JAROSLAV SMÍTAL – CV

Prof. RNDr. Jaroslav Smítal, DrSc. *Born in 1942 in Kroměříž, Czech Republic. Graduated in 1966 at the Comenius University in Bratislava (Mathematical analysis). In 1970 (equivalent of) Ph.D. degree, Associate Professor (hab. Docent) in 1980, the second scientific degree DrSc. in 1985, Full Professor in 1989, all degrees in Mathematical analysis, from the Comenius University in Bratislava, Slovakia. Until l992 Professor of Mathematics at the Comenius University in Bratislava (full position). In 1990 - 91 Vice - Rector of the Comenius University .*

Since 1993 Professor of Mathematics at the Silesian University in Opava, Czech Republic (full position), 1993 – 2005 and since 2008 director of the Mathematical Institute, 1995 - 98 Vice-Rector of the Silesian University in Opava. In 1995 elected a member of the Czech Learned Society. In 1990 - 1995 Vice-President of the Accreditation Board of the Government of Slovak Republic and in 1992 - 1998 Member of the Accreditation Board of the Government of Czech Republic. Member of many other scientific boards, editorial boards etc. both home and abroad.

Main fields of scientific work
real analysis, functional equations and dynamical systems

Principal scientific activities
Invited lectures at tens of conferences and tens of universities abroad (USA, Canada, Portugal, Spain, Italy, Turkey, Austria, Germany, The Nederlands, Danmark, Poland, Hun-gary, Ukraine). Visiting Professor University of Waterloo, Ont., Canada (1982), University of Massachusetts, Amherst, USA (1989, 1990, 1992 and 1995), University of California, Santa Barbara, USA (1991), University of Milan, Italy (1992, 1996, 2003, 2004, 2005, 2006), Universita Autonoma Barcelona, Spain (1994).

Author or coauthor of more than 90 papers, 10 books, with more than 1100 citations. The best doctoral students: *L. Snoha* (B. Bystrica, Slovakia), *K. Janková* (Bratislava, Slovakia), *T. Gedeon* (Montana State University, USA), *J. Bobok* (Czech Technical University, Prague), *D. Pokluda* (Microsoft, Seattle, USA), *M. Štefánková* (Opava), *M. Mlíchová (Čiklová)* (Opava).

Since 2005 14 papers published in journals from database Web of Science, and talks at 25 international conferences and/or universities. Since 2005, more than 300 citations (of that 100 according to the SCI or SSCI, and 80 so called qualified citations). Training of young scientists in Opava: M. Babilonová (Štefánková), Ph.D. (2000, Prize of the Minister), Doc. (2003); D. Pokluda, Ph.D. (2001); M. Málek, Ph.D. (2002); Z. Kočan, Ph.D. (2002); J. Kupka, Ph.D. (2004), M. Mlíchová (Čiklová), Ph.D. (2008). Since 2000 main organizer of i 9 international conferences in Opava with their scope along the proposed research plan.

Main publications relevant to the proposal

[Sm1] B. Schweizer and J. Smítal, Measures of chaos and a spectral decomposition of dynamical systems on the interval, Trans. Amer. Math. Soc. 344 (1994), 737 – 854 (IF 0.824). More than 200 citations, including 40 according to SCI, and 50 qualified citations.

[Sm2] J. Smítal, Chaotic functions with zero topological entropy, Trans. Amer. Math. Soc. 297 (1986), 269 – 282 (IF 0.824). More than 140 citations, including 50 according to SCI, and 70 qualified citations.

[Sm3] K. Janková and J. Smítal, A characterization of chaos, Bull. Austral. Math. Soc. 34 (1986), 283 – 292 (IF 0.297). More than 50 citations, including 15 according to SCI, and 30 qualified citations.

[Sm4] V. V. Fedorenko, A. N. Šarkovskii and J. Smítal, Characterizations of weakly chaotic maps of the interval, Proc. Amer. Math. Soc. 110 (1990), 141-148 (IF 0.520). More than 50 citations, including 15 according to SCI, and 20 qualified citations.

[Sm5] J. Smítal, A chaotic function with some extremal properties, Proc. Amer. Math. Soc. 87 (1983), 54 – 56 (IF 0.520). More than 50 citations, including 30 according to SCI, and 13 qualified citations.

[Sm6] G. - L. Forti, L. Paganoni and J. Smítal, Dynamics of homeomorphisms on minimal sets generated by triangular mappings. Bull. Austral. Math. Soc. 59 (1999), 1 – 20 (IF 0.297). More than 20 citations, including 15 according to SCI, and 8 qualified citations.

