Lectures in Katowice for Ph.D. students

Henrik Stetkær

Lectures on October 1, 2012

Introduction

In this series of lectures we studied functional equations: A functional equation is an equation in which the unknown or unknowns are functions. To avoid a too extensive theory differential, difference and integral equations are not counted as parts of functional equations, these theories being huge separate subjects with their own own life and their own special methods.

We concentrated on special types of functional equations: Trigonometrical functional equations on groups, i.e. equations that extend and generalize classical relations among the trigonometric and hyperbolic functions. So our point of departure was formulas of trigonometry. To take an example, the function cosine satisfies the identity

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y \text{ for all } x, y \in \mathbb{R}.$$

We sought the functions $g : \mathbb{R} \to \mathbb{C}$ that satisfy the corresponding functional equation (called d'Alembert's functional equation or the cosine equation)

$$g(x+y) + g(x-y) = 2g(x)g(y) \text{ for all } x, y \in \mathbb{R}, \tag{1}$$

in which we have replaced cos by g in the identity above. To solve (1) is to find all functions $g : \mathbb{R} \to \mathbb{C}$ for which (1) holds. The equation is a functional equation, because its solutions g are functions, not numbers. Incidentally, the cosine equation has other solutions than $g = \cos$, for instance $g = \cosh$.

However, we are not satisfied with finding the functions on $\mathbb{R}$ that satisfy the classical functional equations above: We want a deeper understanding and to see the functional equations in a wider context, so we extend the scope by replacing the domain of definition $\mathbb{R}$ of the functions by a group G, and instead of the classical range space $\mathbb{R}$ we take $\mathbb{C}$. Thus given a group G we want to describe the solutions $g : G \to \mathbb{C}$ of the cosine equation

$$g(xy) + g(xy^{-1}) = 2g(x)g(y) \text{ for all } x, y \in G,$$

and to find common properties of the solutions on various types of groups (like abelian or compact groups).

The story we told at the lecture series is about the developments in the last 10–20 years of the theory of a number of trigonometric functional equations seen from this advanced point of view.

As a by-product we came to look at relations between the classical trigonometric functions from a higher point of view. We also shed light on how different functional equations on the same group are related.

To solve a functional equation is to express the unknown function or functions in terms of (supposedly known) simple, basic functions. These simple, basic functions are to be our building blocks. The building blocks are formally the same kind of functions on any group: Characters, additive maps, bi-additive maps, representations etc. But such functions vary from group to group according to the nature of the group and should be computed for each group or type of group. Our theory typically presented results that are common for all groups of a certain type, like abelian groups, compact groups or semisimple Lie groups.

The additive Cauchy equation

The additive Cauchy equation (2) was discussed, not so much because its theory is nice, interesting and important (it is), but because the additive functions (= the solutions of the additive Cauchy equation) are among the building blocks, out of which we construct solutions of other functional equations as mentioned above in the introduction. We were in particular interested in the continuous, additive maps.

Definition 1. Let S be a semigroup, and $(H,+)$ an abelian semigroup. A solution of *the additive Cauchy equation*

$$a(xy) = a(x) + a(y), \quad x, y \in S, \tag{2}$$

is a map $a : S \to H$ satisfying (2). A solution is called an *additive map* or an *additive function* from S to H or an additive map/function on S with values in H.

We proved

Lemma 2. *Let V be a vector space over $\mathbb{R}$ and let $a : V \to \mathbb{C}$ be a solution of*

$$a(x + y) = a(x) + a(y), \quad x, y \in V. \tag{3}$$

(a) $a(qx) = qa(x)$ for all $q \in \mathbb{Q}$ and $x \in V$, i.e. a is $\mathbb{Q}$-linear.

(b) If V is a topological vector space and a is continuous at a point then a is continuous.

(c) If V is a topological vector space and a is continuous, then a is linear.

