"REFLEXIVITY IS EQUIVALENT TO THE PERTURBED FIXED POINT PROPERTY FOR CASCADING NONEXPANSIVE MAPS IN BANACH LATTICES."

METRIC FIXED POINT THEORY SEMINAR,
INSTYTUT MATEMATYKI,
UNIWERSYTET MARII CURIE-SKŁODOWSKIEJ,

WED 28/MAY/2014.

CHRIS LENNARD, UNIVERSITY OF PITTSBURGH

Date: May 27, 2014.

Abstract. Using a theorem of Domínguez Benavides and the Strong James Distortion Theorems, Lennard and Nezir recently proved that if a Banach space is a Banach lattice or has an unconditional basis, then it is reflexive if and only if it has an equivalent norm that has the fixed point property for **cascading nonexpansive mappings**. This new class of mappings strictly includes nonexpansive mappings. [*"Reflexivity is equivalent to the perturbed fixed point property for cascading nonexpansive maps in Banach lattices"*, Nonlinear Analysis 95 (2014), 414-420.[10]=[LN]]

1. Introduction

In 1965, Browder [1] proved: [♠] [For every closed, bounded, convex (non-empty) subset C of a Hilbert space $(X, \|\cdot\|)$, for all nonexpansive mappings $T\colon C \to C$ [i.e., $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$], T has a fixed fixed point in C.] Soon after, also in 1965, Browder [2] and Göhde [8] (independently) generalized the result [♠] to uniformly convex Banach spaces $(X, \|\cdot\|)$; e.g., $X = L^p,\ 1 < p < \infty$, with its usual norm $\|\cdot\|_p$.

Later in 1965, Kirk [9] further generalized [♠] to all reflexive Banach spaces X with normal structure: those spaces such that all non-trivial closed, bounded, convex sets C have a smaller radius than diameter.

Spaces $(X, \|\cdot\|)$ with the property of Browder [♠] became known as spaces with the "fixed point property for nonexpansive mappings" (FPP (n.e.)).

Concerning Kirk's theorem, we may ask what further generalizations are possible? After 49 years, it remains an open question as to whether or not every reflexive Banach space $(X, \|\cdot\|)$ has the fixed point property for nonexpansive maps.

Recently, Domínguez Benavides [3] proved that the following intriguing result: [Given a reflexive Banach space $(X, \|\cdot\|)$, there exists an equivalent norm $\|\cdot\|_\sim$ on X such that $(X, \|\cdot\|_\sim)$ has the fixed point property for nonexpansive mappings]. This improves a theorem of van Dulst [6] for separable reflexive Banach spaces.

In contrast to this result, the non-reflexive Banach space $(\ell^1, \|\cdot\|_1)$, the space of all absolutely summable sequences, with the absolute sum norm $\|\cdot\|_1$, fails the fixed point property for nonexpansive mappings. E.g., let $C := \{(t_n)_{n\in\mathbb{N}} :$ each $t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1\}$. This is a closed, bounded, convex subset of ℓ^1. Let $T\colon C \to C$ be the right shift map on C; i.e.,

$$T(t_1, t_2, t_3, \dots) := (0, t_1, t_2, t_3, \dots) \; .$$

T is clearly $\|\cdot\|_1$-nonexpansive (being an isometry) and fixed point free on C.

Recently, in another significant development, Lin [11] provided the first example of a non-reflexive Banach space $(X, \|\cdot\|)$ with the fixed point property for non-expansive mappings. Lin verified this fact for $(\ell^1, \|\cdot\|_1)$ with the equivalent norm $|||\cdot|||$ given by

$$|||x||| = \sup_{k\in\mathbb{N}} \frac{8^k}{1+8^k} \sum_{n=k}^{\infty} |x_n|, \text{ **for all** } x = (x_n)_{n\in\mathbb{N}} \in \ell^1 \ .$$

What about $(c_0, \|\cdot\|_\infty)$, the Banach space of real-valued sequences that converge to zero, with the absolute supremum norm $\|\cdot\|_\infty$? This is another non-reflexive Banach space of great importance in Banach space theory. It also fails the fixed point property for nonexpansive mappings. E.g., let

$$C := \{(t_n)_{n\in\mathbb{N}} : \textbf{ each } t_n \geq 0,\ 1 = t_1 \geq t_2 \geq \cdots \geq t_n \geq t_{n+1} \longrightarrow 0,\ \textbf{as } n \to \infty\} .$$

Let $U\colon C \to C$ be the natural right shift map.

$$U(t_1, t_2, t_3, \dots) := (1, t_1, t_2, t_3, \dots) .$$

Then U is a $\|\cdot\|_\infty$-nonexpansive (isometric, actually) map with no fixed points in the closed bounded convex set C.

