# Homotopy Idempotents and the Bass Conjecture

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## Contents



## 1 Introduction

During the lectures we will give different notions of rank of finitely generated projective modules over integral group rings of finite groups. In particular, we will introduce the notion of the Hattori–Stallings rank of a finitely generated projective module. Using its properties, we will prove the Bass Conjecture for finite groups.

Subsequent lectures will be dedicated to: finitely dominated spaces, the Reidemeister trace, homotopy idempotents, and to a geometric approach to the Bass Conjecture. In order to attack the Bass Conjecture for an arbitrary group using topology, we need a realization of any element of the Wall group of a given integral group ring. Theories of finitely dominated spaces and homotopy idempotents appear automatically at this point. The study of the Reidemeister trace shows a connection with the Bass Conjecture.

## 2 Ranks of projective modules

We assume familiarity with the notion of a projective module. In particular, we will make use of the following:

**Theorem 2.1.** *Let $P$ be an $R$-module.*[1] *The following are equivalent:*

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[1] Throughout these notes, unless specified otherwise, $R$ is an arbitrary ring with identity.

(1) *$P$ is a projective $R$-module.*

(2) *Every exact sequence $0 \to A \to B \to P \to 0$ of $R$-modules splits.*

(3) *$P$ is isomorphic to a direct summand of a free $R$-module. [Additionally, if $P$ is finitely generated, the free module can be taken to be finitely generated.]*

(4) $\mathrm{Hom}_R(P, -)$ *is exact.*

Write $\mathbb{Z}G$ for the integral group ring of $G$. For a $\mathbb{Z}G$-module $P$, set $P_G = \mathbb{Z} \otimes_{\mathbb{Z}G} P$ to be the $\mathbb{Z}$-module obtained from $P$ by extension of scalars via the augumentation homomorphism $\mathbb{Z}G \to \mathbb{Z}$, $g \mapsto 1$.

**Lemma 2.2.** *If $P$ is a finitely generated projective $\mathbb{Z}G$-module, then $P_G$ is a finitely generated free $\mathbb{Z}$-module.*

*Proof.* If $P$ is free, then

$$P_G \cong \mathbb{Z} \otimes_G \Big(\bigoplus_{i=1}^{n} \mathbb{Z}G\Big) \cong \bigoplus_{i=1}^{n} (\mathbb{Z} \otimes_G \mathbb{Z}G) \cong \bigoplus_{i=1}^{n} \mathbb{Z}.$$

In the general case, apply Theorem 2.1: $P \oplus Q \cong F$ for a finitely generated free $\mathbb{Z}G$-module $F$. Thus $P_G \oplus Q_G = F_G$, which is, as we have just seen, a finitely generated free $\mathbb{Z}$-module. Consequently, $P_G$ is a finitely generated projective $\mathbb{Z}$-module, again by Theorem 2.1. Since projective $\mathbb{Z}$-modules are free, the conclusion follows. □

For a finitely generated projective $\mathbb{Z}G$-module $P$ we thus set

$$\epsilon_G(P) = \mathrm{rank}_{\mathbb{Z}}(P_G).$$

Observe that we do not need $G$ to be finite for this definition.

**Lemma 2.3.** *If $G$ is a finite group and $P$ is a finitely generated projective $\mathbb{Z}G$-module, then $P$ is a finitely generated free $\mathbb{Z}$-module.*

*Proof.* It suffices to prove the lemma when $P$ is a finitely generated free $\mathbb{Z}G$-module. In this case, $P \cong \bigoplus_{i=1}^{n} \mathbb{Z}G \cong \bigoplus_{i=1}^{n} \big(\bigoplus_{g \in G} \mathbb{Z}\big)$. □

For a finitely generated projective $\mathbb{Z}G$-module $P$ we can now define

$$\rho_G(P) = \mathrm{rank}_{\mathbb{Z}}\, P/|G|.$$

Our immediate goal is to show that for an arbitrary finite group $G$, the equality $\epsilon_G(P) = \rho_G(P)$ holds for any finitely generated projective $\mathbb{Z}G$-module $P$.