Proposition 3. *Let $a : \mathbb{R} \to \mathbb{R}$ be an additive function, which is not of the form $a(x) = cx$, $x \in \mathbb{R}$ for some constant $c \in \mathbb{R}$. Then its graph $\{(x, a(x)) \mid x \in \mathbb{R}\}$ is dense in $\mathbb{R}^2$.*

Proposition 4. *There exists a discontinuous additive function $a : \mathbb{R} \to \mathbb{R}$.*

Furthermore, by help of Steinhaus' theorem we derived

Proposition 5. *Let $a : \mathbb{R} \to \mathbb{C}$ be additive.*

(a) If a is bounded on a set of positive measure then a is linear.

(b) If a is Lebesgue measurable then a is linear.

Example 6.

Let $h : G \to \mathbb{C}$ be a continuous additive map on the $(ax + b)$-group

$$G := \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0,\ b \in \mathbb{R}\}.$$

We showed that it has the form

$$h \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = h \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = c \log a,$$

where $c \in \mathbb{C}$ is a constant.

Conversely, any function of this form is a continuous additive function on the $(ax + b)$-group.

Bi-additive maps

We may later in ths series of lectures encounter bi-additive functions $B : G \times G \to \mathbb{C}$, where G denotes a group. Here we noted the following simple results for the case of $G = \mathbb{R}^n$.

Lemma 7. *If $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a continuous, bi-additive function, then there exists exactly one matrix $A \in M(n \times n, \mathbb{C})$ such that $B(x, y) = \langle Ax, y\rangle$ for all $x, y \in \mathbb{R}^n$. If B is symmetric, resp. skew-symmetric, then so is the matrix A.*

If B is symmetric, then its continuity is implied by the continuity of the function $f : \mathbb{R}^n \to \mathbb{C}$ defined by $f(x) := B(x, x)$, $x \in \mathbb{R}^n$.

Lectures in Katowice for Ph.D. students

Henrik Stetkær

Lectures on Tuesday, October 2, 2012

THE MULTIPLICATIVE CAUCHY EQUATION

In the lectures on Tuesday we derived some basic properties of group characters, computed the continuous characters of important topological groups and proved Artin's result about linear independence of characters (Corollary 6).

Group characters

As stated in the introduction we will express the solutions of functional equations on a group in terms of certain fundamental building blocks that are related to structure of the group. We met some of these building blocks, the additive functions, in the lectures yesterday. Other building blocks are the group characters that we study below. They are ingredients of the solution formulas of, e.g., the sine addition and d'Alembert's functional equations, but are important in other fields as well. The continuous, unitary characters on a locally compact, abelian group like $\mathbb{R}$ are the building stones for the Fourier analysis on the group: Any square integrable function on the group is according to Fourier's inversion formula a superposition of continuous, unitary characters. Non-unitary exponential functions play an important role in other parts of mathematics, for instance in the theory of partial differential equations with constant coefficients. In our set up solutions are not assumed bounded. Therefore we cannot restrict ourselves to unitary = bounded characters.

Definition 1. A *group character* on G or for brevity just a *character* on G is a homomorphism of G into $\mathbb{C}^*$.

A character γ is said to be *unitary*, if $|\gamma(x)| = 1$ for all $x \in G$.

Definition 2. The *multiplicative Cauchy equation* for a function $\chi : G \to \mathbb{C}$ on a group G is the functional equation

$$\chi(xy) = \chi(x)\chi(y), \quad x, y \in G. \tag{1}$$

It is in the literature also known as the *Cauchy exponential equation*.

The non-zero solutions $\chi : G \to \mathbb{C}$ of (1) are the characters on G (see Lemma 3(a)). The formula (1) expresses that χ is a multiplicative function on G.

The basic properties of characters on groups expressed in Lemma 3 will often be used without explicit mentioning. We recall the notations $\check{\chi}(x) = \chi(x^{-1})$ and $\overline{\chi}(x) = \overline{\chi(x)}$ for any x.

Lemma 3. *Let G be a group, and let $\chi : G \to \mathbb{C}$ be a non-zero multiplicative function.*

(a) χ is a character on G. In particular χ does not vanish at any point of G, and $\chi(e) = 1$.

(b) If χ is bounded, then $|\chi(x)| = 1$ for all $x \in G$, so χ is a unitary character. If χ is unitary, then $\check{\chi} = \overline{\chi}$.

(c) χ is identically 1 on the commutator subgroup $[G, G]$.