[Sm7] F. Balibrea, J. Smítal and M. Štefánková, The three versions of distributional chaos, Chaos, Solitons & Fractals 23 (2005), 1581 – 1583 (IF 3.025). 40 citations, including 20 according to SCI, and 10 qualified citations.

Total number of results since 2005: 17

Total number of citations: about 450 acc. to SCI, the total number of citations is more than 1100 without autocitations

H-index acc. to Web of Science: 14

Research projects from GAČR (principal researcher)

[P1] GA 201/06/0318 Dynamical systems III (standard project for 2006 – 08)

[P2] GD 201/03/H152 Topological and analytical methods in the theory of dynamical systems and mathematical physics (doctoral project, for 2003 – 07), final evaluation: Good

[P3] GA 201/03/1153 Dynamical systems II (standard project for 2003 – 05), final evaluation: Excellent

[P4] GA 201/00/0859 Dynamical systems (standard project for 2000 – 02), final evaluation: Excellent

[P5] GA 201/97/0001 Dynamical systems (standard project for 1997 – 99), final evaluation: Excellent

[P6] GA 201/94/1088 Dynamical systems (standard project for 1994 - 96), final evaluation: Excellent

Opava, October 4, 2010

SOME ASPECTS OF DISCRETE DYNAMICAL SYSTEMS

J. SMÍTAL

Abstract. This is a brief contents of 5 three-hour lectures, presented on October 4 – 8, 2010 to doctoral students at the Silesian University in Katowice. The main aim is to provide the students necessary background based on classical results and then a survey of most recent results from selected fields like distributional and other forms of chaos, dynamics of triangular maps of the square, and dynamics of minimal systems. Also some open problems are given. Taking into account available time, 15 hours, only some theorems are given with complete proofs, some ones only with a sketch of proof, and some without proof. In any case, there is always remark following the theorem, but proofs are not reproduced in this text. The reader may check the references. Many theorems are presented without formal formulation; this fact is also mentioned in the text on proper places.

Approximate time schedule: 3 hours for the first three sections, and 3 hours for every other section.

1. Introduction and notation

Brief historical survey involves contribution of G. D. Birkhoff , H. Poincaré, M. Stone, among others.

Notation: Let (X, ρ) or (Y, ρ) be a compact metric space, $I = [0, 1]$ the unit interval, and $\mathcal{C}(X)$ the system of continuous maps $X \to X$. For $f \in \mathcal{C}(X)$, let $f^n(x)$, $n \in \mathbb{N}$, be the n-th iterate of f, given by $f^0 = Id$, the identity map, and $f^{n+1} = f \circ f^n$. The *trajectory* of a point $x \in X$ is the sequence $\{f^n(x)\}_{n\geq 0}$. The set of limit points of the trajectory of x is the ω-limit set of x, denoted $\omega_f(x)$. An $x \in X$ is *periodic point* of f, with period $n \geq 1$, if $f^n(x) = x$ and $f^j(x) \neq x$ if $0 < j < n$.

2. Dynamics on the interval - basic properties

Lemma 2.1. (Itinerary lemma.) *Let $f \in \mathcal{C}(I)$, and let $\{I_n\}_{n\geq 0}$ be compact intervals in I such that $f(I_n) \supseteq I_{n+1}$. Then there is an $x \in I$ such that $f^n(x) \in I_n$, for every $n \in \mathbb{N}$.*

Proof is given. It is based on the fact that, if $f(J) \supseteq J$ and J is a compact interval, then f has in J a fixed point.

Theorem 2.2. (Sharkovsky theorem, 1964.) *Let $f \in \mathcal{C}(I)$ have a periodic orbit of period m, and let integer n follows (is greater than) m in the Sharkovsky's ordering. Then f has a periodic orbit of period n.*

Sketch of the proof is given. It is based on Itinerary lemma. Application of *Markov diagrams* are explained. The proof is completed by ilustrative examples of graphs of piecewise linear functions of diff erent types relative to the Sharkovsky's ordering. In particular, functions of type 2^n, 2^∞, $2n + 1$, $2^n \cdot 3$, and period dabling functions are given. Historical remarks. Generalizations of Sharkovsky's theorem to other one-dimensional systems like the circle, finite graphs, and skew-product maps of I^n.

3. ω-LIMIT SETS

Theorem 3.1. *Let $f \in \mathcal{C}(X)$, $x \in X$, and F a proper compact subset of $\omega_f(x)$. Then $F \cap \overline{f(\omega_f(x) \setminus F)} \neq \emptyset$.*

Proof is given, along with some consequences (every finite ω-limit set is a cycle, topological structure of ω-limit sets on I and on X, etc).