**Example 2.4.** If $P$ is a finitely generated free $\mathbb{Z}G$-module, then

$$\epsilon_G(P) = \mathrm{rank}_{\mathbb{Z}}(P_G) = \mathrm{rank}_{\mathbb{Z}}\Bigg(\mathbb{Z} \otimes_G \Big(\bigoplus_{i=1}^{n} \mathbb{Z}G\Big)\Bigg) = \mathrm{rank}_{\mathbb{Z}} \bigoplus_{i=1}^{n} \Big(\mathbb{Z} \otimes_G \mathbb{Z}G\Big) = \mathrm{rank}_{\mathbb{Z}} \bigoplus_{i=1}^{n} \mathbb{Z} = n.$$

On the other hand,

$$\rho_G\Big(\bigoplus_{i=1}^{n} \mathbb{Z}G\Big) = \mathrm{rank}_{\mathbb{Z}}\Big(\bigoplus_{i=1}^{n} \mathbb{Z}G\Big)/|G| = \mathrm{rank}_{\mathbb{Z}}\Big(\bigoplus_{i=1}^{n} \bigoplus_{g \in G} \mathbb{Z}\Big)/|G| = n|G|/|G| = n.$$

Hence both $\epsilon_G$ and $\rho_G$ give the 'right' answer if $P$ is the free module $\bigoplus_{i=1}^{n} \mathbb{Z}G$. Otherwise than that, however, it is not clear at all that they have anything to do with each other. In fact, it is not even obvious whether $\rho_G(P)$ is always an integer.

We will now define the Hattori–Stallings rank and show that all these notions of 'rank' coincide for finite groups. It is a conjecture of Bass that they coincide for all groups.[2]

[2]In order to make sense out of that last assertion, one needs a definition of $\rho_G$ for an arbitrary group $G$; Proposition 5.2 provides one.

## 3 The Hattori–Stallings rank

Recall that for a vector space $V$ over a field $k$, $\dim_k(V) = \mathrm{tr}_k(\mathrm{id}_V)$. The Hattori–Stallings rank is a generalization of this idea to finitely generated projective $\mathbb{Z}G$-modules. The big obstacle to overcome is that, in general, group rings are not commutative.

Let $R$ be a ring. Set $T(R) = R/[R,R]$, where $[R,R]$ is the abelian group generated by elements of the form $xy - yx$, $x$, $y \in R$. Warning: $[R,R]$ is not an ideal in general, hence $T(R)$ is only an abelian group.

Let $F$ be a finitely generated free $R$-module and $\alpha\colon F \to F$ an endomorphism. *The Hattori–Stallings trace* of $\alpha$ is defined to be $HS_R(\alpha) = \mathrm{tr}(\alpha) = \sum \bar{\alpha}_{ii}$, where $[\alpha_{ij}]$ is the matrix of $\alpha$ relative to a basis of $F$, and $\bar{\alpha}_{ij} \in T(R)$. We then set

$$HS_R(F) = HS_R(\mathrm{id}_F).$$

It is straightforward to verify that $\mathrm{tr}(\alpha\beta) = \mathrm{tr}(\beta\alpha)$ for $\alpha$ an $m \times n$ matrix and $\beta$ an $n \times m$ matrix. It follows that $\mathrm{tr}(\beta\alpha\beta^{-1}) = \mathrm{tr}(\alpha)$ for any invertible $\beta$, hence the trace is a well-defined element of $T(R)$, independent of the choice of basis.

**Remark 3.1.** We need not worry about the cardinality of a basis of a free $\mathbb{Z}G$-module for any group $G$, since there is an epimorphism $\mathbb{Z}G \to \mathbb{Z}$. In general, in this situation some of the 'good' properties of $\mathbb{Z}$ are inherited by $\mathbb{Z}G$.

We will now extend the definition of the trace to finitely generated projective modules.

Let $P$ be a finitely generated projective $R$-module, $\alpha\colon P \to P$ an endomorphism. Express $Q$ as a direct summand of a finitely generated free $R$-module $F$ via the maps $\pi\colon F \to P$, $i\colon P \to F$ with $\pi i = \mathrm{id}_P$. Then $\alpha$ gives rise to the endomorphism $i\alpha\pi$ of $F$. Define $HS_R(\alpha) = HS_R(i\alpha\pi)$. It is routine to verify that this definition does not depend on the choice of $Q$ and $F$.

For a finitely generated projective $R$-module $P$, we set

$$HS_R(P) = HS_R(\mathrm{id}_P) = HS_R(i\pi).$$

**Remark 3.2.** Note that $(i\pi)^2 = i\pi$. This is a (partial) reason for the 'idempotents' in the title of the lectures: we will be interested in calculating the Hattori–Stallings trace of idempotents.

The next step is to investigate the behaviour of the Hattori–Stallings trace with respect to extension of scalars. The following proposition will play an important role in the next two sections.

**Proposition 3.3.** *Let $\varphi\colon A \to B$ be a ring homomorphism and $T(\varphi)\colon T(A) \to T(B)$ the induced group homomorphism. If $\alpha\colon P \to P$ is an endomorphism of a finitely generated projective $A$-module, then $\mathrm{id}_B \otimes_A \alpha\colon B \otimes_A P \to B \otimes_A P$ is an endomorphism of a finitely generated projective $B$-module and $\mathrm{tr}_B(\mathrm{id}_B \otimes_A \alpha) = T(\varphi)\big(\mathrm{tr}_A(\alpha)\big)$.*

*Proof.* It is clear that it suffices to consider the case when $P$ is free.

