Subsets of the Real Line
Lev Bukovský

1 Introduction

I will essentially follow
L. B., The Structure of the Real Line, due to appear November 15, 2010 at

The lectures are devoted to a survey of recent results on the properties of
special subsets of the real line:
a) related to the convergence of sequences of real continuous functions;
b) covering properties;
c) relationships between those properties;
d) basic properties of hierarchies of the Borel and projective sets.

A topological space \( (X, \mathcal{O}) \): \( X \) a non-empty set, \( \mathcal{O} \) the set of open subsets of \( X \), for simplicity always Hausdorff.

Equivalent definition: a closure operator \( \overline{A} \) satisfying the axioms

\[
\begin{align*}
(C1) \quad & A \subseteq \overline{A}; \\
(C2) \quad & \overline{\emptyset} = \emptyset; \\
(C3) \quad & \overline{A \cup B} = \overline{A} \cup \overline{B}; \\
(C4) \quad & \overline{\overline{A}} = \overline{A}.
\end{align*}
\]

A normal topological space \( X \) is perfectly normal if every open subset of \( X \) is an \( F_\sigma \) set. Recall

**Theorem 1** (P. Urysohn). If \( X \) is a normal topological space and \( A, B \subseteq X \) are disjoint closed subsets, then there exists a continuous function \( f: X \rightarrow \mathbb{R} \) such that \( f(x) = 0 \) for \( x \in A \) and \( f(x) = 1 \) for \( x \in B \).

If \( f: X \rightarrow \mathbb{R} \), then

\[
Z(f) = \{ x \in X : f(x) = 0 \}.
\]

A subset \( A \subseteq X \) is a zero set if there exists a continuous \( f \) such that \( A = Z(f) \).

\( X \) is perfectly normal if and only if every closed subset of \( X \) is a zero set.

If \( A, B \subseteq X \) are closed subsets, then there exist continuous \( f: X \rightarrow [0, 1/2] \)
and \( g: X \rightarrow [1/2, 1] \) such that \( Z(f) = A \) and \( Z(1-g) = B \). If \( A, B \) are disjoint,
then the open sets \( \{ x \in X : f(x) + g(x) < 1 \} \) and \( \{ x \in X : f(x) + g(x) > 1 \} \) separate \( A \) and \( B \).

A convergence structure is a mapping \( \lim : \mathcal{X} \to X \), where \( X \subseteq \mathcal{X} \). A sequence \( \{ x_n \}_{n=0}^\infty \subseteq X \) is convergent, \( \lim(\{ x_n \}_{n=0}^\infty) = \lim_{n \to \infty} x_n \) is the limit. A couple \((X, \lim)\) is an \( L^* \)-space if:

(L1) if \( x_n = x \) for every \( n \), then \( \lim_{n \to \infty} x_n = x \);

(L2) if \( \lim_{n \to \infty} x_n = x \) and \( \{ n_k \}_{k=0}^\infty \) is increasing, then \( \lim_{k \to \infty} x_{n_k} = x \);

(L3) if \( x \neq \lim_{n \to \infty} x_n \), then there exists a subsequence \( \{ x_{n_k} \}_{k=0}^\infty \) such that no subsequence of \( \{ x_{n_k} \}_{k=0}^\infty \) has limit \( x \).

If \( A \subseteq X \), then sequential closure of \( A \) is

\[
\text{scl}(A) = \{ x \in X : x = \lim_{n \to \infty} x_n \text{ for some } x_n \in A \}.
\]

The sequential closure “scl” satisfies the axioms (C1) – (C3). Any topological space is an \( L^* \)-space and \( \text{scl}(A) \subseteq A \). \( X \) is Fréchet if \( \text{scl}(A) = A \).

We can define by transfinite induction:

\[
\text{scl}_0(A) = A, \quad \text{scl}_\xi(A) = \text{scl}(\bigcup_{\eta < \xi} \text{scl}_\eta(A)) \text{ for } \xi > 0.
\]

Note that

\[
\text{scl}_\xi(A) = \text{scl}(\omega_1)A \text{ for any } \xi > \omega_1.
\]

**Theorem 2.** Assume that \( X \) is an \( L^* \)-space. Then the following conditions are equivalent:

a) If \( \lim_{n \to \infty} x_n = x \), then there exist sequences \( \{ n_k \}_{k=0}^\infty \) and \( \{ m_k \}_{k=0}^\infty \) such that \( \lim_{k \to \infty} x_{n_k, m_k} = x \).

b) If \( \lim_{n \to \infty} x_n = x \), then there exist sequences \( \{ n_k \}_{k=0}^\infty \) and \( \{ m_k \}_{k=0}^\infty \) such that \( \{ n_k \}_{k=0}^\infty \) is increasing and \( \lim_{k \to \infty} x_{n_k, m_k} = x \).

c) If \( \lim_{n \to \infty} x_n = x \), then there exist sequences \( \{ n_k \}_{k=0}^\infty \) and \( \{ m_k \}_{k=0}^\infty \) such that \( \lim_{k \to \infty} x_{n_k, m_k} = x \).

d) \( \text{scl}_1(A) = \text{scl}_2(A) \) for any \( A \subseteq X \).

\( X \) possesses the sequence selection property, shortly SSP, if \( X \) possesses any of conditions a) – d).

The Fremlin number \( \Sigma(X) \) of an \( L^* \)-space \( X \) is defined as

\[
\sigma(A, X) = \min \{ \xi : \text{scl}_\xi(A) = \text{scl}_\omega(A) \}, \quad \Sigma(X) = \sup \{ \sigma(A, X) : A \subseteq X \}.
\]

An \( L^* \)-space \( X \) has the property \( (\alpha_i) \), \( i = 1, 2, 3, 4 \), if for any \( x \in X \) and for any sequence \( \{ x_{n_m} \}_{n=0}^\infty \), \( m=0 \) of sequences converging to \( x \), there exists a sequence \( \{ y_m \}_{m=0}^\infty \) such that \( \lim_{m \to \infty} y_m = x \) and

\[
(\alpha_1) \quad \{ x_{n_m} : m \in \omega \} \subseteq^* \{ y_m : m \in \omega \} \text{ for each } n,
\]

\[
(\alpha_2) \quad \{ x_{n_m} : m \in \omega \} \cap \{ y_m : m \in \omega \} \text{ is infinite for each } n,
\]

\[
(\alpha_3) \quad \{ x_{n_m} : m \in \omega \} \cap \{ y_m : m \in \omega \} \text{ is infinite for infinitely many } n,
\]

\[
(\alpha_4) \quad \{ x_{n_m} : m \in \omega \} \cap \{ y_m : m \in \omega \} \neq \emptyset \text{ for infinitely many } n.
\]
It is easy to see that
\[(\alpha_1) \rightarrow (\alpha_2) \rightarrow (\alpha_3) \rightarrow (\alpha_4), \quad \text{SSP} \equiv (\alpha_4).\]

Metatheorem 1 (A. Dow). \((\alpha_1) \equiv (\alpha_2)\) in the Laver’s model of the set theory ZFC.

Metatheorem 2. There exist models of ZFC, in which \((\alpha_2) \not\rightarrow (\alpha_1)\).

2 Pointwise Convergence of Real Functions

In what follows, \(f_n, f\) are functions from a topological space \(X\) into \(\mathbb{R}\).

"\(f_n \rightarrow f\) on \(X\)" means that \(\{f_n\}_{n=0}^{\infty}\) pointwise converges to \(f\) on \(X\), i.e.,
\[
(\forall x \in X) \lim_{n \to \infty} f_n(x) = f(x).
\]

\(C_p(X)\) is the set of all continuous functions from \(X\) into \(\mathbb{R}\).

Topology of \(C_p(X)\) is the subspace topology \(C_p(X) \subseteq X^\mathbb{R}\).

A set \(U \subseteq C_p(X)\) is a neighbourhood of an \(f \in C_p(X)\) if and only if there exist \(x_1, \ldots, x_n \in X\) and \(\varepsilon > 0\) such that
\[
\{g \in C_p(X) : |g(x_1) - f(x_1)| < \varepsilon \land \cdots \land |g(x_n) - f(x_n)| < \varepsilon\} \subseteq U.
\]

Theorem 3. \(f_n \rightarrow f\) in the product topology on \(X^\mathbb{R} \equiv f_n \rightarrow f\) on \(X\).

Theorem 4. If \(C_p(X)\) is a Fréchet space, \(X \subseteq \mathbb{R}\), then \(\lambda(X) = 0\) and \(X\) is meager.

