GRADIENT TRAJECTORIES, QUANTITATIVE ASPECTS

DIDIER D'ACUNTO AND KRZYSZTOF KURDYKA

ABSTRACT. We propose a method, called *of Talweg*, to study quantitative aspects of trajectories of gradient. We show that gradient trajectories of a definable (in an o-minimal structure) family of functions are of uniformly bounded length. We show that the length of a trajectory of the gradient of a polynomial in n variables of degree d in a ball of radius r is bounded by $rA(n,d)$, where $A(n,d) = \nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2})$ and $\nu(n)$ is an explicit constant. We give explicit bounds for the length of gradient trajectories of quasipolynomials and trigonometric quasipolynomials. As an application we give a construction of curves (piece-wise gradient trajectory of a polynomial) joining two points in an open connected semialgebraic set. We give an explicit bound for its length. We also obtain an explicit and quite sharp bound in Yomdin's version of quantitative Morse-Sard Theorem.

This lectures are based partly on PhD thesis [D'A2] **of the first author. The final version will be published elsewhere.**

1. Introduction

The trajectories of a gradient field, or more generally the flow of a gradient field, appear in various branches of mathematics and its applications. The case of the gradient of a polynomial function is of a particular interest. If $f : U \to \mathbb{R}$ is a polynomial of degree d restricted to an open and bounded subset of $\mathbb{R}^n$, then the length of any trajectory of ∇f is bounded by some constant $A > 0$. This follows from the famous Łojasiewicz's inequality [Ło1],[Ło2]: there exists $\rho < 1$ and $C > 0$ such that

$$|\nabla f(x)| \geq C|f(x) - c|^{\rho}, \tag{1.1}$$

for any $x \in U$ such that $f(x)$ is close to c which is a critical value of f. (In fact Łojasiewicz proved this for any analytic function in a neighbourhood of $\overline{U}$.) However, for a given polynomial f, it does not seem possible to obtain by this method the bound A in the case where U is simple, for instance where U is a unit ball in $\mathbb{R}^n$. (The constant C is difficult to control). To our best knowledge no such estimate for A was known explicitly.

In this paper we prove that actually there is a constant $A = A(n,d)$ depending only on the degree d of f and n - the number of variables of f, such that the length of any trajectory of ∇f is bounded by A. More precisely, in Theorem 7.8, we prove that, if U is a unit ball in $\mathbb{R}^n$, then

$$A(n,d) = \nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2}), \tag{1.2}$$

Date: June 30, 2015.
2000 *Mathematics Subject Classification.* Primary 32Bxx, 34Cxx, Secondary 32Sxx, 14P10.
Key words and phrases. gradient trajectories, polynomials, o-minimal structures.
Partially supported by the European research network RAAG HPRN-CT-00271.

where $\nu(n)$ is an explicit constant depending only on the dimension n (see Remark 5.3).

Our method is actually different from the argument of Łojasiewicz: we compare the length of any trajectory to the length of a semialgebraic curve Γ obtained by minimising $|\nabla f|$ on the fibres of f. First, by a transversality argument for quadratic forms, we prove in Proposition 7.2 that for a generic polynomial, the set Γ is actually a curve. Then we prove in Theorem 7.4 that, for a generic f, the bound (1.2) holds and finally that it holds in the general case. The estimate for the length of Γ is based on the Cauchy-Crofton formula which we recall in Chapter 5.

In fact our method applies in the more general situation of definable (in an o-minimal structure) families of C^2 functions. O-minimal structures (see e.g., [Dr],[Co],[Ku]) introduced recently by model theorists, are natural generalisation of semialgebraic or subanalytic geometry. We recall some basic facts on o-minimal structures in Chapter 4.

The second-named author proved in [Ku] that for a C^2 definable function f on a bounded open set U, there is a constant $M > 0$ which bounds the length of the trajectories of the gradient vector field ∇f. Let us now consider a definable family $\mathcal{F} = \{f_p\}_{p \in \mathcal{P}}$ of functions defined on open subsets of $\mathbb{R}^n$ contained in a compact K. One can naturally ask the following question: does there exist a constant $M_{\mathcal{F}}$ such that for all $p \in \mathcal{P}$, the length of the trajectories of the gradient field ∇f_p is bounded by $M_{\mathcal{F}}$? We give the affirmative answer in Theorem 6.2.

Actually, in Theorem 6.2 we also consider the case where f_p cannot be extended to any neighbourhood of $\overline{U}$, for instance f_p may be rational with poles on the boundary of U. So we have to consider not only critical values of f_p, but also generalised critical values which we recall in Chapter **??**.

Some effective bounds for length of trajectories of gradient of quasipolynomials and trigonometric quasipolynomials are given in Chapter 9. They are based on Chapter 7 and Khovanskiĭ's theory.

Some of these results where previously established in the Ph.D. thesis of the first-named author (see [D'A2]). We thank R. Moussu for pointing our attention to the paper [Ch], which was useful in the proof of Proposition 7.2.

The last two chapters (added in the second version of the paper) are due to the second named author. Chapter 10 is dedicated to the memory of Gérard Favre. In this chapter, we consider a problem of joining two points in a connected component A of a set $\{f > 0\} \cap B(x_0, r)$, where f isa polynomial of degree d and $B(x_0, r)$ is a ball of radius r in $\mathbb{R}^n$. This is the simplest situation considered in robotics. We prove in Theorem 10.3 that any two points in A can be joined by a piecewise trajectory of ∇g, where $g(x) = (r^2 - |x - x_0|^2)f(x)$ is a polynomial of degree $d+2$. Moreover we show that the length of the curve can be bounded by $2r\nu(n)(3d+2)^{n-1}$. This bound seems to be quite optimal. In all classical papers on this problem the joining curve was semialgebraic and its construction was hard and quite involved, so the estimates for its length were rather coarse. The new idea here was to use gradient trajectories of a polynomial hence transcendental curves. It seems also that for numerical applications our method is of interest since there are quite efficient algorithms to compute trajectories of gradient of a polynomial.

In the last chapter we consider a quantitative Morse-Sard Theorem. Let $f : B(r) \to \mathbb{R}$ be a polynomial of degree d on a ball $B(r) \subset \mathbb{R}^n$ of radius r. For any $\varepsilon > 0$ we consider the set of nearly critical points $\Sigma_\varepsilon = \{x \in B(r);\, |\nabla f(x)| < \varepsilon\}$.

Yomdin proved in [Yo2] that the set $f(\Sigma_\varepsilon)$ can be covered by $N(n,d)$ segments of length $r\varepsilon$. The number $N(n,d)$ depends only on dimension n and degree d. In many situations it is very important to have a good bound for $N(n,d)$. For instance Donaldson in his important paper [Do] needed the fact that for fixed n the function $d \mapsto N(n,d)$ is bounded by a polynomial in d. We prove in Chapter 11 that $N(n,d) \leq d(2d-1)^{n-1} + A(n,d)$ which is the first (to our knowledge) explicit bound for $N(n,d)$ for any n and d. This bound is not far from optimal in the sense that always $(d-1)^n \leq N(n,d)$.

2. Preliminaries

2.1. **Gradient trajectories.** For simplicity we consider only gradient with respect to the standard Euclidean metric. Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be C^2 function. Let $I \subset \mathbb{R}$ be an interval. We say that $x : I \to U$ is a *trajectory (or an integral curve)* of the vector field ∇f if $t \mapsto x(t)$ is a continuous, piecewise C^1, map satisfying (except at finitely many points) the differential equation

$$x'(t) = \frac{\nabla f(x(t))}{|\nabla f(x(t))|}. \tag{2.1}$$

In fact we consider trajectories of $\frac{\nabla f}{|\nabla f|}$ in the set of regular points of f but we allow trajectories to pass also by singular points (i.e. where $\nabla f = 0$). Under these assumptions, X will denote the image of such a trajectory.

Throughout this paper $\mathbb{B}^n$ stands for the open unit ball in $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ for the unit sphere in $\mathbb{R}^n$, both with respect to the Euclidean metric. For simplicity, $|\cdot|$ denotes here the Euclidean norm corresponding to the standard inner product $\langle\cdot,\cdot\rangle$.

2.2. **The canonical stratification of the incidency variety Σ.** Let us denote by $S_n(\mathbb{R})$ the space of $n\times n$ symmetric matrices with real coefficients. We define

$$\Sigma = \{(V,H) \in \mathbb{R}^n \times S_n(\mathbb{R}) : \exists\, \lambda \in \mathbb{R},\ H\cdot V = \lambda V\}.$$

We shall describe now the canonical stratification of Σ. We denote by $S_n^*(\mathbb{R}) \subset S_n(\mathbb{R})$ the space of $n\times n$ symmetric matrices with real coefficients and simple and nonzero eigenvalues.

Theorem 2.1. *Denote $\Sigma_0 = 0\times S_n(\mathbb{R})$, $\Sigma_0^* = 0 \times S_n^*(\mathbb{R})$, $\Sigma_1 = \Sigma\setminus\Sigma_0$. Then*

(1) Σ is algebraic of codimension $n-1$.
(2) $\Sigma_0\setminus\Sigma_0^*$ is an algebraic set of codimension $n+1$,
(3) $\Sigma_1 = \Sigma\setminus\Sigma_0$ is nonsingular,
(4) (Σ_1,Σ_0^*) satisfies Whitney's condition condition b, (even a stronger condition: (Σ_1,Σ_0^*) is locally analytically trivial along Σ_0^*).

Proof. To prove 2) observe that $\Sigma_0\setminus\Sigma_0^* = 0\times Z$, where $Z\subset S_n(\mathbb{R})$ is a union of Z_1 the zero set of the characteristic polynomial and $Z_2 = \{\det = 0\}$. Hence $\Sigma_0\setminus\Sigma_0^*$ is algebraic of codimension $n+1$.

Let $g : \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R}) \to \Lambda^2(\mathbb{R}^n)$ be the map given by

$$g(V,H) = H\cdot V\wedge V.$$

Note that $(V,H)\in\Sigma$ if and only if V and $H\cdot V$ are collinear. It follows that $\Sigma = g^{-1}(0)$ is algebraic. The differential of g is given by

$$\mathrm{d}_{(V,H)}g(v,h) = H\cdot v\wedge V + H\cdot V\wedge v + h\cdot V\wedge V.$$

Lemma 2.2. *For any $(V,H)\in\mathcal{S}_n(\mathbb{R})$, $V\neq 0$ the map $\mathrm{d}_{(V,H)}g$ has rank $n-1$.*

For simplicity we may assume that the vectors of the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ are eigenvectors of H. We denote by $(\lambda_1, \dots, \lambda_n)$ the eigenvalues of H. Let $h = (h_{ij})$ be a symmetric matrix. Then

$$\begin{aligned} \mathrm{d}_{(e_1,H)} g(e_i, h) &= H \cdot e_i \wedge e_1 + H \cdot e_1 \wedge e_i + h \cdot e_1 \wedge e_1 \\ &= \lambda_i e_i \wedge e_1 + \lambda_1 e_1 \wedge e_i + h \cdot e_1 \wedge e_1 \\ &= (\lambda_i - \lambda_1) e_i \wedge e_1 + h \cdot e_1 \wedge e_1. \end{aligned}$$

Assume that $V = e_1$ is an eigenvector of H associated to the eigenvalue λ_1 of multiplicity $k \geq 1$. Note that the mapping $v \mapsto H \cdot v \wedge V + H \cdot V \wedge v$ has rank $n - k$ and its image is generated by $e_1 \wedge e_{k+1}, \dots, e_1 \wedge e_n$. We have to consider also the mapping

$$L(h) = hV \wedge V = -\sum_{i=2}^{n} h_{1i} e_1 \wedge e_i.$$

Note that the image of L is generated by $e_1 \wedge e_2, \dots, e_1 \wedge e_n$. Thus $\mathrm{d}_{(V,H)} g$ has rank $n-1$. From the constant rank theorem we deduce that Σ_1 is a smooth manifold of codimension $n-1$. This proves property 3) in Theorem 2.1. In fact it proves also property 1) since $\Sigma \setminus \Sigma_1 = \Sigma_0$ is of codimesion n.

Let us fix a symmetric matrix $H_0 \in S_n^*(\mathbb{R})$. We describe now the structure of Σ around the point $(0, H_0) \in \mathbb{R}^n \times S_n(\mathbb{R})$. Clearly, if a symmetric matrix H is close enough to H_0, then it has also only simple eigenvalues $\lambda_1(H) < \cdots < \lambda_n(H)$. Moreover, by the implicit function theorem, the map which associates to H a unit eigenvector $\xi_i(H)$, associated to $\lambda_i(H)$, can be choosen analytic in some neighborhood Ω of H_0. Clearly

$$E_i = \{(V, H) \in \mathbb{R}^n \times \Omega : \, V \in \mathbb{R}\xi_i(H)\}, \tag{2.2}$$

the family of eigenspaces corresponding to ith eigenvalue, is a smooth closed submanifold of $\mathbb{R}^n \times \Omega$. It is easily seen that

$$\Sigma \cap (\mathbb{R}^n \times \Omega) = E_1 \cup \cdots \cup E_n.$$

Moreover $E_i \cap E_j = \{0\} \times \Omega$ and E_i is transverse E_j, for $i \neq j$. This proves 4). $\square$

3. Comparison principle for gradient trajectories

In this chapter we explain an important tool frequently used along this article. Actually we will show that gradient trajectories of a generic smooth function defined in a neigbourhood of a compact set have finite length. The main new idea is that we can bound the length of a gradient trajectory be the length of curve which is chosen in the set where the norm of the gradient (as a function on the fibre) is almost minimal. This technique will be applied in Chapter 6 to establish a uniform bound for the length of gradient trajectories of functions in a definable family. In Chapter 7 we obtain quantitative estimates for the length of gradient trajectories (in a bal) of polynomials. These bounds depend on the degree of a polynomial and the number of variables. In Chapter 8 we give examples showing that our bounds are actually quite sharp. Finally, quantitative bounds for length of gradient trajectories (in a ball) of quasi-polynomials or fewnomials are given in Chapter 9.

3.1. **ε-ridges/valleys.** Let us consider a C^2 function $f : U \to \mathbb{R}$ defined on an open subset $U \subset \mathbb{R}^n$. We assign to f a control function $\varphi : \mathbb{R} \to \mathbb{R}_+$, depending on U, defined by

$$\varphi(t) = \inf\{|\nabla f(x)| : x \in f^{-1}(t)\}$$

if $t \in f(U)$, and $\varphi(t) = +\infty$ otherwise.

