vec{q}$  in the category of compact spaces. Denote by  $q_n: K \rightarrow \mathbb{I}$  the canonical projection onto the  $n$ th element of the sequence. Given a quotient map  $f: K \rightarrow \mathbb{I}$ , for every  $\varepsilon > 0$  there exist  $n \in \omega$  and a piece-wise linear quotient map  $g: \mathbb{I} \rightarrow \mathbb{I}$  such that*

$$\varrho(g \circ q_n, f) < \varepsilon,$$

where  $\varrho$  is the maximum metric on the space of continuous functions.

Let  $\mathfrak{J}$  be the opposite category of non-expansive piece-wise linear quotient mappings of the form

$$f: \langle \mathbb{I}, d_0 \rangle \rightarrow \langle \mathbb{I}, d_1 \rangle,$$

where  $\mathbb{I}$  is the unit interval and  $d_i(s, t) = m_i|s - t|$  for some integer constant  $m_i > 0$ . Let  $\mathfrak{J}_0 = \mathfrak{J}$ , that is, we shall really work in a single category  $\mathfrak{J}$  and all arrows, including approximate arrows of sequences are 0-arrows. In other words, the objects of  $\mathfrak{J}$  are pairs of the form  $\langle \mathbb{I}, d \rangle$ , where  $d$  is the usual metric multiplied by a positive integer constant, needed only for making the maps 1-Lipschitz. In particular, there are only countably many objects.

We could have defined  $\mathfrak{J}$  equivalently by saying that its objects are intervals of the form  $[0, n]$  with  $n \in \mathbb{N}$  endowed with the usual metric, and arrows are 1-Lipschitz quotient maps.

It is clear that  $\mathfrak{J}$  is a metric-enriched category, with the same metric  $\rho$  as in the case of all nonempty compact metric spaces. It is also clear that  $\mathfrak{J}$  is directed. The almost amalgamation property follows from the following well known result, sometimes called the *uniformization principle*:

**Theorem 8.14** (Mountain Climbing Theorem). *Let  $f, g: \mathbb{I} \rightarrow \mathbb{I}$  be quotient maps that are piece-wise monotone and satisfy  $f(i) = i = g(i)$  for  $i = 0, 1$ . Then there exist quotient maps  $f', g': \mathbb{I} \rightarrow \mathbb{I}$  such that  $f \circ f' = g \circ g'$ .*

The result above goes back to Homma [14]; the formulation (actually involving finitely many piece-wise monotone quotient maps) is due to Sikorski & Zarankiewicz [41]. Another version, involving two functions that are constant on no open subintervals of  $\mathbb{I}$  is due to Huneke [15].

The Mountain Climbing Theorem is usually stated for functions  $f, g$  satisfying  $f(0) = 0 = g(0)$  and  $f(1) = 1 = g(1)$ , whose graphs can therefore be interpreted as two slopes of the same mountain. The functions  $f', g'$  can be interpreted as the existence of two “mountain climbings” on these slopes with the property that at each moment of time the travelers have the same altitude (sometimes one of the travelers has to go backwards). This justifies the name of the theorem. It is rather clear that given a quotient map  $f: \mathbb{I} \rightarrow \mathbb{I}$ , there exists a piece-wise linear quotient map  $f_1: \mathbb{I} \rightarrow \mathbb{I}$  such that  $f_1(f(i)) = i$  for  $i = 0, 1$ . As a corollary, we get:

**Proposition 8.15.** *The category  $\mathfrak{J}$  has the strict amalgamation property.*

Let us note that the Mountain Climbing Theorem fails for arbitrary quotient maps of the unit interval; an example was first found by Minagawa (quoted in Homma [14]) and independently by Sikorski & Zarankiewicz [41].

As one can easily guess,  $\mathfrak{J}$  is separable. A natural countable dominating subcategory is described below.

We say that a quotient map  $f: \mathbb{I} \rightarrow \mathbb{I}$  is *rational* if  $f(0) = 0$ ,  $f(1) = 1$ , and there is a decomposition  $0 = t_0 < t_1 < \dots < t_n = 1$  such that  $\{t_i\}_{i < n} \subseteq \mathbb{Q}$ ,  $f \upharpoonright [t_i, t_{i+1}]$  is linear for each  $i < n$  and  $\{f(t_i)\}_{i < n} \subseteq \mathbb{Q}$ . Finally, define  $\mathcal{F} \subseteq \mathfrak{J}$  to be the category of all rational quotient maps. The following fact is rather obvious.

