Entropy methods for linear discrete and continuous structured population models

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Lecture notes

The lectures are devoted to the methods of generalised relative entropy applied to study long-term behaviour of structured population models. Such models describe processes of growth and evolution of a population, which is heterogenous in respect to some physiological properties, such as age, size or stage of cell differentiation, which in turn influence dynamics of the individuals. Analysis of the asymptotic behaviour of such models is an interesting issue even in linear cases, in which we can observe a convergence of solutions to the solutions with a fixed distribution in respect to the structure variable. The method of relative entropy, used before to study equations of mathematical physics, has proven to be a very useful tool to investigate the convergence of rescaled in time solutions to steady states. The idea consists in using qualitative arguments such as strictly convex entropies to provide convergence. We consider models with discrete structure given by systems of ordinary differential equations and models with continuous structure given by transport equations with nonlocal boundary conditions.

Main references:

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- P. Michel, S. Mischler, B. Perthame, General relative entropy inequality: an illustration on growth models, J.Math. Pures Appl. 84, 2005.
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1 Discrete structures: evolution equations with positive matrices

For simplicity we restrict our analysis to matrices with positive coefficients, but it is possible to derive similar results for some systems with nonnegative coefficients.

Let $A \in M_{d\times d}(\mathbb{R}^+)$ a matrix with positive coefficients $a_{ij} > 0$. We are interested in the properties of solutions to linear evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t} n = An \qquad n(0) = n^0. \tag{1}$$

1.1 Eigenvalues and eigenvectors

We know from the PERRON-FROBENIUS-THEOREM that the largest eigenvalue λ_0 of a positive matrix is a simple eigenvalue whose left- and right eigenvectors are positive, namely

$$\begin{aligned} A \cdot N &= \lambda_0 N \qquad N_i > 0 \quad \forall i = 1, \ldots, d, \\ \phi \cdot A &= \lambda_0 \phi \qquad \phi_i > 0 \quad \forall i = 1, \ldots, d. \end{aligned}$$

We can make them unique by the following normalization

$$\sum_{i=1}^{d} N_i = 1 \qquad \sum_{i=1}^{d} N_i \phi_i = 1.$$

Taking the scalar product $\langle\ ,\rangle$ of (1) with ϕ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \phi, n\rangle = \langle \phi, \frac{\mathrm{d}}{\mathrm{d}t} n\rangle = \langle \phi, An\rangle = \langle \phi A, n\rangle = \lambda_0 \langle \phi, n\rangle$$

So $\langle \phi, n(t)\rangle = \langle \phi, n_0\rangle e^{\lambda_0 t}$, which means that we can interpret ϕ as a good weight to analyse the solutions of equation (1).

In the following we define $\tilde{A} := A - \lambda_0 E$, so that $\tilde{A}N = 0$ and $\phi\tilde{A} = 0$ and investigate the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{n} = \tilde{A}\tilde{n} \qquad \tilde{n}(0) = n^0. \tag{2}$$

Obviously it holds $\tilde{n} = ne^{-\lambda_0 t}$ and the following proposition can be deduced.

Proposition 1.1. *For positive matrices and the solutions of the differential equation* (2) *it holds*

a conservation law

$$m^0 := \sum_{i=1}^{d} \phi_i n_i^0 = \sum_{i=1}^{d} \phi_i \tilde{n}_i(t)$$

and a contraction principle

$$\sum_{i=1}^{d} \phi_i |\tilde{n}_i(t)| \leq \sum_{i=1}^{d} \phi_i |n_i^0|.$$

1.2 General relative entropy

Definition: Let H be a realvalued convex function, then General Relative Entropy (abreviated GRE) is the quantity $\sum_{i=1}^{d} \phi_i N_i H\left(\frac{\tilde{n}_i(t)}{N_i}\right)$.

Proposition 1.2. *Let $H(.) \in C^1(\mathbb{R})$ be a realvalued convex function, then $\tilde{n}(t)$ satisfies:*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{d} \phi_i N_i H\left(\frac{\tilde{n}_i(t)}{N_i}\right) \\
&\quad = \sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left[H'\left(\frac{\tilde{n}_i(t)}{N_i}\right) [\frac{\tilde{n}_j(t)}{N_j} - \frac{\tilde{n}_i(t)}{N_i}] - H\left(\frac{\tilde{n}_j(t)}{N_j}\right) + H\left(\frac{\tilde{n}_i(t)}{N_i}\right) \right] \\
&\quad \leq 0.
\end{aligned}
$$

Proof:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{d} \phi_i N_i H\left(\frac{\tilde{n}_i(t)}{N_i}\right) \\
&\quad = \sum_{i=1}^{d} \phi_i N_i H'\left(\frac{\tilde{n}_i(t)}{N_i}\right) \frac{1}{N_i} \sum_{j=1}^{d} \tilde{a}_{ij} \tilde{n}_j(t) \\
&\quad = \sum_{i,j=1}^{d} \phi_i \tilde{a}_{ij} H'\left(\frac{\tilde{n}_i(t)}{N_i}\right) \frac{N_j}{N_j} \tilde{n}_j(t) - \sum_{i=1}^{d} \phi_i \frac{\tilde{n}_i(t)}{N_i} H'\left(\frac{\tilde{n}_i(t)}{N_i}\right) \sum_{j=1}^{d} \tilde{a}_{ij} N_j \\
&\quad = \sum_{i,j=1}^{d} \phi_i \tilde{a}_{ij} N_j H'\left(\frac{\tilde{n}_i(t)}{N_i}\right) \left[\frac{\tilde{n}_j(t)}{N_j} - \frac{\tilde{n}_i(t)}{N_i}\right] \\
&\qquad - \sum_{j=1}^{d} N_j H\left(\frac{\tilde{n}_j(t)}{N_j}\right) \sum_{i=1}^{d} \phi_i \tilde{a}_{ij} + \sum_{i=1}^{d} \phi_i H\left(\frac{\tilde{n}_i(t)}{N_i}\right) \sum_{j=1}^{d} \tilde{a}_{ij} N_j \\
&\quad = \sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left[H'\left(\frac{\tilde{n}_i(t)}{N_i}\right) \left[\frac{\tilde{n}_j(t)}{N_j} - \frac{\tilde{n}_i(t)}{N_i}\right] - H\left(\frac{\tilde{n}_j(t)}{N_j}\right) + H\left(\frac{\tilde{n}_i(t)}{N_i}\right) \right] \\
&\quad \leq 0.
\end{aligned}
$$