We say that $t_0 \in \mathbb{R}$ is a *typical value of* f if and only if there exists $c_0 > 0$ such that $\varphi(t) \geq c_0$ in some neighbourhood of t_0. The complement in $\mathbb{R}$ of the set of typical values of f is the set of *generalised critical values of* f denoted by $K(f)$. Clearly $K_0(f) \subset K(f)$, where $K_0(f)$ stands for the set of all critical values of f, however in general $K_0(f) \neq K(f)$. For instance if f extends to a neighborhood of $\overline{U}$ in such a way that the extension has a critical point at $x \in \partial U$, but f has no critical points in U, then $f(x) \in K(f) \setminus K_0(f)$.

Definition 3.1. Let $\varepsilon \geq 0$, the set

$$V_\varepsilon(f) = \{x \in U : |\nabla f(x)| \leq (1+\varepsilon)\varphi(f(x))\}$$

is called the ε-*ridge/valley* set of f. Clearly $V_\varepsilon(f)$ contains the set of all critical points of f. Note that the set $V_\varepsilon(f)$ depends on U, the domain of f.

The following fact is crucial, if $t \in f(U) \setminus K(f)$, then $\varphi(t) > 0$, so

$$V_\varepsilon(f) \cap f^{-1}(t) \neq \emptyset,$$

for any $\varepsilon > 0$. We will now explain how useful the set of ε-ridges/valleys is when trying to estimate an upper bound for the length of the trajectories of ∇f (in the sense of Chapter 2).

3.2. **Comparison Principle.** First we shall define some particular curves in $V_\varepsilon(f)$, which will "measure" the length of trajectories of ∇f.

Definition 3.2. Let $f : U \to \mathbb{R}$ be a C^2 function defined in an open subset $U \subset \mathbb{R}^n$ such that $K(f)$, the set generalised critical values of f, is finite. Let $\Gamma^\varepsilon(f) \subset V_\varepsilon(f)$ be a curve satisfying the following properties:

(i) $\Gamma^\varepsilon(f)$ is a finite union of smooth connected curves and points;
(ii) for any $t \in f(U) \setminus K(f)$ the set $f^{-1}(t) \cap \Gamma^\varepsilon(f)$ consists of exactly one point;
(iii) $\Gamma^\varepsilon(f)$ intersects transversally $f^{-1}(t)$ for all but finitely many $t \in f(U)$.

Such a curve $\Gamma^\varepsilon(f)$ will be called an ε-*talweg* of f. Note that the ε-talweg of f depends on U, the domain of f.

Under the above assumptions we can state our

Theorem 3.3 (Comparison Principle)**.** *Let $f : U \to \mathbb{R}$ be a C^2 function defined in an open subset U of $\mathbb{R}^n$. Let $x : I \to U$ be a trajectory of ∇f. Then the length of the trajectory $x(t)$ is bounded by* $(1+\varepsilon)$Length $\Gamma^\varepsilon(f)$.

Proof. Since $\Gamma^\varepsilon(f)$ meets transversally all but a finite number of fibres of f, we will assume that the curve $\Gamma^\varepsilon(f)$ is smooth, connected and transverse to every fibre of f. Moreover by deleting finitely many fibres $f^{-1}(t)$, $t \in K(f)$ we may assume that f has no generalised critical values in U.

Let $x(s)$ be the arc-length parametrisation of a trajectory X of ∇f and let $\gamma(u)$ be the arc-length parametrisation of the curve $\Gamma^\varepsilon(f)$. We fix orientations so that both functions $s \mapsto (f \circ x)(s)$ and $u \mapsto (f \circ \gamma)(u)$ are strictly increasing.

Let $\eta : X \to \Gamma^\varepsilon(f)$ be the map $\eta = (f|_{\Gamma^\varepsilon(f)})^{-1} \circ (f|_X)$. We now compute η in our arc-length charts, that is we consider $h(s) = \gamma^{-1} \circ \eta \circ x(s)$. Clearly, to prove Theorem 3.3 it is enough to show that $h'(s) \geq \dfrac{1}{1+\varepsilon}$. Taking derivative with respect to s in the equality $(f \circ \eta \circ x)(s) = (f \circ x)(s)$ we obtain

$$\langle \nabla f((\eta \circ x)(s)), (\eta \circ x)'(s) \rangle = \langle \nabla f(x(s)), x'(s) \rangle.$$

But $x'(s) = \dfrac{\nabla f(x(s))}{|\nabla f(x(s))|}$, hence

$$|\nabla f((\eta \circ x)(s))| \cdot |(\eta \circ x)'(s)| \geq |\nabla f(x(s))|.$$

Since $\eta(x(s)) \in V_\varepsilon(f)$, we have

$$(1+\varepsilon)|\nabla f(x(s))| \geq |\nabla f((\eta \circ x)(s))|,$$

thus $|(\eta \circ x)'(s)| \geq \frac{1}{1+\varepsilon}$. But γ is an arc-length parametrisation, so $h'(s) = (\gamma^{-1} \circ \eta \circ x)'(s) = |(\eta \circ x)'(s)| \geq \frac{1}{1+\varepsilon}$ and Theorem 3.3 follows. □

Remark 3.4. Actually the Comparison Principle applies in a more general situation. Let us assume that f is the restriction of a C^2 function g defined in an open set V containing U. Let Γ^ε be a curve contained in V, but not necessarily in U, satisfying similar conditions to those of Defintion 3.2:

(i') Γ^ε is a finite union of smooth connected curves and points;
(ii') for any $t \in f(U) \setminus K(f)$ the set $g^{-1}(t) \cap \Gamma^\varepsilon$ consists of exactly one point;
(iii') Γ^ε intersects transversally $g^{-1}(t)$ for all but finitely many $t \in f(U)$;
(iv') for any $t \in f(U)$, except finitely many, we have

$$(1+\varepsilon)|\nabla f(x)| \geq |\nabla g(y)|, \text{ for all } x \in f^{-1}(t),$$

where $\{y\} = g^{-1}(t) \cap \Gamma^\varepsilon$.

Then it is easily seen that in this situation the Comparison Principle also applies. That is, the length of any trajectory (in U) of ∇f is bounded by $(1+\varepsilon)$Length Γ^ε. This observation is particularly useful when $\overline{U}$ is compact and $\overline{U} \subset V$. Then, $|\nabla f|$ restricted to a fibre of f attains its minimum at a point of $\overline{U}$, possibly on the boundary of U. In this situation it is natural to take $\varepsilon = 0$.

Remark 3.5. Let M be a C^2 Riemannian manifold of $\mathbb{R}^n$ equipped with the Riemannian metric g induced by the canonical Riemannian metric of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a C^2 function defined over an open bounded subset U of M. It can easily be checked that Theorem 3.3 holds true in this setting.

Finally, let us comment our definition of ε-talweg. It also makes sense when the set $K(f)$ is discrete (but possibly infinite) or even when $K(f)$ is of mesure 0, however it seems that Defintion 3.2 covers the most interesting cases.

3.3. **Talweg lines and gradient extremal lines.** As mentioned above, in Definition 3.1 we allow $\varepsilon = 0$. In this special case $V_0(f)$ is the set of points $x \in U$ at which $|\nabla f|$ has a global minimum on the fibre $f^{-1}(f(x))$. The set $V_0(f)$ depends on U, so it is more natural to consider also local minima of $|\nabla f|$ on fibres of f, that is the set

$$\Gamma_1(f) = \{x \in U : |\nabla f| \text{ has a loc. min. at } x \text{ on the regular fibre } f^{-1}(f(x))\}.$$

We will call this set *talweg* (in ancient spelling *thalweg*) of f or *ridge and valley lines* of f. Clearly $V_0(f) \subset \Gamma_1(f)$.

Actually the notion of ridge or bottom of a valley is not really well determined in the literature. There are several definitions which are used in applied sciences. These "natural" lines appear in classical geomorphology, specially in hydrology, oil recovery [AK1, AK1, GKB] , meteorology and recently in a very spectacular way in artificial vision [**?**].

In the second half of 19th century there were fascinating discussions on how to define mathematically the lines ("talwegs") sketching the drainage pattern of a landscape of the graph of a function f in 2 variables. It involved, among many others people, de Saint-Venant [dSV], Cayley [Cay], Maxwell [Ma], Jordan [Jo1], [Jo2] and [Jo3]. Nowadays, the discussion is still alive and has been imported to image analysis. Some authors insisted that "talwegs" should be composed of trajectories of ∇f, they suggested that stable and unstable manifolds should be good candidates. However to define "talweg" in this way f must have saddle points in U, but on the other hand we "see" ridges or valleys even when we don't "see" saddle points, for instance they are possibly outside of domain U.

The set $\Gamma_1(f)$ is hence one of the possible candidates for "talwegs". It has several advantages, which we explain below. First let us introduce a larger set called *gradient extremal set*

$$\Theta_1(f) = \{x \in U : \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f = 0\}.$$

This is simply the set of critical points of the map $(f, |\nabla f|^2) : U \to \mathbb{R}^2$ and of course

$$\Gamma_1(f) \subset \Theta_1(f).$$

Remark 3.6. Clearly $\Theta_1(f)$ is closed in U.

Let $Hf(x)$ be the Hessian matrix of f at the point x. Note that $\nabla(|\nabla f|^2) = 2Hf \cdot \nabla f$. Hence we have the following

Lemma 3.7. *The set* $\Theta_1(f) = \{x \in \mathbb{R}^n : \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f = 0\}$ *is the set of points* $x \in \mathbb{R}^n$ *such that* $\nabla f(x)$ *is an eigenvector of* $Hf(x)$*.*

Given a C^2 function $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$. We shall consider $T_f : U \to \mathbb{R}^n \times S_n(\mathbb{R})$ a *Gauss-Hesse map of* f given by

$$T_f(x) = (\nabla f(x), Hf(x)).$$

Recall that Σ defined in Section 2 consists of couples eigenvector and symmetric matrix. Hence

$$\Theta_1(f) = (T_f)^{-1}(\Sigma). \tag{3.1}$$

3.4. **The structure of the gradient extremal set of a generic C^∞ function.** Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be C^∞ function. We denote by $C_f = \{x \in U; \nabla f(x) = 0\}$ the critical set of f. Recal that f is Morse at $x \in C_f$ if the Hessian matrix $Hf(x)$ has only nonzero eigenvalues. Suppose that the eigenvalues of $Hf(x)$ are simple, then $Hf(x)$ has n distinct eigenspaces $L_1, \dots, L_n$ each of dimension 1.

Theorem 3.8. *Assume that* T_f*, the Gauss-Hess map of* f*, is transverse to the strata of* Σ*. Then*

(1) *f is Morse function, hence the critical set C_f has only isolated points,*
(2) *$\Theta_1(f) \setminus C_f$ is a C^∞ submanifold of dimension 1,*

(3) for any $x \in C_f$ there exist an open neihborhood U_x such that

$$\Theta_1(f) \cap U_x = S_1 \cup \cdots \cup S_n, \tag{3.2}$$

where;

(a) each S_i is a connected C^∞ submanifold of dimension 1, closed in U_x,

(b) $S_i \cap S_j = \{x\}$, for any $i \neq j$,

(c) L_i is tangent to S_i at x.

Remark 3.9. Possibly $\Theta_1(f) = \emptyset$.

Proof. Note that T_f can meet only Σ_1 and Σ_0^* because the other strata are of codimension strictly greater than n. Since T_f is transverse to the stratum $\Sigma_0 = 0 \times S_n(\mathbb{R})$ it follows that the Hessian matrix $Hf(x)$ is nondegenerate for any $x \in (T_f)^{-1}(\Sigma_0) = C_f$. Thus f is a Morse function. By the transversality to Σ_1 we obtain immediately that $(T_f)^{-1}(\Sigma_1) = \Theta_1(f) \setminus C_f$ is a smooth submanifold of dimension 1 or an empty set.

Now show the last claim, precisely $3a$). Note that actually $C_f = (T_f)^{-1}(\Sigma_0^*)$, so for any $x \in C_f$ the Hessian matrix $Hf(x)$ has only simples eigenvalues. Let us fix $x \in C_f$ an the corresponding Hessian matrix $H_0 = Hf(x)$. Then for some neighborhood $\Omega \subset S_n(\mathbb{R})$ of H_0 we have

$$\Sigma \cap (\mathbb{R}^n \times \Omega) = E_1 \cup \cdots \cup E_n,$$

where E_i are smooth closed submanifolds of $\mathbb{R}^n \times \Omega$, codim$E_i = n-1$, defined by (2.2). Moreover $E_i \cap E_j = \{0\} \times \Omega$ and E_i is transverse to E_j, for $i \neq j$. Since T_f is transverse to $\{0\} \times \Omega = \Sigma_0^* \cap (\mathbb{R}^n \times \Omega)$ it follows that T_f is transverse to each E_i. Hence $S_i = (T_f)^{-1}(E_i)$ is a smooth curve, when restricted to a small neighborhood of x the curve S_i is connected and contains point x. This proves $3a$).

Property $3b$) follows now from $3c$). We are going to prove $3c$) below. For simplicity we assume that $x = 0 \in \mathbb{R}^n$. After an orthogonal change of variables in $\mathbb{R}^n$ we may assume that the Hessian matrix $Hf(0)$ is diagonal. By $a_1 < \cdots < a_n$, we denote the elements on the diagonal. We parametrise each smooth curve S_i by a regular arc $s_i(t)$, $t \in (-\varepsilon, \varepsilon)$ and $s_i(0) = 0$. By Lemma 3.7 we have

$$Hf(s_i(t)) \cdot \nabla f(s_i(t)) = \lambda_i(t) \nabla f(s_i(t)), \tag{3.3}$$

where $\lambda_i(t)$ is the eigenvalue of $Hf(x(t))$ associated to $\nabla f(s_i(t))$. By the implicit function theorem each $\lambda_i(t)$ is an analytic function. After a suitable reordering of S_i we may assume that $\lambda_i(0) = a_i$. We fix one of the curves S_i, say $i = 1$. Let us write $s_1(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$. It follows from (3.3) that

$$a_j^2 x_j(t) = a_j \lambda_1(t) x_j(t) + O(t^2),\ j = 1, \ldots, n. \tag{3.4}$$

Taking derivative, and using $\lambda_1(0) = a_1$ we get

$$a_j(a_j - a_1) x_j'(0) = 0,\ j = 2, \ldots, n. \tag{3.5}$$

Since all a_i are non-zero and distinct we deduce $x_j'(0) = 0$, $j = 2, \ldots, n$. This means that the branch S_1 of the gradient extremal set $\Theta_1(f)$, is tangent (at 0) to the x_1-axis which is the eigenspace L_1 of $Hf(0)$. This proves property $3c$) and terminates the proof of Theorem 3.8. □

Hence the name ridge and valley lines for $\Gamma_1(f)$ is justified also when $n \geq 3$, since $\dim \Gamma_1(f) = 1$ for a "generic" f. However to apply Theorem 3.3 we have to check that we can choose in $\Gamma_1(f)$ a curve $\Gamma^0(f)$ which will satisfy finiteness conditions of Definition 3.2, with $\varepsilon = 0$.