Theorem 3.2. (Bruckner, Smítal, 1991.) *A compact set $F \subseteq I$ is an ω-limit set of a map $f \in \mathcal{C}(I)$ if and only if F is either the union of finitely many compact intervals, or F is nowhere dense.*

No proof, only some illustrations, consequences, and related results like existence of a "universal"function in $\mathcal{C}(I)$ containing homeomorphic copy of every possible ω-limit set, cf. [3].

Theorem 3.2. (Blokh et al. 1996.) *Let $f \in \mathcal{C}(I)$. Then a compact invariant set $W \neq \emptyset$ is an ω-limit set of f if and only if W is locally expanding.*

No proof, only some illustrations. Definition of a locally expanding set as well as the theorem, are taken from [4].

Theorem 3.3. (Blokh et al. 1996.) *Let $\Lambda(f)$ be the collecion of all ω-limit sets of $f \in \mathcal{C}(I)$, equipped with the Hausdorff metric ρ_H. Then $(\Lambda(f), \rho_H)$ is a compact metric space.*

No proof, only information that this result follows from Theorem 3.2, and why. *Hausdorff metric* $\rho_H(U, V)$, $U, V \subseteq X$, is defined as the supremum of $r > 0$ such that $U \setminus B_r(V) \neq \emptyset$ or $V \setminus B_r(U) \neq \emptyset$, where $B_r(U)$ denotes the open r-neighborhood of U.

4. FURTHER PROPERTIES OF ω-LIMIT SETS ON I

Let X, Y be compact metric spaces. Functions $f \in \mathcal{C}(X)$ and $g \in \mathcal{C}(Y)$ are *conjugate* if there is a homeomorphism $\varphi : X \to Y$ such that $\varphi \circ f = g \circ \varphi$. They are *semiconjugate* (in this order) if there is a continuous onto map $\varphi : X \to Y$ with $\varphi \circ f = g \circ \varphi$. In the second case, (Y, g) is a *factor* of (X, f). Finally, (Y, g) is a *subsystem* of (X, f) if $Y \subset X$, $f(Y) = Y$, and $g = f|Y$. Brief information which properties are preserved by conjugacy, and relations between properties of a sysem and its subsystems and factors.

The shift space with k symbols is the set $X_k = \{0, 1, \cdots, k-1\}^{\mathbb{N}}$, and the mapping σ_k, or simply σ such that $\sigma(x_1 x_2 x_3 \cdots) = x_2 x_3 x_4 \cdots$.

An $f \in \mathcal{C}(X)$ is *Li-Yorke chaotic* [5] if there is an $\varepsilon > 0$, and an uncountable set $S \subset X$, called *scrambled set* such that, for every $x, y \in S$, $x \neq y$,

$$\liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n\to\infty} \rho(f^n(x), f^n(y)) \geq \varepsilon. \tag{1}$$

Lemma 4.1. *Let $f \in \mathcal{C}(I)$ be a map such that $f(0) = f(1) = 0$, $f(\frac{1}{3}) = f(\frac{2}{3}) = 1$ and f is linear otherwise. Let Q be the Cantor middle-third set. Then $f(Q) = Q$ is invariant such that, for every $x \in I \setminus Q$ there is an n such that $f^n(x) = 0$, and the system (Q, σ) is conjugate to (X_2, σ) is a subsystem of (I, f). Moreover, (X_2, σ) is Li-Yorke chaotic (briefly, LYC) and hence, f is LYC, with $\varepsilon = 1$.*

Detailed proof is given.

An $f \in \mathcal{C}(I)$ has a *k-horseshoe* if there are non-overlapping intervals $J_1, J_2, \cdots, J_k$ such that $f(J_i) \supseteq J_1 \cup J_2 \cup \cdots \cup J_k$, for every i.

Theorem 4.2. If an $f \in \mathcal{C}(I)$ has a k-horseshoe, with $k \geq 2$ then it has an invariant compact set $Q \subset I$ such that $(Q, f|Q)$ is semiconjugate to (X_k, σ). The semiconjugacy map $\varphi : Q \to X_k$ is almost 1-1, except for a countable set, where it is $2-1$.

No proof, only comments, with reference to proof of Lemma 4.1.

Corollary 4.3. *If $f \in \mathcal{C}(I)$ has a periodic orbit of period different from 2^n, $n \in \mathbb{N}$ then f is LYC.*

Proof follows by 2.2 and 4.2.