Let $[\alpha_{ij}]$ be the matrix of $\alpha$ relative to a basis $\{e_k\}$ of $P$. Then $\{1 \otimes_A e_k\}$ is a basis of $B \otimes_A P$, and

$$\begin{aligned}(\mathrm{id}_B \otimes_A \alpha)(1 \otimes_A e_k) &= 1 \otimes_A \alpha(e_k) = \sum 1 \otimes_A (\alpha_{ik} e_i) = \sum \big((1 \cdot \alpha_{ik}) \otimes_A e_i\big) \\ &= \sum \varphi(\alpha_{ik}) \otimes_A e_i = \sum \varphi(\alpha_{ik})(1 \otimes_A e_i).\end{aligned}$$

Hence the matrix of $\mathrm{id}_B \otimes_A \alpha$ relative to the basis $\{1 \otimes_A e_k\}$ is obtained by applying $\varphi$ to the entries of $[\alpha_{ij}]$, and the conlusion follows. □

# 4 Ranks over commutative rings

Let $A$ be an integral domain with field of fractions $k$. Define $\mathrm{rank}_A(P) = \dim_k(k \otimes_A P)$. [Recall that there is an inclusion $i\colon A \hookrightarrow k$.]

**Proposition 4.1.** *Let $P$ be a finitely generated projective $A$-module. Then*

$$HS_A(P) = \mathrm{rank}_A(P) \cdot 1,$$

*where $1 \in A = T(A)$ is the identity element.*

*Proof.* By Proposition 3.3, $HS_k(k \otimes_A P) = T(i)\big(HS_A(P)\big)$. However, $T(i)$ is simply an inclusion, hence $HS_k(k \otimes_A P) = HS_A(P)$. On the other hand,

$$HS_k(k \otimes_A P) = \dim_k(k \otimes_A P) \cdot 1 = \mathrm{rank}_A(P) \cdot 1.$$

□

Our goal now is to obtain a generalization of Proposition 4.1 to indecomposable rings which are not necessarily domains.[3]

Since $A$ is no longer an integral domain, we do not have a field of fractions to work with. We do, however, have a localization $A_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \lhd A$. The ring $A_{\mathfrak{p}}$ is local, and we have:

**Proposition 4.2.** *Any finitely generated projective module over a local ring is free.*

We can therefore define the *rank of $P$ at $\mathfrak{p}$* to be the rank of the free $A_{\mathfrak{p}}$-module $A_{\mathfrak{p}} \otimes_A P$. What we thus have is a function $\mathscr{R}_A(P)\colon \mathrm{Spec}(A) \to \mathbb{Z}$, whose value at $\mathfrak{p}$ is the rank of $P$ at $\mathfrak{p}$.

**Proposition 4.3.** *The function $\mathscr{R}_A(P)$ is locally constant. If $A$ is indecomposable, then* Spec$(A)$ *is connected and $\mathscr{R}_A(P)$ is constant.*

With this in mind, set $\mathrm{rank}_A(P)$ to be the constant value of the function $\mathscr{R}_A(P)$.

**Proposition 4.4.** *Let $A$ be a commutative indecomposable ring, $P$ a finitely generated projective $A$-module. Then $HS_A(P) = \mathrm{rank}_A(P) \cdot 1$.*

Proof is pretty much the same as that of Proposition 4.1.

# 5 The Bass conjecture; Swan's theorem

Let $\Gamma$ be a finite group. In order to understand the Hattori–Stallings rank $HS_\Gamma$, we need to take a closer look at the group $T(\mathbb{Z}\Gamma)$.

First of all, note that $st - ts = t^{-1}(ts)t - ts = t^{-1}s_1t - s_1$. It follows easily that $T(\mathbb{Z}\Gamma)$ can be identified with the free abelian group on the set of conjugacy classes of elements of $\Gamma$. Thus any element of $T(\mathbb{Z}\Gamma)$ has the form $\sum_{\gamma\in\mathscr{C}} t(\gamma)[\gamma]$, where $\mathscr{C}$ is a set of representatives for the conjugacy classes of elements of $\Gamma$.

Let $P$ be a finitely generated projective $\mathbb{Z}\Gamma$-module. In view of the above, its Hattori–Stallings rank has an expansion

$$HS_\Gamma(P) = \sum_{\gamma\in\mathscr{C}} HS_\Gamma(P)(\gamma)[\gamma].$$

**The Strong Bass Conjecture:** $HS_\Gamma(P) = HS_\Gamma(P)(1)[1]$.

**The Weak Bass Conjecture:** $\sum_{\gamma\in\mathscr{C},\gamma\neq 1} HS_\Gamma(P)(\gamma) = 0$.

Next week we will show how to attack this problem from a topological point of view.