For a real \(x \in \mathbb{R}\) we let \(\|x\|\) to be the distance of \(x\) to the nearest integer.

\[2\|x\| \leq |\sin \pi x| \leq \pi \|x\| \leq \pi\|x\|.
\]

Theorem 5 (P. G. Lévy–Dirichlet). For any \(\varepsilon > 0\) and for any reals \(x_1, \ldots, x_k \in \mathbb{R}\), there exists arbitrarily large \(n\) such that
\[\|nx_l\| < \varepsilon \quad \text{for } l = 1, 2, \ldots, k.\]

Corollary 6. For any non-empty \(X \subseteq \mathbb{R}\) we have \(0 \in \{\|nx\| : n > 0\}\).

Proof of Theorem 4. Assume that \(\|n_x\| \rightarrow 0\) on \(X\). We can assume that \(X = \{x \in \mathbb{R} : \|n_x\| \to 0\}\) and therefore \(X\) is Borel. Then \(\|n_x\| \rightarrow 0\) on \(X - X = \{x - y : x, y \in X\}\).

By Lebesgue Dominated Convergence Theorem
\[
\int_{(X - X)} \|n_x\| \, d\lambda \to 0.
\]

If \(\lambda(X) > 0\) (or if \(X\) is not meager), then by a Steinhaus Theorem there exists reals \(a < b\) such that \((a, b) \subseteq X - X\). For any sufficiently large positive integer \(n\) there exist positive integers \(k, m\) such that
\[
\frac{k - 1}{n} < a < \frac{k}{n} < \frac{m}{n} < b < \frac{m + 1}{n}.
\]

Then
\[
\frac{m - k}{4n} \leq \int_{[a, b]} \|n_x\| \, d\lambda < \frac{m - k + \frac{2}{n}}{4n}.
\]
and therefore \( \lim_{n \to \infty} \int_{[a,b]} \|nx\| \, d\lambda = \frac{b - a}{4} > 0. \)

q.e.d.

The **Baire Hierarchy** of subsets of \( X \mathbb{R} \) with the product topology
\[
C_p(X) \subseteq \text{scl}_1(C_p(X)) \subseteq \cdots \subseteq \text{scl}_\xi(C_p(X)) \subseteq \cdots .
\]

If \( X = [0,1] \) then the hierarchy is proper of length \( \omega_1 \).

**Theorem 7** (D.H. Fremlin). If \( X \) is a topological space, then \( \Sigma(C_p(X)) \) is either 1 or \( \omega_1 \).

If \( \Sigma(C_p(X)) = 1 \), we say that \( X \) is an \( s_1 \)-space.

**Theorem 8** (Essentially D. Fremlin). \( X \) is an \( s_1 \)-space if and only if \( C_p(X) \) possesses SSP.

The set \( \omega \mathbb{R} \) is preordered as
\[
f \leq^* g \equiv \{ n \in \omega : \neg f(n) \leq g(n) \}
\]
is finite.

W. Hurewicz introduced properties of a topological space \( X \):

\( H^* \) for any sequence \( \{ f_n \in C_p(X) : n \in \omega \} \) the family of sequences of reals \( \{\{f_n(x)\}_{n=0}^\infty : x \in X\} \) is not **dominating**, i.e.
\[
(\exists g \in \omega \mathbb{R})(\forall x \in X)(\forall n_0)(\exists n \geq n_0) f_n(x) < g(n).
\]

\( H^{**} \) for any sequence \( \{ f_n \in C_p(X) : n \in \omega \} \) the family of sequences of reals \( \{\{f_n(x)\}_{n=0}^\infty : x \in X\} \) is **eventually bounded**, i.e.
\[
(\exists g \in \omega \mathbb{R})(\forall x \in X)(\exists n_0)(\forall n \geq n_0) f_n(x) \leq g(n).
\]

Evidently
\[
H^{**} \rightarrow H^*.
\]

Fact: 

*Any \( \sigma \)-compact topological space possesses the property \( H^{**} \).*

**Theorem 9.** Both properties are preserved by passing to a closed subset and continuous image.

### 3 Quasi-normal Convergence

Let \( (Y, \rho) \) be a metric space, \( f_n, f : X \rightarrow Y, n \in \omega \). A sequence \( \{f_n\}_{n=0}^\infty \) **converges quasi-normally** to \( f \) on \( X \) if there exists a sequence \( \{\varepsilon_n\}_{n=0}^\infty \) of positive reals converging to zero and such that
\[
(3) \quad (\forall x \in X)(\exists k)(\forall n \geq k) \rho(f_n(x), f(x)) < \varepsilon_n.
\]

\( \{\varepsilon_n\}_{n=0}^\infty \) is a **control sequence** or \( \{\varepsilon_n\}_{n=0}^\infty \) **witnesses** the quasi-normal convergence. We shall write \( \{f_n\}_{n=0}^\infty \text{ QN} \to f \) on \( X \).
A sequence \( \{f_n\}_{n=0}^{\infty} \) converges \textit{discretely} to \( f \) on \( X \) if
\[
(\forall x \in X)(\exists k)(\forall n \geq k) f_n(x) = f(x)
\]

We shall write “\( f_n \overset{\text{DS}}{\to} f \) on \( X \)”.  

**Theorem 10.** Let \( f_n \overset{\text{QN}}{\to} f \) on \( X \). For any sequence \( \varepsilon_n \to 0 \) of positive reals there exists an increasing sequence \( \{n_k\}_{k=0}^{\infty} \) such that \( f_{n_k} \overset{\text{QN}}{\to} f \) on \( X \) with the control \( \{\varepsilon_k\}_{k=0}^{\infty} \).

**Theorem 11.** Let \( \{f_n\}_{n=0}^{\infty} \) and \( f \) be functions from \( X \) into a metric space \( Y \). Then the following conditions are equivalent:

a) \( f_n \overset{\text{QN}}{\to} f \) on \( X \).

b) There exists a sequence \( \{X_k\}_{k=0}^{\infty} \) of subsets of \( X \) such that \( X = \bigcup_k X_k \) and \( f_n \Rightarrow f \) on \( X_k \) for each \( k \).

c) There exists a non-decreasing sequence \( \{X_k\}_{k=0}^{\infty} \) of subsets of \( X \) such that \( X = \bigcup_k X_k \) and \( f_n \Rightarrow f \) on \( X_k \) for each \( k \).

Moreover, if \( X \) is a topological space and \( \{f_n\}_{n=0}^{\infty} \) are continuous, then conditions a)–c) are equivalent with

b) There exists a non-decreasing sequence \( \{X_k\}_{k=0}^{\infty} \) of closed subsets of \( X \) such that \( X = \bigcup_k X_k \) and \( f_n \to f \) on \( X_k \) for each \( k \).

c) There exists a non-decreasing sequence \( \{X_k\}_{k=0}^{\infty} \) of subsets of \( X \) such that \( X = \bigcup_k X_k \) and \( f_n(x) = f(x) \) for each \( x \in X_k \) for each \( n \geq k \).

Moreover, if \( X \) is a topological space and \( \{f_n\}_{n=0}^{\infty} \) are continuous, then conditions a)–c) are equivalent with

b) There exists a non-decreasing sequence \( \{X_k\}_{k=0}^{\infty} \) of closed subsets of \( X \) such that \( X = \bigcup_k X_k \) and \( f_n(x) = f(x) \) for each \( x \in X_k \) for each \( n \geq k \).

**Theorem 12.** Let \( \{f_n\}_{n=0}^{\infty} \) and \( f \) be functions from \( X \) into a metric space \( Y \). Then the following conditions are equivalent:

a) \( f_n \overset{\text{QN}}{\to} f \) on \( X \).

b) There exists a sequence \( \{X_k\}_{k=0}^{\infty} \) of subsets of \( X \) such that \( X = \bigcup_k X_k \) and \( f_n(x) = f(x) \) for each \( x \in X_k \) for each \( n \geq k \).

c) There exists a non-decreasing sequence \( \{X_k\}_{k=0}^{\infty} \) of subsets of \( X \) such that \( X = \bigcup_k X_k \) and \( f_n(x) = f(x) \) for each \( x \in X_k \) for each \( n \geq k \).

Moreover, if \( X \) is a topological space and \( \{f_n\}_{n=0}^{\infty} \) are continuous, then conditions a)–c) are equivalent with

b) There exists a non-decreasing sequence \( \{X_k\}_{k=0}^{\infty} \) of closed subsets of \( X \) such that \( X = \bigcup_k X_k \) and \( f_n(x) = f(x) \) for each \( x \in X_k \) for each \( n \geq k \).

X is called a \textit{QN-space} if every sequence of continuous real functions converging pointwise to 0 on \( X \) converges quasi-normally to 0 on \( X \) as well.

X is called a \textit{wQN-space} if for every sequence \( \{f_n\}_{n=0}^{\infty} \) of continuous real functions converging pointwise to 0 on \( X \) there exists an increasing sequence of integers \( \{n_k\}_{k=0}^{\infty} \) such that \( f_{n_k} \overset{\text{QN}}{\to} 0 \) on \( X \).