Theorem 4.4. (Feigenbaum's conjecture 1978, computer-asisted proof by Lanford III 1982, analytic proof by M. Jakobson $\approx$ 1985.) *Let $g \in \mathcal{C}^1(I)$ be such that $g(0) = g(1) = 1$, g has unique maximum at $\xi \in (0,1)$, $g''(\xi)$ exists, g is strictly increasing on $[0, \xi]$ and strictly decreasing otherwise. Let $g_\lambda(x) = \lambda g(x)$, for $x \in I$. Then there is an increasing sequence $\lambda_1 < \lambda_2 < \cdots$ such that g_λ has a periodic point of period 2^k if and only if $\lambda > \lambda_k$. Then, for $\lambda_\infty := \lim_{k\to\infty} \lambda_k$, g_{λ_∞} is a function of type 2^∞, and $\lim_{k\to\infty}(\lambda_k - \lambda_{k-1})/(\lambda_{k+1} - \lambda_k) = \delta := 4,669201\cdots$.*

Without proof, but with comments.

A *simple priodic orbit* of an $f \in \mathcal{C}(I)$ of period 2^k is defined as follows: Any periodic orbit of period 2 is simple, and a periodic orbit $P = p_1 < p_2 < \cdots < p_{2^{k+1}}$ is simple if f exchanges the left and the right half of P, and if either of the halves is a simple periodic orbit of period 2^k for f^2. *Simple periodic orbit of a compact interval* is defined similarly.

Let $W = \omega_f(x)$. A system of compact periodic intervals $J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$ of $f \in \mathcal{C}(I)$ such that J_k has a simple periodic orbit of period 2^k, and $Q = \bigcap_{k\geq 0} \bigcup_{0\leq j<2^k} f^j(J_k)$ contains W, is *solenoidal system associated with* W.

A function $f \in \mathcal{C}(X)$ is *of type* 2^∞ if, for every n, it has a periodic orbit of period 2^n, and if it has no other periods.

Let τ be a mapping $X_2 \to X_2$ such that $\tau(x) = x + 1$ where the adding is mod 2 from the left to the right (e.g., $\tau(110100\cdots) = 001100\cdots$). Then (X_2, τ) is *2-adding machine* or *2-odometer*, or *2-solenoid*, simply adding machine, odometer or solenoid. General odometer, where adding is mod a given sequence of prime numbers, is defined similarly.

Theorem 4.5. (Smítal 1986.) *Let $f \in \mathcal{C}(I)$ be of type 2^∞. Then, for every infinite $W = \omega_f(x)$ there is an associated solenoidal system $J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$. If $Q = \bigcap_{k\geq 0} \bigcup_{0\leq j<2^k} f^j(J_k)$ has empty interior then W is a Cantor-type set, $Q = W$ and $f|Q$ is conjugate to (X_2, τ). Otherwise, there is a Cantor-type set $\widetilde{Q} \subset Q \cap W$ such that $(\widetilde{Q}, f|\widetilde{Q}$ is a subsystem of (I, f) semiconjugate to (X_2, τ). The semiconjugacy is almost 1-1.*

Finally, f is LYC if and only if there is an ω-limit set as above such that Q contains an interval J with $\#J \cap W = 2$.

No proof, but many examples and comments.

Theorem 4.6. (Smítal 1986.) *A map $f \in \mathcal{C}(I)$ of type 2^∞ is LYC if and only if it has an infinite ω-limit set W such that the assocated solenoidal system has intersection Q containing an intercal J such that $\# J \cap W = 2$.*

No proof, only explanation.

Let $g \in \mathcal{C}(I)$, $g(0) = g(1) = 0$, $g(x) = 1$ for x in an interval $[a, b]$, $a < b$, and let g be increasing on $[0, a]$ and decreasing on $[b, 1]$. Let g_λ be as in Theorem 4.4. g with these properties exists in $\mathcal{C}^\infty(I)$.

Theorem 4.7 (Misiurewicz and Smítal 1988.) *For some λ_∞, g_{λ_∞} is of type 2^∞, and LYC, with $\varepsilon = b - a$.*

Without proof, but with comments how this is related to Theorem 4.5.

5. Topological entropy

Historical comments, brief description of the original definition of topological entropy by Adler, Konheim an McAndrew [9]. Topological entropy of a map f is denoted by $h(f)$. A measure μ on X is an *invariant measure* of a map $f : X \to X$ if, for every measurable set A, $\mu(f^{-1}(A) = \mu(A)$. Brief description of measure entropy of a map f with respect to a measure μ; it is denoted by $h_\mu(f)$. Main properties of entropy, in particular, $h(f^n) = nh(f)$ and $h_\mu(f^n) = nh_\mu(f)$. Relations to statistical physics.