[3]Indecomposability is a necessary restriction on the base ring in order to have a 'reasonable' notion of a $\mathbb{Z}$-valued rank for finitely generated projective modules over it. See [Brown, Chapter IX, Section 1, Exercise 1].

**Proposition 5.1.** *Let $\Gamma$ be a group and $\Gamma' \subseteq \Gamma$ a subgroup of finite index. For any $\gamma \in \Gamma'$ there is an integer $n(\gamma) > 0$ such that for every finitely generated projective $\mathbb{Z}\Gamma$-module $P$ we have*

$$HS_{\Gamma'}(P)(\gamma) = n(\gamma) HS_{\Gamma}(P)(\gamma).$$

*Moreover, $n(1) = [\Gamma \colon \Gamma']$.*

**Proposition 5.2.** *Let $\Gamma$ be a finite group. Both $\epsilon_\Gamma$ and $\rho_\Gamma$ can be expressed in terms of the Hattori–Stallings rank. Precisely, given a finitely generated projective $\mathbb{Z}\Gamma$-module $P$,*

$$\epsilon_\Gamma(P) = \sum_{\gamma \in \mathscr{C}} HS_\Gamma(P)(\gamma),$$
$$\rho_\Gamma(P) = HS_\Gamma(P)(1).$$

*Proof.* Proposition 3.3 applied to the augumentation homomorphism $\varphi \colon \mathbb{Z}\Gamma \to \mathbb{Z}$ gives:

$$\begin{aligned}\epsilon_\Gamma(P) &= \operatorname{rank}_{\mathbb{Z}}(\mathbb{Z} \otimes_\Gamma P) = \operatorname{tr}_{\mathbb{Z}}(\mathrm{id}_{\mathbb{Z}} \otimes_\Gamma \mathrm{id}_P) = T(\varphi)\big(\operatorname{tr}_\Gamma(\mathrm{id}_P)\big) \\ &= T(\varphi)\big(HS_\Gamma(P)\big) = \sum_{\gamma \in \mathscr{C}} HS_\Gamma(P)(\gamma).\end{aligned}$$

The second equality is a consequence of Proposition 5.1: taking $\Gamma' = \{1\}$ simply says that $\operatorname{rank}_{\mathbb{Z}}(P) = |\Gamma| HS_\Gamma(P)(1)$, and the conclusion follows. □

**Theorem 5.3.** *If $\Gamma$ is finite and $P$ is a finitely generated projective $\mathbb{Z}\Gamma$-module, then*

$$HS_\Gamma(P)(\gamma) = 0 \text{ for any } \gamma \in \Gamma,\ \gamma \neq 1.$$

*Thus there exists an integer $r$ such that $HS_\Gamma(P) = r \cdot [1]$ and one has $\epsilon_\Gamma(P) = \rho_\Gamma(P) = r$.*

*Proof.* Let $\gamma \in \Gamma$, $\gamma \neq 1$ and consider $\Gamma' = \langle \gamma \rangle$, the finite cyclic group generated by $\gamma$. By Proposition 5.1, $HS_{\Gamma'}(P)(\gamma) = n(\gamma) HS_\Gamma(P)(\gamma)$ for a positive integer $n(\gamma)$, hence is is enough to prove the theorem for $\Gamma'$.

Obviously $\mathbb{Z}\Gamma'$ is commutative, and it follows from formal properties of $\rho$ that $\mathbb{Z}\Gamma'$ is indecomposable. Proposition 4.4 is therefore applicable and gives $HS_{\Gamma'}(P) = \operatorname{rank}_{\mathbb{Z}\Gamma'}(P)[1]$, which shows that $HS_{\Gamma'}(P)$ is an integral multiple of $[1]$, exactly as desired.

The second assertion follows from the first one in view of Proposition 5.2.

□

# 6 Finitely dominated spaces

Recall that a space $X$ is said to be *dominated* by a space $Y$ if there exist maps $X \xrightarrow{i} Y \xrightarrow{r} X$ with $ri \simeq \mathrm{id}_X$.

**Problem 1.** Suppose $Y$ is CW-complex which dominates a space $X$. Does $X$ have the homotopy type of a CW-complex?

Whitehead: yes. (See, for example, [Hatcher, Proposition A.11].)

**Problem 2.** Suppose additionally $Y$ is a finite CW-complex. Does $X$ have the homotopy type of a finite CW-complex?

Wall: in general, no. There is, however, a result of Ferry which says that in this situation $X$ has the homotopy type of a compact metric space. [This leads to the following (very) general question: is there an 'algebraic topology' for compact metric spaces? There is one, in fact – shape theory, although we will not dwell on this right now.]

**Open problem.** Suppose $Y$ is a finite CW-complex which dominates an $H$-space $X$. Does $X$ have the homotopy type of a finite CW-complex?