It is natural to ask whether there is a c_0-analogue of Lin's theorem about ℓ^1. It remains an open question as to whether or not there exists an equivalent norm $\|\cdot\|_\sim$ on $(c_0, \|\cdot\|_\infty)$ such that $(c_0, \|\cdot\|_\sim)$ has the fixed point property for nonexpansive mappings. However, if we weaken the nonexpansive condition to "asymptotically nonexpansive", then the answer is "no". In 2000, Dowling, Lennard and Turett [5] showed that for every equivalent renorming $|||\cdot|||$ of $(c_0, \|\cdot\|_\infty)$, there exists a closed, bounded, convex set C and an asymptotically nonexpansive mapping $T\colon C \to C$ [i.e., there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $[1,\infty)$ such that $k_n \xrightarrow[n]{} 1$, and for all $n \in \mathbb{N}$, for all $x, y \in C$, $|||T^n x - T^n y||| \leq k_n |||x-y|||$] such that T has no fixed point.

In contrast to this, note that in 1972, Goebel and Kirk [7] proved that for all uniformly convex spaces $(X, \|\cdot\|)$, for every closed, bounded, convex set $C \subseteq X$, for all eventually asymptotically nonexpansive maps $T : C \to C$, T has a fixed point in C. We call a mapping $T : C \to C$ *eventually asymptotically nonexpansive* **if there exists $\nu \in \mathbb{N}$ and a sequence $(k_n)_{n \geq \nu}$ in $[1, \infty)$ such that $k_n \xrightarrow[n]{} 1$, and for all $n \geq \nu$, for all $x, y \in C$, $\|T^n x - T^n y\| \leq k_n \|x - y\|$.**

In the above-mentioned paper [10]**, using the above-described theorem of Domínguez Benavides and the Strong James' Distortion Theorems, we proved that if a Banach space is a Banach lattice or has an unconditional basis, then it is reflexive if and only if it has an equivalent norm that has the fixed point property for** *cascading nonexpansive mappings* **(see Definition 2.1).**

This new class of mappings strictly includes nonexpansive mappings. *Cascading nonexpansive mappings* **are analogous to** *asymptotically nonexpansive mappings*, **but examples show that neither of these two classes of mappings contain the other. One of these examples is due to Łukasz Piasecki. He invented an asymptotically nonexpansive map that is not cascading nonexpansive. This example will be described in the paper currently being prepared by Lennard, Nezir and Piasecki [LNP].**

Note that via Lin's example (described above) of an equivalent renorming $|||\cdot|||$ of $(\ell^1, \|\cdot\|_1)$ such that $(\ell^1, |||\cdot|||)$ has the fixed point property for nonexpansive mappings, in our theorem one cannot replace "cascading nonexpansive mappings" by "nonexpansive mappings".

Also, using a theorem of Mil'man and Mil'man, we show in [LN] that for Banach lattices or Banach spaces with an unconditional basis, reflexivity is equivalent to the fixed point property for affine cascading nonexpansive mappings.

Finally, we prove (see [LN]) that in any Banach space, for every closed bounded convex subset C and every cascading nonexpansive mapping $T : C \longrightarrow C$, T has an approximate fixed point sequence;
i.e., $\inf_{x \in C} \|Tx - x\| = 0$**.**

On the other hand, it remains an open question as to whether or not every asymptotically nonexpansive mapping on a closed bounded convex set in a Banach space has an approximate fixed point sequence.