Theorem 5.1. (Variational principle; Dinaburg 1970.) *$f \in \mathcal{C}(X)$ then $h(f) = \sup_\mu h_\mu(f)$, where supremum is taken over all probability invariant measures of f.*

Without proof.

The following is an equivalent definition of topological entropy by R. Bowen [11]. Let $f \in \mathcal{C}(X)$. A set $S \subseteq X$ is *(n, ε)-separated* if, for every $x \neq y$ in S there is a j, $0 \leq j < n$ such that $\rho(f^j(x), f^j(y)) \geq \varepsilon$. A set $T \subseteq X$ is an *(n, ε)-span*, if for every $x \in X$ there is a $y \in T$ such that $\rho(f^j(x), f^j(y)) < \varepsilon$ whenever $0 \leq j < n$. Let $S(f, n, \varepsilon)$ be a maximal (n, ε)-separated set for f, and $T(f, n, \varepsilon)$ a minimal (n, ε)-span for f. Then

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \# S(f, n, \varepsilon) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \# T(f, n, \varepsilon). \tag{2}$$

Lemma 5.2. *For the shift σ_k on X_k, $h(\sigma_k) = \log k$.*

With the proof.

Theorem 5.3. *If $f \in \mathcal{C}(I)$ has a k-horseshoe then $h(f) \geq \log k$.*

Proof. See Theorem 4.2 and Lemma 5.2. Comments and examples are given.

Theorem 5.4. (Misiurewicz 1979.) *For a map $f \in \mathcal{C}(I)$, $h(f) > 0$ if and only if f has a periodic orbit of period different from 2^n, $n \in \mathcal{N}$.*

One implication follows by Theorems 2.2 and 4.2, the main idea of the converse implication is sketched.

Theorem 5.5. (Misiurewicz and Szlenk 1980.) *Let $f \in \mathcal{C}(I)$ be piecewise monotone (with finitely many monotone pieces), and let c_n denote the number of monotone pieces of f^n. Then $h(f) = \limsup_{n \to \infty} \frac{1}{n} \log c_n$.*

Main idea of the proof is given.

We see that, by Corollary 4.3 and Theorem 5.4 that, for maps in $\mathcal{C}(I)$, positive topological entropy implies *LYX*. This is the general principle.

Theorem 5.6. (Blanchard et al. 2002.) *If $f \in \mathcal{C}(X)$ has positive topological entropy then f is LYC.*

Without proof.

Let $f \in \mathcal{C}(X)$. A point $x \in X$ is *recurrent*, denoted $x \in R(f)$ if, for every neighborhood U of x, the set $N(f, U)$ of $n \in \mathbb{N}$ with $f^n(x) \in U$ is infinite. Thus, $x \in R(f)$, if and only if $x \in \omega_f(x)$. A point x is *uniformly recurrent*, denoted $x \in UR(f)$ if, for every neighborhood U of x there is a $K > 0$ such that the gaps between neighbor members of $N(f, U)$ are less than K. It is easy to prove that $x \in UR(f)$ if and only if $\omega_f(x)$ is a minimal set for f. A point $x \in X$ is *regularly recurrent* if, for every neighborhood U of x, there is a $k \in \mathbb{N}$ such that the set $N(f, U)$ contains all numbers kn, $n \in \mathbb{N}$. Finally, $x \in X$ is *a transitive point*, $x \in Tran(f)$, and the system is *transitive*, if $\omega_f(x) = X$. It can be shown that $Tran(f)$, if non-empty, is a dense G_δ, hence, a residual set in X.

In (X_k, σ) a point x is recurrent if and only if every finite block $B = x_n x_{n+1} \cdots x_m$, $m > n$, appears in x infinitely many times. Point x is uniformly recurrent if, in addition, the distances between the same blocks in x are bounded. And $x \in RR(\sigma)$ if every finite block appears in x periodically.

Theorem 5.7. *Let $Y \subseteq X_k$ be an invariant compact subset. Let $p(n)$ be the number of the blocks $c = c_1 c_2 \cdots c_n$ such that a point $y \in Y$ begins with c. Then $h(\sigma|Y) = \lim_{n\to\infty} \frac{1}{n} \log p(n)$.*

Without proof, only comments and examples.

Theorem 5.8. (Furstenberg 1967.) *There is a point $x \in X_4$ such that $x \in RR(\sigma)$, and $h(\sigma|\omega_\sigma(x)) > 0$. Thus, $\omega_\sigma(x)$ is a minimal set supporting positive topological entropy.*

Without proof, with comments and remark that this is the first such example.