**Remark 6.1.** A domination $X \xrightarrow{i} Y \xrightarrow{r} X$ gives rise to a homotopy idempotent:

$$(ir)^2 = i(ri)r \simeq ir.$$

## 7 CW-complexes of finite type; Milnor's construction

Let $K$, $X$ be CW-complexes, $\varphi\colon K \to X$ a map. Set $M_\varphi = X \cup_\varphi (K \times I)$ to be the mapping cylinder of $\varphi$, and $\pi_i(\varphi) = \pi_i\big(M_\varphi, K \times \{1\}\big)$, $i \geq 0$.

Recall that if both $K$ and $X$ are connected, and $\pi_i(\varphi) = 0$ for $0 \leq i \leq n$, then $\varphi$ is said to be *$n$-connected*.

**Remark 7.1.** If $\varphi\colon K \to X$ is an $n$-connected map with $n \geq 2$, then $\varphi_*\colon \pi_1(K) \to \pi_1(X)$ is an isomorphism, so we can regard $\pi_{n+1}(\varphi)$ as a $\mathbb{Z}\pi_1(X)$-module.

Suppose $X$ has the homotopy type of a connected CW-complex. Following Wall, we shall describe various conditions on $X$.

**Definition 7.2.** $(F_1)$ The group $\pi_1(X)$ is finitely generated.

$(F_2)$ The group $\pi_1(X)$ is finitely presented, and for any finite CW-complex $K^2$ and any map $\varphi\colon K^2 \to X$ inducing an isomorphism of fundamental groups, $\pi_2(\varphi)$ is a finitely generated $\mathbb{Z}\pi_1(X)$-module.

$(F_n)$ Condition $F_{n-1}$ holds, and for any finite CW-complex $K^{n-1}$ and any $(n-1)$-connected map $\varphi\colon K^{n-1} \to X$, $\pi_n(\varphi)$ is a finitely generated $\mathbb{Z}\pi_1(X)$-module.

Recall that a CW-complex $X$ is said to be of *finite type* if each $n$-skeleton $X^n$ of $X$ is finite. The aim of this section is to give a sketch of proof of the following theorem.

**Theorem 7.3.** *Suppose $X$ has the homotopy type of a connected CW-complex. The following are equivalent:*

(1) *$X$ is homotopy equivalent to a CW-complex of finite type.*

(2) *$X$ is dominated by a CW-complex of finite type.*

(3) *$X$ satisfies $F$, where $F$ means $F_n$ for all $n \geq 1$.*

**Milnor's construction.** Let $\varphi\colon K \to X$ be an $(n-1)$-connected map with $n \geq 2$. If $n = 2$, let $\{\alpha_j\}_{j \in J}$ denote a set of generators of the group $\pi_n(\varphi)$; if $n \geq 3$, let $\{\alpha_j\}_{j \in J}$ denote a set of generators of the $\mathbb{Z}\pi_1(X)$-module $\pi_n(\varphi)$. Use $\alpha_{j|S^{n-1}}$ to attach $n$-cells to $K$, and denote the resulting space as $L$. Write $\psi\colon L \to X$ for the obvious extension of $\varphi$ over $L$.

**Lemma 7.4.** *The map $\psi$ is $n$-connected.*

*Proof.* Consider the following portion of the exact sequence of homotopy groups:

$$\cdots \to \pi_n(L, K) \xrightarrow{\alpha} \pi_n(\varphi) \to \pi_n(\psi) \to \pi_{n-1}(L, K) \to \cdots$$

Since $\pi_k(L, K) = 0$ for $0 \leq k \leq n-1$ and $\alpha$ is onto by contruction, $\pi_k(\psi)$ vanishes for $0 \leq k \leq n$, and so $\psi$ is $n$-connected. □

This gives an idea how to make a map between CW-complexes a homotopy equivalence.

**Proposition 7.5.** *Let $K$ be a CW-complex, $X$ have the homotopy type of one, and suppose $\varphi\colon K \to X$ is an $(n-1)$-connected map. Then there exists a CW-complex $L$, obtained from $K$ by attaching cells of dimension $\geq n$, and a homotopy equivalence $\psi\colon L \to X$ extending $\varphi$.*

*Proof.* Proposition 7.4 allows us to attach $n$-cells to $K$ to make $\varphi$ $n$-connected. Repeat the process indefinitely and apply the Whitehead Theorem. □

**Theorem 7.6.** *Suppose $X$ has the homotopy type of a connected CW-complex. The following are equivalent:*

(1) *$X$ is homotopy equivalent to a CW-complex with finite $n$-skeleton.*

(2) *X is dominated by a CW-complex with finite n-skeleton.*

(3) *X satisfies $F_n$.*

*Proof.* Because of time restrictions, we will confine ourselves to only a partial proof.