We used mainly the eigenvalue equations $\tilde{A}N = 0$ and $\phi\tilde{A} = 0$ to add terms, which are in fact zero. Further by definition it holds that $\tilde{a}_{ij} = a_{ij}$ for $i \neq j$, while the summands containing $\tilde{a}_{ii}$ are zero. Using the convexity of H we obtain that the general relative entropy is decaying.

Remark: The general relative entropy is decaying also for a general convex function. A convex function is locally Lipschitz continuous and therefore absolute continuous. Therefore, the derivative exists almost everywhere and it is integrable and we obtain for all $x_1, x_2 \in \mathbb{R}$

$$H(x_2) - H(x_1) = \int_{x_1}^{x_2} H'(\tilde{x}) d\tilde{x}$$

Therefore, for $t_1 \leq t_2$ it holds

$$H\left(\frac{\tilde{n}_i(t_2)}{N_i}\right) - H\left(\frac{\tilde{n}_i(t_1)}{N_i}\right) = \int_{t_1}^{t_2} H'\left(\frac{\tilde{n}_i(t)}{N_i}\right) \frac{\mathrm{d}}{\mathrm{d}t}\frac{\tilde{n}_i(t)}{N_i} dt$$

Now we can multiply this equation by ϕ_i and N_i and sum over all i. Furthermore, we can manipulate the right hand side in the same way as in Proposition 1.2 and obtain that the general relative entropy is decaying.

Remark: Now we can prove Proposion 1.1 by applying the GRE property. The choice of convex functions $H(u) = u$, resp. $H(u) = -u$ lead together to $\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^d \phi_i \tilde{n}_i(t) = 0$. Similarly, the convex function $H(u) = |u|$ leads to $\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^d \phi_i |\tilde{n}_i(t)| \leq 0$.

1.3 Asymptotic behaviour

Next, we use the entropy to show maximum principle and the exponential decay towards the steady state solution.

Proposition 1.3. *Let c, C constants given by $cN_i \leq n_i^0 \leq CN_i$, then it holds*

$$cN_i \leq \tilde{n}_i(t) \leq CN_i.$$

Furthermore, there is a constant $\alpha > 0$ such that

$$\sum_{i=1}^d \phi_i N_i \left(\frac{\tilde{n}_i(t) - m^0 N_i}{N_i}\right)^2 \leq \sum_{i=1}^d \phi_i N_i \left(\frac{n_i^0 - m^0 N_i}{N_i}\right)^2 e^{-\alpha t},$$

where m^0 is defined in Proposition 1.1.

Proof: Using the positive convex function $H(u) = (u - C)_+$, we see that $\sum_{i=1}^d \phi_i N_i H(\frac{n_i^0}{N_i}) = 0$, but as the general relative entropy is nonnegative and decaying, it remains zero for all times

$$\sum_{i=1}^d \phi_i N_i H(\frac{\tilde{n}_i(t)}{N_i}) = 0.$$

Because of the positivity of ϕ and N we find $H(\frac{\tilde{n}_i(t)}{N_i}) = 0$, which leads to the upper bound. In a similar fashion we find the lower bound using $H(u) = (c - u)_+$.
To show the second statement of Proposition 1.3, we define (remember that $m^0 = \sum_{i=1}^d \phi_i n_i^0$)

$$h(t) = \tilde{n}(t) - m^0 N.$$

Then $h(t)$ fulfills the evolution equation (2) because of $\frac{\mathrm{d}}{\mathrm{d}t}h(t) = \tilde{A}\tilde{n}(t) - m^0\tilde{A}N = \tilde{A}h(t)$ and furthermore $h(t)$ verifies

$$\sum_{i=1}^{d} \phi_i h_i(t) = \sum_{i=1}^{d} \phi_i \tilde{n}_i(t) - m^0 \sum_{i=1}^{d} \phi_i N_i = 0,$$

because of the conservation law. Using the quadratic entropy function $H(u) = u^2$ and Proposition 1.2, we find that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{d} \phi_i N_i \left(\frac{h_i(t)}{N_i}\right)^2 \\
&= \sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left[2\left(\frac{h_i(t)}{N_i}\right)[\frac{h_j(t)}{N_j} - \frac{h_i(t)}{N_i}] - \left(\frac{h_j(t)}{N_j}\right)^2 + \left(\frac{h_i(t)}{N_i}\right)^2\right] \\
&= -\sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{h_j(t)}{N_j} - \frac{h_i(t)}{N_i}\right)^2.
\end{aligned}$$

Now we will use the discrete Poincare inequality, which reads

Lemma 1.4. *Let $\phi, N > 0$ positive vectors, $a_{ij} > 0$ for all $i, j = 1, \ldots, d$ with $i \neq j$. Then, there is a constant $\alpha > 0$ such that it holds*

$$\sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{m_j}{N_j} - \frac{m_i}{N_i}\right)^2 \geq \alpha \sum_{i}^{d} \phi_i N_i \left(\frac{m_i}{N_i}\right)^2,$$

for all vectors fulfilling $\langle \phi, m \rangle = 0$.