2. Reflexivity iff the perturbed cascading nonexpansive FPP in Banach lattices

Let C **be a closed bounded convex subset of a Banach space** $(X, \|\cdot\|)$**. Let** $T : C \longrightarrow C$ **be a mapping. Let** $C_0 := C$ **and**

$$C_1 := \overline{\text{co}}\big(T(C)\big) \subseteq C \ .$$

Clearly C_1 **is a closed bounded convex set in** C**. Let** $x \in C_1$**. Then**

$$Tx \in T(C_1) \subseteq T(C) \subseteq \overline{\text{co}}\,(T(C)) = C_1 \ .$$

So, T **maps** C_1 **into** C_1**. Inductively, for all** $n \in \mathbb{N}$ **we define**

$$C_n := \overline{\text{co}}\big(T(C_{n-1})\big) \subseteq C_{n-1} \ .$$

Similarly to above, it follows that T **maps** C_n **into** C_n**.**

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space and C be a closed bounded convex subset of X. Let $T : C \longrightarrow C$ be a mapping and $(C_n)_{n\in\mathbb{N}}$ be defined as above. We say that T is *cascading nonexpansive* if there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $[1,\infty)$ such that $\lambda_n \underset{n}{\longrightarrow} 1$, and for all $n \in \mathbb{N}$, for all $x, y \in C_n$,

$$\|Tx - Ty\| \leq \lambda_n \|x - y\| .$$

Note that every cascading nonexpansive mapping is norm-to-norm continuous; and every nonexpansive map is cascading nonexpansive.

Example 2.2. [Not all cascading nonexpansive mappings are eventually asymptotically nonexpansive] Let $(\gamma_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence in $(0,1)$ that converges to 1. Also assume that $(\gamma_{n+1}/\gamma_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $(1,2]$. An example of such a sequence is $\big[\,\gamma_n := 8^n/(1+8^n)$, for all $n \in \mathbb{N}\,\big]$, mentioned in the Introduction above. Consider the Banach space ℓ^1, endowed with the equivalent norm

$$\|x\|_\sim := \sup_{\nu\in\mathbb{N}} \gamma_\nu \sum_{k=\nu}^{\infty} |x_k| \,, \text{ for all } x \in \ell^1 \,.$$

Let $K := \{x \in \ell^1 : \text{ each } x_j \geq 0 \text{ and } \sum_{j=1}^{\infty} x_j = 1\}$. Also, define the mapping $R : K \longrightarrow K$ by $Rx := (0, x_1, x_2, x_3, \dots)$, for all $x \in K$. The set $K_0 := K$ is closed, bounded and convex in ℓ^1. Let $K_n := \overline{\mathrm{co}}\big(R(K_{n-1})\big)$, for all $n \in \mathbb{N}$. It is easy to check that each

$$K_n = \left\{ (\overbrace{0,\ldots,0}^{n}, x_{n+1}, x_{n+2}, \ldots) \in \ell^1 \right.$$
$$\left. : \quad \text{each } x_j \geq 0 \text{ and } \sum_{j=n+1}^{\infty} x_j = 1 \right\}.$$

Fix $n \in \mathbb{N}_0$. Fix $u, v \in K_n$. Then $u = (\overbrace{0,\ldots,0}^{n}, x_{n+1}, x_{n+2}, \ldots)$ and $v = (\overbrace{0,\ldots,0}^{n}, y_{n+1}, y_{n+2}, \ldots)$, for some x, y as described above. Thus, $Ru = (\overbrace{0,\ldots,0}^{n+1}, x_{n+1}, x_{n+2}, \ldots)$ and

$Rv = (\overbrace{0, \ldots, 0}^{n+1}, y_{n+1}, y_{n+2}, \ldots)$. Then,

$$\|u - v\|_\sim = \sup_{\nu \geq n+1} \gamma_\nu \sum_{k=\nu}^{\infty} |x_k - y_k|$$

and

$$\|Ru - Rv\|_\sim = \sup_{\nu \geq n+2} \gamma_\nu \sum_{k=\nu-1}^{\infty} |x_k - y_k| \; .$$

Therefore,

$$\|Ru - Rv\|_\sim \leq \|u - v\|_\sim \sup_{\nu \geq n+2} \frac{\gamma_\nu}{\gamma_{\nu-1}} = \frac{\gamma_{n+2}}{\gamma_{n+1}} \|u - v\|_\sim \; .$$