Continuous characters on selected groups

In this part of the lectures we computed the continuous characters of some important topological groups. Exponential functions played prominent roles. As mentioned above the characters will come forth later as building blocks of solutions of functional equations.

Example 4. (a) For any $\lambda \in \mathbb{C}$ the function

$$\chi_\lambda(x) := e^{\lambda x}, \ x \in \mathbb{R}, \tag{2}$$

is a continuous character on $(\mathbb{R}, +)$. Conversely, for any continuous character χ on $(\mathbb{R}, +)$ there exists exactly one $\lambda \in \mathbb{C}$ such that $\chi = \chi_\lambda$.

The unitary, continuous characters are the ones for which $\lambda \in i\mathbb{R}$.

(b) More generally, the continuous characters on $(\mathbb{R}^n, +)$, $n \in \mathbb{N}$, are the functions of the form

$$\chi_\lambda(x) := e^{\langle \lambda, x \rangle}, \ x \in \mathbb{R}^n, \tag{3}$$

where λ ranges over $\mathbb{C}^n$. Here we use for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ the notation $\langle \lambda, x \rangle := \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$.

The unitary, continuous characters are the ones for which $\lambda_k \in i\mathbb{R}$ for $k = 1, 2, \dots, n$.

Linear independence of multiplicative functions

We proved extensions of Artin's result (Corollary 6) that characters are linearly independent. Artin's result will be very useful for us at many later occasions, because it means that a function can be written as a linear combination of characters in at most one way. From an advanced point of view the proper place of Artin's result is in the framework of non-commutative harmonic analysis.

The set of exponential functions $\{e^{\lambda x} \mid \lambda \in \mathbb{C}\}$ is a linearly independent subset of the vector space $C(\mathbb{R})$. So, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are different complex numbers and $a_1, a_2, \ldots, a_n$ are complex numbers such that

$$a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + \cdots + a_n e^{\lambda_n x} = 0 \text{ for all } x \in \mathbb{R},$$

then $a_1 = a_2 = \cdots = a_n = 0$. The listener should try to prove this fact on his own, because he will then appreciate the simplicity of the proof of Theorem 5 that we presented at the lectures.

Theorem 5. *Let S be a semigroup and $n \in \mathbb{N}$. Let $\chi_1, \chi_2, \ldots, \chi_n : S \to \mathbb{C}$ be n different multiplicative functions, and let $a_1, a_2, \ldots, a_n \in \mathbb{C}$. Define $f := a_1\chi_1 + a_2\chi_2 + \cdots + a_n\chi_n : S \to \mathbb{C}$.*

(a) If $f = 0$, then $a_1\chi_1 = a_2\chi_2 = \cdots = a_n\chi_n = 0$. In other words, the multiplicative functions form a direct sum.

(b) The set of non-zero multiplicative functions is a linearly independent subset of the complex-valued functions on S.

(c) If f is bounded, then each of the functions $a_1\chi_1$, $a_2\chi_2$, $\ldots$, $a_n\chi_n$ is also bounded.

(d) If S is a topological semigroup and f is continuous, then each of the functions $a_1\chi_1$, $a_2\chi_2$, $\ldots$, $a_n\chi_n$ is also continuous.

(e) If S is a topological semigroup and f is Borel measurable, then each of the functions $a_1\chi_1$, $a_2\chi_2$, $\ldots$, $a_n\chi_n$ is also Borel measurable.

(f) If S is a locally compact group and f is measurable with respect to the left Haar measure on S, then the functions $a_1\chi_1$, $a_2\chi_2$, $\ldots$, $a_n\chi_n$ are continuous.

Corollary 6 (Artin). *The set of characters on a group G is a linearly independent subset of the vector space of all complex-valued functions on G.*

As we saw above in Theorem 5, the method of the proof of Artin's result is so versatile that it has several consequences. Thus we have got a proof with results, not just a result with a proof!

Lectures in Katowice for Ph.D. students

Henrik Stetkær

Lectures on Wednesday, October 3, 2012

ADDITION AND SUBTRACTION FORMULAS

The main topic of the lectures on Wednesday was the sine addition formula.