The proof of this inequality is deferred to the Appendix.
Using the Poincare inequality, we estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{d} \phi_i N_i \left(\frac{h_i(t)}{N_i}\right)^2 \leq -\alpha \sum_{i=1}^{d} \phi_i N_i \left(\frac{h_i(t)}{N_i}\right)^2$$

and we can finish the proof by the use of Gronwall Lemma.

2 Linear models with a continuous age structure - renewal equation

Now we will consider a model of dynamics of a population structured by a continuous parameter. For simplicity, we consider a closed population without migration or death. We assume that the population is aging and growing due to some reproduction.

Let $n(t,x)$ be a population density of individuals of age $x > 0$ at time $t \in (0,\infty)$. Because of the process of aging, it holds

$$n(t+s, x+s) = n(t,x), \qquad \forall s \geq 0.$$

In consequence, differentiating in s and setting $s = 0$, we find

$$\frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) = 0 \quad \forall t \geq 0, \forall x \geq 0.$$

This equation is completed by a boundary condition at $x = 0$. The number of new-borns at time t is given by

$$n(t, x=0) = \int_0^\infty B(y) n(t,y) dy,$$

where $B(x)$ is an age-depending birth rate of the population. Furthermore, we need an initial age distribution of the population at time $t = 0$

$$n(t=0, x) = n^0(x).$$

We assume on B that

$$B(.) \geq 0, \quad B \in L^\infty(\mathbb{R}^+), \quad 1 < \int_0^\infty B(x) dx < \infty. \tag{3}$$

The last assumption is necessary to ensure an expanding population.

Altogether, we study the behavior of the following equation

$$\begin{aligned} &\frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) = 0 \quad \forall t \geq 0, \forall x \geq 0, \\ &n(t, x=0) = \int_0^\infty B(y) n(t,y) dy, \\ &n(t=0, x) = n^0(x). \end{aligned} \tag{4}$$

This is can be written as an evolution equation

$$\frac{\partial}{\partial t} n = An \quad n(t=0,x) = n^0(x)$$

with the operator $An = -\frac{\partial}{\partial x} n$ defined on the space

$$X = \left\{ n(t,x) \in \mathcal{D}'\left((0,\infty) \times (0,\infty)\right) \mid n(t,0) = \int_0^\infty B(y) n(t,y) dy \right\}.$$

2.1 Properties of the eigenvalue problem

Similar to the finite dimensional case, it is convenient to consider the eigenvalue problem of the operator A and its dual A^* (which corresponds to the left eigenvector). We are interested in solutions of the form $n(t,x) = N(x)e^{\lambda_0 t}$, where λ is the largest eigenvalue of A and $N(x)$ is the corresponding eigenfunction. We shall show that such positive eigenvalue exists.
To identify a proper weight function (to find a space in which the model has a conservative property), we take a a function $\varphi(x)$ be (smooth enough and fast decaying at infinity), and calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty n(t,x)e^{-\lambda_0 t}\varphi(x)dx$$
$$= e^{-\lambda_0 t}\int_0^\infty n(t,x)\left(\frac{\partial}{\partial x}\varphi(x) + B(x)\varphi(0) - \lambda_0\varphi(x)\right)dx$$

using integration by parts and the initial condition. Consequently, choosing the dual eigenfunction as the weight we observe that the integral above is an conserved quantity.

Now we formulate the EIGENVALUE PROBLEM: we search for $(\lambda_0, N(x), \phi(x))$ with λ_0 maximal such that $N(x)$ is the normalized positive eigenfunction of the operator A

$$\begin{aligned}
&\frac{\partial}{\partial x}N(x) + \lambda_0 N(x) = 0 \qquad \forall x \geq 0,\\
&N(0) = \int_0^\infty B(y)N(y)dy,\\
&N(.) > 0,\\
&\int_0^\infty N(y)dy = 1
\end{aligned} \tag{5}$$

and $\phi(x)$ is the normalized positive eigenfunction of the dual operator A^*

$$\begin{aligned}
&-\frac{\partial}{\partial x}\phi + \lambda_0\phi = B(x)\phi(0) \qquad \forall x \geq 0,\\
&\phi(.) > 0,\\
&\int_0^\infty N(y)\phi(y)dy = 1.
\end{aligned} \tag{6}$$

The value λ_0, which is the maximal eigenvalue of the operator is called MALTHUS PARAMETER.

Lemma 2.1. *Under the assumptions* (3) *there is a unique solution* $(\lambda_0, N(x), \phi(x))$ *to the* EIGENPROBLEM *and we have the estimation*

$$\phi(x) \leq ||B||_{L^\infty}\left(\lambda_0^2\int_0^\infty yB(y)e^{-\lambda_0 y}dy\right)^{-1}.$$

Proof: The function $N(x) = \lambda_0 e^{-\lambda_0 x}$ fulfills the differential equation for N and also the normalization. Then, it is enough to show that there exists a unique λ_0, which fulfills the boundary conditions that means

$$1 = \int_0^\infty B(x) e^{-\lambda_0 x} dx. \tag{7}$$

We observe that $x \mapsto B(x)e^{-\lambda x}$ is integrable for all $\lambda > 0$ and $\lambda \mapsto B(x)e^{-\lambda x}$ is continuous and monotone decreasing. Thus, $\lambda \mapsto \int_0^\infty B(x)e^{-\lambda x} dx$ is continuous and the theorem of monotone convergence yields

$$\begin{aligned} &\lim_{\lambda \to 0^+} \int_0^\infty B(x)e^{-\lambda x} dx = \int_0^\infty B(x) dx > 1 \quad \text{and} \\ &\lim_{\lambda \to \infty} \int_0^\infty B(x)e^{-\lambda x} dx = 0. \end{aligned}$$