Also, $e_{n+1}, e_{n+2} \in K_n$ and

$$\begin{aligned} \|Re_{n+1} - Re_{n+2}\|_\sim &= \|e_{n+2} - e_{n+3}\|_\sim = 2\,\gamma_{n+2} \\ &= \frac{\gamma_{n+2}}{\gamma_{n+1}} \|e_{n+1} - e_{n+2}\|_\sim \; . \end{aligned}$$

So, R is a cascading nonexpansive mapping on K, with best constants $\lambda_n = \dfrac{\gamma_{n+2}}{\gamma_{n+1}}$, for all $n \in \mathbb{N}_0$.

On the other hand, $\|e_1 - e_2\|_\sim = 2\,\gamma_1$; and for all $n \in \mathbb{N}$,

$$\|R^n e_1 - R^n e_2\|_\sim = \|e_{n+1} - e_{n+2}\|_\sim = 2\,\gamma_{n+1} = \frac{\gamma_{n+1}}{\gamma_1}\,\|e_1 - e_2\|_\sim\,.$$

But $\dfrac{\gamma_{n+1}}{\gamma_1} \underset{n}{\longrightarrow} 1/\gamma_1 > 1$. Therefore, R is not eventually asymptotically nonexpansive on K.

Example 2.3 (Not all eventually asymptotically nonexpansive mappings are cascading nonexpansive)**.** Consider the Banach space $(\mathbb{R}, |\cdot|)$ and the closed, bounded, convex set $K_0 := K := [0, 1/\sqrt{2}]$. Let $a \wedge b := \min\{a, b\}$, for all $a, b \in \mathbb{R}$. Also, let $\mathbb{Q}$ denote the set of all rational numbers and $\mathbb{I} := \mathbb{R} \backslash \mathbb{Q}$ be the set of all irrational numbers. We define the mapping $U : K \longrightarrow K$ by setting $Ux := \sqrt{2}\,x \wedge (1/\sqrt{2})$, for all $x \in \mathbb{Q} \cap K$; and $Ux := 0$, for all $x \in \mathbb{I} \cap K$.

It is easy to check that $K_n := \overline{\mathrm{co}}\big(U(K_{n-1})\big) = K$, for all $n \in \mathbb{N}$. Further, $0, 1/2 \in K$ and

$$|U(0) - U(1/2)| = |0 - 1/\sqrt{2}| = 1/\sqrt{2} = \sqrt{2}\,(1/2) = \sqrt{2}\,|0 - 1/2|\,.$$

Thus, U fails to be a cascading nonexpansive mapping on K.

On the other hand, for all $n \geq 2$, for all $x \in K$, $U^n x = 0$. Therefore, U is eventually asymptotically nonexpansive on K.

Remark 2.4. Note that in the previous example, U is not a continuous mapping on K, and so U is *not* asymptotically nonexpansive.

Recently, Łukasz Piasecki [14] has invented an asymptotically nonexpansive mapping T on a closed, bounded, convex set K in a Banach space $(X, \|\cdot\|)$ such that T is not cascading nonexpansive.

Cascading nonexpansive mappings arise naturally in Banach spaces $(X, \|\cdot\|)$ that contain an isomorphic copy of ℓ^1 or c_0. Examples of such spaces are Banach spaces isomorphic to a nonreflexive Banach lattice, and nonreflexive Banach spaces with an unconditional basis. (See, for example, Lindenstrauss and Tzafriri [13] 1.c.5 and [12] 1.c.12.)

Theorem 2.5. *Let $(X, \|\cdot\|)$ be a Banach space that contains an isomorphic copy of ℓ^1 or c_0. Then there exists a closed bounded convex set $C \subseteq X$ and an affine cascading nonexpansive mapping $T : C \longrightarrow C$ such that T is fixed point free.*

The proof uses the Strong James' Distortion Theorem for ℓ^1 and c_0 ([4], [5]).