“\( f_n \searrow 0 \) on \( X \)” means that \( f_n \to f \) on \( X \) and \( f_n(x) \geq f_{n+1}(x) \) for each \( x \in X \) and each \( n \).

X is called an \textit{mQN-space} if for every sequence \( \{f_n\}_{n=0}^{\infty} \) of continuous functions such that \( f_n \searrow 0 \) on \( X \) also \( f_n \overset{\text{QN}}{\to} 0 \) on \( X \).

\textit{Every compact space is an mQN-space.}
Theorem 13. Each of introduced properties is preserved by passing to an $F_\sigma$ subset and by passing to an image by a function, that is a quasi-normal limit of continuous functions.

Theorem 14. There exists a sequence $\{h_n\}_{n=0}^\infty$ of continuous real functions defined on Cantor middle-third set $C$ such that $h_n \to 0$ on $C$ and for any increasing sequence $\{n_k\}_{k=0}^\infty$ of natural numbers there exists a $z \in C$ such that

\[
\sum_{k=0}^\infty h_{n_k}(z) = +\infty.
\]

Proof. For $x = \sum_{k=0}^\infty x_k3^{-k-1} \in C$, $x_k \in \{0,2\}$, we set

\[h_n(x) = \begin{cases} 
0 & \text{if } x_n = 0, \\
1/m_n & \text{where } m_n = |\{k < n : x_k = 2\}| + 1, \text{ otherwise.}
\end{cases} \]

If $\{n_k\}_{k=0}^\infty$ is increasing, we set $z = \sum_{k=0}^\infty z_k3^{-k}$, where $z_{n_k} = 2$ and $z_n = 0$ otherwise. Then $h_{n_k}(z) = 1/(k+1)$. q.e.d.

Corollary 15. If $X$ contains topologically $C$, then $X$ is not a $wQN$-space.

Theorem 16. Let $X$ be a perfect separable metric space. Then there exists a sequence $\{f_n\}_{n=0}^\infty$ of continuous real-valued functions such that

i) $f_n \to 0$ on $X$;

ii) if $A \subseteq X$, $\{n_k\}_{k=0}^\infty$ is an increasing sequence such that $f_{n_k} \equiv 0$ on $A$, then $A$ is nowhere dense.

Proof. Let $Q = \{r_i : i \in \omega\}$ be a countable dense subset of $X$. Since no point $r_i$ is isolated there exists a sequence $x_{i,n} \to r_i$, such that $x_{i,n} \notin Q$ for each $n \in \omega$. Let $h_{i,n} : X \to [0,2^{-i}]$ be continuous and such that $h_{i,n}(x_{i,n}) = 2^{-i}$ and $h_{i,n}(x) = 0$ if $\rho(x,x_{i,n}) \geq 1/2\rho(r_i,x_{i,n})$. Denote

\[f_n(x) = \sum_{i=0}^\infty h_{i,n}(x) \text{ for } x \in X, n \in \omega.\]

Each $f_n$ is continuous function from $X$ into $[0,2]$. Evidently $h_{i,n} \to 0$ on $X$ for every fixed $i$. If $x \in X$ and $\varepsilon > 0$, one can find an $i_0$ such that $\sum_{i>i_0}2^{-i} < \varepsilon/2$ and such $n_0$ that $\sum_{i \leq i_0}h_{i,n}(x) < \varepsilon/2$ for every $n \geq n_0$. Thus $f_n \to 0$ on $X$.

Assume that $\{n_k\}_{k=0}^\infty$ is increasing and $f_{n_k} \equiv 0$ on a set $A$. We can assume that $A$ is closed. If $\text{Int}(A) \neq \emptyset$ then there exist an $r_i \in \text{Int}(A)$. Hence there exists an $m$ such that $x_{i,n} \in \text{Int}(A)$ for $n \geq m$. Since for $n \geq m$ we have

\[\sup\{f_n(x) : x \in A\} \geq f(x_{i,n}) \geq h_{i,n}(x_{i,n}) = 2^{-i},\]

we get a contradiction. q.e.d.

Corollary 17. Let $\{f_n\}_{n=0}^\infty$ be the sequence of the Theorem. If $f_{n_k} \xrightarrow{QN} 0$ on $A$, then $A$ is meager.
A subset $A \subseteq X$ of a topological space $X$ is **perfectly meager** if for any perfect set $P \subseteq X$ the intersection $A \cap P$ is meager in the subspace $P$, i.e. if for any perfect set $P$ there are closed sets $F_n$ such that $A \cap P \subseteq \bigcup F_n$ and the inclusion $P \cap U \subseteq F_n$ implies $P \cap U = \emptyset$ for any $n \in \omega$ and any open set $U$. If $X$ is a perfect Polish space, then any perfectly meager subset of $X$ is meager in $X$. Moreover, the family $\mathcal{P}_M(X)$ of perfectly meager subsets of $X$ is a $\sigma$-ideal.

**Theorem 18.** Any wQN-subset $A$ of a separable metric space $X$ is perfectly meager.

**Theorem 19.** $X$ is a normal topological space, $\{f_n\}_{n=0}^{\infty}$, $f$ are real functions. If $f_n \overset{\text{QN}}{\to} f$ on $X$ and all $f_n$ are continuous, then there exist $\{g_n\}_{n=0}^{\infty}$ continuous functions such that $g_n \overset{\text{DS}}{\to} f$ on $X$. Thus $f$ is $\Delta^0_2$-measurable.

**Proof.** If $\{U_n : n \in \omega\}$ is a countable base of the topology on $\mathbb{R}$, then for any open set $U \subseteq \mathbb{R}$ we have

$$f(x) \in U \equiv (\exists m)(\exists k) \left( \overline{U_n} \subseteq U \land (\forall m \geq k) f_m(x) \in U_n. \right)$$

Thus, $f$ is $F_\sigma$-measurable.

Take closed subsets $\{X_k : k \in \omega\}$ such that $X = \bigcup X_k$ and $f_n \equiv f$ on $X_k$ for each $k$. Thus $f|X_k$ is continuous. By Tietze-Urysohn Theorem there exist real continuous functions $\{g_k\}_{k=0}^{\infty}$ defined on $X$ such that $g_k|X_k = f|X_k$ for every $k$. Then $g_n \overset{\text{DS}}{\to} f$ on $X$.

Since for any open $U$ we have

$$f(x) \in U \equiv (\forall n)(\exists m > n) g_m(x) \in U,$$

the function $f$ is $G_\delta$-measurable.

q.e.d.

**Theorem 20** (M. Scheepers – D.H.Fremlin). For a topological space $X$ the following are equivalent:

a) $X$ is a wQN-space.

b) The space $C_p(X)$ possesses the sequence selection property.

c) The space $C_p(X)$ possesses the property $(\alpha_2)$.

d) The space $C_p(X)$ possesses the property $(\alpha_3)$.

e) The space $C_p(X)$ possesses the property $(\alpha_4)$.

**Proof.**

e) $\to$ a). Assume that $\{f_n : n \in \omega\}$ are continuous, $f_n \to 0$ on $X$ and $f_n > 0$ for every $n$. We denote $f_{n,m} = 2^n \cdot f_{n+m}$. Then $f_{n,m} \to 0$ on $X$ for every $n$. By $(\alpha_4)$, there exist increasing sequences $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=0}^{\infty}$ such that $f_{n_k,m_k} \to 0$ on $X$. We claim that $f_{n_k+m_k} \overset{\text{QN}}{\to} 0$ on $X$ with control $\{2^{-n_k}\}_{k=0}^{\infty}$.

Indeed, if $f_{n_k,m_k} \to 0$ on $X$, then for any $x \in X$ there exists an $n_0$ such that $f_{n_k,m_k}(x) < 1$ for $n \geq n_0$. Hence $f_{m_k}(x) < 2^{-n_k}$ for any $n \geq n_0$. 