Consequently, there exists a unique λ_0 fulfilling equation (7). To show the existence and uniqueness of ϕ, we consider $Q(x) := \frac{\phi(x)N(x)}{\phi(0)N(0)}$. This is a solution of the differential equation

$$\frac{\partial}{\partial x} Q(x) = -\frac{B(x)N(x)}{N(0)} = -B(x)e^{-\lambda_0 x} \quad Q(0) = 1,$$

so using the boundary condition for $N(x)$ (equation (7)) we can write

$$Q(x) = 1 - \int_0^x B(y)e^{-\lambda_0 y} dy = \int_x^\infty B(y)e^{-\lambda_0 y} dy.$$

Obviously we obtain $0 \le Q(x) \le 1$. Furthermore we can calculate the integral

$$\begin{aligned} \int_0^\infty Q(x) dx &= \int_{(x,y) \in [0,\infty] \times [0,\infty], y \ge x} B(y)e^{-\lambda_0 y} dy dx \\ &= \int_0^\infty \int_0^y B(y)e^{-\lambda_0 y} dx dy \\ &= \int_0^\infty y B(y) e^{-\lambda_0 y} dy < \infty. \end{aligned}$$

Choosing $\phi(0) = \lambda_0^{-1} \left(\int_0^\infty y B(y)^{-\lambda_0 y} dy \right)$, we obtain

$$\frac{1}{\phi(0)N(0)} = \int_0^\infty Q(x) dx,$$

which ist equivalent to the desired normalization

$$\int_0^\infty \phi(x) N(x) dx = 1.$$

Finally we can estimate

$$Q(x) \leq ||B||_{L^\infty} \int_x^\infty e^{-\lambda_0 y} dy = \frac{||B||_{L^\infty}}{\lambda_0} e^{-\lambda_0 x}$$

what yields

$$\phi(x) \leq \phi(0) \frac{||B||_{L^\infty}}{\lambda_0} = ||B||_{L^\infty} \left(\lambda_0^2 \int_0^\infty y B(y) e^{-\lambda_0 y} dy \right)^{-1}.$$

Remark: When B has a compact support, then ϕ is has a compact support as well and $\mathrm{supp}(\phi) = \mathrm{conv}(0, \mathrm{supp}(B))$.

2.2 Existence theory

Knowing the Malthus parameter λ_0, we can formulate a rescaled renewal equation,

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{n}(t,x) + \frac{\partial}{\partial x}\tilde{n}(t,x) + \lambda_0 \tilde{n}(t,x) = 0 \qquad \forall t \geq 0, \forall x \geq 0, \\ &\tilde{n}(t,0) = \int_0^\infty B(y)\tilde{n}(t,x)dy, \\ &\tilde{n}(0,x) = n^0(x). \end{aligned} \tag{8}$$

It holds $\tilde{n}(t,x)e^{\lambda_0 t} = n(t,x)$.

Theorem 2.2. *Under the assumptions* (3) *and for initial conditions satisfying*

$$\exists C_0 \quad \textit{with} \quad |n^0(x)| \leq C_0 N(x),$$

there is an unique solution of the rescaled renewal equation (8) *in the distribution sense* $\tilde{n} \in C(\mathbb{R}^+, L^1(\mathbb{R}^+; \phi(x)dx))$ *and it holds:*

1. *the maximum principle*

$$|\tilde{n}(t,x)| \leq C_0 N(x) \quad \forall t \geq 0,$$

2. *the comparison principle*

$$n_1^0(x) \leq n_2^0(x) \quad \Rightarrow \quad \tilde{n}_1(t,x) \leq \tilde{n}_2(t,x),$$

3. *the conservation law and the* $L^1(\mathbb{R}^+, \phi(x)dx)$ *contraction principle*

$$\begin{aligned} \int_0^\infty \tilde{n}(t,x)\phi(x)dx &= \int_0^\infty n^0(x)\phi(x)dx \\ \int_0^\infty |\tilde{n}(t,x)|\phi(x)dx &\leq \int_0^\infty |n^0(x)|\phi(x)dx. \end{aligned}$$

Proof: The proof is divided in several steps. After showing the existence of a solution for an initial condition in $L^1(\mathbb{R}^+; dx)$, we can directly deduce the comparison and the maximum principle. Then we show the contraction principle. Since $L^1(\mathbb{R}^+; dx)$ is dense in $L^1(\mathbb{R}^+; \phi(x)dx)$, we can construct a solution in the general case. At last we can show the conservation law.

First step. We show existence of a solution for $n^0 \in L^1(\mathbb{R}^+; dx)$ using the Banach-Fix Point-Theorem in the Banach space

$$X = C([0,T]; L^1(\mathbb{R}^+, dx)) \quad \text{endowed with norm} \quad ||n||_X = \sup_{0\le t\le T} ||n(t)||_{L^1(\mathbb{R}^+)}.$$

We choose T so that

$$T \cdot ||B||_{L^\infty(\mathbb{R}^+)} \le \frac{1}{2},$$

then $\tilde{n}$ is the fixed point of the operator $\mathcal{T}$, defined as follows:

$$\begin{aligned} \mathcal{T}:\ & X && \to X \\ & m && \mapsto n = \mathcal{T}[m], \end{aligned}$$

where n is the solution of

$$\begin{aligned} \frac{\partial}{\partial t} n(t,x) &+ \frac{\partial}{\partial x} n(t,x) + \lambda_0 n(t,x) = 0 \\ n(t,0) &= \int_0^\infty B(x) m(t,x) dx \\ n(0,x) &= n^0(x). \end{aligned}$$