The sine addition formula

In the lectures on Wednesday we examined the sine addition formula in an abstract setting, in which the underlying space needed not even be an abelian group. We used merely the associative property $x(yz) = (xy)z$, i.e. that the underlying space is a semigroup.

We shall in Theorem 1(a) below in particular see that solutions of the generalized sine addition formula (1) automatically also satisfy a version of the cosine addition formula. A converse result holds for the general cosine addition formula. So you cannot have the sine addition formula without the cosine addition formula.

The sine addition formula (1) is our first functional equation apart from the ones defining our building blocks. It illustrates an interesting common aspect of functional equations, namely that one equation may determine more than just one unknown function: (1) contains two unknown functions, f and g.

The functional equation (1) will be called the *sine addition formula.*

Theorem 1. *Let S be a topological semigroup. Let $f, g \in C(S)$ satisfy the sine addition formula*

$$f(xy) = f(x)g(y) + f(y)g(x) \text{ for all } x, y \in S. \tag{1}$$

Assume furthermore that $f \neq 0$.

(a) There exists a constant $\alpha \in \mathbb{C}$ such that

$$g(xy) = g(x)g(y) + \alpha^2 f(x)f(y) \text{ for all } x, y \in S. \tag{2}$$

(b) $\chi_1 := g + \alpha f : S \to \mathbb{C}$ *and* $\chi_2 := g - \alpha f : S \to \mathbb{C}$ *are continuous multiplicative functions such that*

$$g = \frac{\chi_1 + \chi_2}{2}. \tag{3}$$

Given g *then* χ_1 *and* χ_2 *are the only multiplicative functions satisfying* (3)*, except that they may be interchanged.*

(c) Assume that $\chi_1 \neq \chi_2$*. Then* $\alpha \neq 0$ *and* $f = \frac{1}{2\alpha}(\chi_1 - \chi_2)$*.*

(d) Assume that $\chi_1 = \chi_2$*. Then* $g = \chi_1 = \chi_2$*. With* $\chi := g$ *the sine addition formula* (1) *becomes to*

$$f(xy) = f(x)\chi(y) + f(y)\chi(x) \text{ for all } x, y \in S, \tag{4}$$

where $\chi : S \to \mathbb{C}$ *is a continuous, multiplicative function. Furthermore:*

If S *is a monoid with neutral element* e*, then* $\chi(e) = 1$ *and* $f(e) = 0$*.*

If S *is a group, then* χ *is a continuous character on* S*, and there exists an additive function* $a \in C(S)$ *such that* $f = \chi a$*.*

(e) Both f *and* g *are abelian functions.*

(f) Define $d(x) := 2g(x)^2 - g(x^2)$*,* $x \in S$*. Then* $d = \chi_1\chi_2$*, so* $d : S \to \mathbb{C}$ *is multiplicative.*

THE CASORATI DETERMINANT

We stated and generalized a classical criterion that determines when a set of functions is linearly dependent (their Casorati determinant should vanish). We used the classical criterion to discuss functions F of the form $F(x, y) = \sum_{l=1}^{N} g_l(x)h_l(y)$ (Section 3). The generalization is a constituent of the proof of the important fact that continuous solutions of trigonometric functional equations are smooth.

The Casorati determinant

Definition 2. Let X be a set and $f_1, f_2, \ldots, f_N : X \to \mathbb{C}$. The *Casorati matrix* of $(f_1, f_2, \ldots, f_N)$ at $(x_1, x_2, \ldots, x_N) \in X \times X \times \cdots \times X$ is

$$\begin{pmatrix} f_1(x_1) & f_2(x_1) & \ldots & f_N(x_1) \\ f_1(x_2) & f_2(x_2) & \ldots & f_N(x_2) \\ \vdots & \vdots & \ldots & \vdots \\ f_1(x_N) & f_2(x_N) & \ldots & f_N(x_N) \end{pmatrix}$$

The *Casorati determinant* of $(f_1, f_2, \ldots, f_N)$ at $(x_1, x_2, \ldots, x_N) \in X \times X \times \cdots \times X$ is the determinant of the Casorati matrix.