7
a) \(\Rightarrow\) b). Let \(f_{n,m} \to 0\) on \(X\) for every \(n\). We set

\[
g_m(x) = \sum_{n=0}^{\infty} \min\{2^{-n}, f_{n,m}(x)\}, \quad x \in X.
\]

Let \(x \in X, \varepsilon > 0\) be a real. Then there exists an \(n_0\) such that \(2^{-n_0+2} < \varepsilon\). For every \(n < n_0\) there exists an \(m_n\) such that \(f_{n,m}(x) < \varepsilon/(2n)\) for every \(m \geq m_n\). We denote \(k = \max\{m_n : n < n_0\}\). For \(m \geq k\) we have

\[
g_m(x) \leq \sum_{n < n_0} \varepsilon + \sum_{n \geq n_0} 2^{-n} < \varepsilon.
\]

Thus \(\lim_{m \to \infty} g_m(x) = 0\). Since \(X\) is a \(w\)-QN-space there exists an increasing sequence \(\{m_n\}_{n=0}^{\infty}\) such that \(g_{m_n} \to 0\) on \(X\) with the control \((2^{-n})_{n=0}^{\infty}\). However by (6), if \(g_{m_n}(x) < 2^{-n}\), then also \(f_{n,m_n}(x) < 2^{-n}\). Thus \(f_{n,m_n} \to 0\) on \(X\).

b) \(\Rightarrow\) c). Let \(f_{n,m} \to 0\) on \(X\) for every \(n\). We set \(h_{n,m} = f_{\lambda(n),m}\) where \(\lambda\) is the left inverse to a pairing function\(^1\). Therefore the sequence \(\{h_{n,m}\}_{n=0}^{\infty}\) contains every \(f_{i,m}\) infinitely many times. Let \(g_{n,m} = \max\{h_{i,m} : i \leq n\}\). Evidently \(g_{n,m} \to 0\) for every \(n\). Then there exist increasing sequences \(\{n_k\}_{k=0}^{\infty}\) and \(\{m_k\}_{k=0}^{\infty}\) such that \(\lim_{m \to \infty} g_{n_k,m_k} = 0\). We set \(f_i = h_{i,m_k}\) for \(n_{k-1} < i \leq n_k\). Then \(f_i \leq g_{n_k,m_k}\) for \(n_{k-1} < i \leq n_k\) and therefore \(f_i \to 0\) on \(X\). Every sequence \(\{f_{n,m}\}_{m=0}^{\infty}\) contains infinitely many members of \(\{f_i\}_{i=0}^{\infty}\).

q.e.d.

Since a Fréchet space possesses the sequence selection property we obtain:

**Corollary 21.** If \(C_p(X)\) is a Fréchet space, then \(X\) is a \(w\)-QN-space.

There is a similar result for a \(Q\)-space.

**Theorem 22 (M. Scheepers).** If \(C_p(X)\) has the property \((\alpha_1)\), then the topological space \(X\) is a \(Q\)-space.

**Proof.** If \(\{f_m\}_{m=0}^{\infty}\) is a sequence of continuous functions converging pointwise to \(0\) on \(X\), we set \(f_{n,m}(x) = 2^n \cdot |f_m(x)|\). Then \(\{f_{n,m}\}_{m=0}^{\infty}\) converges pointwise to \(0\) on \(X\) for each \(n\). By \((\alpha_1)\) there exists a sequence \(\{h_k\}_{k=0}^{\infty}\) converging to \(0\) on \(X\) and such that the sequence \(\{h_k\}_{k=0}^{\infty}\) contains all but finitely many members of the sequence \(\{f_{n,m}\}_{m=0}^{\infty}\) for every \(n\). Thus, there exists an increasing sequence \(\{m_n\}_{n=0}^{\infty}\) such that \(\{f_{n,m}\}_{m=m_n}^{\infty}\) is a subsequence of \(\{h_k\}_{k=0}^{\infty}\) for each \(n\). Moreover, we can assume that

\[
(\forall m \geq m_n) (\forall i) \quad (h_i = f_{n,m} \to i \geq n)
\]

for every \(n\). Let

\[
\varepsilon_m = 2^{-k} \text{ for } m_k \leq m < m_{k+1}, \quad \varepsilon_m = 1 \text{ for } m < m_0.
\]

Let \(x \in X\), then there exists a \(k_0\) such that \(|h_k(x)| < 1\) for \(k \geq k_0\). For any \(m \geq m_{k_0}\) there exists a \(k_0\) such that \(m_k \leq m < m_{k+1}\). Then \(f_{k,m} = h_i\) for some \(i \geq k \geq k_0\). Hence \(f_{k,m}(x) < 1\). Since \(\varepsilon_m = 2^{-k}\) we obtain \(|f_m(x)| < \varepsilon_m\).

q.e.d.

---

\(^1\)A bijection \(\pi : \omega \times \omega^{\omega} \to \omega\) is a *pairing function*, \(\lambda, \rho\) are the left and the right inverse to \(\pi\). Thus \(\pi(\lambda(n), \rho(n)) = n, \lambda(\pi(n,m)) = n\) and \(\rho(\pi(n,m)) = m\) for any \(n, m\).
Theorem 23 (L. Bukovský – J. Haleš – M. Sakai). If $X$ is a QN-space, then $C_p(X)$ possesses the property $(\alpha_1)$.

Proof. Let $\{\{f_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ be a sequence of sequences converging to 0 on $X$. We can assume that values of each $f_{n,m}$ are in $[0,1]$. We define the functions $g_m$ by (6). Then $g_m$ are continuous and $g_m \to 0$ on $X$.

Since $X$ is a QN-space, there exist positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$, $\varepsilon_n \to 0$ such that

$$(\forall x)(\exists l_x)(\forall m \geq l_x) g_m(x) < \varepsilon_m.$$  

There are also natural numbers $m_k$ such that

$$(\forall k)(\forall m \geq m_k) \varepsilon_m < 2^{-k}.$$  

We can assume that $m_k < m_{k+1}$ for any $k$. We claim that the sequence

$$\{f_{n,m} : n \in \omega \land m \geq m_n\}$$

converges to 0 on $X$.

Corollary 24. A topological space $X$ is a QN-space if and only if $C_p(X)$ possesses the property $(\alpha_1)$.

$X$ has the quasi-normal sequence selection property, QSSP, if for any functions $f, f_n, f_{n,m} : X \to \mathbb{R}$, $n, m \in \omega$, such that

1. $f_n \text{ QN } f$ on $X$,
2. $f_{n,m} \text{ QN } f_n$ on $X$ for every $n \in \omega$,
3. every $f_{n,m}$ is continuous,

there exists an increasing $\beta \in \omega^\omega$ such that $f_{\beta(n)} \text{ QN } f$ on $X$.

Theorem 25 (L. Bukovský – J. Šupina). Any QN-space possesses QSSP.

Proof. We assume that the control of the quasi-normal convergences in (2) is $\{2^{-2m-1}\}_{m=0}^{\infty}$ and the control of that in (1) is $\{\varepsilon_n\}_{n=0}^{\infty}$.

We set

$$g_m^n(x) = \min \{|f_{n,m}(x) - f_{n+1,m}(x)| \cdot 2^m, 1\}.$$  

Evidently for a fixed $n \in \omega$ we have $g_m^n \to 0$ on $X$. Since the space $C_p(X)$ satisfies $(\alpha_1)$, there exists an increasing function $\beta \in \omega^\omega$ such that the set $\{g_m^n : m \geq \beta(n) \land n \in \omega\}$ converges to 0.

Then $f_{\beta(n)} \text{ QN } f$ with the control $\{2^{-\beta(n)+1} + \varepsilon_n\}_{n=0}^{\infty}$.

A topological space is a $\sigma$-space, if every $F_\sigma$ subset is also a $G_\delta$ subset. Then every Borel subset is an $F_\sigma$ set.

Theorem 26 (I. Recław). If a perfectly normal topological space $X$ has the property QSSP, then $X$ is a $\sigma$-space. Therefore every perfectly normal topological QN-space is a $\sigma$-space.
Proof. Let $F = \bigcup_n F_n$, $F_n$ being closed, $F_n \subseteq F_{n+1}$ for any $n \in \omega$. We show that the characteristic function $\chi_F$ is $G_\delta$-measurable.

Since $X$ is perfectly normal, there exist closed sets $(F_{n,m} : n,m \in \omega)$ such that $F_{n,m} \subseteq F_{n,m+1}$ and $X \setminus F_n = \bigcup_k F_{n,k}$ for any $n$ and $m$. For any $n$ and $m$, there exists a continuous function $f_{n,m} : X \rightarrow [0,1]$ such that $f_{n,m}(x) = 1$ for $x \in F_n$ and $f_{n,m}(x) = 0$ for $x \in F_{n,m}$. Evidently $f_{n,m} \overset{\text{DS}}{\rightarrow} \chi_{F_n}$ on $X$. Moreover, $\chi_{F_n} \overset{\text{DS}}{\rightarrow} \chi_F$ on $X$.

By QSSP there exists a $\beta$ such that $f_{n,\beta(n)} \overset{\text{QN}}{\rightarrow} \chi_F$. Since $f_{n,\beta(n)}$ are continuous, by Theorem 19 the function $\chi_F$ is $G_\delta$-measurable.

q.e.d.

Corollary 27. Every subset of a metric separable QN-space is a QN-space.

Lemma 28. If $\text{Ind}(X) = 0$, then every simple $\Delta^0_2$-measurable function $g : X \rightarrow [0,1]$ is a discrete limit of a sequence $\{g_n\}_{n=0}^\infty$ of simple continuous functions.