3.5. **Applying comparison principle for generic C^∞ functions.** We explain below that the talweg of a generic function satisfies condition iii) of Definition 3.2. It follows from Thom's transversality theorem (see [Go-Gu]) that the set Ω_Σ of C^∞ functions $f : U \to \mathbb{R}$ such that the associated Gauss-Hess map T_f is transverse to the strata of Σ, is dense in the set of all C^∞ functions $f : U \to \mathbb{R}$, (in Whitney's C^∞ topology). Since the stratification of Σ can be choosen in such a way that it satisfies Whitney's condition (a), it follows from [Tr] see also [Go-Ma] p. 38, that the set Ω_Σ of is also open. For the later use we state explicitely (in a weaker version) the result of Trotman [Tr]. Let M_1, M_2 be two C^∞ manifolds. We denote by $C^\infty(M_1, M_2)$ the space of C^∞ maps from M_1 to M_2, equipped with Whitney's C^∞ topology.

Theorem 3.10 ([Tr])**.** *Suppose that $Z \subset M_2$ is a closed set with a Whitney (a) regular stratification. Then the set*

$$\mathcal{T}_Z = \{g \in C^\infty(M_1, M_2) : g \text{ transverse to } Z\}$$

is open and dense in $C^\infty(M_1, M_2)$.

Since the natural map $C^\infty(U, \mathbb{R}) \ni f \mapsto j^k f \ni C^\infty(U, J^k(U, \mathbb{R}))$ is continous in Whitney's C^∞ topology (cf. [Go-Gu] Proposition 3.4) it follows that Ω_Σ of is also open. Here $j^k f(x)$ stands for k-jet of f at x and $J^k(U, \mathbb{R})$ is the space of k jets (cf. [Go-Gu]).

Theorem 3.11. *Let U be an open subset of $\mathbb{R}^n$ (or more generally a Riemannian manifold). Then there exists an open set $\mathcal{S}_1 \subset C^\infty(U, \mathbb{R})$ which is dense in Whitney's C^∞ topology, such that for every $f \in \mathcal{S}_1$ if the set $\Theta_1(f) = (T_f)^{-1}(\Sigma)$ is nonempty then there exists a discrete set $D_f \subset U$ such that $\Theta_1(f) \setminus D_f$ is a smooth submanifold of U, of dimension 1. Moreover f has no critical points on $\Theta_1(f) \setminus D_f$.*

We denote by $\overline{T}\Sigma \subset (\mathbb{R}^n \times S_n(\mathbb{R})^2$ the Zariski closure of the tangent boundle $T\Sigma_1$. Clearly $\overline{T}\Sigma$ is of codimension $2n - 2$ in $(\mathbb{R}^n \times S_n(\mathbb{R})^2$. We associate to a smooth function $f : U \to \mathbb{R}$ the following map $\Phi_f : U \times S^{n-1} \to (\mathbb{R}^n \times S_n(\mathbb{R})^2 \times \mathbb{R}$, defined by

$$\Phi_f(x, \xi) = (T_f(x), d_x T_f(\xi), d_x f(\xi)). \tag{3.6}$$

Here S^{n-1} stands for the unit sphere in $\mathbb{R}^n$. From classical theorems in transversality theory [Go-Ma] it follows that

Lemma 3.12. *Assume $\Theta_1(f) = (T_f)^{-1}(\Sigma) \neq \emptyset$. If T_f is transverse to Σ and Φ_f is transverse to $\overline{T}\Sigma \times 0$, then there exists a discrete set $D_f \subset U$ such that $\Theta_1(f) \setminus D_f$ is smooth submanifold of U and f has no critical points on $\Theta_1(f) \setminus D_f$.*

Proof. Observe that $\overline{T}\Sigma \times 0$ is of codimension $2n-1$. So transversality (to a Whitney stratified set) implies that $B_f = (T_f)^{-1}(\overline{T}\Sigma)$ is a stratified set of dimension 0, hence it is a discrete subset of $U \times S^{n-1}$. Thus also its projection A_f on U is discrete, since S^{n-1} is compact. Also $C_f = T_f^{-1}(0 \times S_n(\mathbb{R}))$ the set of critical points of f must be discrete. We put $D_f = A_f \cup C_f$. Let $x \in \Theta_1(f) \setminus C_f$, then x is a critical point of f on $\Theta_1(f)$ if and only if there exists $\xi \in S^{n-1}$ such that $(x, \xi) \in B_f$. Note that x is a smooth point of $\Theta_1(f)$ and ξ is a unit tangent vector to $\Theta_1(f)$ at x. Hence f has no critical points on $\Theta_1(f) \setminus D_f$.

We are going to show now that the set $\mathcal{S}$ of functions such that T_f is transverse to Σ and Φ_f is transverse to $\overline{T}\Sigma \times 0$ is dense as claimed in Theorem 3.11. More precisely we prove below that generic perturbation, by a polynomial of degree ≤ 3, is transverse to $\overline{T}\Sigma \times 0$. We denote by $\mathbb{R}_3[x]$ the space of polynomials in n variables of degree ≤ 3, (and vanishing at the origin). We fix also a norm $\| \|$ on $\mathbb{R}_3[x]$.

Proposition 3.13. *Assume that $f \in C^\infty(U, \mathbb{R})$ is such that T_f transverse to Σ. Let $K \subset U$ be a compact and $\varepsilon > 0$. Then there exists $h \in \mathbb{R}_3[x]$ such that,*

(1) $\|h\| < \varepsilon$,
(2) T_{f+h} is transverse to Σ on K,
(3) Φ_{f+h} is transverse to $\overline{T}\Sigma \times 0$ on $K \times S^{n-1}$.

Proof. Any $h \in \mathbb{R}_3[x]$ is of the form $h(x) = \alpha(x) + \beta(x) + \gamma(x)$ where α, β, γ are homogeneous polynomials of degree $1, 2, 3$ respectively. We consider now the map

$$\Psi : U \times S^{n-1} \times \mathbb{R}_3[x] \to (\mathbb{R}^n \times S_n(\mathbb{R})^2 \times \mathbb{R},$$

defined by $\Psi(x, \xi, h) = \Phi_{f+h}(x, \xi) = (T_{f+h}(x), d_x T_{f+h}(\xi), d_x(f + h)(\xi))$. Note that the first componenet $T_{f+h}(x)$ is transverse to Σ on K for h small enough, since transversality is an open property. The differential of the second component $d_x T_{f+h}(\xi)$ with respect to (β, γ) is submersive, also the differential of the third component $d_x(f + h)(\xi)$ with respect to α is submersive with respect to α. So Ψ is is transverse to $\overline{T}\Sigma \times 0$ on $K \times S^{n-1}$, for h small enough. By the transversality with parameters (c.f. [Go-Gu]) it follows that there exists aribitrary small h such that Φ_{f+h} is transverse to $\overline{T}\Sigma \times 0$ on $K \times S^{n-1}$.

□

Now, by routine arguments in transversality theory [Go-Gu], Theorem 3.11 follows from Lemma 3.12 and Proposition 3.13.

□

We explain now how our Comparison Principle 3.3 can be applied for generic smooth functions. Let U be an open subset of $\mathbb{R}^n$, or more generally a Riemannian manifold. Let V be an open subset of U such that $\overline{V} \subset U$ is compact moreover, the boundary B of V is smooth. For any $f \in C^\infty(U, \mathbb{R})$ we consider the set of the critical points of $|\nabla f|^2$, on fibers of f restricted to the boundary B of V. This set is given by

$$\Theta_2(f) = \{x \in B : \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f \wedge \mathrm{d}r = 0\},$$

where r is a local equation for B. We state an analogue of Theorem 3.11 for $\Theta_2(f)$.

Theorem 3.14. *There exists an open set $\mathcal{S}_2 \subset C^\infty(U, \mathbb{R})$ which is dense in Whitney's C^∞ topology, such that, if $f \in \mathcal{S}_2$ then set $\Theta_2(f)$ is a finite union of points E_f and finitely many smooth connected curves. The length of $\Theta_2(f)$ is finite. Moreover f has no critical points on $\Theta_2(f) \setminus E_f$.*

In Chapter 7 we give a full proof of this result in the polynomial case (Proposition 7.3) which is important for the main quantitative result of this paper. In the general case the proof of this result is rather long and technically involved so will not include it in this paper. Note that $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$ is open and dense in $C^\infty(U, \mathbb{R})$. Finally we obtain

Theorem 3.15. *If $f \in \mathcal{S}$, then the length of any trajectory of ∇f in V is bounded by the length of $(\overline{V} \cap \Theta_1(f)) \cup \Theta_2(f)$, in particular this is finite.*

Proof. Let $\tilde{\Gamma}(f) = \{x \in \overline{V} : |\nabla f| \text{ has a minimum at } x \text{ on } f^{-1}(f(x)) \cap \overline{V}\}$. Clearly $\tilde{\Gamma}(f) \subset (\overline{V} \cap \Theta_1(f)) \cup \Theta_2(f) := \Theta(f)$ By Theorems 3.11 and 3.14 there is a finite set $X_f \subset \Theta(f)$ such that f is a local diffeomorphims on $\Theta(f) \setminus X_f$. Let $\Gamma(f)$ be a union of some componenets of $\tilde{\Gamma}(f) \setminus X_f$ such that f is injective on $\Gamma(f)$ and $f(V) \subset f(\Gamma(f))$, except a finite set. Now $\Gamma(f)$ satisfies all conditions in Definition 3.2, so the theorem follows from Comparison Principle.

□

4. O-MINIMAL STRUCTURES

We recall below the definition of o-minimal structures expanding the real field. The rest of this chapter is devoted to some examples and properties of o-minimal structures that will be useful for the proof of Theorem 6.2.

Definition 4.1. Let $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$, where each $\mathcal{M}_n$ is a family of subsets of $\mathbb{R}^n$. We say that the collection $\mathcal{M}$ is an o-minimal structure expanding $(\mathbb{R}, +, \cdot, <)$ if:

- (i) each $\mathcal{M}_n$ is closed under finite unions, finite intersections and taking complement;
- (ii) if $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_m$, then $A \times B \in \mathcal{M}_{n+m}$;
- (iii) let $A \in \mathcal{M}_{n+m}$ and $\pi_n : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection on the first n coordinates, then $\pi_n(A) \in \mathcal{M}_n$;
- (iv) let $f, g_1, \dots, g_k \in \mathbb{R}[X_1, \dots, X_n]$, then the set $\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0\}$ belongs to $\mathcal{M}_n$;
- (v) $\mathcal{M}_1$ consists of all finite unions of open intervals and points.

For a fixed o-minimal structure $\mathcal{M}$ expanding $(\mathbb{R}, +, \cdot, <)$ we say that A is an $\mathcal{M}$-set if $A \in \mathcal{M}_n$ for some $n \in \mathbb{N}$. We say that a function $f : A \to \mathbb{R}^m$, where $A \subset \mathbb{R}^n$, is an $\mathcal{M}$-function if its graph is an $\mathcal{M}$-set. The property expressed by axiom (v) is called the o-minimality of $\mathcal{M}$. We write for short *definable (in $\mathcal{M}$)* instead of *$\mathcal{M}$-set*.

Example 4.2. We give below a non exhaustive list of o-minimal structures on the real field $(\mathbb{R}, +, \cdot, <)$ (see also [Dr-Mi] for detailed definitions and comparisons between the above examples) with examples of functions definable in those o-minimal structures:

- (i) Semialgebraic sets (by Tarski-Seidenberg); $f(x, y) = \sqrt{x^4 + y^4}$.
- (ii) Global subanalytic sets (by Gabrielov); $f(x, y) = \frac{y}{\sin x}$, $x \in (0, \pi)$.
- (iii) $\mathbb{R}_{\exp}$-definable sets (by Wilkie [Wi]); $f(x, y) = x^2 \exp(-\frac{y^2}{x^4+y^2}) \ln x$.
- (iv) $(\mathbb{R}^{\mathbb{R}}_{an})$-definable sets (by Miller [Me]); $f(x, y) = x^{\sqrt{2}} \exp(\frac{x}{y})$, $0 < x < y < 1$.
- (v) $\mathbb{R}_{an,\exp}$-definable sets (by van den Dries, Macintyre, Marker [D-M-M]); $f(x, y) = x^{\sqrt{2}} \ln(\sin y)$, $x > 0, y \in (0, \pi)$.

In the remaining of this paper $\mathcal{M}$ will denote some fixed, but arbitrary, o-minimal structure expanding $(\mathbb{R}, +, \cdot, <)$. We will give now several elementary properties of definable sets and definable functions.

Remark 4.3. Let E be a definable set in $\mathbb{R}^{n+1}$. Axioms (i) and (iii) imply that the sets

$$\{x \in \mathbb{R}^n : \exists x_{n+1}\ (x, x_{n+1}) \in E\} \text{ and } \{x \in \mathbb{R}^n : \forall x_{n+1}\ (x, x_{n+1}) \in E\}$$

are definable sets. Actually the first set is the image of E by projection, the second is the complement of the image of the complement of E by projection.

Remark 4.4. The sum, product, inverse, composition of definable functions is again a definable function. Also the image and inverse image of a definable set by a definable mapping are again definable sets. The proofs of these facts are quite standard applications of Remark 4.3 and axioms (i)-(iv) and actually the same as in the semialgebraic case (see e.g. [B-C-R]).

Lemma 4.5. *Let $f : A \to \mathbb{R}$ be a definable function such that $f(x) \geq 0$ for all $x \in A$. Let $G : A \to \mathbb{R}^m$ be a definable mapping and define a function $\varphi : G(A) \to \mathbb{R}$ by*

$$\varphi(y) = \inf_{x \in G^{-1}(y)} f(x).$$

Then φ is definable.