**Proposition 8.16.** *The category  $\mathcal{F}$  is countable and dominating in  $\mathfrak{J}$ .*

It is clear that the category  $\mathfrak{J}$  has a canonical “limiting” functor, which assigns the inverse limit to a sequence. From now on, let  $\vec{u}$  be a Fraïssé sequence in  $\sigma\mathfrak{J}$  and let

$$P = \varprojlim \vec{u}$$

in the category of compact spaces. It turns out that  $P$  is the pseudo-arc. We explain the details below.

First of all, recall that formally the *pseudo-arc* is defined to be a hereditarily indecomposable chainable continuum, which by a result of Bing [5] is known to be unique. Recall that  $K$  is *indecomposable* if it cannot be written as  $A \cup B$  where

$A, B$  are proper subcontinua. A continuum  $K$  is *hereditarily indecomposable* if every subcontinuum of  $K$  is indecomposable.

**Lemma 8.17.** *Let  $\vec{v} = \{v_n^m\}_{n < m < \omega}$  be an inverse sequence of quotient maps of the unit interval. Then  $\varprojlim \vec{v}$  is homeomorphic to  $P$  if and only if  $\vec{v}$  satisfies the following condition:*

( $\heartsuit$ ) *Given  $n \in \omega$ ,  $\varepsilon > 0$ , given a quotient map  $f: \mathbb{I} \rightarrow \mathbb{I}$ , there exist  $m > n$  and a quotient map  $g: \mathbb{I} \rightarrow \mathbb{I}$  such that  $\varrho(f \circ g, v_n^m) < \varepsilon$ .*

Note that in fact condition ( $\heartsuit$ ) does not depend on the metric on  $\mathbb{I}$ , since all metrics on a compact space are uniformly equivalent.

*Proof.* First, by Proposition 8.12, we may assume that all maps  $v_n^m$  are piece-wise linear. Next, by an easy induction, we can “convert”  $\vec{v}$  to a sequence in  $\mathfrak{J}$ . It is clear that ( $\heartsuit$ ) is preserved under the equivalence of sequences, therefore the “corrected” sequence still satisfies ( $\heartsuit$ ). Now it is obvious that ( $\heartsuit$ ) is equivalent to condition (A) of the Fraïssé sequence, since the map  $f$  can be approximated by a piece-wise linear quotient map which in turn can be made 1-Lipschitz by multiplying the metric of  $\mathbb{I}$  by a large enough constant.

Thus, if  $\vec{v}$  satisfies ( $\heartsuit$ ) then it is equivalent to a Fraïssé sequence which is uniquely determined, showing that  $\varprojlim \vec{v} \approx P$ . Finally, if  $\varprojlim \vec{v} \approx P$ , then  $\vec{v}$  is equivalent to  $\vec{u}$ , therefore it satisfies ( $\heartsuit$ ).  $\square$

Lemma 8.17 allows us to work in the monoidal category of quotient maps of the unit interval, endowed with the standard metric. This is formally not a metric-enriched category, because the composition operator is not 1-Lipschitz, but in practice this does not cause any trouble.

**Lemma 8.18.** *Every non-degenerate subcontinuum of  $P$  is homeomorphic to  $P$ .*

*Proof.* Let  $K$  be a subcontinuum of  $P$  and let  $K_n = u_n[K]$ , where  $u_n: P \rightarrow \mathbb{I}$  is the canonical  $n$ -th projection. From some point on,  $K_n$  is a non-degenerate interval. Without loss of generality, we may assume that this is the case for all  $n \in \omega$ . Given  $n < m$ , let  $v_n^m: K_m \rightarrow K_n$  be the restriction of  $u_n^m$ . Then  $\vec{v} = \{v_n^m\}_{n < m < \omega}$  is a sequence in  $\mathfrak{J}$ , an inverse sequence of piece-wise linear quotient maps of closed intervals. Furthermore,  $K = \varprojlim \vec{v}$ . It suffices to check that  $\vec{v}$  is a Fraïssé sequence in  $\mathfrak{J}$ .