(1) ⇒ (2) is trivial; (2) ⇒ (1) relies on the fact that a retract of a finitely presented group is again finitely presented.

(3) ⇒ (1). If $X$ satisfies $F_1$, we can find a finite bouquet of circles $K^1$ and a map $K^1 \to X$ which induces an epimorphism on fundamental groups, and so is 1-connnected. Now apply Proposition 7.5.

If $X$ satisfies $F_2$, there exist: a finite CW-complex $K^2$ and a map $\varphi\colon K^2 \to X$ inducing an isomorphism on fundamental groups. The group $\pi_2(\varphi)$ is finitely generated by hypothesis, hence we can attach a finite collection of 2-cells to $K^2$ – forming $L^2$ – and extend $\varphi$ to a 2-connected map $L^2 \to X$. Applying Proposition 7.5 completes the construction.

For $n \geq 3$, proceed inductively along the same lines. □

Theorem 7.3 is a straightforward consequence of Theorem 7.6.

## 8 CW-complexes of finite dimension

The algebraic conditions imposed by finite dimension are much clearer than in the case of finite type.

**Definition 8.1.** We say that $X$ satisfies $D_n$ if:

(1) $H_i(\tilde{X}) = 0$ for $i > n$. [Here $\tilde{X}$ denotes the universal cover of $X$.]

(2) $H^{n+1}(X, \mathscr{B}) = 0$ for any system of local coefficients $\mathscr{B}$.

Clearly, if $X$ is homotopy equivalent to a CW-complex $K$ with $\dim K \leq n$, then $X \in D_n$.

**Proposition 8.2.** *Let $n \geq 3$ and $\varphi\colon K^{n-1} \to X$ be an $(n-1)$-connected map. Suppose $H^{n+1}(X, \mathscr{B}) = 0$ for any local coefficient system $\mathscr{B}$. Then $\pi_n(\varphi)$ is a projective $\mathbb{Z}\pi_1(X)$-module.*

The main result of this section is given by:

**Theorem 8.3.** *Let $n \geq 3$. $X$ is homotopy equivalent to an $n$-dimensional CW-complex $K$ if and only if $X$ satisfies $D_n$. Moreover, if $1 \leq r \leq n-2$ and $X$ satisfies $D_n$ and $F_r$ simultaneously, $K$ can be taken to have finite $r$-skeleton.*

## 9 The obstruction to finiteness

Let $R$ be a ring. We define $K_0(R)$ to be the quotient of the free abelian group generated by the isomorphism classes of finitely generated projective $R$-modules by the subgroup generated by elements of the form $[A] + [B] - [A \oplus B]$. $K_0(R)$ is called the *Grothendieck group*.

**Example 9.1.** Since all projective modules over $\mathbb{Z}$ are free, $K_0(\mathbb{Z}) \cong \mathbb{Z}$.

Consider the map $\mathbb{Z} \to K_0(R)$, $1 \mapsto [R]$; write $\tilde{K}_0(R)$ for its cokernel.

**Remark 9.2.** Let $P$ and $Q$ be finitely generated projective $R$-modules. One easily sees that $[P] = [Q]$ in $K_0(R)$ if and only if there exists a free $R$-module $F$ such that $P \oplus F \cong Q \oplus F$. Likewise, $[P] = [Q]$ in $\tilde{K}_0(R)$ if and only if there exist free $R$-modules $F_1$, $F_2$ such that $P \oplus F_1 \cong Q \oplus F_2$.

Let $n \geq 3$ and suppose $X$ satisfies both $D_n$ and $F_n$. Take a finite CW-complex $K^{n-1}$ and an $(n-1)$-connected map $\varphi\colon K^{n-1} \to X$; then $\pi_n(\varphi)$ is a finitely generated projective $\mathbb{Z}\pi_1(X)$-module. Consequently, $X$ determines an element of $\tilde{K}_0(\mathbb{Z}\pi_1(X))$. Verifying that the element depends only on the homotopy type of $X$ requires some work.

Write $P$ for $\pi_n(\varphi)$ and let $F$ be finitely generated free, $F \cong P \oplus Q$. Attach cells to $K^{n-1}$ corresponding to $F$; this yields a finite CW-complex $L^n$ and an $n$-connected map $\psi\colon L^n \to X$ with $\pi_{n+1}(\psi) \cong Q$.