Proof. Write a formula for the graph of the function φ and apply Remark 4.3.

The two following properties of definable sets are essential for the proof of Theorem 6.2. The proofs can be found in [Dr] or [Co].

Lemma 4.6 (Definable choice). *Let $S \subseteq \mathbb{R}^{m+n}$ be a definable set and $\pi_m : \mathbb{R}^{m+n} \to \mathbb{R}^m$ be the natural projection on the first m coordinates. Then there exists a definable map $f : \pi_m(S) \to \mathbb{R}^n$ such that the graph of f in contained in S.*

Lemma 4.7 (Uniform finiteness). *Let $A \subset \mathbb{R}^{n+m}$ be a definable set and let $A_y = \{x \in \mathbb{R}^n : (x, y) \in A\}$. Assume that for all $y \in \mathbb{R}^m$ the set A_y is finite, then there exists an integer N such that for all $y \in \mathbb{R}^m$, $\mathrm{Card} A_y \leq N$.*

5. Cauchy-Crofton Formula

Let Γ be a compact definable curve and let $\mathcal{H}$ denote the set of affine hyperplanes in $\mathbb{R}^n$. Then, for almost every $H \in \mathcal{H}$ (that is except maybe for a definable subset $\mathcal{H}_1 \subset \mathcal{H}$ of codimension greater than or equal to 1), the set $\Gamma \cap H$ is finite. Let $i(\Gamma, H)$ denote the cardinality of $\Gamma \cap H$.

Cauchy-Crofton formula 5.1 (see e.g. [Fe]). *There exists a normalisation of the canonical measure $\mathrm{d}\mu$ on $\mathcal{H}$ such that the length of Γ can be expressed with the following formula (*Cauchy-Crofton*):*

$$\mathrm{Length}(\Gamma) = \int_{\mathcal{H}} i(\Gamma, H)\mathrm{d}\mu.$$

Note that the Cauchy-Crofton formula is still valid if we just assume Γ to be a rectifiable curve. However, the definability hypothesis is crucial for the following Corollary:

Corollary 5.2. *Let $K \subset \mathbb{R}^n$ be a compact set and let $\mathcal{G} = \{\Gamma_p\}_{p \in \mathcal{P}}$ be a definable family of definable curves contained in K. Then, there exists a constant $m_{\mathcal{G}} > 0$, depending only on the family $\mathcal{G}$, such that for any $p \in \mathcal{P}$,*

$$\mathrm{Length}(\Gamma_p) \leq m_{\mathcal{G}}.$$

Proof. Note that $\mathcal{C} = \{(p, H) \in \mathcal{P} \times \mathcal{H} : i(\Gamma_p, H) < \infty\}$ is a definable family. Moreover by the uniform finiteness Lemma there exists an integer i such that $i(\Gamma_p, H) < i$ for any $(p, H) \in \mathcal{C}$. Now it is enough to apply the Cauchy-Crofton formula. □

Remark 5.3. Note that in Corollary 5.2 the constant m is the product of some integer i by the normalised volume of the hyperplanes that intersect the compact set K. Let us denote by $\nu(n)$ this volume when $K = \overline{\mathbb{B}^n}$. Then we have the following formula (see [Mo])

$$\nu(n) = \frac{nVol_n(\mathbb{B}^n)}{Vol_{n-1}(\mathbb{B}^{n-1})}.$$

Note that $\nu(n)$ can be computed in a different way using the Euler Γ function (see for instance [Fe]). We thus obtain the following alternative formula:

$$\nu(n) = 2\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})^{-1}.$$

Note that in dimension 2, we obtain $\nu(2) = \pi$ and, for any $n \in \mathbb{N}$, $\nu(n) \leq 2n$. A simple computation shows that $\nu(n) \sim \sqrt{2\pi n}$ for sufficiently large n.

6. Gradient trajectories of definable functions

In Subsection 3.1 we have defined the set $K(f)$ of generalised critical values of a C^1 function f. In the o-minimal setting, we have the following result (cf. [Ku]):

Proposition 6.1. *Let $f : U \to \mathbb{R}$ be a C^1 function defined on an open bounded subset U of $\mathbb{R}^n$. Assume that f is definable in an o-minimal structure, then the set $K(f)$ is finite.*

Let $F : U \to \mathbb{R}$ be a definable function defined on a definable set $U \subset \mathbb{R}^n \times \mathbb{R}^k$. Let us denote by $\mathcal{P} \subset \mathbb{R}^k$ the projection of U on the last coordinate. Let $p \in \mathcal{P}$, and define $U_p = \{x \in \mathbb{R}^n : (x, p) \in U\}$. For any $p \in \mathcal{P}$ we denote by f_p the definable function $F(\cdot, p)$ defined on the definable open set U_p. Throughout this chapter, $\mathcal{F} = \{f_p\}_{p\in\mathcal{P}}$ denotes the definable family of such functions f_p. Under these hypotheses, we state the main result of this chapter:

Theorem 6.2. *Let $\mathcal{F}$ be a definable family of functions as above. Assume that for each $p \in \mathcal{P}$, the function f_p is of class C^2 on U_p. Let K be a compact set in $\mathbb{R}^n$ such that $U_p \subset K$. Then there exists a constant $M_{\mathcal{F}} > 0$ such that, for all $p \in \mathcal{P}$, the length of any trajectory of ∇f_p is bounded by $M_{\mathcal{F}}$.*

Proof. Let us fix $\varepsilon > 0$. Let $p \in \mathcal{P}$, in Chapter 3 we defined the ε-ridge/valley set of f_p as $V_\varepsilon(f_p) = \{x \in U_p : |\nabla f_p(x)| \leq (1 + \varepsilon)\varphi_p(f_p(x))\}$, where

$$\varphi_p(t) = \inf\{|\nabla f_p(x)| : x \in f_p^{-1}(t)\}$$

if $t \in f_p(U_p)$ and $\varphi_p(t) = +\infty$ otherwise. We shall also consider the set

$$V_\varepsilon = \{(x, p) \in U : p \in \mathcal{P}, x \in V_\varepsilon(f_p)\}.$$

By Lemma 4.5, the sets V_ε and $V_\varepsilon(f_p)$ are definable. Recall that by the definition of $K(f_p)$, if $t \notin K(f_p)$, then $\varphi_p(t) > 0$. Hence, $V_\varepsilon(f_p) \cap f_p^{-1}(t) \neq \emptyset$. We consider a definable map

$$\Phi : V_\varepsilon \ni (x, p) \longmapsto (f_p(x), p) \in \mathbb{R} \times \mathcal{P}.$$

Then, according to the definable choice Lemma 4.6, there exists a definable section γ of Φ, that is a definable mapping $\gamma : \Phi(V_\varepsilon) \to V_\varepsilon$ satisfying $\Phi \circ \gamma = \mathrm{Id}_{\Phi(V_\varepsilon)}$.

Clearly, we can write $\gamma(t,p) = (\gamma_p(t), p)$, where $\gamma_p : f_p(U_p) \setminus K(f_p) \to V_\varepsilon(f_p) \subset U_p$ is a definable mapping.

Let Γ_p denote the image of γ_p. For any $p \in \mathcal{P}$, by routine o-minimality arguments, the definable curve Γ_p is a finite union of points and connected C^1 submanifolds. On each such component of Γ_p, the function f_p is C^1 and injective, hence has only finitely many critical points (by o-minimality). This implies that Γ_p is transverse to the level sets of f_p, except maybe for finitely many levels. Hence, finiteness conditions of Definition 3.2 are fulfilled and Γ_p is an ε-talweg of f_p.

Comparison Theorem 3.3 implies that if X_p is a trajectory of ∇f_p, then

$$\text{Length}(X_p) \leq (1+\varepsilon)\ \text{Length}(\Gamma_p). \tag{6.1}$$

Note that in fact we have proved a more precise estimate for the length of any trajectory of ∇f_p.

Remark 6.3. Let I be a finite union of open intervals in $\mathbb{R}$ and let $x : I \to U$ be a trajectory of ∇f_p. Assume that $x(I) \subset f_p^{-1}((t_1, t_2))$. Then the length of $x(I)$ is bounded by $(1+\varepsilon)$ times the length of $\Gamma_p \cap f_p^{-1}((t_1,t_2))$.

We can now complete the proof of Theorem 6.2. Note that the definable family $\{\Gamma_p\}_{p\in\mathcal{P}}$ satisfies the hypotheses of Corollary 5.2, so there exists a constant $m_\mathcal{F} > 0$ such that for any $p \in \mathcal{P}$, the length of Γ_p is bounded by $m_\mathcal{F}$. Hence, inequality (6.1) implies that the length of any trajectory of ∇f_p is bounded by $M_\mathcal{F} = (1+\varepsilon)m_\mathcal{F}$. □

In fact, the hypotheses of Theorem 6.2 can be weakened. Let $\mathcal{F}$ denote a definable family of functions verifying the same assumptions as in Theorem 6.2 except that the definable set U_p are bounded but no more contained in a fixed compact set K. Let d_p denote the diameter of U_p. Then we have the following Corollary:

Corollary 6.4. *There exists a constant $M_\mathcal{F} > 0$ such that for every $p \in \mathcal{P}$ the length of any trajectory of ∇f_p is bounded by $M_\mathcal{F} \cdot d_p$.*

Proof. Let $f_p \in \mathcal{F}$ and $x_p \in \mathbb{R}^n$ such that $U_p \subset B(x_p, d_p)$. We may assume that the mappings $p \mapsto x_p$ and $p \mapsto d_p$ are definable. Let $T_p : \mathbb{B}^n \ni X \mapsto x_p + d_pX \in B(x_p, d_p)$. Define $g_p = f_p \circ T_p$. The bound is obtained by applying Theorem 6.2 to the definable family $\mathcal{G} = \{g_p\}_{p\in\mathcal{P}}$. □

7. Bounds for gradient trajectories of polynomials

Throughout this Chapter $f : \mathbb{R}^n \to \mathbb{R}$ denotes a polynomial function of degree d. We will give an explicit upper bound for the length of a trajectory of ∇f restricted to the unit ball $\mathbb{B}^n$.

First observe that the following Corollary follows easily from Theorem 6.2.

Corollary 7.1. *Let f be a polynomial in n variables of degree d. Then the length of any trajectory of ∇f in a ball of radius r is bounded by $rA(n,d)$, where $A(n,d)$ is a constant depending only on d and n.*

Proof. The family $\mathcal{F}$ of polynomial functions in n variables and of degree at most d is a semialgebraic family and thus definable in any o-minimal structure. The space of parameters is the space of coefficients of these polynomials. Let $\mathcal{F}_1$ be the semialgebraic family formed with the restriction to $\mathbb{B}^n$ of all functions in $\mathcal{F}$. Theorem 6.2 applied to the family $\mathcal{F}_1$ and $K = \overline{\mathbb{B}^n}$ provides a bound $A(n,d)$ on the length of the trajectories independent on the coefficients. Finally, by Corollary

6.4 we get the desired upper bound for the trajectories of ∇f in any ball of radius r. $\square$

In order to compute $A(n, d)$ explicitly we will use the method described in Chapter 3. We shall construct explicitly a talweg of f, i.e. a semi-algebraic curve Γ with the following property: if $y \in \overline{\mathbb{B}^n}$, then

$$|\nabla f(y)| \geq |\nabla f(x)|, \text{ for some } x \in \Gamma \cap f^{-1}(f(y)). \tag{7.1}$$

In other words we have to minimise $|\nabla f|^2$ on the fibres of f restricted to $\overline{\mathbb{B}^n}$. More precisely, we shall prove that, for a generic polynomial f of degree d, the set

$$\Gamma_1 = \{x \in \mathbb{B}^n : |\nabla f|^2 \text{ has a loc. min. at } x \text{ on the regular fibre } f^{-1}(f(x)) \cap \mathbb{B}^n\}$$

is of dimension at most 1. We shall also prove that the set $\Gamma_2 \subset \mathbb{S}^{n-1}$ defined by

$$\Gamma_2 = \{x \in \mathbb{S}^{n-1} : |\nabla f|^2 \text{ has a loc. min. at } x \text{ on the regular fibre } f^{-1}(f(x)) \cap \mathbb{S}^{n-1}\}$$

is of dimension 1. Then we take $\Gamma = \Gamma_1 \cup \Gamma_2$ and for a generic polynomial we shall give explicit formulae for polynomials describing Γ_1 and Γ_2.

The following Proposition shows that the points at which the level sets of the polynomials f and $|\nabla f|^2$ are non transverse, generically define an algebraic curve. Let $\mathbf{X} = (X_1, \ldots, X_n)$ and denote by $\mathbb{R}_d[\mathbf{X}]$ the space of polynomials in n variables of degree less than or equal to d.

Proposition 7.2. *Let $n \geq 2$ and $d \geq 2$. Then, there exists a semialgebraic set $E_d \subset \mathbb{R}_d[\mathbf{X}]$ of codimension greater than or equal to 1 such that, for any polynomial $f \in \mathbb{R}_d[\mathbf{X}] \setminus E_d$, the set*

$$\Theta_1(f) = \{x \in \mathbb{R}^n : \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f = 0\}$$

is either empty or consists of a finite union of real algebraic curves.

Proof. Recall that by (3.1) we have $\Theta_1(f) = (T_f)^{-1}(\Sigma)$, where

$$\Sigma = \{(V, H) \ : \exists\ \lambda \in \mathbb{R} : H \cdot V = \lambda V\}$$

is an algebraic subset of $\mathbb{R}^n \times S_n(\mathbb{R})$ and $T_f = (\nabla f, Hf)$ is the Gauss-Hesse map of f. Recall that $codim \Sigma = n - 1$, by Theorem 2.1. So it is enough to show that the set

$$E_d = \{f \in \mathbb{R}_d[\mathbf{X}] : T_f \, is \, transverse \, to \, \Sigma\}$$

is dense. By routine arguments E_d is semialgebraic.