6. Distributional chaos

Distributional chaos, briefly *DC*, is defined as follows: For $f \in \mathcal{C}(X)$, $u, v \in X$, and every $t \in \mathbb{R}$, $0 < t < \operatorname{diam} X$, put

$$\Phi_{uv}(t) = \liminf_{n\to\infty} \frac{1}{n} \# \left\{0 \le j < n; \rho(f^j(u), f^j(v)) < t\right\}, \tag{3}$$

and

$$\Phi^*_{uv}(t) = \limsup_{n\to\infty} \frac{1}{n} \# \left\{0 \le j < n; \rho(f^j(u), f^j(v)) < t\right\}. \tag{4}$$

Functions $\Phi_{uv}, \Phi^*_{uv} : (0, \operatorname{diam} X) \to I$ are *the lower and the upper distribution functions* generated by f, u and v; obviously, these functions are nondecreasing and, by convention, we can made them to be left-continuous. The three types, *DC1 – DC3*, of distributional chaos are given by

$$\Phi_{uv}(\varepsilon) = 0 \text{ for some } \varepsilon > 0, \text{ and } \Phi^*_{uv} \equiv 1, \text{ for some } u, v \in X, \qquad (DC1) \tag{5}$$

(6) $\Phi_{uv}(0+) < 1$ and $\Phi^*_{uv} \equiv 1$, for some $u, v \in X$, $(DC2)$

(7) $\Phi_{uv} < \Phi^*_{uv}$, for some $u, v \in X$. $(DC3)$

The inequality in (7) means that $\Phi_{uv}(t) < \Phi^*_{uv}(t)$, for every t in a non-degenerate interval. The original notion of distributional chaos, *DC1*, was introduced in 1994 in [16] for maps in $\mathcal{C}(I)$, the weaker types, *DC2* and *DC3* were introduced later in [17] and [18]. It turns out that, for $X = I$ all three types of DC are equivalent to positive topological entropy.

Theorem 6.1. (Schweizer and Smítal 1994.) *Let* $f \in \mathcal{C}(I)$.

(A) If h(f)=0 then $\Phi_{xy} = \Phi^*_{xy}$, *for all* $x, y \in I$.

(B) If $h(f) > 0$ *then there are finite sets* $\Sigma(f) = \{F_1, F_2, \cdots, F_m\}$ *and* $\Sigma_w(f)$, *the spectrum, and the weak spectrum of* f *consisting of lower distribution function such that* $\Sigma_w(f) \setminus \Sigma(f) = \{F_{m+1}, \cdots, F_n\}$, *with* $n \geq m$. *Moreover,*

(B1) both the spectrum $\Sigma(f)$ *and the weak spectrum* $\Sigma_w(f)$ *are finite and non-empty;*

(B2) for every i, $1 \leq i \leq n$ *there is an* $\varepsilon_i > 0$ *such that* $F_i(\varepsilon_i) = 0$.

For any positive integer $k \leq n$, *let* π_k *be the system of sets* P *(scrambled sets) such that* $\#P \geq 2$ *and, for any distinct* u, v *in* P, $F_k = \Phi_{uv} < \Phi^*_{uv} \equiv 1$.

(B3) If $k \leq m$ *then* π_k *contains a non-empty perfect set* P_k.

(B4) If $m < k \leq n$ *then* π_k *is nonempty and any* P *in* π_k *is a two-point set.*

(B5) If S *is a Li-Yorke scrambled set for* f *then there are* $i, j \leq m$, *and a decomposition* $S = S_i \cup S_j$ *such that* $\Phi_{uv} \geq F_i$ *if* $u, v \in S_i$, *and* $\Phi_{uv} \geq F_j$ *if* $u, v \in S_j$.

No proof, only remarks and examples. If f is transitive, like the tent map, then $\Sigma(f) = \Sigma_w(f)$ contains a single distribution functions, which is piecewise constant, with infinitely many pieces, and with points of discontinuity at the maximal periodic points $p_k = 2^k/(2^k+1)$ of period k. Then $p_1 = \frac{2}{3} < p_2 = \frac{4}{5} < p_3 = \frac{8}{9} < \longrightarrow 1$, and the gaps, $\lim_{x\to p_k+} f(x) - \lim_{x\to p_k-} f(x) = \frac{k}{k+1} - \frac{k-1}{k}$, $k \geq 1$, are rational numbers.

Let, for $f \in \mathcal{C}(I)$, $\mu(f) = \sup_{x,y} \int_0^1 (\Phi^*_{uv}(t) - \Phi_{uv}(t))dt$. Then $\mu(f)$ is good *measure of the size of distributional chaos.*

Theorem 6.2. (Balibrea et al. 2003.) *Let* $f, g \in \mathcal{C}(I)$.