Given $m_1, m_2 \in X$, we define $n_i = \mathcal{T}[m_i]$ for $i = 1,2$. The difference $n = n_2 - n_1$ fulfills

$$\begin{aligned} \frac{\partial}{\partial t} n(t,x) &+ \frac{\partial}{\partial x} n(t,x) + \lambda_0 n(t,x) = 0 \\ n(t,0) &= \int_0^\infty B(x) m(t,x) dx \\ n(0,x) &= 0 \end{aligned}$$

where $m = m_2 - m_1$. It holds also

$$\begin{aligned} \frac{\partial}{\partial t} |n(t,x)| &+ \frac{\partial}{\partial x} |n(t,x)| + \lambda_0 |n(t,x)| = 0 \\ |n(t,0)| &= |\int_0^\infty B(x) m(t,x) dx| \\ |n(0,x)| &= 0. \end{aligned}$$

To estimate the X-norm of n we observe at first that

$$\frac{\partial}{\partial t}\left(e^{\lambda_0 t} n(t,x)\right) + \frac{\partial}{\partial x}\left(e^{\lambda_0 t} n(t,x)\right) = 0.$$

Thus, using the theorem of characteristics we can calculate that

$$e^{\lambda_0 t} n(t,x) = \begin{cases} e^{\lambda_0 (t-x)} n(t-x,0) & \text{if} \quad t > x \\ n(0,x-t) = 0 & \text{if} \quad t \leq x. \end{cases}$$

For fixed t, we can divide through $e^{\lambda_0 t}$ and find

$$\int_0^\infty n(t,x)dx = \int_0^t n(t,x)dx = \int_0^t e^{\lambda_0(s-t)} n(s,0)ds.$$

Finally, for all t we can estimate

$$\begin{aligned} ||n(t,.)||_{L^1(\mathbb{R}^+)} &\leq \int_0^t |n(s,0)|ds \\ &= \int_0^t | \int_0^\infty B(x) m(s,x) dx | ds \\ &\leq t ||B||_{L^\infty(\mathbb{R}^+)} \sup_{0 \leq s \leq t} ||m(s)||_{L^1(\mathbb{R}^+)}. \end{aligned}$$

Taking now the supremum over $0 \leq t \leq T$ and using the assumption on T, we obtain

$$||n_1 - n_2||_X \leq \frac{1}{2} ||m_1 - m_2||_X.$$

This means that $\mathcal{T}$ is a strict contraction in the Banach space X, which proves the existence of the fixed point. We can iterate this process on the intervals $[T, 2T], [2T, 3T], \ldots$ and built a solution in $C(\mathbb{R}^+, L^1(\mathbb{R}^+; dx))$.

Second step. The comparison principle follows from the above construction. For two different initial data $n_1^0 \leq n_2^0$, it holds $\mathcal{T}_1[m] \leq \mathcal{T}_2[m]$ for all m, and therefore also the fixed point fulfills $n_1 \leq n_2$. As a direct consequence the maximum principle holds, since $\pm C_0 N(x)$ is a solution and can be used in the comparison principle.

Third step. Now we consider an initial age distribution $n^0(x) \in L^1(\mathbb{R}^+, \phi(x)dx)$. Because of the boundedness of ϕ, the space $L^1(\mathbb{R}^+, dx)$ is dense in $L^1(\mathbb{R}^+, \phi(x)dx)$, and thus, there is a sequence $n_k^0 \in L^1(\mathbb{R}^+, dx)$ with

$$n_k^0 \underset{k \to \infty}{\longrightarrow} n^0 \qquad \text{in} \quad L^1(\mathbb{R}^+, \phi(x)dx).$$

For $\tilde{n}_k(t,x) \in L^1(\mathbb{R}^+, dx)$ being the corresponding solution of the rescaled renewal equation (8), we consider $\tilde{n} = \tilde{n}_k - \tilde{n}_l$, which is still a solution of (8). Combining $\tilde{n}$ with the dual equation we obtain

$$\frac{\partial}{\partial t}(\tilde{n}(t,x)\phi(x)) + \frac{\partial}{\partial x}(\tilde{n}(t,x)\phi(x)) = -\phi(0)B(x)\tilde{n}(t,x) \tag{9}$$

and so

$$\frac{\partial}{\partial t}(|\tilde{n}(t,x)|\phi(x)) + \frac{\partial}{\partial x}(|\tilde{n}(t,x)|\phi(x)) = -\phi(0)B(x)|\tilde{n}(t,x)|.$$

After integration in x we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty |\tilde{n}(t,x)|\phi(x)dx &= |\tilde{n}(t,0)|\phi(0) - \phi(0)\int_0^\infty B(x)|\tilde{n}(t,x)|dx \\ &= \phi(0)\left(|\int_0^\infty B(x)\tilde{n}(t,x)dx| - \int_0^\infty B(x)|\tilde{n}(t,x)|dx\right) \\ &\leq 0\end{aligned} \tag{10}$$

Finally,

$$\int_0^\infty |\tilde{n}_k(x,t) - \tilde{n}_l(x,t)|\phi(x)dx \leq \int_0^\infty |n_k^0(x) - n_l^0(x)|\phi(x)dx.$$

Therefore $\tilde{n}$ is a Cauchysequence in $C(\mathbb{R}^+; L^1(\mathbb{R}^+, \phi(x)dx))$. Furthermore, it is uniformly bounded $|\tilde{n}_k(x,t)| \leq C_0 N(x)$, because of the maximum principle provided in the second step. Thus the sequence is converging in $C(\mathbb{R}^+; L^1(\mathbb{R}^+, \phi(x)dx))$ and weakly in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ to a solution of (8) in the distribution sense.

As equation (10) is valid for arbitrary $\tilde{n}$ which are solutions of (8), we deduce the contraction principle.
The solution is unique, because for two possible solution with the same intitial conditions we have from the contraction principle

$$\int_0^\infty |\tilde{n}_1(t,x) - \tilde{n}_2(t,x)|\phi(x)dx \leq 0.$$

Thus, they agree on the support of ϕ, and therefore also on the support of B. We arrive at a transport equation with the same initial and boundary conditions, and thus the solutions are equal.