Let us fix a polynomial f of degree at most d. We consider a deformation $\tilde{f}$ of f by adding quadratic polynomials. For $\alpha = (\alpha_i) \in \mathbb{R}^n$ and $\varepsilon = (\varepsilon_{jk}) \in \mathbb{R}^{\frac{n(n+1)}{2}}$ we put

$$\tilde{f}_{\alpha,\varepsilon}(x) = f(x) + \sum_{i=1}^{n} \alpha_i x_i + \sum_{1 \leq j \leq k \leq n} \varepsilon_{jk} x_j x_k.$$

Let $\Psi : \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}} \times \mathbb{R}^n \to \mathbb{R}^n \times S_n(\mathbb{R})$ be the map defined as follows:

$$\Psi(\alpha, \varepsilon, x) = (\nabla \tilde{f}_{\alpha,\varepsilon}(x), H\tilde{f}_{\alpha,\varepsilon}(x)),$$

where $H\tilde{f}_{\alpha,\varepsilon}$ stands for Hessian matrix of $\tilde{f}_{\alpha,\varepsilon}$. Note that the differential of Ψ with respect to (α, ε) is an isomorphism. Hence Ψ is a submersion. By the theorem on transversality with parameters (see e.g. [Gu-Po] or [Go-Ma] Chapter 1), we can deduce that there exists an open dense semialgebraic set $\mathcal{E}_1 \subset \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}}$ such that, for any $(\alpha, \varepsilon) \in \mathcal{E}_1$, the map

$$\Psi_{\alpha,\varepsilon} = \Psi(\alpha, \varepsilon, \cdot)$$

is transverse to Σ. Hence E_d is dense and Proposition 7.2 follows. □

We now study Γ_2, the set of points of $\mathbb{S}^{n-1}$ where $|\nabla f|^2$ has a local minimum on the fibres of f restricted to the sphere. Let $r(x) = |x|^2$ and define

$$\Theta_2(f) = \{x \in \mathbb{S}^{n-1} : \mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f \wedge \mathrm{d}r = 0\}.$$

Note that $\Gamma_2 \subset \Theta_2(f)$ and $\Theta_2(f)$ is the set of critical points of $|\nabla f|^2$ on the fibres of f restricted to the sphere. For a generic polynomial f the set $\Theta_2(f)$ is a curve, namely we have:

Proposition 7.3. *There exists a semialgebraic set $F_d \subset \mathbb{R}_d[\mathbf{X}]$ of codimension greater than or equal to 1 such that, for any polynomial $f \in \mathbb{R}_d[\mathbf{X}] \setminus F_d$, the set $\Theta_2(f)$ is nonempty and consists of a finite union of real algebraic curves.*

Proof. We proceed like before and define

$$\tilde{\Sigma} = \left\{(V, H, p) \in \mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1} : (H \cdot V) \wedge V \wedge p = 0\right\}.$$

Let $\pi : \mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1} \to \mathbb{R}^n \times S_n(\mathbb{R})$ denote the projection on the first two coordinates. Let $H, V \in \mathbb{R}^n \times S_n(\mathbb{R})$ be such that $(H \cdot V) \wedge V = 0$. Then $\pi^{-1}(H, V) = \{H\} \times \{V\} \times \mathbb{S}^{n-1}$ and by Theorem 2.1 the set

$$\tilde{\Sigma}_1 = \left\{(V, H, p) \in \mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1} : (H \cdot V) \wedge V = 0\right\}.$$

is of codimension $n-1$ in $\mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1}$. Consider now the set $\tilde{\Sigma}_2 \subset \tilde{\Sigma}$ defined by

$$\tilde{\Sigma_2} = \left\{(V, H, p) \in \mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1} : (H \cdot V) \wedge V \wedge p = 0,\ (H \cdot V) \wedge V \neq 0\right\}.$$

Note that conditions $(H \cdot V) \wedge V \wedge p = 0$ and $(H \cdot V) \wedge V \neq 0$ just mean that p is contained in the plane spanned by the vectors $H \cdot V$ and V. Hence, $\pi^{-1}(H, V) \cap \tilde{\Sigma}_2$ has codimension $n-2$ in $\mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1}$. Moreover, condition $(H \cdot V) \wedge V \neq 0$ is generic in $\mathbb{R}^n \times S_n(\mathbb{R})$. Thus $\tilde{\Sigma}_2$ has codimension $n-2$ in $\mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1}$. Hence, $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$ is of codimension greater than or equal to $n-1$.
Once again we use a quadratic perturbation of polynomials of fixed degree d and the transversality Theorem with parameters. More precisely, we fix $f \in \mathbb{R}_d[\mathbf{X}]$, for $\alpha = (\alpha_i) \in \mathbb{R}^n$ and $\varepsilon = (\varepsilon_{jk}) \in \mathbb{R}^{\frac{n(n+1)}{2}}$ we put

$$\tilde{f}_{\alpha,\varepsilon}(x) = f(x) + \sum_{i=1}^{n} \alpha_i x_i + \sum_{1 \le j \le k \le n} \varepsilon_{jk} x_j x_k.$$

Let $\tilde{\Psi} : \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}} \times \mathbb{S}^{n-1} \to \mathbb{R}^n \times S_n(\mathbb{R}) \times \mathbb{S}^{n-1}$ be the map defined as follows:

$$\tilde{\Psi}(\alpha, \varepsilon, p) = (\nabla \tilde{f}_{\alpha,\varepsilon}(p), H\tilde{f}_{\alpha,\varepsilon}(p), p),$$

where $H\tilde{f}_{\alpha,\varepsilon}$ stands for Hessian matrix of $\tilde{f}_{\alpha,\varepsilon}$. Note that the differential of $\tilde{\Psi}$ is an isomorphism and thus $\tilde{\Psi}$ is a submersion. This means that $\tilde{\Psi}$ is transverse to $\tilde{\Sigma}$. By the theorem on transversality with parameters there exists an open dense semialgebraic set $\mathcal{E}_2 \subset \mathbb{R}^n \times \mathbb{R}^{\frac{n(n+1)}{2}}$ such that for any $(\alpha, \varepsilon) \in \mathcal{E}_2$, the map $\tilde{\Psi}_{\alpha,\varepsilon}$ is transverse to $\tilde{\Sigma}$. Thus, $\operatorname{codim}_{\mathbb{S}^{n-1}}(\tilde{\Psi}_{\alpha,\varepsilon})^{-1}(\tilde{\Sigma}) \ge n-2$. Hence $\dim \theta_2(\tilde{f}_{\alpha,\varepsilon}) \le 1$. This completes the proof of Proposition 7.3. □

At this point we are able to bound the length of any trajectory of the gradient field for a generic polynomial. According to Propositions 7.2 and 7.3 we can give an upper bound by estimating the lengths of $\Theta_1(f)$ and $\Theta_2(f)$ in the case where these sets are of dimension at most equal to 1. This gives rise to the following:

Theorem 7.4. *Let $n \geq 2$ and $d \geq 2$ be integers. Then there exists a semialgebraic set $G_d \subset \mathbb{R}_d[X_1, \ldots, X_n]$, of codimension greater or equal to 1, such that for any polynomial $f \in \mathbb{R}_d[X_1, \ldots, X_n] \setminus G_d$, the length of any trajectory of ∇f in $\mathbb{B}^n$ is bounded by*

$$A(n, d) = \nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2})$$

where $\nu(n)$ is a constant depending only on the dimension.

Proof. Take $G_d = E_d \cup F_d$. Assume that $f \in \mathbb{R}_d[\mathbf{X}] \setminus G_d$, then the sets $\Theta_1(f)$ and $\Theta_2(f)$ are curves, by Propositions 7.2 and 7.3. Moreover their union contains the points at which the function $|\nabla f|^2$ restricted to $\overline{\mathbb{B}^n}$ is minimal on the fibres of f. Note that $\Theta_1(f)$ is the zero set of a 2-form with polynomial coefficients of degree at most $3d-4$. Reordering, if necessary, the variables we may assume that $\frac{\partial f}{\partial x_1}$ vanishes on $\Theta_1(f)$ only at a finite number of points. Then $\tilde{\Theta}_1(f)$ - the common zeroes of the coefficients of $\mathrm{d}x_1 \wedge \mathrm{d}x_i$, $i = 2, \ldots, n$ - contains $\Theta_1(f)$ and possibly some other components which cut $\Theta_1(f)$ only at isolated points. So for a generic affine hyperplane H, the intersection $H \cap \tilde{\Theta}_1(f)$ contains all the points of $H \cap \Theta_1(f)$ which are nondegenerate (hence isolated also in the complexification) and possible other components. By the general version of Bezout's Theorem (see e.g. [Fu]) the number of irreducible components of $H \cap \tilde{\Theta}_1(f)$ is not greater than the product of degrees, that is $(3d-4)^{n-1}$. Thus we have

Lemma 7.5. *For a generic affine hyperplane H the set $H \cap \Theta_1(f)$ has at most $(3d-4)^{n-1}$ points.*

In the same way, $\Theta_2(f)$ is defined by the equation of the unit sphere in $\mathbb{R}^n$ and by the coefficients of the corresponding 3-form. The coefficients of the 3-form associated to $\Theta_2(f)$ are polynomials of degree at most $3(d-1)$. As above we can choose $n-2$ coefficients (and the equation of the sphere) to describe $\Theta_2(f)$ generically. So again by the general version of Bezout's Theorem we have

Lemma 7.6. *For a generic affine hyperplane H the set $H \cap \Theta_2(f)$ has at most $2(3d-3)^{n-2}$ points.*

Now we are in the position to finish the proof of Theorem 7.4. For a generic polynomial f of degree d we have

$$\Gamma_1 \subset \Theta_1(f) \text{ and } \Gamma_2 \subset \Theta_2(f),$$

where Γ_1 and Γ_2, were defined in the beginning of this chapter as the set of points where $|\nabla f|^2$ has the minimum on fibres of f restricted to the open ball and respectively on fibres of f restricted to the sphere. We put $\Gamma = \Gamma_1 \cup \Gamma_2$. Let X be a trajectory of ∇f in the unit ball $\mathbb{B}^n$. We shall prove that the length of the trajectories of ∇f is bounded by the length of Γ. Actually we prove a more general result.

Let D be an open subset of $\mathbb{R}^n$ and let $f : D \to \mathbb{R}$ be a C^2 function in some neighbourhood of $\overline{D}$. Let $\Gamma \subset \overline{D}$ be a C^1 curve. We assume that for each $t \in f(D)$, the set $f^{-1}(t) \cap \Gamma$ consists of exactly one point and that, for all but finitely many $t \in f(D)$, the curve Γ is transverse to $f^{-1}(t)$. Assume that, for any $x \in \Gamma$ and for all $y \in f^{-1}(f(x))$, we have $|\nabla f(x)| \leq |\nabla f(y)|$.

By Lemmas 7.5 and 7.6, the Cauchy-Crofton formula applied to the curve Γ yields

$$\text{Length}(\Gamma) \;\leq\; \nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2}) = A(n,d),$$

where $\nu(n)$ is the measure of the set of affine hyperplanes which meet the unit ball (see Remark 5.3 for the explicit value of $\nu(n)$).

Applying now Theorem 3.3, or more precisely the related Remark 3.4, to an arbitrary trajectory $X \subset D$ of ∇f and the curve Γ yields Theorem 7.4. □

A careful reading of the proof of Theorem 7.4 gives a more precise result. Let $f : \mathbb{R}^n \to \mathbb{R}$, with $n \geq 2$, be a polynomial of degree $d \geq 2$ and let $\bar{B}(r)$ be a closed ball of radius r in $\mathbb{R}^n$. Let $D_i, i \in I$ be all connected components of $f^{-1}(t,s) \cap \bar{B}(r)$. Let $\Gamma = \Gamma(f)$ be the curve associated to a generic polynomial f of degree d, that is Γ contains all points at which $|\nabla f|^2$ attains a local minimum on the fibres of f. Thus we have:

Corollary 7.7. *Let $\lambda_i : [\alpha_i, \beta_i] \to D_i$ be a trajectory of ∇f in D_i. Then*

$$\textit{length } (\lambda_i) \leq \textit{length } (\Gamma \cap D_i).$$

In particular, if we have a trajectory λ_i in each D_i, then

$$\sum_{i \in I} \textit{length } (\lambda_i) \leq \textit{length } (\Gamma \cap B(r)) \leq rA(n,d),$$

where $A(n,d) = \nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2})$ is the constant of theorem 7.8.

We complete this chapter by extending the bound to any polynomial function f in n variables and degree less or equal to d.

Theorem 7.8. *If $f : \mathbb{R}^n \to \mathbb{R}$, with $n \geq 2$, is a polynomial of degree $d \geq 2$, then the length of any trajectory of ∇f in a ball of radius r is bounded by*

$$\nu(n)((3d-4)^{n-1} + 2(3d-3)^{n-2}) \cdot r.$$

Proof. Clearly, by Corollary 6.4 it is enough to prove the estimate for the unit ball $\mathbb{B}^n$. Let us fix a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree $d \geq 2$ and $\varepsilon > 0$. Let $K_0(f) = \{c_1, \ldots, c_k\}$ be the set of critical values of f. For any $\eta > 0$ let us put $K_\eta(f) = \bigcup_{i=1}^k (c_i - \eta, c_i + \eta)$ and $C_\eta = f^{-1}(K_\eta(f))$. We have the following

Lemma 7.9. *For every positive ε, there exists $\eta > 0$ such that the length of any trajectory of ∇f in the set $C_\eta \cap \mathbb{B}^n$ is bounded by $\varepsilon/2$. (See our definition of trajectory in Chapter 2).*

By the proof of Theorem 6.2 there exists a semialgebraic curve $\Gamma \subset \overline{\mathbb{B}^n}$ such that the length of any trajectory of ∇f in $\mathbb{B}^n$ is bounded by $2\text{Length}\,\Gamma$. More precisely, as in Remark 6.3 we have: let I be finite a union of open intervals in $\mathbb{R}$ and let $x : I \to \mathbb{B}^n$ be a trajectory of ∇f. Assume that $x(I) \subset f^{-1}((t_1, t_2))$. Then the length of $x(I)$ is bounded by twice the length of $\Gamma \cap f^{-1}((t_1, t_2))$. Also recall that f is injective on Γ, so the length of $\Gamma \cap f^{-1}((t_1, t_2))$ tends to 0 as t_1 tends to t_2 (or t_2 tends to t_1). We can take as t_2 (respectively as t_1) a critical value c_i, now it is easy to conclude since there are finitely many critical values. This ends Lemma 7.9.