Proof. Assume that $g = \sum_{i=0}^k a_i \chi_{A_i}$, where $A_i \in \Delta^0_2$ are pairwise disjoint, $\bigcup_{i=0}^k A_i = X$ and $0 = a_0 < a_1 < \cdots < a_k < 1$. Then for every $i = 0, \ldots, k$ there exist non-decreasing and non-increasing sequences $\{F_i\}_{n=0}^\infty$ and $\{G_i\}_{n=0}^\infty$ of $F_\sigma$ and $G_\delta$ sets, respectively, such that $A_i = \bigcup_n F_i^n = \bigcap_n G_i^n$ for $i = 0, \ldots, k$.

Since $\text{Ind}(X) = 0$, there exists clopen sets $C_i^n$ such that $F_i^n \subseteq C_i^n \subseteq G_i^n$ for every $i \leq k$ and every $n \in \omega$. Replacing eventually $C_i^n$ by $X \setminus \bigcup_{0 \leq i < k} C_i^n = X$ we can assume that $\bigcup_{0 \leq i \leq k} C_i^n = X$. Let $D_i^n = C_i^n \setminus \bigcup_{j \leq i} C_j^n$. Then $D_i^n$ are pairwise disjoint and $\bigcup_{0 \leq i \leq k} D_i^n = X$. Set $g_n = \sum_{i=0}^k a_i \chi_{D_i^n}$. Since each $D_i^n$ is clopen, $g_n$ is continuous.

q.e.d.

Theorem 29. If $X$ is a normal topological space possessing property QSSP, then any Borel measurable function $f : X \rightarrow [0,1]$ is a quasi-normal limit of a sequence of continuous functions.

Proof. If $f : X \rightarrow \omega \subseteq [0,1]$ is Borel measurable, then by Reclaw’s Theorem 26 the function $f$ is $\Delta^0_2$-measurable. For any $n$ and any $i < 2^n - 1$, we set

$$A_i^n = \{x \in X : \frac{i}{2^n} \leq f(x) < \frac{i+1}{2^n}\}, \quad A_i^{2^n-1} = \{x \in X : \frac{2^n-1}{2^n} \leq f(x)\}.$$ 

Then the sequence of simple $\Delta^0_2$-measurable functions

$$f_n = \sum_{i=0}^{2^n-1} \frac{i}{2^n} \chi_{A_i^n}$$

converges uniformly to $f$ with control $2^{-n}$. By the Lemma, for every $n$ there exists a sequence $\{g_{n,i}^{\alpha}\}_{i=0}^\infty$ of simple continuous functions such that $g_{n,i}^{\alpha} \overset{\text{DS}}{\rightarrow} f_n$ on $X$. Thus, by Theorem 25 there exists an increasing $\alpha \in \omega^\omega$ such that $g^{\alpha(n)}_{\alpha(n)} \overset{\text{QN}}{\rightarrow} f$.

q.e.d.
Theorem 30 (B. Tsaban – L. Zdomskyy). The image of a perfectly normal topological QN-space $X$ by a Borel measurable function into $\omega^\omega$ is eventually bounded.

Proof. Assume that $f : X \to \omega^\omega \subseteq [0,1]$ is Borel measurable. By Theorem 29 there exists a sequence of continuous functions $\langle f_n : X \to [0,1] : n \in \omega \rangle$ such that $f_n \overset{QN}{\to} f$ on $X$. By Theorem 13 the set $f(X) \subseteq \omega^\omega$ is eventually bounded. q.e.d.

Corollary 31. For a perfectly normal topological space $X$ the following are equivalent:

a) $X$ is a QN-space.

b) If $\langle f_n : n \in \omega \rangle$ are Borel measurable function from $X$ into $[0,1]$ and $f_n \to f$ on $X$, then $f_n \overset{QN}{\to} f$ on $X$.

c) Any Borel measurable image of $X$ into $\omega^\omega$ is eventually bounded.

Proof. a) $\to$ c) follows by Tsaban-Zdomskyy Theorem, the implication b) $\to$ a) is trivial.

We show that c) $\to$ b). Let $\langle f_n : n \in \omega \rangle$ be Borel measurable functions from $X$ into $[0,1]$ and $f_n \to f$ on $X$. Set $g_n(x) = \sup\{|f_n(x) - f(x)| : m \geq n\}$. Then $g_n$ is Borel measurable and $g_n \downarrow 0$. The function $\psi : X \to \omega^\omega$ defined as $\psi(x)(m) = \min\{n : g_n(x) < 2^{-m}\}$ is Borel measurable. By c), the set $\psi(X)$ is eventually bounded by a $\beta \in \omega^\omega$. Then $g_{\beta(n)} \overset{QN}{\to} 0$ with the control $\langle 2^{-n} \rangle_n$. Since $\langle g_n \rangle_n$ is non-increasing we obtain $g_n \overset{QN}{\to} 0$ and also $f_n \overset{QN}{\to} f$.

q.e.d.

Lemma 32 (J. Haleš). If $f_n \downarrow 0$ on $X$ are continuous then there exists a continuous function $h : X \to \mathbb{R}$ such that

$$(\forall x \in X)(\forall n > h(x)) f_n(x) < 1.$$ 

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \min\{1, f_n(x)\} \cdot 2^{-n}.$$ 

The function $g : X \to [0,2]$ is continuous. Set $h(x) = -\log_2(2 - g(x))$. \quad \Box

Theorem 33 (L.B. – J. Haleš). A topological space $X$ is an mQN-space if and only if $X$ possesses the property $H^{**}$.

Theorem 34. If $A$ is a subset of a Polish space $X$ with property $H^{**}$, then there exists a $\sigma$-compact set $B \subseteq X$ such that $A \subseteq B$.

Proof. Let $\{r_n : n \in \omega\}$ be a countable dense subset of $X$. Denote

$$f_n(x) = \min\{\rho(x, r_i) : i = 0, \ldots, n\}.$$ 

Then $f_n \downarrow 0$ on $X$ and therefore $f_n \overset{QN}{\to} 0$ on $A$. If $\{\varepsilon_n\}_{n=0}^{\infty}$ is a control sequence for the quasi-normal convergence of $\{f_n\}_{n=0}^{\infty}$ on $A$, then the set

$$B_n = \{x \in X : (\forall m \geq n) f_m(x) \leq \varepsilon_m\}$$

is a $\sigma$-compact set.
is closed. If \( m \geq n \) is such that \( \varepsilon_m \leq \varepsilon \), then the set \( \{m, \ldots, r_m\} \) is an \( \varepsilon \) net on \( B_n \). Thus, \( B_n \) is totally bounded and therefore, \( B_n \) is compact. Evidently \( \mathcal{A} \subseteq \bigcup n \mathcal{B}_n \).

\( \text{q.e.d.} \)

**Corollary 35** (Hurewicz Theorem). Assume that \( D \) is an analytic subset of a Polish space \( X \) such that \( D \) is not an \( \mathcal{F}_\varepsilon \) set. Then there exists a closed subset of \( D \) homeomorphic to \( \omega^\omega \).

## 4 Covering Properties

A **cover** \( \mathcal{U} \) of \( X \) is a set of subsets of \( X \) such that \( \bigcup \mathcal{U} = X \) and \( X \notin \mathcal{U} \).

A cover \( \mathcal{U} \) of \( X \) is called **essentially infinite** if no finite subset of \( \mathcal{U} \) is a cover of \( X \). A cover \( \mathcal{V} \subseteq \mathcal{U} \) is said to be a **subcover** of \( \mathcal{U} \). An infinite cover \( \mathcal{U} \) is a **\( \gamma \)-cover** if every point \( x \in X \) is in all but finitely many sets from \( \mathcal{U} \). \( \Gamma(X) \) or simply \( \Gamma \) is the family of all open \( \gamma \)-covers of \( X \). \( \Gamma_\omega \) is the family of all countable open \( \gamma \)-covers. A cover \( \mathcal{U} \) is an **\( \omega \)-cover** if for every finite \( A \subseteq X \) there is a \( U \in \mathcal{U} \) such that \( A \subseteq U \). \( \Omega(X) \) or simply \( \Omega \) is the family of all open \( \omega \)-covers of \( X \).

A cover \( \mathcal{V} \) is a **refinement** of a cover \( \mathcal{U} \) if

\[
(\forall V \in \mathcal{V})(\exists U \in \mathcal{U}) V \subseteq U.
\]

A countable open \( \gamma \)-cover \( \mathcal{U} \) is **shrinkable** if there exists a closed \( \gamma \)-cover that is a refinement of \( \mathcal{U} \). \( \Gamma^{sh}(X) \) or simply \( \Gamma^{sh} \) is the family of shrinkable \( \gamma \)-covers of \( X \).