Forth step. Equation (9) can be obtained by x-integration

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \tilde{n}(t,x)\phi(x)dx = \tilde{n}(t,0)\phi(0) - \phi(0)\int_0^\infty B(x)\tilde{n}(t,x)dx = 0$$

and find the conservation law.

2.3 Regularity of solutions

Theorem 2.3. *Under assumptions* (3) *and Lipschitz continuous initial data* $n^0(x)$ *satisfying*

$$|n^0(x)| \leq C_0 N(x), \qquad |\frac{\partial}{\partial x} n^0(x)| \leq C_1 N(x)$$

and the compatibility condition $n^0(0) = \int_0^\infty B(x)n^0(x)dx,$

the solution of equation (8) *fulfills the estimations*

$$
\begin{aligned}
|\frac{\partial}{\partial t}\tilde{n}(t,x)| &\leq (C_1+\lambda_0)N(x) \quad \forall t\geq 0, \forall x \geq 0 \\
|\frac{\partial}{\partial x}\tilde{n}(t,x)| &\leq (C_1+\lambda_0+\lambda_0 C_0)N(x) \quad \forall t\geq 0, \forall x \geq 0.
\end{aligned}
$$

Proof: Because of linearity of the model, the time derivation $\frac{\partial}{\partial t}\tilde{n}(t,x)$ is also fulfilling the rescaled renewal equation. Furthermore, we can estimate the initial condition

$$
\begin{aligned}
&\frac{\partial}{\partial t}\tilde{n}(t,x)\big|_{t=0} = -\frac{\partial}{\partial x}\tilde{n}(0,x) - \lambda_0\tilde{n}(0,x) = -\frac{\partial}{\partial x}n^0(x) - \lambda_0 n^0(x), \\
&|\frac{\partial}{\partial t}\tilde{n}(t,x)\big|_{t=0} \leq (C_1+\lambda_0)N(x).
\end{aligned}
$$

Using the maximum principle we can deduce

$$
|\frac{\partial}{\partial t}\tilde{n}(t,x)| \leq (C_1+\lambda_0)N(x) \quad x>0.
$$

Because of the compatibility condition there is no jump at $t=x=0$ and the estimation is also valable for $x=0$.

The second estimation we get again by using equation (8)

$$
\begin{aligned}
|\frac{\partial}{\partial x}\tilde{n}(t,x)| &= |\frac{\partial}{\partial t}\tilde{n}(t,x) + |\lambda_0\tilde{n}(0,x)| \\
&\leq (C_1+\lambda_0)N(x) + \lambda_0 C_0 N(x).
\end{aligned}
$$

2.4 Generalized relative entropy

Theorem 2.4. *We consider again equation* (8) *under the assumptions* (3)*, then*

1. *for all convex functions H and for all $t>0$, it holds*

$$
\int_0^\infty \phi(x)N(x)H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)dx \leq \int_0^\infty \phi(x)N(x)H\left(\frac{n^0(x)}{N(x)}\right)dx, \tag{11}
$$

2. *for the probability measure $d\mu(x) = B(x)\frac{N(x)}{N(0)}dx$ and for all convex functions $H:\mathbb{R}^+\to\mathbb{R}^+$, it holds*

$$
\begin{aligned}
\int_0^\infty \Big[\int_0^\infty H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)d\mu(x) - H\left(\int_0^\infty \frac{\tilde{n}(t,x)}{N(x)}d\mu(x)\right)\Big]dt \\
\leq K\int_0^\infty \phi(x)N(x)H\left(\frac{n^0(x)}{N(x)}\right)dx.
\end{aligned} \tag{12}
$$

Proof: Using the equations (5) and (8) we obtain that

$$\frac{\partial}{\partial t}\left(\frac{\tilde{n}(t,x)}{N(x)}\right)+\frac{\partial}{\partial x}\left(\frac{\tilde{n}(t,x)}{N(x)}\right)=0.$$

Now multiplying the last equation with $H'\left(\frac{\tilde{n}(t,x)}{N(x)}\right)$ (which makes sense, as H is absolute continuous and we are interested in weak solutions), we obtain that

$$\frac{\partial}{\partial t}H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)+\frac{\partial}{\partial x}H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)=0.$$

Finally, it holds

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\phi(x)N(x)H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)\right]+\frac{\partial}{\partial x}\left[\phi(x)N(x)H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)\right]\\
&=\phi(x)N(x)\left(\frac{\partial}{\partial t}H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)+\frac{\partial}{\partial x}H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)\right)\\
&\qquad+\frac{\partial}{\partial x}\left(\phi(x)N(x)\right)H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)\\
&=-\phi(0)B(x)N(x)H\left(\frac{\tilde{n}(t,x)}{N(x)}\right).
\end{aligned}$$

After integration in x we find, using the notation $d\mu(x)=B(x)\frac{N(x)}{N(0)}dx$, which is a probability measure in view of equation (7)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\phi(x)N(x)H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)dx\\
&\quad=-\int_0^\infty\frac{\partial}{\partial x}\phi(x)N(x)H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)dx-\phi(0)\int_0^\infty B(x)N(x)H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)dx\\
&\quad=\phi(0)N(0)H\left(\frac{\tilde{n}(t,0)}{N(0)}\right)-\phi(0)N(0)\int_0^\infty H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)d\mu(x)\\
&\quad=\phi(0)N(0)\left[H\left(\int_0^\infty\frac{\tilde{n}(t,x)}{N(x)}d\mu(x)\right)-\int_0^\infty H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)d\mu(x)\right].
\end{aligned}\tag{13}$$

The last quantity is negative because of Jensen's inequality. That shows that $\int_0^\infty\phi(x)N(x)H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)dx$ is decaying and so we find inequality (11).
Time integration of equation (13) leads to

$$\begin{aligned}
&\int_0^\infty\left[\int_0^\infty H\left(\frac{\tilde{n}(t,x)}{N(x)}\right)d\mu(x)-H\left(\int_0^\infty\frac{\tilde{n}(t,x)}{N(x)}d\mu(x)\right)\right]dt\\
&\quad=\int_0^\infty\frac{\phi(x)N(x)}{\phi(0)N(0)}H\left(\frac{n^0(x)}{N(x)}\right)dx-\int_0^\infty\frac{\phi(x)N(x)}{\phi(0)N(0)}H\left(\frac{\tilde{n}(\infty,x)}{N(x)}\right)dx
\end{aligned}$$

and the inequality (12) is satisfied.