The classical result about linear dependence in a space of functions is

Lemma 3. *Let X be a set and $f := (f_1, f_2, \dots, f_N)^t : X \to \mathbb{C}^N$ for some $N \in \mathbb{N}$.*

(a) $f_1, f_2, \dots, f_N$ are linearly dependent if and only if their Casorati determinant vanishes at all points $(x_1, x_2, \dots, x_N) \in X \times X \times \cdots \times X$.

(b) $f_1, f_2, \dots, f_N$ are linearly independent if and only if there exists a point $(x_1, x_2, \dots, x_N) \in X \times X \times \cdots \times X$ where their Casorati determinant does not vanish.

(c) $f_1, f_2, \dots, f_N$ are linearly independent if and only if span$\{f(x) \mid x \in X\} = \mathbb{C}^N$.

Proof. (a) is an easy consequence of Proposition 4 below: Take Φ as the set of point evaluations (the point evaluation at $x \in X$ is the map $f \mapsto f(x)$, $f \in \mathbb{C}^X$). (b) is immediate from (a), and (c) is a consequence of (b). $\square$

A generalization of the Casorati determinant

As said we need a modification of Lemma 3. The following proposition encompasses both versions. It also reveals that the key property of the point evaluations is that they separate functions.

Proposition 4. *Let $\mathcal{F}$ be a real or complex vector space and let $\mathcal{F}^*$ be its dual space. The value of $\phi \in \mathcal{F}^*$ at $f \in \mathcal{F}$ is denoted $\langle f, \phi \rangle$. Let Φ be a subset of $\mathcal{F}^*$ that separates the points of $\mathcal{F}$, i.e. has the property*

$$f \in \mathcal{F} \text{ and } \langle f, \phi \rangle = 0 \text{ for all } \phi \in \Phi \Rightarrow f = 0.$$

Let finally $f_1, f_2, \dots, f_N \in \mathcal{F}$, where $N \in \mathbb{N}$.

Then $f_1, f_2, \dots, f_N$ are linearly dependent if and only if

$$\det \begin{pmatrix} \langle f_1, \phi_1 \rangle & \langle f_2, \phi_1 \rangle & \dots & \langle f_N, \phi_1 \rangle \\ \langle f_1, \phi_2 \rangle & \langle f_2, \phi_2 \rangle & \dots & \langle f_N, \phi_2 \rangle \\ \vdots & \vdots & \dots & \vdots \\ \langle f_1, \phi_N \rangle & \langle f_2, \phi_N \rangle & \dots & \langle f_N, \phi_N \rangle \end{pmatrix} = 0 \tag{5}$$

for all $\phi_1, \dots, \phi_N \in \Phi$.

Lectures in Katowice for Ph.D. students

Henrik Stetkær

Lectures on Thursday, October 4, 2012

Most of the lecturing today was spent on the Levi-Civita functional equation, but at the end we introduced d'Alembert's functional equation.

Levi-Civita's equation

A *solution of Levi-Civita's functional equation* on a semigroup S is an ordered set of functions $f, g_1, \dots, g_N, h_1, \dots, h_N : S \to \mathbb{C}$ satisfying *Levi-Civita's functional equation*

$$f(xy) = \sum_{l=1}^{N} g_l(x) h_l(y) = g(x)^t h(y), \ x, y \in S, \tag{1}$$

where we on the right hand side have introduced the vector valued functions

$$g := (g_1, g_2, \dots, g_N)^t : S \to \mathbb{C}^N \text{ and } h := (h_1, h_2, \dots, h_N)^t : S \to \mathbb{C}^N.$$

The expression on the right hand side of Levi-Civita's functional equation (1) has a special form, being a finite sum of terms in each of which the variables x and y are separated. Nevertheless (1) encompasses interesting and important equations like the Cauchy equations and the sine and cosine addition formulas, so it is a natural sequel to the topics treated earlier in this course of lectures.

In the lectures we described the structure of any solution of Levi-Civita's functional equation by help of the close connection between the solutions of (1) and the matrix-coefficients of N-dimensional subrepresentations of the right regular representation R of S. The most important result on Thursday was Theorem 2 that describes this connection.