A topological space \( X \) is a **\( \gamma \)-space** if from every open \( \omega \)-cover of \( X \) one can choose a \( \gamma \)-subcover.

Let \( \mathcal{A}(X), \mathcal{B}(X) \) be families of covers of a topological space \( X \). \( X \) is said to be an \( S_1(\mathcal{A}, \mathcal{B}) \)-**space** if for every sequence \( \langle \mathcal{U}_n : n \in \omega \rangle \) of covers from \( \mathcal{A}(X) \) there exist sets \( U_n \in \mathcal{U}_n \) such that \( \{U_n : n \in \omega\} \) is a cover belonging to \( \mathcal{B}(X) \).

**Theorem 36.** Assume that the families \( \mathcal{A}(X) \) and \( \mathcal{B}(X) \) of open covers have the following property:

i) if \( \mathcal{V} \in \mathcal{B}(X) \) is a refinement of an open cover \( \mathcal{U} \), then there exists a subcover of \( \mathcal{U} \) which belongs to \( \mathcal{B}(X) \),

ii) every two covers of \( \mathcal{A}(X) \) have a common refinement which belongs to \( \mathcal{A}(X) \).

Then \( X \) is an \( S_1(\mathcal{A}, \mathcal{B}) \)-space if and only if for every sequence \( \{\mathcal{U}_n\}_{n=0}^\infty \) of covers from \( \mathcal{A}(X) \) such that \( \mathcal{U}_{n+1} \) is a refinement of \( \mathcal{U}_n \) for every \( n \), there exist sets \( U_n \in \mathcal{U}_n \) such that \( \{U_n : n \in \omega\} \) is a cover belonging to \( \mathcal{B}(X) \).

\( X \) is an \( U_{\text{fin}}(\mathcal{A}, \mathcal{B}) \)-**space** if for any sequence \( \{\mathcal{U}_n\}_{n=0}^\infty \) of essentially infinite \( \mathcal{A} \)-covers of \( X \) there exist finite \( \mathcal{V}_n \subseteq \mathcal{U}_n \) such that \( \{\mathcal{V}_n : n \in \omega\} \) is a \( \mathcal{B} \)-cover.

**Theorem 37** (J. Gerlits – Z. Nagy). A topological space \( X \) is a **\( \gamma \)-space** if and only if \( X \) is an \( S_1(\Omega, \Gamma) \)-space.
Proof. Let \( \{U_n : n \in \omega \} \) be a sequence of \( \omega \)-covers such that \( U_{n+1} \) is a refinement of \( U_n \) for every \( n \). Choose distinct \( x_n \in X : n \in \omega \). Then
\[
U = \{ U \setminus \{ x_n \} : U \in U_n \land n \in \omega \}
\]
is an \( \omega \)-cover. Then there exists a \( \gamma \)-subcover \( \{ V_k : k \in \omega \} \subseteq U \). Let \( n_k \) be such that \( V_k = U \setminus \{ x_{n_k} \} \), where \( U \in U_{n_k} \). If \( \{ x_0, \ldots, x_n \} \subseteq V_k \), then \( n_k > n \). Thus the set \( \{ n_k : k \in \omega \} \) is infinite and therefore we can assume that the sequence \( \{ n_k \}_{k=0}^{\infty} \) is increasing and \( n_0 = 0 \). For any \( m < n_k \), \( m \geq n_{k-1} \), \( k > 0 \) take \( U_m \in U_m \) such that \( V_k \subseteq U_m \setminus \{ x_{n_k} \} \). One can easily see that \( \{ U_m : m \in \omega \} \) is a \( \gamma \)-cover.

q.e.d.

Corollary 38. A topological \( \gamma \)-space \( X \) is an \( S_1(\Gamma, \Gamma) \)-space.

A topological \( U_{fin}(\mathcal{O}, \mathcal{O}) \)-space is said to have the Menger Property. A topological space \( U_{fin}(\mathcal{O}, \Gamma) \)-space is said to have the Hurewicz Property. A topological \( S_1(\mathcal{O}, \mathcal{O}) \)-space is said to have the Rothberger Property or to be a \( C'' \)-space.

Rothberger \( \rightarrow \) Menger, \quad \text{Hurewicz} \rightarrow \text{Menger}.

Theorem 39 (D.H. Fremlin – A.W. Miller). If \( (X, \mathcal{O}) \) is a separable metrizable topological space, then \( X \) has the Rothberger Property if and only if \( X \) has strong measure zero with respect to any metric compatible with the topology \( \mathcal{O} \).

Theorem 40 (W. Hurewicz). Let \( X \) be a perfectly normal space. Then
\begin{enumerate}
  \item \( X \) has the property \( H^* \) if and only if \( X \) has the Menger Property,
  \item \( X \) has the property \( H^{**} \) if and only if \( X \) has the Hurewicz Property.
\end{enumerate}

Proof. If \( \{V_n : n \in \omega \} \) is a sequence of countable open covers of \( X \) we take continuous \( f_{n,k} : X \to [0, 1] \) such that \( Z(f_{n,k}) = X \setminus V_n \cup \{ x \} \neq \emptyset \). Set
\[
f_n = \sum_{k=0}^{\infty} 2^{-k} \cdot f_{n,k}.
\]

\( f_n > 0 \) is continuous.

If \( X \) has the property \( H^* \), there exists a sequence \( \{a_n\}_{n=0}^{\infty} \) of positive reals such that
\[
(\forall x \in X)(\forall m)(\exists n > m) 1/f_n(x) < a_n.
\]

Let \( k_n \) be such that
\[
\sum_{i>k_n} 2^{-i} < 1/a_n.
\]

We denote \( V_n = \{ U_{n,i} : i \leq k_n \} \). Then \( \{ \bigcup V_n : n \in \omega \} \) is a cover of \( X \).

If \( X \) has the property \( H^{**} \) then there exists a sequence \( \{a_n\}_{n=0}^{\infty} \) of positive reals such that \( \{1/f_n(x)\}_{n=0}^{\infty} \leq* \{a_n\}_{n=0}^{\infty} \). Let \( k_n \) and \( V_n \) be as above. If \( f_n(x) \geq 1/a_n \), then \( \sum_{i=0}^{k_n} 2^{-i} \cdot f_{n,i}(x) > 0 \) and therefore \( x \in \bigcup V_n \). Then \( \{ \bigcup V_n : n \in \omega \} \) is a \( \gamma \)-cover of \( X \).
Now, let \( \{ f_n : X \to \mathbb{R} \}_{n=0}^{\infty} \) be a sequence of continuous real functions. Set \( U_{n,m} = \{ x \in X : |f_n(x)| < m \} \). Then for every \( n \), \( U_n = \{ U_{n,m} : m \in \omega \} \) is an open cover of \( X \) and \( U_{n,m} \subseteq U_{n,m+1} \). Also \( U_{n,k} = \{ U_{n,m} : m \geq k \} \) is a cover.

If \( X \) has the Menger Property, then there exist finite sets \( V_n \subseteq W_n = U_{\pi(n), \lambda(n)} \) (\( \lambda, \rho \) are the inverse functions to the pairing function \( \pi \)), such that \( \{ \bigcup V_n : n \in \omega \} \) is a cover of \( X \). Since \( \bigcup V_n = U_{\rho(n), m_n} \) for some \( m_n \geq \lambda(n) \), we obtain for every \( x \in X \) that \( |f_n(x)| < m_n \) for infinitely many \( n \).

If \( X \) has the Hurewicz Property, then there exists a finite subset \( V_n \subseteq U_n \) such that \( \{ \bigcup V_n : n \in \omega \} \) is a \( \gamma \)-cover of \( X \). Then for every \( x \in X \) we have \( |f_n(x)| < m_n \) for all but finitely many \( n \).

\( \text{q.e.d.} \)

**Corollary 41.** A \( \gamma \)-space has the property \( H^{**} \).

**Theorem 42.** If \( X \) has a countable base of the topology and \( |X| < p \), then \( X \) is a \( \gamma \)-space.

**Proof.** Let \( U \) be a countable open \( \omega \)-cover of \( X \). Let

\[ U_x = \{ U \in U : x \in U \}. \]

The family \( \{ U_x : x \in X \} \) possesses the finite intersection property. Thus, there exists an infinite set \( V \subseteq U \) such that \( V \setminus U_x \) is finite for every \( x \in X \). Then \( V \) is desired \( \gamma \)-cover.

\( \text{q.e.d.} \)

**Theorem 43.** There exists a subset \( A \) of \( \omega^2 \) of cardinality \( p \) that is not a \( \gamma \)-space.

**Proof.** Let \( F \) be a family of subsets of \( \omega \) of cardinality \( p \) such that \( F \) has f.i.p. and has no pseudointersection. Let \( A \subseteq \omega^2 \) be the set of all characteristic functions of sets from the family \( F \). We claim that \( A \) is not a \( \gamma \)-space.