**Lemma 9.3.** $[Q] = -[P]$ *in* $\tilde{K}_0(\mathbb{Z}\pi_1(X))$.

*Proof.* Consider the following portion of the long exact sequence of homology groups:

$$\cdots \to H_{n+1}(\tilde{X}, \tilde{K}) \to H_{n+1}(\tilde{X}, \tilde{L}) \to H_n(\tilde{L}, \tilde{K}) \to H_n(\tilde{X}, \tilde{K}) \to H_n(\tilde{X}, \tilde{L}) \to \cdots$$

Since $\psi$ is $n$-connected, $\pi_n(\psi) = \pi_n(X, L) \cong H_n(\tilde{X}, \tilde{L}) = 0$. Moreover, $X$ satisfies $D_n$, so $H_{n+1}(\tilde{X}, \tilde{K}) \cong H_{n+1}(\tilde{X}) = 0$. Thus there is a short exact sequence

$$0 \to H_{n+1}(\tilde{X}, \tilde{L}) \to H_n(\tilde{L}, \tilde{K}) \to H_n(\tilde{X}, \tilde{K}) \to 0.$$

Now observe that $\pi_n(\tilde{X}, \tilde{K}) = \pi_n(\varphi) = P$ and $H_n(\tilde{L}, \tilde{K}) = C_n(\tilde{L}) \cong F$, hence we have

$$0 \to H_{n+1}(\tilde{X}, \tilde{L}) \to F \to P \to 0.$$

Consequently, $F \cong H_{n+1}(\tilde{X}, \tilde{L}) \oplus P$. It follows that $[H_{n+1}(\tilde{X}, \tilde{L})] = [\pi_{n+1}(\psi)] = -[P]$. □

The remaining part of the argument factors through:

**Lemma 9.4.** *Let $X$ satisfy $D_n$ and $\psi\colon L \to X$ be $n$-connected. Then $\psi$ has a homotopy right inverse, so $L$ dominates $X$.*

**Lemma 9.5.** *Suppose $X$ satisifes both $F_n$ and $D_n$. Let $L_i^n$ be finite, $\psi\colon L_i \to X$ be $n$-connected and $Q_i = \pi_{n+1}(\psi)$, $i = 1, 2$. Then $[Q_1] = [Q_2]$ in $\tilde{K}_0(\mathbb{Z}\pi_1(X))$.*

We have thus constructed the so called 'obstruction to finiteness'.

**Theorem 9.6.** *$X$ is dominated by a finite CW-complex of dimension $n$ if and only if it satisfies both $F_n$ and $D_n$. When this holds, and $n \geq 2$, there is an obstruction in $\tilde{K}_0(\mathbb{Z}\pi_1(X))$, depending only on the homotopy type of $X$, which vanishes if $X$ is finite, and whose vanishing is necessary and sufficient for $X$ to be homotopy equivalent to a finite complex.*

**Theorem 9.7.** *Let $K^{n-1}$ be a finite CW-complex, $n \geq 3$, and $\alpha \in \tilde{K}_0(\mathbb{Z}\pi_1(K))$ be arbitrary. Then there exists a finitely dominated CW-complex $X$ with $X^{n-1} = K$ and $\tilde{\omega}(X) = \alpha$.*

## 10 The Reidemeister trace and homotopy idempotents

We will now show how to attack the Bass Conjecture using topology.

Let $G$ be an arbitrary group and $[P]$, $[Q] \in K_0(\mathbb{Z}G)$. By linearity of the Hattori–Stallings rank, if $[P] = [Q]$, then $HS_G(P) = HS_G(Q)$. Hence the strong version of the Bass conjecture says that for any $[P] \in K_0(\mathbb{Z}G)$, the equality $HS_G([P]) = r \cdot [1]$ holds for an integer $r$.

**The Reidemeister trace.** Let $X$ be a finite CW-complex, a base point $x \in X$ implicit; $\tilde{X}$ its universal cover. It is well-known that the cellular chain complex $C_*(\tilde{X})$ is a finitely generated free $\mathbb{Z}\pi_1(X)$-module.

Let $f\colon (X, x) \to (X, x)$ be a map. Define elements $\alpha$, $\beta \in \pi_1(X)$ to be *$f_*$-conjugate* if

$$\alpha = z\beta f_*(z)^{-1} \text{ for some } z \in \pi_1(X).$$

Let $\pi_1(X)_{f_*}$ denote the set of $f_*$-conjugacy classes, making $\mathbb{Z}\pi_1(X)_{f_*}$ a quotient of $\mathbb{Z}\pi_1(X)$.

Lift $f$ to a map $\tilde{f}\colon \tilde{X} \to \tilde{X}$ of the universal cover. The *Reidemeister trace* is defined as

$$R(f, x) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}\big(\tilde{f}_*\colon C_i(\tilde{X}) \to C_i(\tilde{X})\big),$$

where $\operatorname{tr}\big(\tilde{f}_*\colon C_i(\tilde{X}) \to C_i(\tilde{X})\big) \in \mathbb{Z}\pi_1(X)_{f_*}$.