Proof. **Case 1.** $(X, \|\cdot\|)$ contains an isomorphic copy of ℓ^1. By the Strong James' Distortion Theorem for ℓ^1 ([4], [5]), there exists a normalized sequence $(x_j)_{j\in\mathbb{N}}$ in X and a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$ such that for all $n \in \mathbb{N}$, for all $t = (t_j)_{j\in\mathbb{N}} \in c_{00}$,

$$(1-\varepsilon_n)\sum_{j=n}^{\infty}|t_j| \leq \left\|\sum_{j=n}^{\infty} t_j\, x_j\right\| \leq \sum_{j=n}^{\infty}|t_j|\,.$$

Define the closed bounded convex subset C of X by

$$C := \overline{\mathrm{co}}\{x_j : j \in \mathbb{N}\} = \left\{\sum_{j=1}^{\infty} t_j\, x_j :\ \text{each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1\right\}.$$

Further define $T : C \longrightarrow C$ by

$$T\left(\sum_{j=1}^{\infty} t_j\, x_j\right) := \sum_{j=1}^{\infty} t_j\, x_{j+1}\ .$$

The mapping T is affine and fixed point free. Inductively, we see that for all $n \in \mathbb{N}$, $C_n := \overline{\mathrm{co}}\big(T(C_{n-1})\big) = T(C_{n-1})$ is given by

$$C_n = \left\{ \sum_{j=n+1}^{\infty} t_j\, x_j : \text{ each } t_j \geq 0 \text{ and } \sum_{j=n+1}^{\infty} t_j = 1 \right\}\ .$$

Fix $n \in \mathbb{N}_0$ and fix $x, y \in C_n$. So, $x = \sum_{j=n+1}^{\infty} t_j\, x_j$ and $y = \sum_{j=n+1}^{\infty} s_j\, x_j$, where each $t_j, s_j \geq 0$ and $\sum_{j=n+1}^{\infty} t_j =$

$\sum_{j=n+1}^{\infty} s_j = 1$. Let $\alpha_j := t_j - s_j$, for all $j \geq n+1$. Then

$$\|x - y\| = \left\| \sum_{j=n+1}^{\infty} \alpha_j \, x_j \right\| \geq (1 - \varepsilon_{n+1}) \sum_{j=n+1}^{\infty} |\alpha_j| \, ,$$

and since each $\|x_k\| = 1$,

$$\begin{aligned} \|Tx - Ty\| \; &= \left\| \sum_{j=n+1}^{\infty} \alpha_j \, x_{j+1} \right\| \leq \sum_{j=n+1}^{\infty} |\alpha_j| \\ &\leq \frac{1}{(1 - \varepsilon_{n+1})} \|x - y\| \, . \end{aligned}$$

Consequently, T is cascading nonexpansive on C.

Case 2. $(X, \|\cdot\|)$ contains an isomorphic copy of c_0. By a strengthening of the Strong James' Distortion Theorem for c_0 ([5], Theorem 8), there exists a normalized sequence $(x_j)_{j\in\mathbb{N}}$ in X and a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$ such that for all $n \in \mathbb{N}$, for all $t = (t_j)_{j\in\mathbb{N}} \in c_{00}$,

$$\max_{j\geq n} |t_j| \leq \left\| \sum_{j=n}^{\infty} t_j\, x_j \right\| \leq (1+\varepsilon_n) \max_{j\geq n} |t_j| \ .$$

Define $f_k := x_1 + \cdots + x_k$, for all $k \in \mathbb{N}$. Next we define the closed bounded convex subset C of X by

$$C := \overline{\mathrm{co}}\{f_k : k \in \mathbb{N}\} = \left\{ \sum_{j=1}^{\infty} t_j\, x_j : 1 = t_1 \geq t_2 \geq \cdots \geq t_k \downarrow_k 0 \right\} .$$

Also, we define the function $T : C \longrightarrow C$ by

$$T\left(\sum_{j=1}^{\infty} t_j\, x_j \right) := x_1 + \sum_{j=1}^{\infty} t_j\, x_{j+1} \ .$$

The mapping T is affine and fixed point free. Inductively, it follows that for all $n \in \mathbb{N}$, $C_n := \overline{\text{co}}\big(T(C_{n-1})\big) = T(C_{n-1})$ is given by

$$C_n = \left\{ \sum_{j=1}^{\infty} t_j\, x_j : 1 = t_1 = t_2 = \cdots = t_{n+1} \geq t_{n+2} \geq \cdots \geq t_k \downarrow_k 0 \right\} .$$