Definition 1. Let S be a semigroup. For any $y \in S$ we define $R(y) : \mathbb{C}^S \to \mathbb{C}^S$ by $[R(y)F](x) := F(xy)$, $x \in S$, for $F \in \mathbb{C}^S$. $R(y)$ is a linear operator for each $y \in S$.

Theorem 2. *Let M be a monoid with unit element $e \in M$ and let $N \in \mathbb{N}$. Let $\{f, g, h\}$ be a solution of (1) such that both $g_1, \dots, g_N$ and $h_1, \dots, h_N$ are linearly independent. Let $V := span\{R(y)f \mid y \in M\}$.*

(a) $V = span\{g_1, g_2, \dots, g_N\}$, *so the right regular representation R leaves $span\{g_1, \dots, g_N\}$ invariant.*

(b) *Let $\rho(x) = \{\rho_{ij}(x)\}_{i,j=1}^N$ for $x \in M$ denote the matrix of $R(x) : V \to V$ with respect to the basis $\{g_1, \dots, g_N\}$ of V, i.e. $R(x)g_i = \sum_{j=1}^N \rho_{ji}(x) g_j$. Then ρ is a representation of M on $\mathbb{C}^N$ with $h(e) \in \mathbb{C}^N$ as a cyclic vector.*

(c) *We have the solution formulas*

$$f(x) = g(e)^t \rho(x) h(e),\ \ g(x) = \rho(x)^t g(e),\ \ h(x) = \rho(x) h(e) \tag{2}$$

for all $x \in M$. In particular f and the components of g and h are matrix-coefficients of the N-dimensional representation ρ.

A kind of converse to the theorem exists:

Proposition 3. *Let S be a semigroup and let $N \in \mathbb{N}$. Let $\rho : S \to \mathrm{Mat}(N \times N, \mathbb{C})$ satisfy $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in S$. Then for any $\eta, \zeta \in \mathbb{C}^N$ the following triple of functions on S*

$$f(x) := \eta^t \rho(x) \zeta,\ \ g(x) := \rho(x)^t \eta,\ \ h(x) := \rho(x)\zeta,\ \ x \in S,$$

is a solution of Levi-Civita's functional equation (1).

Regularity

It follows from the formulas in Theorem 2(c) and Theorem 4 that the continuous solutions of the Levi-Civita functional equation on $G = \mathbb{R}^n$ are not just continuous, but actually are very smooth (being exponential polynomials).

Theorem 4. *Matrix-coefficients of continuous, finite-dimensional representations of $\mathbb{R}^n$ are exponential polynomials, i.e. linear combinations of functions of the form $x \mapsto p(x)e^{\langle \lambda, x\rangle}$, $x \in \mathbb{R}^n$, where $\lambda \in \mathbb{C}^n$ and p is a polynomial.*

d'Alembert's functional equation

d'Alembert's functional equation

$$g(x+y) + g(x-y) = 2g(x)g(y) \text{ for all } x, y \in \mathbb{R}, \tag{3}$$

for functions $g : \mathbb{R} \to \mathbb{C}$ on the real line has it roots back in d'Alembert's investigations of vibrating strings from 1750, but the equation is also closely connected to the trigonometrical functions and hence to the group structure of the real line. Indeed, one solution of (3) is $g(x) = \cos x$, another is

$g(x) = \cosh x$. As we shall see below, the general continuous solution $g \neq 0$ of (3) is

$$g(x) = \frac{e^{\alpha x} + e^{-\alpha x}}{2}, \ x \in \mathbb{R}, \text{ where } \alpha \in \mathbb{C}.$$

With $\alpha = i$ we get the solution $\cos x$, and with $\alpha = 1$ the solution $\cosh x$.

The obvious extension of the functional equation (3) from $\mathbb{R}$ to a group G is

$$g(xy) + g(xy^{-1}) = 2g(x)g(y) \text{ for all } x, y \in G, \tag{4}$$

and our aim is to find the solutions $g : G \to \mathbb{C}$ of (4) and some of their properties.

Lemma 5 below introduces functions of a special form that are abelian solutions of d'Alembert functional equation on any semigroup with involution. These special d'Alembert functions will be put into perspective in our later discussion of Kannappan's result, because we show there that all abelian solutions have this form. So they are important.