(i) Then $\lim_{||f-g||\to 0} \mu(g) \geq \mu(f)$. *So,* $\mu(f)$ *is lower semicontinuous; distributional chaos can explode, but cannot be destroyed by small perturbations of the function.*

(ii) Any $F \in \Sigma(f)$ *can be approximated by distributional functions of pairs of periodic points: There are periodic points* p_n, q_n, $n \in \mathbb{N}$ *such that* $F = \inf_n \Phi_{p_n q_n}$.

No proof, only remarks. Stability of distributional chaos for maps in $\mathcal{C}(I)$, or other one-domensional spaces like $\mathbb{S}^1$ or finite topological graphs is implied by the stability of positive topological entropy. In higher dimensions this is not the case, even in the class $\mathcal{T}(I^2)$ of triangular maps $I^2 \to I^2$, $(x, y) \longmapsto (f(x), g_x(y))$.

Theorem 6.3. *There is a map* $F \in \mathcal{T}(I^2)$ *which is DC1, but* $h(F) = 0$.

Sketch of the proof, with comments.

Theorem 6.4. (Kolyada 1992.) *There is an* $F \in \mathcal{T}(I^2)$ *of type* 2^∞, *with* $h(F) > 0$.

Sketch of the proof from [18], which is easier than the original one. Following [18] it is also shown that there are $G_n \in \mathcal{T}(I^2)$ with $||F - G_n|| \to 0$ and $h(G_n) = 0$, for every n, and that $F \in DC2 \setminus DC1$. The following is important but difficult open problem.

Problem 6.5. *Let $f \in \mathcal{C}(X)$. Does $h(f) > 0$ imply $f \in DC2$ or $f \in DC3$?*

The class of maps $f \in \mathcal{C}(X)$ with positive topological entropy, but without being $DC1$ is relatively large. Other examples follow from the next results. Recall that a system (X, f) is *almost 1–1 extension* of (Y, g), if (Y, g) is a factor of (X, f) via semiconjugacy map $\varphi : X \to Y$ with the property that the set of points $y \in Y$ which have only one preimage with respect to φ, is dense in Y.

Theorem 6.6. (Huang and Ye, 2004.) *Let (X, f) be a minimal system. Then $RR(f) \neq \emptyset$ if and only if (X, f) is an almost 1–1 extension of a solenoid (i.e., adding machine).*

No proof.

Lemma 6.7. *Let $f \in \mathcal{C}(X)$, and let, for every $\varepsilon > 0$, there is a compact periodic subset M of X with $\operatorname{diam}(M) < \varepsilon$. Then $f \notin DC1$.*

Proof. Assume $\Phi*_{xy} \equiv 1$, for some $x, y \in X$, $\operatorname{diam}(M) < \varepsilon$, and let m be the period of Y. Then $\liminf \rho(f^n(x), f^n(y)) = 0$ so that, for some k, $f^k(x), f^k(y) \in Y$. But then $f^{k+jm}(x), f^{k+jm}(y) \in Y$, for every $j \in \mathbb{N}$. It follows that $\Phi_{xy}(\varepsilon) > \frac{1}{m}$. Since ε is arbitrary, $f \notin DC1$. □

Corollary 6.8. *A minimal system (X, f) containing a regularly recurrent point cannot be DC1. Consequently, there are minimal systems with positive topological entropy, which are not DC1.*

Proof. The first statement follows by Theorem 6.6, and Lemma 6.7. The second statement follows by Furstenber's Theorem 5.8. □

Kolyada's example in Theorem 6.4 and other examples show that the maps in the class $\mathcal{T}(I^2)$ have properties impossible in the class $\mathcal{C}(I)$. Actually, there are more than 50 mutually equivalent conditions characterizing functions in $\mathcal{C}(I)$ with zero topological entropy. Only few of them are equivalent for maps in $\mathcal{T}(I^2)$. Therefore, in 1989, A. N. Sharkovsky formulated a program of classification of these conditions for maps in $\mathcal{T}(I^2)$: for every possible implication decide whether it is true or not. Tens of mathematicians considered 32 conditions applicable to triangular maps, which amount 992 possible implications. After 20 yearsthe program is almost completed. Only in 18 cases it is not known whether the corresponding implication holds or not. All but two concern the weaker forms of distributional chaos, *DC2* and *DC3*. Survey of known results can be found in [22]. Here we provide the most important unsolved problems (compare with more general Problem 6.4.)