Fix $n \in \mathbb{N}_0$ and fix $x, y \in C_n$. Hence, $x = \sum_{j=1}^{\infty} t_j\, x_j$ and $y = \sum_{j=1}^{\infty} s_j\, x_j$, where $1 = t_1 = \cdots = t_{n+1} \geq t_{n+2} \geq \cdots \geq t_k \downarrow_k 0$ and $1 = s_1 = \cdots = s_{n+1} \geq s_{n+2} \geq \cdots \geq s_k \downarrow_k 0$. Let $\alpha_j := t_j - s_j$, for all $j \in \mathbb{N}$. We have that

$$\|x - y\| = \left\| \sum_{j=n+2}^{\infty} \alpha_j\, x_j \right\| \geq \max_{j \geq n+2} |\alpha_j| \,,$$

and

$$
\begin{aligned}
\|Tx - Ty\| &= \left\| \sum_{j=1}^{\infty} \alpha_j \, x_{j+1} \right\| = \left\| \sum_{j=n+2}^{\infty} \alpha_j \, x_{j+1} \right\| \\
&= \left\| \sum_{k=n+3}^{\infty} \alpha_{k-1} \, x_k \right\| \\
&\leq (1 + \varepsilon_{n+3}) \max_{k \geq n+3} |\alpha_{k-1}| = (1 + \varepsilon_{n+3}) \max_{j \geq n+2} |\alpha_j| \\
&\leq (1 + \varepsilon_{n+3}) \|x - y\| \, .
\end{aligned}
$$

Consequently, T is cascading nonexpansive on C. $\square$

Theorem 2.6. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then there exists an equivalent norm $\|\cdot\|_\sim$ on X such that for every closed bounded convex subset C of X, for all $\|\cdot\|_\sim$-cascading nonexpansive mappings $T : C \longrightarrow C$, T has a fixed point in C.*

Proof. By Domínguez Benavides [3], there exists an equivalent norm $\|\cdot\|_\sim$ on X such that for every closed bounded convex subset E of X, for all $\|\cdot\|_\sim$-nonexpansive mappings $U : E \longrightarrow E$, U has a fixed point in E. Fix an arbitrary closed bounded convex subset C of X. Let $T : C \longrightarrow C$ be a $\|\cdot\|_\sim$-cascading nonexpansive mapping. As above, let $C_0 := C$ and $C_n := \overline{\mathrm{co}}\big(T(C_{n-1})\big)$, for all $n \in \mathbb{N}$. By hypothesis there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $[1, \infty)$ such that $\lambda_n \underset{n}{\longrightarrow} 1$, and for all $n \in \mathbb{N}$, for all $x, y \in C_n$,

$$\|Tx - Ty\|_\sim \leq \lambda_n \|x - y\|_\sim.$$

Now X is reflexive, and so C is weakly compact. By Zorn's Lemma there exists a closed bounded convex (non-empty) set $D \subseteq C$ such that D is a *minimal invariant set for* T. I.e., $T(D) \subseteq D$, and if F is a closed bounded convex (non-empty) subset of D with $T(F) \subseteq F$, then $F = D$. It follows that $\overline{\mathrm{co}}\big(T(D)\big) = D$. Let $D_0 := D$ and $D_n := \overline{\mathrm{co}}\big(T(D_{n-1})\big)$, for all $n \in \mathbb{N}$. Inductively, we see that for all $n \in \mathbb{N}$, $D = D_n \subseteq C_n$. Therefore, by our hypotheses on T, for all $x, y \in D$, for all $n \in \mathbb{N}$,

$$\|Tx - Ty\|_\sim \leq \lambda_n \|x - y\|_\sim \, .$$

But $\lambda_n \underset{n}{\longrightarrow} 1$. Consequently, for all $x, y \in D$,

$$\|Tx - Ty\|_\sim \leq \|x - y\|_\sim \, ;$$

i.e., T is $\|\cdot\|_\sim$-nonexpansive on D. By Domínguez Benavides [3], T has a fixed point in $D \subseteq C$. $\square$

Combining Theorem 2.5, the remarks preceding that theorem, and Theorem 2.6, we get the following "fixed point property" characterization of reflexivity in Banach lattices and Banach spaces with an unconditional basis.