Lemma 5. *Let $\tau : S \to S$ be an involution of a semigroup S. If $\chi : S \to \mathbb{C}$ is multiplicative, then*

$$g := \frac{\chi + \chi \circ \tau}{2} \tag{5}$$

is an abelian solution of d'Alembert's functional equation (4)*.*

Proposition 6. *The continuous, non-zero solutions of d'Alembert's functional equation* (3) *on the real line are the functions $g_\lambda(x) := \cos(\lambda x)$, $x \in \mathbb{R}$, where λ ranges over $\mathbb{C}$.*

The proposition extends to greater dimensions as follows:

Proposition 7. *The non-zero, continuous solutions of d'Alembert's classical functional equation* (4) *on $\mathbb{R}^n$ are the functions of the form*

$$g_\lambda(x) := \frac{e^{i\langle \lambda, x\rangle} + e^{-i\langle \lambda, x\rangle}}{2} = \cos\langle \lambda, x\rangle, \ x \in \mathbb{R}^n,$$

where λ ranges over $\mathbb{C}^n$.

Lectures in Katowice for Ph.D.- students

Henrik Stetkær

Lectures on Friday, October 5, 2012

d'Alembert's functional equation

We continued our lectures about d'Alembert's functional equation, but supplemented them by introductions to d'Alembert's long functional equation and to Wilson's functional equation.

Throughout the lectures we let G denote a group.

Yesterday we saw examples of abelian solutions of d'Alembert's functional equation. We presented another example:

Example 1. Davison noted that any classical d'Alembert function g on the group $(\mathbb{Z},+)$ is given by the values of the Chebyshev polynomials $\{T_n\}_{n=0}^{\infty}$ at a point of $\mathbb{C}$. More precisely that the formula $g(n) = T_n(g(1))$ holds for $n \geq 0$. We derived Davison's formula.

The *Chebyshev polynomials* (of the first kind) T_n, $n = 0, 1\ldots$, are defined by $T_0(s) = 1$, $T_1(s) = s$ and the recurrence relation $T_{n+1}(s) = 2sT_n(s) - T_{n-1}(s)$ for $s \in \mathbb{C}$ and $n \geq 1$. T_n is a polynomial of degree n with leading coefficient 2^{n-1} for $n \geq 1$. An easy induction on n gives the formula

$$T_n(\frac{z+z^{-1}}{2}) = \frac{z^n + z^{-n}}{2} \quad \text{for } z \in \mathbb{C} \setminus \{0\}, \tag{1}$$

of which a particular case is

$$T_n(\cos x) := \cos(nx),\ x \in \mathbb{R},\ n = 0, 1, \ldots \tag{2}$$

(2) is often taken as the definition of the Chebyshev polynomials.

Our theory (more precisly Kannappan's theorem below) will tells us that the classical d'Alembert functions on $\mathbb{Z}$ are the functions of the form $g(n) := (z^n + z^{-n})/2$, $n \in \mathbb{Z}$, where z ranges over $\mathbb{C}^*$. From this we obtain Davison's formula $g(n) = T_n(g(1))$ by (1). We conclude that the classical d'Alembert functions g on $\mathbb{Z}$ are the functions $g(n) = T_n((z + z^{-1})/2)$, $n = 0, 1, 2, \ldots$, where z ranges over $\mathbb{C}^*$. An example is $g(n) = (-1)^n$ corresponding to $z = -1$.

We next gave an example of a non-abelian solution of d'Alembert's functional equation.

Example 2. The normalized trace

$$g(x) := \frac{1}{2}\operatorname{tr}(x),\ \ x \in M(2\times 2, \mathbb{C}),$$

restricts to a continuous, classical d'Alembert function on the matrix group $SL(2,\mathbb{C})$. To see this we noted by the Cayley-Hamilton theorem that any $y \in SL(2,\mathbb{C})$ is a root in its characteristic polynomial, i.e. $y^2-(\operatorname{tr} y)y+I = 0$. We multiplied this by xy^{-1} and took the trace, which gave the desired result that g is a solution of d'Alembert's functional equation. We checked directly that g is non-abelian: With

$$a := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ b := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

we find that $g(abc) = -1$ is different from $g(acb) = 1$.