Problem 6.9. *For a map $F \in \mathcal{T}(I^2)$ does any of the following conditions imply existence of DC2 or DC3?*

(i) *$h(F) > 0$;*
(ii) *F has a minimal set M such that $h(F|M) > 0$;*
(iii) *$h(F|RR(F)) > 0$;*
(iv) *$R(F) \neq UR(F)$.*

7. Other forms of chaos

A map $f \in \mathcal{C}(X)$ is *Lyapunov stable* at a point x if, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\rho(x,y) < \delta$ implies $\rho(f^n(x), f^n(y)) < \varepsilon$, for every $n \in \mathbb{N}$. If f is not Lyapunov stable at an x then it is *sensitive* at x. Let $Eq_\varepsilon(f)$ be the union of all open sets $V \subset X$ such that $\mathrm{diam} f^n(V) \le \varepsilon$ for every $n \in \mathbb{N}$. Then any $x \in Eq_\varepsilon(f)$ is an *ε-equicontinuity point*, and $Eq(f) = \bigcap_{\varepsilon>0} Eq_\varepsilon(f)$ the set of *equicontinuity points*.

Theorem 7.1. (Glasner and Weiss 1993, cf. [23].) *Let $f \in \mathcal{C}(X)$ be transitive. Then one of the following two conditions hods:*

(i) $Eq(f) \neq \emptyset$. *Then $Eq(f)$ equals to the set of transitive points of f, and there is a sequence $n_< n_2 < n_3 < \cdots$ in $\mathbb{N}$ such that the sequence $\{f^{n_k}\}$ converges uniformly to the identity.*

(ii) $Eq(f) = 0$. *Then (X, f) is sensitive, i.e., there is an $\varepsilon > 0$ such that $Eq_\varepsilon(f) = \emptyset$.*
In particular, every minimal system is either equicontinuous or sensitive.

Without proof, with comments and examples of minimal sensitive and minimal equicontinuous systems.

An $f \in \mathcal{C}(X)$ is *Li-Yorke sensitive*, briefly *LYS*, (Akin and Kolyada 2003) if there is an $\varepsilon > 0$ such that, for every $x \in X$ and every neighborhood U of x, there is a $y \in U$ such that x, y is a Li-Yorke pair, i.e., satisfies (1). An $f \in \mathcal{C}(X)$ is *weak mixing* if the map $f \times f : X \times X \to X \times X$, is transitive. The set $Prox(x)$, *proximal cell of* x is the set of y satisfying the first condition in (1).

Theorem 7.2. (Akin and Kolyada 2003.) *If $f \in \mathcal{C}(X)$ is weak mixing then, for every x, $Prox(x)$ is dense in X. Consequently, f is LYS.*

Without proof, but with comments, based on alternative definitions of weak mixing.

Theorem 7.3. *For a minimal $f \in \mathcal{C}(X)$ the following conditions are equivalent:*

(i) *f is weak mixing;*

(ii) *For every $x \in X$, the proximal cell $Prox(x)$ is dense in X;*

(iii) *For some $x \in X$, the proximal cell $Prox(x)$ is dense in X;*

(iv) *The set of proximal pairs is densi in $X \times X$.*

Without proof, with comments.

Problems 7.4. (Akin and Kolyada 2003.) *Let $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$.*

(i) *Does LYS imply LYC?*

(ii) *Does every LYS minimal system have a nontrivial weak mixing factor?*

(iii) *If f and g are minimal, and g is a LYS factor of f, is then f LYS?*

(iv) *If f and g are minimal, and g is a mixing factor of f, is then f LYS?*

Theorem 7.5. (Čiklová - Mlíchová 2006, 2009.) *Let $X = Q \times \mathbb{S}^1$ where Q is the Cantor set, and $\mathbb{S}^1$ the unit circle. Then there are minimal skew-product maps $F, G : X \to X$ such that (Y, G) is a factor of (X, F), (Y, G) is LYS without having a weak mixing factor, and (X, F) is not LYS.*

The maps F, G are such that $F(x,y) = (\tau(x), f_x(y))$ and $G(x,y) = (\tau(x), g_x(y))$, where τ is the adding machine map on Q, and f_x, g_x are either rational rotations, or homeomorphisms of $\mathbb{S}^1$ with exactly three fixed points.

This result solves in the negative problems 7.4 (ii) and (iii). There was given sketch of the construction, in particular, of the (original) method of extension of an adding machine to more complicated systems. Some consequences are also given.

Problems 7.4 (i) and (iv) are still open, the latter even in the case when g satisfies a stronger condition - mixing property.

In the final part a survey of other notions of chaos applicable to minimal sets, with known relations, is given.

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