**Remark 10.1.** Observe that if $\tilde{X} = X$, then $R$ is just the ordinary Lefschetz trace.

**Homotopy idempotents.** Recall that a map $f\colon X \to X$ of a finite CW-complex $X$ is said to *split* if there exist: a CW-complex $Y$ and maps $X \xrightarrow{u} Y \xrightarrow{v} X$ such that $vu = f$ and $uv \simeq \mathrm{id}_Y$.

**Theorem 10.2.** *If $f$ is a homotopy idempotent of a finite CW-complex, then $f$ splits.*

**Remark 10.3.** If one removes the finiteness hypothesis from the theorem, it does not hold. The counterexample is given by the Thomson group $T$ and the homomorphism $h\colon T \to T$, $h(x_i) = x_{i+1}$. This induces a non-splitting map $K(T,1) \to K(T,1)$.

Suppose $f\colon X \to X$ splits. It follows from the definition that $Y$ is finitely dominated by $X$ and $\omega(Y) \in \tilde{K}_0\big(\mathbb{Z}\pi_1(Y)\big)$. Furthermore, $u_*v_*$ is an isomorphism, hence $v_*$ is a monomorphism.

Consider $\tilde{K}_0(v_*)\colon \tilde{K}_0\big(\pi_1(Y)\big) \to \tilde{K}_0\big(\pi_1(X)\big)$, $[P] \mapsto \pi_1(X) \otimes_{\pi_1(Y)} [P]$.

**Definition 10.4.** Set $\omega(f) = \tilde{K}_0(v_*)\big(\omega(Y)\big)$. If $\omega(f)$ is trivial, then $f$ splits via a finite CW-complex.

Since $uv \simeq \mathrm{id}_Y$, $f\colon X^n \to X^n$ is a homotopy idempotent.

**Theorem 10.5.** *Let $G$ be a finitely presented group. For any $\alpha \in \tilde{K}_0(\mathbb{Z}G)$ there exist: a finite-dimensional CW-complex $X$ with $\pi_1(X) \cong G$ and a homotopy idempotent $f\colon X \to X$ such that $f_*\colon \pi_1(X) \to \pi_1(X)$ is the identity and $\omega(f) = \alpha$.*

## 11 A topological approach to the Bass conjecture

(1) Let $G$ be an arbitrary group; fix $\alpha \in K_0(\mathbb{Z}G)$. We want to show that that $HS_G(\alpha) = r \cdot [1]$.

(2) A key observation: If the Bass Conjecture is true for finitely presented groups, then it is true for all groups.

(3) Suppose $G$ is a finitely presented group. Theorem 10.5 is applicable, hence we have a finite-dimensional CW-complex $X$ and a homotopy idempotent $f\colon X \to X$ such that $f_* = \mathrm{id}_{\pi_1(X)}$ and $\omega(f) = \alpha$.

(4) **Lemma 11.1** (Geoghegan)**.** $HS_G\big(\omega(f)\big) = R(f,x)$.

(5) The trick is to calculate the Reidemeister trace $R(f,x) = \sum_{s\in\mathscr{C}} t(s)[s]$. Recall that the *Nielsen number* of $f$ is given by $N(f) = \#\{s \in \mathscr{C} \mid t(s) \neq 0\}$.

(5) **Proposition 11.2.** *If $N(f) = 0$ or $1$, then the Bass Conjecture follows.*

*Proof.* It is clear when $N(f) = 0$. If $N(f) = 1$, then $R(f,x) = r \cdot [s]$ for some $s \in \mathscr{C}$. If $[s] = [1]$, the conclusion follows. Assume $[s] \neq [1]$. Let $f' = f \vee \mathrm{id}_{S^2}\colon X \vee S^2 \to X \vee S^2$. Then $\omega(f') = \omega(f) \oplus [\mathbb{Z}G]$, which means that $\omega(f') = \omega(f)$ in $\tilde{K}_0(\mathbb{Z}G)$. On the other hand,

$$\begin{aligned}
R(f',1) &= \sum_{i=0}^{n} (-1)^i \mathrm{tr}\big(\tilde{f}'_*\colon C_i(\widetilde{X \vee S^2}) \to C_i(\widetilde{X \vee S^2})\big) \\
&= \sum_{i=0}^{n} (-1)^i \mathrm{tr}\big(\tilde{f}'_*\colon C_i(\tilde{X}) \oplus C_i(S^2) \to C_i(\tilde{X}) \oplus C_i(S^2)\big) \\
&= R(f,x) + \sum_{i=0}^{n} (-1)^i \mathrm{tr}\big(\mathrm{id}\colon C_i(S^2) \to C_i(S^2)\big) = R(f,x) + 1[e] + 1[e] \\
&= 2[e] + r[s].
\end{aligned}$$

Consequently, $N(f) = 2$. A contradiction. ∎

## References


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