Theorem 2.7. *Let $(X, \|\cdot\|)$ be a Banach lattice or a Banach space with an unconditional basis. Then the following are equivalent.*

(1) X is reflexive.

(2) There exists an equivalent norm $\|\cdot\|_\sim$ on X such that for all closed bounded convex sets $C \subseteq X$ and for all $\|\cdot\|_\sim$-cascading nonexpansive mappings $T : C \longrightarrow C$, T has a fixed point in C.

Corollary 2.8. *Let $(X, \|\cdot\|)$ be a Banach lattice or a Banach space with an unconditional basis. Then the following are equivalent. (1) X is reflexive.*
(2) There exists an equivalent norm $\|\cdot\|_\sim$ on X such that for all c.b.c. sets $C \subseteq X$ and for all $\|\cdot\|_\sim$-cascading nonexpansive mappings $T : C \longrightarrow C$, T has a fixed point.
(3) There exists an equivalent norm $\|\cdot\|_\sim$ on X such that for all c.b.c. sets $C \subseteq X$ and for all affine $\|\cdot\|_\sim$-cascading nonexpansive mappings $T : C \longrightarrow C$, T has a fixed point.
(4) For all equivalent norms $|||\cdot|||$ on X, for all c.b.c. sets $C \subseteq X$ and for all affine $|||\cdot|||$-cascading nonexpansive mappings $T : C \longrightarrow C$, T has a fixed point.
(5) For all c.b.c. sets $C \subseteq X$ and for all affine norm-to-norm continuous mappings $T : C \longrightarrow C$, T has a fixed point.

3. Reflexivity iff the perturbed FPP for mappings of cascading nonexpansive type

Theorem 3.1 (L, Veysel Nezir, Łukasz Piasecki, in preparation, 2014)**.** *Let $(X, \|\cdot\|)$ be a Banach space. Then the following are equivalent.*

(1) X is reflexive.

(2) There exists an equivalent norm $\|\cdot\|^{\sim}$ on X such that for all closed bounded convex sets $C \subseteq X$ and for all mappings $T : C \longrightarrow C$ of $\|\cdot\|^{\sim}$-cascading nonexpansive type, T has a fixed point in C.

This second new class of mappings, mappings of cascading nonexpansive type, strictly includes both nonexpansive mappings and cascading nonexpansive mappings.

Dziękuję !!!

Thank-you for your invitation to visit Lublin, and for your wonderful company and hospitality !!

References

1. F.E. Browder, *Fixed-point theorems for noncompact mappings in Hilbert space*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), no. 6, 1272–1276.
2. ______, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), no. 4, 1041–1044.
3. T. Domínguez Benavides, *A renorming of some nonseparable Banach spaces with the Fixed Point Property*, J. Math. Anal. Appl. **350** (2009), no. 2, 525–530.
4. P.N. Dowling, W.B. Johnson, C.J. Lennard, and B. Turett, *The optimality of James's distortion theorems*, Proc. Amer. Math. Soc. **125** (1997), no. 1, 167–174.
5. P.N. Dowling, C.J. Lennard, and B. Turett, *Some fixed point results in l^1 and c_0*, Nonlinear Analysis **39** (2000), 929–936.
6. D. van Dulst, *Equivalent norms and the fixed point property for nonexpansive mappings*, J. London Math. Soc. **25** (1982), no. 2, 139–144.
7. K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.
8. D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr. **30** (1965), 251–258.
9. W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, The American Mathematical Monthly **72** (1965), no. 9, 1004–1006.
10. C. Lennard and V. Nezir, *Reflexivity is equivalent to the perturbed fixed point property for cascading nonexpansive maps in Banach lattices*, Nonlinear Analysis **95** (2014), 414–420.
11. P.K. Lin, *There is an equivalent norm on ℓ_1 that has the fixed point property*, Nonlinear Analysis **68** (2008), 2303–2308.
12. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I: Sequence Spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 92, Springer-Verlag, 1977.
13. ______, *Classical Banach spaces II: Function Spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 97, Springer-Verlag, 1979.
14. Ł. Piasecki, *Personal communication*, 2013.