\( \text{q.e.d.} \)

**Theorem 44** (F. Galvin – A.W. Miller). If \( p = \mathfrak{c} \), then there exists a \( \gamma \)-space \( A \) of cardinality \( \mathfrak{c} \).

**Theorem 45** (J. Gerlits – Z. Nagy). A completely regular topological space \( X \) is a \( \gamma \)-space if and only if the topological space \( C_p(X) \) is Fréchet.

**Proof.** If \( X \) is an infinite completely regular topological space, we fix mutually distinct elements \( \{ x_n \in X : n \in \omega \} \). For \( h \in C_p(X) \) and \( n \in \omega \) we let

\[ U_{h,n} = \{ x \in X : |h(x)| < 2^{-n} \land x \neq x_n \}. \]

Evidently \( U_{h,n} \) is an open set.

Assume that \( X \) is a \( \gamma \)-space. Let \( A \subseteq C_p(X) \), \( f \in \overline{A} \setminus \{ f \} \).

The family

\[ U = \{ U_{f-g,n} : n \in \omega, g \in A \} \]

is an open \( \omega \)-cover of \( X \).

If \( G = \{ G_k : k \in \omega \} \) is a countable \( \gamma \)-subcover of \( U \), then there are \( g_k \in A \) and \( n_k \in \omega \) such that

\[ G_k = U_{f-g_k,n_k} \quad \text{for each } k \in \omega. \]
We can assume that \( \{n_k\}_{k=0}^{\infty} \) is increasing (otherwise some \( x_{n_k} \) does not belong to infinitely many \( G_k \)).

We claim that \( f = \lim_{k \to \infty} g_k \) on \( X \): if \( x \in X \), then \( |f(x) - g_k(x)| < 2^{-n_k} \) for all but finitely many \( k \).

Assume now that \( C_p(X) \) is Fréchet space. Let \( U \) be an open \( \omega \)-cover of \( X \). We set

\[
A = \{ f \in C_p(X) : (\exists U \in U) \{ x \in X : |f(x)| < 1 \} \subseteq U \}.
\]

Then \( 0 \in A \).

Since \( C_p(X) \) is a Fréchet space, there exists a sequence \( \langle f_n \in A : n \in \omega \rangle \) such that \( \lim_{n \to \infty} f_n = 0 \). By definition of the set \( \Lambda \), for every \( n \) there exists a set \( U_n \in U \) such that \( \{ x \in X : |f_n(x)| < 1 \} \subseteq U_n \). Then \( \{ U_n : n \in \omega \} \) is an open \( \gamma \)-cover of \( X \).

\[\text{q.e.d.}\]

**Theorem 46** (L. Bukovský – J. Haleš). Every \( S_1(\Gamma^{sh}, \Gamma) \)-space is a \( wQN \)-space.

**Proof.** If \( \langle f_n : n \in \omega \rangle \) are continuous, \( f_n \to 0 \) on \( X \), \( f_n > 0 \), we set

\[
(10) \quad U_{n,m} = \{ x \in X : f_m(x) < 2^{-n} \}, \quad U_n = \{ U_{n,m} : m \in \omega \}.
\]

Let \( L = \{ n \in \omega : X \not\in U_n \} \).

If \( \omega \setminus L = \{ n_k : k \in \omega \} \) is infinite, then there exist increasing sequences \( \{ n_k \}_{k=0}^{\infty}, \{ m_k \}_{k=0}^{\infty} \) such that \( U_{n_k, m_k} = X \) for every \( k \). Since \( f_{m_k}(x) < 2^{-n_k} \) for every \( x \in X \) we obtain that \( f_{m_k} \equiv 0 \) on \( X \).

Assume that \( \omega \setminus L \) is finite. We can assume that \( L = \omega \). Then \( U_n \) is a shrinkable \( \gamma \)-cover for every \( n \). Since \( X \) is a \( S_1(\Gamma^{sh}, \Gamma) \)-space there exist \( V_n \in U_n \) such that \( \{ V_n : n \in \omega \} \) is a \( \gamma \)-cover. Let \( m_n \) be such that \( V_n = U_{n, m_n} \). The sequence \( \{ m_n \}_{n=0}^{\infty} \) is unbounded and we can assume that \( \{ m_n \}_{n=0}^{\infty} \) is increasing. Then \( f_{m_n} \to 0 \) on \( X \) with the control \( \{ 2^{-n} \}_{n=0}^{\infty} \).

\[\text{q.e.d.}\]

**Corollary 47** (M. Scheepers). Every \( S_1(\Gamma, \Gamma) \)-space is a \( wQN \)-space.

**Theorem 48** (L. Bukovský – J. Haleš). A normal topological space \( X \) is a \( wQN \)-space if and only if \( X \) is an \( S_1(\Gamma^{sh}, \Gamma) \)-space.

**Conjecture 49** (M. Scheepers). A normal topological space \( X \) is a \( wQN \)-space if and only if \( X \) is an \( S_1(\Gamma, \Gamma) \)-space.

**Metatheorem 3.** The Scheepers Conjecture 49 is consistent with \( \text{ZFC} \).

**Theorem 50.** If \( t = b \), then there exists a set of reals \( X \subseteq \omega^2 \) of cardinality \( b \) such that \( X \) is an \( S_1(\Gamma, \Gamma) \)-space, therefore also a \( wQN \)-space, and \( X \setminus [\omega]^{<\omega} \) is not a \( wQN \)-space. Hence \( X \) is not a \( QN \)-space.

**Metatheorem 4.** The equivalences \( \text{QN} = S_1(\Gamma, \Gamma) \) and \( \text{QN} = wQN \) are undecidable in \( \text{ZFC} \).
5 Borel and Analytic Sets

Let \( (X, \mathcal{O}) \) be a topological space. \( \text{Borel}(X, \mathcal{O}) \) will denote the smallest \( \sigma \)-algebra of subsets of \( X \) containing all open subsets of \( X \).

\[
\Sigma^0_\eta(X, \mathcal{O}) = \bigcup_{n \in \omega} \Pi^0_n(X, \mathcal{O}),
\]

\[
\Pi^0_\eta(X, \mathcal{O}) = \{ A \subseteq X : A \text{ is closed} \}.
\]

Any of the inclusions \( \Sigma^0_\eta \subseteq \Sigma^0_\zeta \) and \( \Pi^0_\eta \subseteq \Pi^0_\zeta \) is equivalent to the condition that \( (X, \mathcal{O}) \) is perfectly normal.

**Theorem 51.** If \( (X, \mathcal{O}) \) is perfectly normal, especially metric, space then

\[
\Sigma^0_\eta(X, \mathcal{O}) \subseteq \Delta^0_\eta(X, \mathcal{O}) \subseteq \Sigma^0_\zeta(X, \mathcal{O}), \quad \Pi^0_\eta(X, \mathcal{O}) \subseteq \Delta^0_\eta(X, \mathcal{O}) \subseteq \Pi^0_\zeta(X, \mathcal{O})
\]

for any \( \eta < \xi < \omega_1 \). Therefore

\[
\text{Borel}(X, \mathcal{O}) = \bigcup_{\xi < \omega_1} \Sigma^0_\xi(X, \mathcal{O}) = \bigcup_{\xi < \omega_1} \Pi^0_\xi(X, \mathcal{O}) = \bigcup_{\xi < \omega_1} \Delta^0_\xi(X, \mathcal{O}).
\]

**Borel Hierarchy:**

\[
\begin{align*}
\Sigma_1 &\subseteq \Sigma_2 \subseteq \cdots \subseteq \Sigma_\eta \subseteq \cdots \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \cdots \subseteq \Delta_\eta \subseteq \cdots \\
\Pi_1 &\subseteq \Pi_2 \subseteq \cdots \subseteq \Pi_\eta \subseteq \cdots \subseteq \Pi_\zeta \subseteq \Delta_\zeta \subseteq \cdots \subseteq \Delta_{\zeta+1} \subseteq \cdots
\end{align*}
\]

For an uncountable Polish space \( X \) all inclusions are proper.

**Theorem 52.** Any Polish space \( X \) is a continuous image of \( \omega^\omega \).

**Theorem 53.** For any Polish space \( X \) there exists a closed set \( C \subseteq \omega^\omega \) and a continuous bijection \( f : C \xrightarrow{\text{onto}} X \).

**Theorem 54.** If \( B \) is a Borel subset of a Polish space \( X \), then there exist a closed set \( C \subseteq \omega^\omega \) and a continuous bijection \( f : C \xrightarrow{\text{onto}} B \).