2.5 Long time asymptotics - entropy method

Theorem 2.5. *Under the assumptions* (3) *and initial data fulfilling* $|n^0(x)| \leq CN(x)$*, the solution to equation* (8) *satisfies*

$$\int_0^\infty |\tilde{n}(t,x) - m^0 N(x)|\phi(x)dx \to 0 \qquad as \quad t \to \infty \tag{14}$$

where $m^0 = \int_0^\infty n^0(x)\phi(x)dx$ *is defined in 1.1.*

Proof: First step. We regularize the initial data $n^0 \in L^1(\phi(x)dx)$, so that they satisfy the assumptions of theorem 2.3. Call n^0_ϵ the regularized initial data, $\tilde{n}_\epsilon(t,x)$ the corresponding solution of equation (8) and $m^0_\epsilon = \int_0^\infty n^0_\epsilon(x)\phi(x)dx$, then the contraction principle yields

$$\int_0^\infty |n(t,x) - \tilde{n}_\epsilon(t,x)|\phi(x)dx \leq \int_0^\infty |n^0(x) - n^0_\epsilon(x)|\phi(x)dx := r_\epsilon \to 0,$$

and also $|m^0 - m^0_\epsilon| \leq r_\epsilon$. Using the triangle inequality we find

$$\int_0^\infty |\tilde{n}(t,x) - m^0 N(x)|\phi(x)dx \leq 2r_\epsilon + \int_0^\infty |\tilde{n}_\epsilon(t,x) - m^0_\epsilon N(x)|\phi(x)dx.$$

Showing the theorem for regularized initial data, we obtain $\lim_{t\to\infty} \int_0^\infty |\tilde{n}(t,x) - m^0 N(x)|\phi(x)dx \leq 2r_\epsilon$ for all $\epsilon > 0$. Therefore, the limit vanishes as $r_\epsilon \to 0$ and the theorem is true in the general case.

Second step. In the regularized case, we set $h(t,x) = \tilde{n}(t,x) - m^0 N(x)$, which is also a solution of (8) because of linearity and it satisfies $|h(t,x)| \leq C_0 N(x)$ and

$$\int_0^\infty h(t,x)\phi(x)dx = \int_0^\infty \tilde{n}(t,x)\phi(x)dx - m^0 \int_0^\infty N(x)\phi(x)dx = 0$$

by the conservation law. The contraction principle yields that $\int_0^\infty |h(t,x)|\phi(x)dx$ is decaying and it is positive, so it converges to some value $L \geq 0$. Then, it remains to show that $L = 0$.

Third step. We now define solutions $h_k(t,x) \in C(\mathbb{R}^+, L^1(\mathbb{R}^+; \phi(x)dx))$ to equation (8) by setting

$$h_k(t,x) := h(t+k,x)$$

and thus $|h_k(t,x)| \leq C_0 N(x)$. Let H be a nonnegative convex function, then we see with Theorem 2.4 *ii*) that the quantity I_k defined by

$$\begin{aligned} I_k &= \int_0^\infty \left[\int_0^\infty H\left(\frac{h_k(t,x)}{N(x)}\right) d\mu(x) - H\left(\int_0^\infty \frac{h_k(t,x)}{N(x)} d\mu(x)\right)\right] dt \\ &= \int_k^\infty \left[\int_0^\infty H\left(\frac{h(t,x)}{N(x)}\right) d\mu(x) - H\left(\int_0^\infty \frac{h(t,x)}{N(x)} d\mu(x)\right)\right] dt \end{aligned}$$

is bounded. As the integrand is nonnegative and integrable we can deduce $\lim_{k\to\infty} I_k = 0$.
Moreover h_k satisfies the equation

$$\frac{\partial}{\partial t}h_k(t,x) + \frac{\partial}{\partial x}h_k(t,x) + \lambda_0 h_k(t,x) = 0 \qquad \forall t \geq 0, \forall x \geq 0,$$
$$h_k(t,0) = \int_0^\infty B(y)h(t,y)dy,$$
$$\int_0^\infty \phi(x)h_k(x,t)dx = 0.$$

Next using the regularity of h (via Theorem 2.3) we may extract a subsequence (still denoted h_k) such that, for all $T > 0$,

$$h_k \to g \text{ in } C([0,T]\times\mathbb{R}^+), \qquad 0 \leq |g| \leq C_0 N(x),$$
$$\int_0^\infty B(y)h_k(t,y)dy \to \int_0^\infty B(y)g(t,y)dy \quad \text{in } C([0,T]),$$
$$\int_0^\infty \phi(x)g(t,x)dx = 0, \qquad \int_0^\infty \phi(x)|g(t,x)|dx = L.$$

We pass to the limit in the definition of I_k and obtain by convexity in weak limits

$$\int_0^\infty\int_0^\infty H\left(\frac{g(t,x)}{N(x)}\right)d\mu(x)dt \leq \lim \int_0^\infty\int_0^\infty H\left(\frac{h_k(t,x)}{N(x)}\right)d\mu(x)dt$$
$$= \int_0^\infty H\left(\int_0^\infty \frac{g(t,x)}{N(x)}d\mu(x)\right)dt.$$