Fix  $n \in \omega$ ,  $\varepsilon > 0$  and fix a piece-wise linear quotient map  $f: \mathbb{I} \rightarrow F_n$ . Assume  $F_n = [a, b]$ , where  $0 \leq a < b \leq 1$ . Composing  $f$  with a suitable quotient map, we may assume that  $f(0) = a$  and  $f(1) = b$ . Extend  $f$  to a piece-wise linear map  $f': [-1, 2] \rightarrow \mathbb{I}$  in such a way that  $f'[-1, 0] = [0, a]$  and  $f'[1, 2] = [b, 1]$ . We can treat  $[-1, 2]$  as the unit interval with multiplied metric. Thus, using the fact that  $\vec{u}$  is Fraïssé, we find  $m > n$  and a piece-wise linear quotient map  $g: \mathbb{I} \rightarrow [-1, 2]$  such that  $\varrho(f' \circ g, u_n^m) < \varepsilon$ . Note that  $g[F_m]$  is  $\varepsilon$ -close to  $[a, b]$ , therefore we can “correct”  $g$  so that  $g[F_m] = [a, b]$ , replacing  $\varepsilon$  by  $3\varepsilon$ . Finally,  $g \upharpoonright F_m$  witnesses that  $\vec{v}$  satisfies condition (A) of the definition of a Fraïssé sequence.  $\square$

**Lemma 8.19.**  *$P$  is hereditarily indecomposable.*

*Proof.* In view of Lemma 8.18, it suffices to show that  $P$  is indecomposable. For this aim, suppose  $P = A \cup B$ , where  $A, B$  are proper subcontinua of  $P$ . Let  $A_n = u_n[A]$ ,  $B_n = u_n[B]$ , where as before,  $u_n$  is the canonical  $n$ th projection. Fix  $n \in \omega$  such that both  $A_n$  and  $B_n$  are non-degenerate proper intervals. Without loss of generality, we may assume that  $A_n = [0, a]$ ,  $B_n = [b, 1]$ , where  $0 < b \leq a < 1$ . Let  $f: \mathbb{I} \rightarrow \mathbb{I}$  be the tent map, that is,  $f(t) = 2t$  for  $t \in [0, \frac{1}{2}]$  and  $f(t) = \frac{1}{2} - 2t$  for  $t \in [\frac{1}{2}, 1]$ . Then  $f^{-1}[A_n] = [0, s] \cup [t, 1]$ , where  $s = \frac{a}{2}$  and  $t = 1 - \frac{a}{2} > s$ . Furthermore,  $f$  is 2-Lipschitz with respect to the standard metric, therefore multiplying the metric in the domain of  $f$  by 2 we obtain a 1-Lipschitz piece-wise linear quotient map. Fix a small enough  $\varepsilon > 0$ . Using property (A) of the Fraïssé sequence, we find  $m > n$  and a quotient map  $g: \mathbb{I} \rightarrow \mathbb{I}$  such that  $\varrho(f \circ g, u_n^m) < \varepsilon$ .

Notice that  $J = g[A_m]$  is an interval, therefore if  $\varepsilon$  is small enough then either  $J \cap [0, s] = \emptyset$  or  $J \cap [t, 1] = \emptyset$ . This means that either  $[0, s] \subseteq g[B_m]$  or  $[t, 1] \subseteq g[B_m]$ . In particular, there is  $r \in B_m$  such that  $g(r) \in \{0, 1\}$ . On the other hand,  $d(f(g(r)), u_n^m(r)) = d(0, u_n^m(r)) < \varepsilon$ , where  $d$  is the metric in the  $n$ th interval of the sequence  $\vec{u}$ . Notice that  $u_n^m(r) \in u_n^m[B_m] = B_n$ . Thus, if  $\varepsilon < d(0, b)$  then we get a contradiction.  $\square$

The two lemmas above together with Bing's uniqueness result [5] give

**Corollary 8.20.**  *$P$  is the pseudo-arc.*

Applying Theorem 7.4, we obtain another proof of the result of Mioduszewski [34]:

**Theorem 8.21.** *Every chainable continuum is a continuous image of the pseudo-arc.*

Almost homogeneity can be easily strengthened, obtaining the result of Irwin & Solecki [16] which in turn improves a result of Lewis (sketched after Thm. 4.2 in [30]):

**Theorem 8.22.** *Let  $K$  be a chainable continuum with some fixed metric and let  $p, q: P \rightarrow K$  be quotient maps. Then for each  $\varepsilon > 0$  there exists a homeomorphism  $h: P \rightarrow P$  such that  $\varrho(q \circ h, p) < \varepsilon$ .*

*Proof.* Using the fact that  $K$  is the inverse limit of unit intervals, there is a quotient map  $f: K \rightarrow \mathbb{I}$  such that all  $f$ -fibers have diameter  $< \varepsilon$ . A standard compactness argument shows that  $f$  satisfies the following condition:

$$(\star) \quad (\forall s, t \in K) \quad |f(s) - f(t)| < \delta \implies d(s, t) < \varepsilon.$$

Now let  $p' = f \circ p$  and  $q' = f \circ q$ . By Lemma 8.13, both  $p', q'$  come from approximate arrows, therefore by Theorem 7.3, there is a homeomorphism  $h: P \rightarrow P$  such that

$\varrho(q' \circ h, p') < \delta$ . We have the following diagram, in which the upper triangle is  $\delta$ -commutative and the side-triangles are commutative.

$$\begin{array}{ccccc}
 & & P & \xrightarrow{h} & P \\
 & & \downarrow p & \searrow p' & \swarrow q' \\
 & & K & \xrightarrow{f} & \mathbb{I} & \xleftarrow{f} & K \\
 & & & & & & \downarrow q
 \end{array}$$

Finally, condition  $(\star)$  gives  $\varrho(q \circ h, p) < \varepsilon$ . □

As one can guess, the property in Theorem 8.22 characterizes the pseudo-arc among chainable continua. This has already been proved by Irwin & Solecki [16]. Our results actually provide a new, even simpler, characterization of the pseudo-arc:

**Theorem 8.23.** *A chainable continuum  $K$  is homeomorphic to  $P$  if and only if it satisfies the following condition:*

- (P) *Given  $\varepsilon > 0$ , given quotient maps  $q: K \rightarrow \mathbb{I}$ ,  $f: \mathbb{I} \rightarrow \mathbb{I}$ , there exists a quotient map  $g: K \rightarrow \mathbb{I}$  such that  $\varrho(f \circ g, q) < \varepsilon$ .*

*Proof.* By Theorem 8.22,  $P$  satisfies condition (P). Now suppose that  $K$  satisfies (P) and choose a sequence  $\vec{v}$  in  $\mathfrak{J}$  whose inverse limit is  $K$ . We shall check that  $\vec{v}$  is a Fraïssé sequence. In fact, only condition (A) requires a proof.

Fix  $n \in \omega$ ,  $\varepsilon > 0$  and fix a piece-wise linear quotient map  $f: \mathbb{I} \rightarrow \mathbb{I}$ . Let, as usual,  $v_n: K \rightarrow \mathbb{I}$  be the  $n$ th canonical projection. Using (P), we find a quotient map  $p: K \rightarrow \mathbb{I}$  such that  $\varrho(f \circ p, v_n) < \varepsilon/2$ . By Lemma 8.13, there exist  $m > n$  and a piece-wise linear quotient map  $g: \mathbb{I} \rightarrow \mathbb{I}$  such that  $\varrho(g \circ v_m, p) < \varepsilon/2$ . Thus we get

$$\varrho(f \circ g \circ v_m, v_n^m \circ v_m) \leq \varrho(f \circ g \circ v_m, f \circ p) + \varrho(f \circ p, v_n) < \varepsilon$$

and so  $\varrho(f \circ g, v_n^m) < \varepsilon$ , because  $v_m$  is a quotient map. This shows (A) and completes the proof. □

As noticed at the beginning of the proof above, condition (P) easily follows from Theorem 8.22. A direct proof of the converse implication would require the approximate back-and-forth argument, which is hidden in the proof of Lemma 7.1 above.

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