Let \( X \) be Polish space. \( A \subseteq X \) is **analytic** if there exist a Polish space \( Y \), a Borel subset \( B \subseteq Y \) and a continuous mapping \( f : Y \to X \) such that \( A = f(B) \). The family of all analytic subsets of the space \( X \) is denoted by \( \Sigma_1^1(X) \). A subset \( A \subseteq X \) of a Polish space \( X \) is called **co-analytic** if \( X \setminus A \) is analytic. The family of all co-analytic subsets of the space \( X \) is denoted by \( \Pi_1^1(X) \).
Theorem 55. Let $A$ be a subset of a Polish space $X$. Then the following are equivalent:

a) $A$ is analytic.

b) $A$ is sifted by a closed Souslin scheme with vanishing diameter.

c) There exists a continuous $f : \omega^\omega \to X$ such that $A = f(\omega)$. 

d) There exist a Polish space $Y$, a Borel subset $B \subseteq Y$ and a continuous surjection $f : B \onto A$.

e) There exist a Polish space $Y$ and a Borel subset $B \subseteq Y \times X$ such that $A = \text{proj}_2(B)$.

h) There exists a closed set $B \subseteq \omega^\omega \times X$ such that $\text{proj}_2(B) = A$.

i) There exists a Borel measurable $f : \omega^\omega \to X$ such that $A = f(\omega)$. 

Theorem 56. An uncountable analytic subset of a Polish space contains a perfect subset homeomorphic copy to the Cantor middle-third set. Thus every analytic set is either countable or of cardinality continuum.

Theorem 57. There exists a model of ZFC, in which there exists an uncountable co-analytic set that does not contain a perfect subset.

Theorem 58 (N. N. Luzin). Assume that $A \subseteq X$ is a Borel subset of a Polish space $X$. If $f : X \to Y$ is continuous and injective on $A$, then $f(A)$ is Borel.

We shall call two disjoint subsets $A, B \subseteq X$ Borel separable if there exists a Borel set $C \subseteq X$ such that $A \subseteq C$ and $B \subseteq X \setminus C$.

Theorem 59 (Luzin Separation Theorem). Any two disjoint analytic subsets of a Polish space are Borel separable.

Corollary 60 (M. J. Souslin). Let $X$ be a Polish space. If $A \subseteq X$ is both analytic and co-analytic, then $A$ is Borel. Thus

\[(11) \quad \text{Borel} = \Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1.\]

Theorem 61. Every analytic and therefore every co-analytic set of reals is Lebesgue measurable.

**Lebesgue Decomposition:**

A real $x \in [0,1]$ can be identified with a subset $p(x)$ of $\omega$: take an infinite binary expansion $x = \sum_{n=0}^\infty x_n 2^{-n-1}$ and let $p(x) = \{ n \in \omega : x_n = 1 \}$. For any $a \subseteq \omega$, the set $\pi^{-1}(a) \subseteq \omega \times \omega$ is a binary relation on $\omega$. Set

$L_\xi = \{ x \in [0,1] : (\omega, \pi^{-1}(p(x))) \text{ is a well-ordered set of the type } \xi \}.$

The family

\[\{ L_\xi : \omega \leq \xi < \omega_1 \} \cup \{ [0,1] \cup \bigcup_{\omega \leq \xi < \omega_1} L_\xi \}\]

is the the **Lebesgue decomposition** of $[0,1]$. 

17
Theorem 62. The set $\bigcup_{\omega \leq \xi < \omega_1} L_\xi$ is a co-analytic set, that is not analytic. Every $L_\xi$ is a Borel set.

Moreover, if $A \subseteq \bigcup_{\omega \leq \xi < \omega_1} L_\xi$ is analytic, then there is an ordinal $\xi_0 < \omega_1$ such that $A \subseteq \bigcup_{\omega \leq \xi < \xi_0} L_\xi$. Thus $A$ is analytic.

The Lebesgue decomposition plays an important role in the investigation of projective sets.

6 Projective Sets

The families of projective sets are defined as follows:

\begin{align*}
\Pi_0^1(X) &= \Sigma_0^1(X) = \text{Borel}(X), \\
\Sigma_{n+1}^1(X) &= \exists \omega [\Pi_n^1(\omega \times X)], \\
\Pi_{n+1}^1(X) &= \neg \Sigma_n^1(X), \\
\Delta_{n+1}^1(X) &= \Sigma_{n+1}^1(X) \cap \Pi_{n+1}^1(X),
\end{align*}

for any $0 < n \in \omega$.

The Projective Hierarchy

\[
\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \Sigma_n \subseteq \cdots \\
\Delta_1 \subseteq \Delta_2 \subseteq \cdots \subseteq \Delta_n \subseteq \cdots \\
\Pi_1 \subseteq \Pi_2 \subseteq \cdots \subseteq \Pi_n \subseteq \cdots
\]

Theorem 63.

a) The families $\Sigma_n^1(X)$, $\Pi_n^1(X)$ and $\Delta_n^1(X)$ are closed under Borel measurable inverse images, especially under continuous inverse images.

b) Borel measurable image of a $\Sigma_n^1$ set is a $\Sigma_n^1$ set for any $n > 0$.

c) The families $\Sigma_n^1(X)$ and $\Pi_n^1(X)$ are closed under countable unions and countable intersections. Thus the family $\Delta_n^1(X)$ is a $\sigma$-field of subsets of $X$.

d) If $f : X \overset{1 \to}{\longrightarrow} Y$ is a continuous bijection from a Polish space $X$ into a Polish space $Y$ and $A \subseteq X$ is a $\Sigma_n^1$, $\Pi_n^1$ or $\Delta_n^1(X)$ set, $n > 0$, then $f(A)$ is a $\Sigma_n^1$, $\Pi_n^1$ or $\Delta_n^1(X)$ set, respectively, as well.

Theorem 64. There exists a model of ZFC, in which there exists a $\Delta_2^1(X)$ set, that is not Lebesgue measurable.

There exists a model of ZFC, in which every $\Sigma_2^1(X)$ set is Lebesgue measurable.

A family $\Gamma$ of subsets of $X$ has the separation property if for any sets $A_1, A_2 \in \Gamma$, $A_1 \cap A_2 = \emptyset$ there exists a set $B \in \Gamma$ such that $X \setminus B \in \Gamma$, $A_1 \subseteq B$, and $A_2 \cap B = \emptyset$. A family $\Gamma$ of subsets of $X$ has the reduction property if
for any sets $A_1, A_2 \in \Gamma$, there exist sets $B_1, B_2 \in \Gamma$ such that $B_1 \cap B_2 = \emptyset$, $B_1 \subseteq A_1$, $B_2 \subseteq A_2$ and $A_1 \cup A_2 = B_1 \cup B_2$.

Luzin Separation Theorem 59 says that $\Sigma^1_1(X)$ has the separation property.

**Theorem 65.**

a) If a family $\Gamma \subseteq \mathcal{P}(X)$ has the reduction property, then the dual family $\neg \Gamma = \{ X \setminus A : A \in \Gamma \}$ has the separation property.

b) If $X$ is an uncountable Polish space, then for any $n > 0$ neither $\Sigma^1_n(X)$ nor $\Pi^1_n(X)$ has both the separation and reduction property.

**Theorem 66.** If $X$ is an uncountable Polish space, then $\Pi^1_1(X)$ and $\Sigma^1_1(X)$ possess the reduction property and therefore $\Pi^1_2(X)$ possesses the separation property.

What about $\Sigma^1_2(X)$ and $\Pi^1_2(X)$?

N.N. Luzin: "the domain of projective sets is a domain where the principle of excluded third cannot be applied more".

A boxed family in the next pictures possesses the reduction property.

If every set is constructible (consistent to assume):

\[
\begin{array}{cccccccc}
\Sigma^1_1 & \Sigma^1_2 & \Sigma^1_3 & \ldots & \Sigma^1_{2n-1} & \Sigma^1_{2n} & \Sigma^1_{2n+1} & \ldots \\
\Pi^1_1 & \Pi^1_2 & \Pi^1_3 & \ldots & \Pi^1_{2n-1} & \Pi^1_{2n} & \Pi^1_{2n+1} & \ldots \\
\end{array}
\]

Assuming the Axiom of Determinacy:

\[
\begin{array}{cccccccc}
\Sigma^1_1 & \Sigma^1_2 & \Sigma^1_3 & \ldots & \Sigma^1_{2n-1} & \Sigma^1_{2n} & \Sigma^1_{2n+1} & \ldots \\
\Pi^1_1 & \Pi^1_2 & \Pi^1_3 & \ldots & \Pi^1_{2n-1} & \Pi^1_{2n} & \Pi^1_{2n+1} & \ldots \\
\end{array}
\]

References


Institute of Mathematics,
P. J. Šafárik University,
Jesenná 5, 040 01 Košice, Slovakia.
e-mail: bukovsky@kosice.upjs.sk