Note that $|\nabla f| > 0$ on the compact set $B_\eta = \overline{\mathbb{B}^n} \setminus C_\eta$. Hence

$$\inf\{|\nabla f(x)|; x \in B_\eta\} = \delta > 0.$$

Thus

Lemma 7.10. *For any $\rho > 0$ there exists $\theta > 0$ such that for any polynomial $g : \mathbb{R}^n \to \mathbb{R}$ of degree d we have:*

$$\text{if } \|f - g\| \leq \theta \quad \text{then}, \quad \measuredangle(\nabla f(x), \nabla g(x)) \leq \rho, \quad \text{for any} \quad x \in B_\eta.$$

Here $\measuredangle(\cdot,\cdot)$ stands for the (unoriented) angle measure between vectors and $\|\cdot\|$ for a norm on the space of polynomials of degree not greater than d. For instance $\|\cdot\|$ may be the maximum of the absolute value of coefficients.

To conclude the Theorem we need to compare the length of a trajectory of ∇f with the length a trajectory of ∇g where g is a generic polynomial close to f.

Lemma 7.11. *Let g be a generic polynomial (in the sense of Theorem 7.4). Assume that $\measuredangle(\nabla f(x), \nabla g(x)) \leq \rho < \pi/2$, for any $x \in B_\eta$. Then the length of any trajectory of ∇f in the set B_η is not greater than $\frac{1}{\cos\rho} A(n, d)$.*

Let $\Theta(g)$ be the restriction to B_η of the curves constructed in Lemmas 7.5 and 7.6 for the polynomial g. Let $X \subset \mathbb{B}^n$ be a trajectory of ∇f. Similarly to the proof of Comparison Principle (Theorem 3.3) we can compare the length of the trajectory X with the length of $\Theta(g)$. We thus obtain that the length of the trajectory X is bounded by $\frac{1}{\cos\rho} \cdot \text{Length}\ \Theta(g)$. By Theorem 7.4 we get the bound $\frac{1}{\cos\rho} A(n, d)$.

At this point, we can complete the proof of Theorem 7.8. For any $\varepsilon > 0$, by Lemma 7.9, there exists $\eta > 0$ and thus two sets C_η and B_η such that the length of X restricted to $C_\eta \cap \mathbb{B}^n$ is bounded by $\varepsilon/2$. Choose $\rho > 0$ such that

$$\frac{1}{\cos\rho} A(n, d) \leq A(n, d) + \varepsilon/2.$$

Both Lemma 7.10 and Theorem 7.4 provide a polynomial $g \in \mathbb{R}_d[\mathbf{X}] \setminus G_d$ such that

$$\measuredangle(\nabla f(x), \nabla g(x)) \leq \rho$$

in B_η. By Lemma 7.11 the length of X restricted to B_η is bounded by $A(n,d) + \varepsilon/2$. Thus finally $\text{Length}(X) \leq A(n,d) + \varepsilon$. Hence Theorem 7.8 follows. □

8. Examples

Let us fix integers d and n, we will denote by $D(d, n)$ the supremum of length of gradient trajectories of polynomials of degree d in the unit ball in $\mathbb{R}^n$. We will now show that

Theorem 8.1. *For any integers $n, d \geq 2$*

$$D(2d, n) \geq 2d^{n-1}, \ d \in \mathbb{N}.$$

Proof. We are going to construct a sequence of examples which will confirm the estimate from below. First we recall "sinusoidal-like" properties of Chebyshev's polynomials. Recall that the d-th Chebyshev polynomial (of the first kind) $T_d(x)$ is determined by

$$T_d(\cos\theta) = \cos(d\theta). \tag{8.1}$$

In particular it has the following properties:

Lemma 8.2.

(1) $|T_d(x)| \leq 1$, *for* $x \in [-1, 1]$;

(2) T_d *has* $d+1$ *extrema on* $[-1, 1]$, *and the values at each extremum is* ± 1.

Hence the length of the graph of T_d *restricted to* $[-1, 1]$, *is greater than* $2d$.

Now return to the construction and, for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, put

$$f_i(x) = x_{i+1} - T_d(x_i).$$

Next we define a polynomial

$$p = \sum_{i=1}^{n-1} f_i^2.$$

Note that p is a polynomial of degree $2d$ in n variables. The zero set of p is a smooth curve which is the intersection of the hypersurfaces $\{x_{i+1} = T_d(x_i)\}$, for $i = 1, \dots, n-1$. It is not difficult to see, by Lemma 8.2, that the length of the curve $C = p^{-1}(0) \cap [-1, 1]^n$ is at least $2d^{n-1}$. Now we define our polynomial by

$$f(x) = \alpha(1 - x_1^2) - p(x),$$

where $\alpha > 0$ is small enough. We will show below that f has only one critical point a which is in fact the global maximum. Note that the set $\{f > 0\}$ is a thin neighbourhood of the curve C. Let b be a point in $\{f > 0\}$ very close to the end of the curve C. Here the end of C is precisely the point $(1, T_d(1), T_d((T_d(1)), \dots)$. Let $x(t)$ be a trajectory of ∇f starting from the point b, so the trajectory must stay in the set $\{f > 0\}$ and it must end at the critical point of f. The critical point a is in fact the middle of the curve C. Hence the length of the trajectory $x(t)$ is not less than half the length of C which is at least $2d^{n-1}$, by Lemma 8.2. We are almost done except that the trajectories is included in $[-1, 1]^n$ instead of the unit ball. The trick, due to J-M. Lion, is to slightly modify the functions f_i in the following way. Let $\varepsilon < 1$ and for $i = 1, \dots, n-2$ let $f_i^\varepsilon(x) = x_{i+1} - \varepsilon T_d(x_i/\varepsilon)$ and $f_{n-1}^\varepsilon(x) = x_n - T_d(x_{n-1}/\varepsilon)$. Then we put

$$p_\varepsilon = \sum_{i=1}^{n-1} (f_i^\varepsilon)^2$$

and $C = p_\varepsilon^{-1}(0) \cap [-\varepsilon, \epsilon]^{n-1} \times [-1, 1]$. Finally we define $f^\varepsilon(x) = \alpha(\varepsilon^2 - x_1^2) - p_\varepsilon(x)$. In order to stay in the unit ball, we make the substitution

$$\tilde{f^\varepsilon}(x) = f^\varepsilon((1 + (n-1)\varepsilon^2)^{-1/2} x),$$

so Theorem 8.1 holds when taking $\varepsilon \to 0$.

We are left with computing the critical points of f, the case of $\tilde{f^\varepsilon}$ is similar. We have to solve the system

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = -2 f_2 \dfrac{\partial f_2}{\partial x_1} - 2\alpha x_1 = 0, \\ \dfrac{\partial f}{\partial x_i} = -2 f_i - 2 f_{i+1} \dfrac{\partial f_{i+1}}{\partial x_i} = 0,\ i = 2, \dots, n-1, \\ \dfrac{\partial f}{\partial x_n} = -2 f_n = 0. \end{cases} \tag{8.2}$$

Observe that if $f_{i+1}(x) = 0$ then $2f_{i+1}\frac{\partial f_{i+1}}{\partial x_i}(x) = \frac{\partial f_{i+1}^2}{\partial x_i}(x) = 0$. So (8.2) reduces to a very simple system

$$x_1 = 0 \text{ and } f_i = 0,\ i = 2, \dots, n, \tag{8.3}$$

So $a = (0, T_d(0), T_d((T_d(0)), \dots)$ is the only critical point of f. □

9. Bounds for gradient trajectories of quasipolynomials and fewnomials

Our method can give an explicit bound for the length of trajectories of gradients in many other cases. For instance, by Khovanskiĭ's theory [Kh], for exponential polynomials, trigonometric polynomials or fewnomials. Let us first recall Khovanskiĭ's definition of quasipolynomials and trigonometric quasipolynomials.

Definition 9.1. Let P be a polynomial of degree d in $n+k$ variables and let $a_1, \dots, a_k \in \mathbb{R}^n$. Then the function f defined by $f(x) = P(x, y_1, \dots, y_k)$ and $y_j = \exp\langle a_j, x\rangle$ for $j = 1, \dots, k$ is called a quasipolynomial of degree d.

Let $a_1, \dots, a_k \in \mathbb{R}^n$ and $P_1, \dots, P_n$ be polynomials in $n+k$ variables of degree $\deg P_i = d_i$. Let $f_1, \dots, f_n$ be the corresponding quasipolynomials. Then Khovanskiĭ proved the following

Theorem 9.2. *(*[Kh] §1.2*) The number of non-degenerate solutions of $f_1 = \cdots = f_n = 0$ is finite and at most $d_1 \cdots d_n (\sum d_i + 1)^k 2^{\frac{k(k-1)}{2}}$.*

Let f be a quasipolynomial of degree d in n variables and k exponentials. Assume that n and d are greater than or equal to 2. Note that the function f is definable in $\mathbb{R}_{\exp}$, and if $k = 0$, then f satisfies the hypotheses of Theorem 7.8. So we may assume that $k \geq 1$. Using Theorem 9.2 we can compute a bound for the length of the trajectories of ∇f inside a ball of radius r. Thus, we have the following

Theorem 9.3. *Let f be a quasipolynomial of degree $d \geq 2$ in n variables and k exponentials. The length of any trajectory of ∇f in a ball of radius r is bounded by*

$$r\nu(n)[(3d)^{n-1}(3d(n-1)+2)^k + 2(3d+1)^{n-2}((3d+1)(n-2)+3)^k]2^{\frac{k(k-1)}{2}}.$$

Proof. As in Chapter 7 we define $\Theta_1(f)$ and $\Theta_2(f)$ as the set of points at which the level sets of the functions $|\nabla f|^2$ and f are tangents. Again, a deformation of f by quadratic polynomials proves that for almost all f the sets $\Theta_1(f)$ and $\Theta_2(f)$ are either empty or consist of a finite union of definable (in $\mathbb{R}_{\exp}$) curves and points. Note that in this case the coefficients of ∇f and of the Hessian H_f are exponential polynomials of degree less or equal to d. Thus the coefficients of the 2-form $\mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f$ are also exponential polynomials of degree less or equal to $3d$. Similarly, the degree of the coefficients of $\mathrm{d}(|\nabla f|^2) \wedge \mathrm{d}f \wedge \mathrm{d}r$ does not exceed $3d+1$. Theorem 9.2 implies that the number of non-degenerate intersection points of $\Theta_1(f)$ with a generic hyperplane $H \subset \mathbb{R}^n$ is bounded by

$$N_1(k,n,d) = (3d)^{n-1}(3d(n-1)+2)^k 2^{\frac{k(k-1)}{2}}.$$

Similarly the number of non-degenerate intersection points of $\Theta_2(f) \cap H$ is bounded by

$$N_2(k,n,d) = 2(3d+1)^{n-2}((n-2)(3d+1)+3)^k 2^{\frac{k(k-1)}{2}}.$$

At this point, it suffices to follow the proofs of Theorems 7.4 and 7.8 to complete the proof of Corollary 9.3. □

Let us continue with Khovanskiĭ's theory. We extend our computations to a more general class of function. We shall now consider trigonometric quasipolynomials. According to [Kh] we have the following definition

Definition 9.4. Let P be a polynomial of degree d in $n+k+2p$ variables and let $a_1,\dots a_k, b_1,\dots,b_p \in \mathbb{R}^n$. Then the function f defined by

$$f(x) = P(x, y_1,\dots,y_k,u_1,\dots,u_p,v_1,\dots,v_p)$$

where $y_j = \exp\langle a_j, x\rangle$ for $j=1,\dots,k$ and $u_q = \sin\langle b_q, x\rangle$, $v_q = \cos\langle b_q, x\rangle$ for $q = 1,\dots,p$ is called a trigonometric quasipolynomial of degree d.

If f be a trigonometric quasipolynomial of degree d in $n+k+2p$ variables, then the restriction of f to a ball of radius r is definable in $\mathbb{R}_{an,\exp}$, the o-minimal structure of globally subanalytic sets augmented with the exponential function. Note that in general f, as a function defined on $\mathbb{R}^n$, is not definable in any o-minimal structure. Let $a_1,\dots,a_k \in \mathbb{R}^n$, $b_1,\dots,b_p \in \mathbb{R}^n$ and $P_1,\dots,P_n$ be polynomials in $n+k+2p$ variables of degree $\deg P_i = d_i$. Let $f_1,\dots f_n$ be the corresponding trigonometric quasipolynomials. The following Theorem gives a bound on the number of roots of a system of trigonometric quasipolynomial.

Theorem 9.5. *([Kh] §1.3) If $|\langle b_q, x\rangle| < \frac{\pi}{2}$, then the number of nonsingular solutions of $f_1 = \cdots = f_n = 0$ is finite and at most*

$$d_1\cdots d_n(\sum d_i + 1 + p)^{p+k} 2^{p+\frac{(p+k)(p+k-1)}{2}}.$$

Note that the assumptions of Theorem 9.5 are more restrictive than those of Theorem 9.2. In order to apply our method of computation for the trigonometric quasipolynomials, we thus need to strengthen the hypotheses. Consider the semialgebraic set $\mathcal{B} = \{b \in \mathbb{R}^n : \forall x \in \overline{\mathbb{B}^n}, |\langle b, x\rangle| < \frac{\pi}{2}$. Let $a_1,\dots,a_k \in \mathbb{R}^n$ and $b_1,\dots,b_p \in \mathcal{B}$ and P be a polynomial in $n+k+2p$ variables of degree d. Let f be the corresponding trigonometric quasipolynomial then we have the following

Theorem 9.6. *Let f be a trigonometric quasipolynomials of degree $d \geq 2$ in $n+k+2p$ variables with $b \in \mathcal{B}$. The length of any trajectory of ∇f in the unit ball is bounded by*

$$\nu(n)\left[(3d)^{n-1}M_1 + 2(3d+1)^{n-2}M_2\right] 2^{p+\frac{(p+k)(p+k-1)}{2}},$$

where $M_1 = (3d(n-1)+2+p)^{p+k}$ and $M_2 = ((3d+1)(n-2)+3+p)^{p+k}$.

We conclude this chapter with an estimate on the length of gradient trajectories of fewnomials. Let f_i, $i = 1,\dots,n$ be polynomials in n variables with only K monomials. Then, Khovanskiĭ's classical result states that the number of non degenerate solutions of the system $f_1 = \ldots = f_n = 0$ is bounded by

$$2^n(1+n)^K 2^{\frac{K(K-1)}{2}}.$$

Theorem 9.7. *Let f be a polynomial in n variables containing only k monomials. Then the length of the trajectories of ∇f inside a ball of radius r is bounded by*

$$r\nu(n)N(n,k),$$

where $N(n,k) = 2^n[(1+n)^{K_1}2^{\frac{K_1(K_1-1)}{2}} + (1+n)^{K_2}2^{\frac{K_2(K_2-1)}{2}}]$, $K_1 = 2n(k+\frac{n(n+3)}{2})^3$ *and* $K_2 = 6n(k+\frac{n(n+3)}{2})^3$.