Theorem 3. *Any solution g of d'Alembert's functional equation on G satisfies the pre-d'Alembert functional equation*

$$\begin{aligned} g(xyz) + g(xzy) &= 2g(x)g(yz) + 2g(y)g(zx) + 2g(z)g(xy) \\ &\quad - 4g(x)g(y)g(z) \text{ for all } x, y, z \in G, \end{aligned} \tag{3}$$

and as a consequence it also satisfies the symmetrized sine addition formula in the sense that

$$g_x(yz) + g_x(zy) = 2g_x(y)g(z) + 2g_x(z)g(y) \text{ for all } y, z \in G. \tag{4}$$

On an abelian group G the identity (4) reduces to the sine addition formula, the solutions of which we found in earlier lectures. As a consequence we derived Kannappan's theorem, in which we use the notation $\check{F}(x) := F(x^{-1})$ for any function $F : G \to \mathbb{C}$ and any $x \in G$.

Theorem 4 (Kannappan). *The abelian d'Alembert functions on G are the functions*

$$g = \frac{\chi + \check{\chi}}{2}, \tag{5}$$

where χ ranges over the characters of G.

Given g of the form (5) where χ is a character of G, then

(a) χ is unique, except that it can be replaced by $\check{\chi}$.

(b) g is bounded if and only if χ is unitary.

(c) If G is a topological group and g is continuous, then χ is also continuous.

(d) Let G be a locally compact group. If g is measurable with respect to the (left) Haar measure on G, then χ and g are continuous.

We cited Davison's result for non-abelian solutions of d'Alembert's functional equation:

Theorem 5. *Let G be a topological group and $g \in C(G)$ a non-abelian d'Alembert's function.*

(a) g can be written in the form $g = \frac{1}{2}\operatorname{tr}\rho$, where ρ is a continuous, irreducible representation of G on $\mathbb{C}^2$ such that $\rho(x) \in SL(2,\mathbb{C})$ for all $x \in G$.

(b) If g is bounded, we may choose the ρ from (a) *such that $\rho(x) \in SU(2)$ for all $x \in G$.*

The converse of Theorem 5 holds.

Wilson's functional equation

Lemma 6. *Any solution (f,g) such that $f \neq 0$ of Wilson's functional equation*

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x,y \in G, \tag{6}$$

satisfies d'Alembert's long functional equation

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y) \text{ for all } x,y \in G. \tag{7}$$

We used the lemma to solve Wilson's functional equation in the abelian case:

Corollary 7. *Let G be an abelian, topological group. Let the pair (f,g), where $f : G \to \mathbb{C}$ and $g \in C(G)$, be a solution of Wilson's classical functional equation*

$$f(x+y) + f(x-y) = 2f(x)g(y), \; x,y \in G. \tag{8}$$

If $f \neq 0$, then there exists a character $\chi \in C(G)$ on G such that g has the form $g = (\chi+\check{\chi})/2$. The character χ is uniquely determined by $g = (\chi+\check{\chi})/2$ except that it can be interchanged with $\check{\chi}$.

Given $g = (\chi+\check{\chi})/2$ where $\chi \in C(G)$ is a character we have

(a) When $\chi \neq \check{\chi}$ there exist constants $\alpha, \beta \in \mathbb{C}$ such that

$$f = \alpha\,\frac{\chi - \check{\chi}}{2} + \beta\,\frac{\chi + \check{\chi}}{2}.$$

(b) When $\chi = \check{\chi}$ there exist an additive function $a \in C(G)$ and a constant $\beta \in \mathbb{C}$ such that $f = (a+\beta)\chi$.

If furthermore $f \neq 0$ is bounded, then g is bounded, χ is unitary and the additive function $a : G \to \mathbb{C}$ in (b) *vanishes.*

Conversely, if $\chi \in C(G)$ is a character on G then $g = (\chi+\check{\chi})/2$ and the formulas for f from (a) *and* (b) *define solutions of* (8)*.*