The last equality is valid because $\lim_{k\to\infty} I_k = 0$. But from the Jensen inequality, we also find the reverse inequality, so that we finally obtain:

$$\int_0^\infty\int_0^\infty H\left(\frac{g(t,x)}{N(x)}\right)d\mu(x)dt = \int_0^\infty H\left(\int_0^\infty \frac{g(t,x)}{N(x)}d\mu(x)\right)dt.$$

As this equality is true for all convex H, thus also for strictly convex, we showed that it holds, for almost all $t > 0$,

$$\frac{g(t,x)}{N(x)} = C(t) \quad \text{on the support of } B.$$

Inserting this information in the limit in the distribution sense of equation (8), we obtain

$$\frac{\partial}{\partial t}g(t,x) + \frac{\partial}{\partial x}g(t,x) + \lambda_0 g(t,x) = 0 \qquad t \geq 0, x \geq 0$$
$$g(t,0) = \int_0^\infty B(y)g(t,y)dy$$

and we can deduce

$$\frac{\partial}{\partial t}\frac{g(t,x)}{N(x)} + \frac{\partial}{\partial x}\frac{g(t,x)}{N(x)} = 0.$$

Therefore $\frac{g(t,x)}{N(x)} \equiv C(t)$ and we finally find

$$0 = \int_0^\infty g(t,x)\phi(x)dx = C(t)\int_0^\infty N(x)\phi(x)dx = C(t)$$

Now we can conclude that L vanishes, because $L = \int_0^\infty |g(t,x)|\phi(x)dx \to 0$ for $t \to 0$.

2.6 Long time asymptotic: exponential decay

Theorem 2.6. *Under the assumptions* (3) *and the supplementary condition*

$$\exists \mu_0 > 0, \quad \textit{such that} \quad B(x) \geq \mu_0 \frac{\phi(x)}{\phi(0)}, \tag{15}$$

the solution to (8) *satisfies*

$$\int_0^\infty |\tilde{n}(t,x) - m^0 N(x)|\phi(x)dx \leq e^{-\mu_0 t}\int_0^\infty |n^0(x) - m^0 N(x)|\phi(x)dx.$$

Proof: We consider again $h(t,x) = \tilde{n}(t,x) - m^0 N(x)$. As this fulfills equation (8), we can combine it with the dual equation and we find

$$\begin{aligned}\frac{\partial}{\partial t}(h(t,x)\phi(x)) \;&+\; \frac{\partial}{\partial x}(h(t,x)\phi(x)) = -\phi(0)B(x)h(t,x) \qquad \forall t \geq 0, \forall x \geq 0,\\ \phi(0)h(t,0) \;&=\; \phi(0)\int_0^\infty B(y)h(t,y)dy\end{aligned}$$

and also

$$\begin{aligned}\frac{\partial}{\partial t}(|h(t,x)|\phi(x)) \;&+\; \frac{\partial}{\partial x}(|h(t,x)|\phi(x)) = -\phi(0)B(x)|h(t,x)| \qquad \forall t \geq 0, \forall x \geq 0,\\ \phi(0)|h(t,0)| \;&=\; \phi(0)|\int_0^\infty B(y)h(t,y)dy|.\end{aligned}$$

And now after integration in x we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty |h(t,x)|\phi(x)dx &= -\phi(0)\int_0^\infty B(x)|h(t,x)|dx + |h(t,0)|\phi(0)\\ &= -\phi(0)\int_0^\infty B(x)|h(t,x)|dx + |\int_0^\infty \phi(0)B(x)h(t,x) - \mu_0\phi(x)h(t,x)dx|\\ &\leq -\phi(0)\int_0^\infty B(x)|h(t,x)|dx + \int_0^\infty (\phi(0)B(x) - \mu_0\phi(x))\,|h(t,x)|dx\\ &= -\mu_0\int_0^\infty |h(t,x)|\phi(x)dx.\end{aligned}$$

We conclude using Gronwall's lemma.

3 Appendix

Proof of Lemma 1.4.

Proof: For $m = 0$ the assertion is trivial, so we can assume $m \neq 0$ and normalize

$$\sum_{i=1}^{d} \phi_i N_i \left(\frac{m_i}{N_i} \right)^2 = 1. \tag{16}$$

Moreover, we have

$$\sum_{i=1}^{d} \phi_i m_i = 0.$$

Then we argue by contradiction. If no such α exists, we can construct a sequence of vectors $(m^k)_{k \geq 1}$ with

$$\sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{m_j^k}{N_j} - \frac{m_i^k}{N_i} \right)^2 \leq \frac{1}{k}$$

and also fulfilling equation (16). Then, we we can extract a converging subsequence, still called $(m^k)_{k \geq 1}$ with $\lim_{k \to \infty} m^k = \bar{m}$. The limit fulfills (16) and furthermore

$$\sum_{i,j=1}^{d} \phi_i a_{ij} N_j \left(\frac{\bar{m}_j}{N_j} - \frac{\bar{m}_i}{N_i} \right)^2 = 0.$$

Because of the positivity of ϕ_i, N_i, a_{ij}, this is only possible when for all $i, j = 1, \ldots, d$ we have

$$\frac{\bar{m}_i}{N_i} = \frac{\bar{m}_j}{N_j} = \nu.$$

But because of $0 = \sum_{i=1}^{d} \phi_i N_i \frac{\bar{m}_i}{N_i} = \nu \sum_{i=1}^{d} \phi_i N_i$, this is only possible for $\nu = 0$ which contradicts $\sum_{i=1}^{d} \phi_i N_i \left(\frac{\bar{m}_i}{N_i} \right)^2 = 1$.