Proof. First note that the genericity condition used in Chapter 7 involves a quadratic perturbation of f. But the space of polynomial in n variables containing at most k monomials is not stable under quadratic perturbations. The polynomial

$$\tilde{f}_{\alpha,\varepsilon}(x) = f(x) + \sum_{i=1}^{n} \alpha_i x_i + \sum_{1\le j\le k\le n} \varepsilon_{jk} x_j x_k$$

has at most $K = k + \frac{n(n+3)}{2}$ monomials. Moreover, for almost all $(\{\alpha_i\}_i, \{\varepsilon_{jk}\}_{j,k})$, the sets $\Theta_1(\tilde{f}_{\alpha,\varepsilon})$ and $\Theta_2(\tilde{f}_{\alpha,\varepsilon})$ have dimension at most 1. One can check that the coefficients of the 2-form $\mathrm{d}(|\nabla\tilde{f}_{\alpha,\varepsilon}|^2)\wedge \mathrm{d}\tilde{f}_{\alpha,\varepsilon}$ have at most $2nK^3$ monomials while those of $\mathrm{d}(|\nabla f|^2)\wedge \mathrm{d}\tilde{f}_{\alpha,\varepsilon}\wedge \mathrm{d}(|x|^2-r^2)$ have at most $6nK^3$ monomials. Hence, with arguments similar to those of Chapter 7 we deduce that the length of $\Theta_1(\tilde{f}_{\alpha,\varepsilon})$ is bounded by $2r\nu(n)N_1(n,k)$ and the length of $\Theta_2(\tilde{f}_{\alpha,\varepsilon})$ is bounded by $2r\nu(n)N_2(n,k)$. This implies the bound announced for gradient trajectories of a generic $\tilde{f}_{\alpha,\varepsilon}$. This estimate remains valid for any polynomial with at most k monomials arguing like in the end of Chapter 7.

□

10. Application I: Joining two points in a connected semialgebraic set

In many situations e.g. robotics or quantitative transversality (see [Do]) it is important to find a path joining two points x and y in a given connected component A of a semialgebraic set in $\mathbb{R}^n$. The usual manner it is done (see Yomdin [Yo1], [Yo-C]; Canny [Ca1],[Ca2], Donaldson [Do], Basu-Pollack Roy [B-P-R]) is the following: one first constructs so called "roadmap" i.e. a connected semialgebraic curve $C \subset A$ in such a way that it is easy to join each point of A with the curve C. Since C is arc-connected we can join x with y via C. Actually the constructed path is semialgebraic. The construction of C is in general hard and not efficient; one applies induction on n which requires several quantifiers eliminations. There are several algorithms to compute a roadmap, but they don't give a realistic estimate for the length of path in terms of n and degrees of polynomials involved in a description of A. This is due to the fact that the description of C is very complex.

We propose below a new way of joining two points in a semialgebraic set A which is a connected component of $\{f > 0\} \cap B(r)$, where f is a polynomial of degree d and $B(r)$ is a ball of radius r in $\mathbb{R}^n$. We will show that for a generic f we can join in A any two points of A by a curve which is piecewise trajectory of ∇f or of $-\nabla f$.

Examining carefully our proof of Theorem 7.4 we will show that any two points in A may be joined in A by a curve of the length bounded by $2rA(n, d+2)$. It seems that from the numerical viewpoint this method is very promising. In fact there are quite efficient algorithms which numerically compute trajectories of gradients.

Let A be a connected and semialgebraic subset of $\mathbb{R}^n$. For any pair of points $x, y \in A$ we denote by $d_g(x,y)$ the infimum of lengths (for the Euclidean metric) of arcs joining, in A, x with y. Clearly $d_g(x,y) < \infty$, since A is semialgebraic. In fact d_g is a distance on A. We call

$$\mathrm{diam}_g(A) = \sup_{x,y\in A} d_g(x,y)$$

the *geodesic diameter* of the set A.

We now explain the context of our work. First we recall the following result of Yomdin [Yo1],[Yo-C]

Theorem 10.1. *Let $A \subset \mathbb{R}^n$ be a semialgebraic set. Then, for any ball $B(r)$ of radius r in $\mathbb{R}^n$, the geodesic diameter of every connected component of $A \cap B(r)$ is bounded by rK. The constant $K = K(D)$ depends only on D the diagram of A that is: the dimension n and the number and degrees of polynomials describing A.*

In particular for any integers d, n there exists a constant $K(n, d)$, such that if $f : \mathbb{R}^n \to \mathbb{R}$ is polynomial of degree d, then the geodesic diameter of every connected component of $\{f > 0\} \cap B(r)$ is bounded by $rK(n, d)$.

The construction of arcs joining two points in a connected component of $A \cap B(r)$ proposed by Yomdin (also by other authors in a more general setting, see references to Theorem 1.1 in [Yo-C]) gives a semialgebraic arc of "bounded complexity" or in other words of controlled diagram and its length can by estimated by Cauchy-Crofton formula. However it does not give any realistic bound for K, in particular for $K(n, d)$.

S. K. Donaldson in his famous paper [Do] on existence of symplectic submanifolds used in a crucial way the above special case of Theorem 10.1. In fact what he needed (and what he proved in [Do]) was the following:

Theorem 10.2. *For any integer n there exists $C(n) > 0$ and $k(n) > 0$ such that*

$$K(n, d) \le C(n) d^{k(n)}, \text{ for any } d \in \mathbb{N}.$$

A thorough examination of his construction can give explicit estimates, but still far from realistic. In fact the two Theorems mentioned above were used by the authors to obtain a quantitative version of the Morse-Sard theorem, and next by Donaldson [Do] for a quantitative transversality. We explain this at the end of the paper. From the robotics viewpoint the main goal is to construct explicitly a path joining two points.

Our first main result is the following.

Theorem 10.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$, with $n \ge 2$, be a polynomial of degree $d \ge 2$ and let $B(r)$ be a ball of radius r in $\mathbb{R}^n$. Let $D_i, i \in I$ be all connected components of $\{f > 0\} \cap B(r)$. Then*

$$\sum_{i \in I} diam_g(D_i) \ \le \ 2r\nu(n)(3d + 2)^{n-1},$$

In particular $diam_g(D_i) \le 2r\nu(n)(3d + 2)^{n-1}$ for any connected component D_i. More precisely, there is a polynomial g of degree $d + 2$ such that any two points in D_i can be joined in D_i by an arc which is a piecewise trajectory of ∇g or $-\nabla g$ of length not greater than $2r\nu(n)(3d + 2)^{n-1}$.

Proof. Let us fix a ball $B(r) = B(x_0, r) = \{x \in \mathbb{R}^n; |x - x_0|^2 < r\}$ and put

$$g(x) = (r - |x - x_0|^2) f(x).$$

Then any connected component of $\{f > 0\} \cap B(r)$ is a connected component of $\{g > 0\}$. Of course g is a polynomial of degree $d + 2$.

By a small perturbation (very slightly modifying $g^{-1}(0) \cap B(r)$) we may assume that g is generic in the sense of Proposition 7.2 and is a Morse function. This means

g has only isolated non-degenerate critical points $c_1, \dots, c_k$ in the ball and moreover that the corresponding critical values $y_j = g(c_j)$ for $j = 1, \dots, k$ are distinct.

Let x, y be two points in D_i a connected component of $\{g > 0\}$. We first outline the general idea of the construction of an arc joining x with y in D_i.

We may assume that $\nabla g(x) \neq 0$, if not we slightly move x. We take the trajectory $\lambda_1 : [0, a_1) \to D_i$ of ∇g starting at x, that is $\lambda_1(0) = x$ and $\lambda_1'(t) = \nabla g(\lambda_1(t))$. We consider $\lambda_1(t)$ maximal (to the right), hence

$$\lim_{t \to a_1} \lambda_1(t) = c_{j_1},$$

where c_{j_1} is one of the critical points of g. If c_{j_1} is a local maximum of g we stop at that point. If not we continue: we take a point x_1 close to c_{j_1} and such that $g(x_1) > y_{j_1} = f(c_{j_1})$. We can start the trajectory $\lambda_2 : [a_1, a_2) \to D_i$ of ∇g such that $\lambda_2(a_1) = x_1$. (In fact we could take $x_1 = c_{j_1}$, since by a result of Łojasiewicz (cf. [Ło1], [Ło2]) there exists a trajectory of $-\nabla g$ which tends to c_{j_1}). Again $\lim_{t \to a_2} \lambda_1(t) = c_{j_2}$ is a critical point of g. Thus, by finitely many steps we obtain a trajectory

$$\lambda_x : [0, a_x] \to D_i, \text{such that}, \lambda_x(0) = x, \text{ and } \lambda_x(a_x) \text{ is a local maximum of } g.$$

The curve λ_x is continuous, analytic except at finitely many points and at each of those points has left-hand side and right-hand side derivative (by the Gradient Conjecture [K-M-P]).

In this way we have shown that any two points in D_j can be joined by a trajectory of ∇g with a local maximum of g. Clearly, if there is only one local maximum in D_i we obtain the desired bound for the geodesic diameter of D_i directly from Theorem 7.8.

If there are several local maxima of g in a connected component D_i of $\{g > 0\}$ we shall proceed in a more systematic way, which we describe below.

Let $\Gamma = \Gamma(g)$ be the curve associated in Chapter 7 to our polynomial g of degree $d + 2$, that is Γ contains all points at which $|\nabla g|^2$ attains a local minimum on the fibres of g. Note that the fibres of g are compact and thus $\Gamma = \Gamma_1$ (cf. Chapter 7). Hence the length of Γ is bounded by $r\nu(n)(3d + 2)^{n-1}$. Let $D_i, i \in I$ be all connected components of $f^{-1}((t, s)) \cap B(r)$ and let $\lambda_i : [\alpha_i, \beta_i] \to D_i$ be a trajectory of ∇g in D_i. Then, by Corollary 7.7, length $(\lambda_i) \leq$ length $(\Gamma \cap D_i)$. In particular, if we have a trajectory λ_i in each D_i, then

$$\sum_{i \in I} \text{length } (\lambda_i) \leq \text{length } (\Gamma \cap B(r) \cap f^{-1}((t, s))) \leq r\nu(n)(3d + 2)^{n-1}.$$

We shall also consider the connected components of sets $\{g > t\}$, where t is a variable. We write decomposition into connected components

$$\{g > t\} \cap B(r) = \bigcup_{i \in I_t} A_i^t.$$

Now we have

Lemma 10.4. *Any two points in A_i^t can be joined in A_i^t by an arc whose length is not greater than 2length$(\Gamma \cap A_i^t)$.*

We prove the Lemma by "descending" t to 0. If A_i^t contains only one critical point of g, then, as we already explained above, we can join (in A_i^t) any point of A_i^t to the maximum of g in A_i^t, by a trajectory of ∇g. So by Proposition 7.7 we can join any two points in A_i^t by a curve whose length is not greater than 2length $(\Gamma \cap A_i^t)$.

In the general case we will proceed by induction on the number of critical points of g in A_i^t.

Recall that g has only Morse singularities which we denote by c_j, $j = 1, \dots, k$.

Let us fix $0 < t < s$ and assume that the interval (t, s) contains only one critical value y_j. Also assume that the component A_i^t contains the critical point c_j such that $g(c_j) = y_j$. Note that there are most two connected components of $\{g > s\}$ contained in A_i^t, since g has a Morse singularity at c_j. Let us denote them by A_1^s and A_2^s and suppose that at least A_1^s is non-empty. We will take $s > y_j$ but very close to y_j.

Take two points $x, y \in A_i^t$. We shall consider several cases.

Case 1. Assume $g(x) < y_j$ and $g(y) < y_j$. We take the trajectory λ_x of ∇g starting at the point x and ending at a point x' such that $g(x') > s$. If the trajectory passes by the critical point c_j we extend it starting from this point. Note that c_j is not a local maximum of g, so we can pass over the level $\{g = y_j\}$. At this point we could also take an arbitrary short segment joining a point on trajectory at which $g < y_j$ with a point at which $g > y_j$.

In the same way we take the trajectory λ_y of ∇g starting at the point y and ending at a point y' such that $g(y') > s$. Note that by Proposition 7.7 we have

$$\text{Length}\,(\lambda_x) + \text{Length}\,(\lambda_y) \leq 2\text{Length}\,(\Gamma \cap A_i^t \cap g^{-1}(t, s)). \tag{10.1}$$

Case 1.1. If x' and y' belong to the same component A_1^s or A_2^s, then by our induction hypothesis we can join x' with y' in A_1^s, respectively in A_2^s, by a piece-wise trajectory of ∇g of length at most $2\text{Length}\,(\Gamma \cap A_1^s)$, respectively at most $2\text{Length}\,(\Gamma \cap A_2^s)$. Hence, by (10.1) the total length of the curve joining x with y in A_i^t is not greater than $2\text{Length}\,(\Gamma \cap A_i^t)$ as claimed in Lemma 10.4.

Case 1.2. Assume now that $x' \in A_1^s$ and $y' \in A_2^s$. By induction hypothesis we can join, in A_1^s, x' with a point x'' very close to c_j. Respectively we can join, in A_2^s, y' with a point y'' very close to c_j. Note that the total length of both curves is not greater

$$2(\text{Length}\,(\Gamma \cap A_1^s) + \text{Length}\,(\Gamma \cap A_2^s))$$

Finally we can join, in A_i^t, the point x'' with y'' by a short segment (or, via c_j by a short piece of a trajectory of ∇g). So again, by (10.1), we deduce that the total length of the curve joining x with y in A_i^t is not greater than $2\text{Length}\,(\Gamma \cap A_i^t)$ as claimed in Lemma 10.4.

The remaining cases where $g(x) < y_j$ and $g(y) > y_j$ or $g(x) > y_j$ and $g(y) > y_j$ can be handled analogously. So Lemma 10.4 follows.

Taking $t = 0$ in Lemma 10.4 we obtain Theorem 10.3. □

We complete this chapter by an example based on the one proposed in Chapter 8.

Let us fix integers d and n, we will denote by $D_g(d, n)$ the supremum of geodesic diameters of connected components, included in a unit ball of $\mathbb{R}^n$, of sets $\{f > 0\}$, where f is a polynomial of degree d. We will now show that

Theorem 10.5. *For any $n, d \in \mathbb{N}$,*

$$D_g(d, n) \geq 2d^{n-1},\ d \in \mathbb{N}.$$

Proof. Let T_d be the d-th Chebyshev polynomial (of the first kind) defined in Chapter 8. Recall that the length of the graph of T_d restricted to $[-1,1]$, is greater than $2d$. Now return to the construction in $\mathbb{R}^n$, and as before, for $\varepsilon \in (0,1]$ define

$$p_{d,n}(x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n-2}(x_{i+1} - \varepsilon T_d(x_i/\varepsilon))^2 + (x_n - T_d(x_{n-1}/\varepsilon))^2.$$

which is a polynomial of degree $2d$ in n variables. The zero set of $p_{d,n}$ is a smooth curve which is the intersection of the hypersurfaces $\{x_{i+1} = \varepsilon T_d(x_i/\varepsilon)\}$, for $i = 1,\ldots,n-2$ and$\{x_n = T_d(x_{n-1}/\varepsilon)\}$. Note that the length of the curve $p_{d,n}^{-1}(0) \cap [-\varepsilon,\epsilon]^{n-1} \times [-1,1]$ is at least $2d^{n-1}$. Now choose an $\alpha > 0$ small enough and put

$$\tilde{P}_{d,n}(x) = -p_{d,n}(x) + \alpha(x_1^2 - \varepsilon^2).$$

The set $\{\tilde{P}_{d,n} > 0\}$ is a thin neighbourhood of the curve $p_{d,n}^{-1}(0) \cap [-\varepsilon,\varepsilon]^{n-1} \times [-1,1]$. So its geodesic diameter is almost equal to the length of $p_{d,n}^{-1}(0) \cap [-\varepsilon,\varepsilon]^{n-1} \times [-1,1]$ which is at least $2d^{n-1}$. Finally, to obtain the desired semialgebraic set in the unit ball we put $s(\varepsilon) = 1 + (n-1)\varepsilon^2$ and $P_{d,n}(x) = \tilde{P}_{d,n}(\sqrt{s(\varepsilon)}x)$. So Theorem 10.5 holds when taking $\varepsilon \to 0$. □

11. Application II: Quantitative Morse-Sard Theorem

Let us first recall the quantitative version of Morse-Sard Theorem due to Yomdin (cf. [Yo2]), which was previously mentioned.

Theorem 11.1. *Let $B(r)$ be a ball of radius r in $\mathbb{R}^n$, and let $f : B(r) \to \mathbb{R}$ be the restriction of a polynomial of degree d. For any $\varepsilon > 0$ we put $\Sigma_\varepsilon = \{x \in B(r); |\nabla f(x)| < \varepsilon\}$. Then the set $f(\Sigma_\varepsilon)$ can be covered by $N(n,d)$ segments of length $r\varepsilon$. The number $N(n,d)$ depends only on dimension n and degree d.*

This entropic version of Morse-Sard Theorem has the advantage of being stable under small smooth perturbations and has many applications, see [Yo2], [Yo-C]. Another important application appears in a paper of Donaldson [Do], it is so-called controlled transversality (in fact this notion goes back to Gromov). Indeed Theorem 11.1 allows to find a ball of controlled radius in the set of regular values of f. Yomdin insisted in [Yo2] that for most actual applications it is important to have an explicit and realistic estimate for the number $N(n,d)$. Actually he stated in [Yo2] an explicit estimate in the case $n = 2$ which was $N(2,d) \leq \frac{7d^2-15d}{4} + 3$, but for general n no such estimate was known.

The usual way to prove Theorem 11.1 is to apply Theorem 10.1 in the following way. We consider the set

$$\Sigma_\varepsilon = \{x \in B(r); |\nabla f|^2 < \varepsilon^2\} = B(r) \cap \{p > 0\},$$

where $p(x) = \varepsilon^2 - |\nabla f(x)|^2$. Clearly p is of degree $2(d-1)$. Let A_i be a connected component of Σ_ε. Note that $f(A_i)$ is an interval in $\mathbb{R}$, assume that $\overline{f(A_i)} = [\alpha_i,\beta_i]$. Let us take points $x_i, y_i \in A_i$ such that $f(x_i)$ is very close to α_i and $f(y_i)$ is very close to β_i. Let λ_i be a piece-wise smooth arc, constructed according to Theorem 10.3, joining x_i with y_i in A_i. So we may assume that $f(\lambda_i) = [\alpha_i,\beta_i]$. By the Mean Value Theorem we obtain that

$$\beta_i - \alpha_i = \text{Length }(f(\lambda_i)) \leq \varepsilon\text{Length }(\lambda_i)$$

Recall that by Theorem 10.3 the total length of all λ_i is not greater $2rA(n, 2d)$. Note that each connected component of Σ_ε is actually a connected component of $\{\tilde{p} > 0\}$, where $\tilde{p}(x) = p(x)(r^2 - |x - x_0|^2)$. Here x_0 is the center of the ball $B(r)$. The classical bounds for the topology of semi-algebraic sets (cf. [Mi], see also [Yo-C]) yields that the number of bounded connected components of $\{\tilde{p} > 0\}$ is at most

$$B(n, d) = \frac{1}{2}(\deg \tilde{p})(\deg \tilde{p} - 1)^{n-1} = d(2d - 1)^{n-1}.$$

So we have at most $B(n, d)$ segments of the total length not greater $\varepsilon r 2A(n, 2d)$, so they can be covered by $B(n, d) + 2A(n, 2d)$ intervals of length εr. Thus we obtained the following estimate

$$N(n, d) \leq d(2d - 1)^{n-1} + 2\nu(n)((6d - 4)^{n-1} + 2(6d - 3)^{n-2}). \tag{11.1}$$

This is a first (to our knowledge) explicit bound for $N(n, d)$, note that $N(n, d) \geq (d - 1)^n$ since there are real polynomials of degree d which have $(d - 1)^n$ critical values.

We will give now a better estimate for $N(n, d)$ based on a direct analysis of the curve Γ associated in Chapter 7 to a generic polynomial of degree d. Recall that the curve Γ contains all local minima of $|\nabla f|$ on fibres of f restricted to $\overline{B(r)}$. By abuse of notation we denote $\Sigma_\varepsilon = \{x \in \overline{B(r)}; |\nabla f(x)| < \varepsilon\}$ which is slightly bigger than the set in the Theorem 11.1 because we also take into account the boundary of $B(r)$.

Note that, if $t \in f(\Sigma_\varepsilon)$, then $\inf\{|\nabla f(x)|; x \in f^{-1}(t) \cap \overline{B(r)}\} < \varepsilon$, hence

$$f^{-1}(t) \cap \Gamma \cap \Sigma_\varepsilon \neq \emptyset.$$

Consequently

$$f(\Sigma_\varepsilon) = f(\Gamma \cap \Sigma_\varepsilon).$$

By Theorem 7.8 we know that the total length of $\Gamma \cap \overline{B(r)}$, hence also of $\Gamma \cap \Sigma_\varepsilon$, is bounded by $rA(n, d)$. So the total length of $f(\Sigma_\varepsilon)$ is not greater than $\varepsilon r A(n, d)$. We showed above that the number of connected components of $f(\Sigma_\varepsilon)$ is bounded by $B(n, d) = \frac{1}{2}(2d)(2d - 1)^{n-1}$. Thus we have a better bound

Theorem 11.2. *Let $N(n, d)$ be the constant in Yomdin's Theorem 11.1. Then*

$$N(n, d) \leq d(2d - 1)^{n-1} + \nu(n)((3d - 4)^{n-1} + 2(3d - 3)^{n-2}).$$

References

[AK1] Antipov M.V., Kireitov V.R. , *Some mathematical problems of surface topography,* Russian Geology and Gephysics, 10, No. 4,(1979) pp. 124-131.

[AK1] Antipov M.V., Kireitov V.R. , *Construction of a T(W) set for a relief surface,* Russian Geology and Gephysics, 10, No. 11,(1979) pp. 90-97.

[B-P-R] S. Basu, R. Pollack and M-F. Roy, *Computing roadmaps of semi-algebraic sets on a variety,* J. Am. Math. Soc. 13, No.1 (2000), pp. 55-82.

[B-C-R] J. Bochnak, M. Coste and M.F. Roy, *Real algebraic geometry,* E.M.G vol. 36 (1998) Springer.

[Ca1] J.Canny *Constructing roadmaps of semi-algebraic sets. I: Completeness,* Artif. Intell. 37, No.1-3 (1988), pp. 203-222 .

[Ca2] J.Canny *Computing roadmaps of general semi-algebraic sets,* Comput. J. 36 No.5 (1993), pp. 504-514.

[Cay] A. Cayley, *On contour and slope lines,* The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, 18 (1859) pp. 264-268.

[Ch] F. Chazal, *Structure locale et globale des feuilletages de Rolle, un théorème de fibration* Ann. Inst. Fourier 48-2, (1998), pp. 553-592.

[Co] M. Coste, *An introduction to o-minimal geometry,* Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa (2000).
[D'A1] D. D'Acunto, *Valeurs Critiques Asymptotiques d'une Fonction Définissable dans une Structure o-minimale,* Ann. Pol. Math, 35 (2000), pp. 35-45.
[D'A2] D. D'Acunto, *Sur les courbes intégrales du champ de gradient,* Thèse de l'Université de Savoie (2001).
[Do] S. K. Donaldson, *Symplectic submanifold and almost-complex geometry,* J. Diff. Geometry 44 (1996), pp. 666-705.
[Dr] L. van den Dries, *Tame topology and o-minimal structures,* London Math. Soc Lecture Notes 248, Cambridge University Press.
[D-M-M] van den Dries, A. Macintyre and D. Marker *The elementary theory of restricted analytic fields with exponentiation* Ann. of Math. 140 (1994), pp. 183–205.
[Dr-Mi] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures,* Duke Math. Journal, 84 (1996), pp. 497-540.
[Fe] H. Federer, *Geometric measure theory,* Grundlehren Math. Wiss, vol. 153 Springer-Verlag (1969).
[Fu] W. Fulton, *Intersection Theory,* Springer (1998).
[GKB] Golubyatnikv V.P., Krasavchikov V.O., Bukreeva G.F., Kontorovich V.A. *Geometrical approach to modeling of Hydrocarbon migration pathways",* 3-D Forum, 15 No 4, (2001) pp. 140-145.
[Go-Gu] M. Golubitsky and V. Guillemin, *Stable Mappings and their Singularities,* Grad. Texts in Math., Springer-Verlag, New-York (1974).
[Go-Ma] M. Goresky and R. Mac Pherson, *Stratified Morse theory,* E.M.G vol. 14 Springer-Verlag (1988).
[Gu-Po] V. Guillemin and A. Pollack *Differential topology,* Prentice-Hall (1974).
[Jo1] C. Jordan, *Sur les lignes de faîte et de thalweg,* C. R. Acad. Sci. 74 (1872), pp. 1457-1459.
[Jo2] C. Jordan, *Sur les lignes de faîte et de thalweg; réponse aux observations de M. Boussinesq,* C. R. Acad. Sci. 75 (1872), pp. 625-627.
[Jo3] C. Jordan, *Nouvelles observations sur les lignes de faîte et de thalweg,* C. R. Acad. Sci. 75 (1872), pp. 1023-1025.
[Kh] A.G. Khovanskiĭ, *Fewnomials,,* Translations of Mathematical Mongraphs 88, AMS Providence, R.I. (1991).
[Ku] K. Kurdyka, *On gradients of functions definable in o-minimal structures,* Ann. Inst. Fourier, 48-3 (1998), pp. 769-783.
[K-M-P] K. Kurdyka, T. Mostowski and A. Parusiński, *Proof of the gradient conjecture of R. Thom,* Annals of Math. 152 (2000), pp. 763- 792.
[Ło1] S. Łojasiewicz, *Une propriété topologique des sous-ensembles analytiques réels*, Colloques Internationaux du CNRS, No 117, (1962), 87–89.
[Ło2] S. Łojasiewicz, *Sur les trajectoires du gradient d'une fonction analytique,* Semin. Geom., Univ. Studi Bologna 1982/1983 (1984), pp. 115-117.
[Ma] J. C. Maxwell, *On hills and dales*, The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, 40 (1870), pp. 421-425.
[Me] C. Miller, *Expansion of the real field with power functions* Ann. Pure Appl. Logic 68 (1994), pp. 79–94.
[Mi] J. Milnor, *On the betti numbers of real varieties,* Proc. Amer. Math. Soc. 15 (1964), pp. 275-280.
[Mo] M. Moszyńska, *Geometria zbiorów wypukłych (Polish),* Wydawnictwa Naukowo-Techniczne (2001).
[dSV] de Saint-Venant, *Surfaces à plus grande pente constituées sur des lignes courbes,* Bulletin de la Soc. Philomath. de Paris, March 6th (1852), pp. 24-30.
[Tr] D. Trotman, *Stability of transversality implies Whitney (a) regularity,* Invent. Math. 50 (1979), pp. 273-277.
[Wi] A. Wilkie, *Model completness results for expansions of the ordered field of reals by restricted Pffafian functions and the exponential function* J. Amer. Math. Soc. 9 (1996), pp. 1051-1094.
[Yo1] Y. Yomdin, *The geometry of critical and near-critical values of differentiable mappings,* Math. Ann. 4 (1983), 495-515.
[Yo2] Y. Yomdin, *Metric properties of semialgebraic sets an mappings and their applications in smooth analysis* Proc. 2nd International Conference on Algebraic Geometry, La Rabida

(Spain) 1984, J.M. Aroca, T. Sanchez-Geralda, J.L. Vincente eds., Travaux en Course, Hermann, Paris (1987), pp. 165-183.

[Yo-C] Y. Yomdin and G. Comte *Tame geometry with applications in smooth analysis,* Springer Lecture Notes in Mathematics, vol. 1834 (2004).

D. D'Acunto, Laboratoire de Mathématiques, UMR 5127 CNRS, Université de Savoie, Campus scientifique, F-73376 Le Bourget du Lac cedex, France
E-mail address: didier.dacunto@univ-savoie.fr

K. Kurdyka, Laboratoire de Mathématiques, UMR 5127 CNRS, Université de Savoie, Campus scientifique, F-73376 Le Bourget du Lac cedex, France
E-mail address: kurdyka@univ-savoie.fr