Subsets of the Real Line

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1 Introduction

I will essentially follow
L.B. Štruktúra Reálnej Osi, (The Structure of the Real Line, Slovak), Veda, Bratislava 1979.
L. B., The Structure of the Real Line, due to appear November 15, 2010 at Springer/Birkhäuser, Monografie Matematyczne, Vol. 71.

The lectures are devoted to a survey of recent results on the properties of special subsets of the real line:
a) related to the convergence of sequences of real continuous functions;
b) covering properties;
c) relationships between those properties;
d) basic properties of hierarchies of the Borel and projective sets.

A topological space $\langle X, \mathcal{O}\rangle$: X a non-empty set, $\mathcal{O}$ the set of open subsets of X, for simplicity always **Hausdorff**.

Equivalent definition: a closure operator $\overline{A}$ satisfying the axioms

(C1) $A \subseteq \overline{A}$;

(C2) $\overline{\emptyset} = \emptyset$;

(C3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;

(C4) $\overline{\overline{A}} = \overline{A}$.

A normal topological space X is **perfectly normal** if every open subset of X is an F_σ set. Recall

Theorem 1 (P. Urysohn). *If X is a normal topological space and $A, B \subseteq X$ are disjoint closed subsets, then there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

If $f : X \longrightarrow \mathbb{R}$, then

$$\mathrm{Z}(f) = \{x \in X : f(z) = 0\}.$$

A subset $A \subseteq X$ is a **zero set** if there exists a continuous f such that $A = \mathrm{Z}(f)$.

X is perfectly normal if and only if every closed subset of X is a zero set.

If $A, B \subseteq X$ are closed subsets, then there exist continuous $f : X \longrightarrow [0, 1/2]$ and $g : X \longrightarrow [1/2, 1]$ such that $\mathrm{Z}(f) = A$ and $\mathrm{Z}(1-g) = B$. If A, B are disjoint, then the open sets $\{x \in X : f(x) + g(x) < 1\}$ and $\{x \in X : f(x) + g(x) > 1\}$ separate A and B.

A **convergence structure** is a mapping $\lim : \mathcal{X} \longrightarrow X$, where $\mathcal{X} \subseteq {}^{\omega}X$. A sequence $\{x_n\}_{n=0}^{\infty} \in \mathcal{X}$ is **convergent**, $\lim(\{x_n\}_{n=0}^{\infty}) = \lim_{n\to\infty} x_n$ is the **limit**. A couple $\langle X, \lim\rangle$ is an **$\mathcal{L}^*$-space** if:

(L1) if $x_n = x$ for every n, then $\lim_{n\to\infty} x_n = x$;

(L2) if $\lim_{n\to\infty} x_n = x$ and $\{n_k\}_{k=0}^{\infty}$ is increasing, then $\lim_{k\to\infty} x_{n_k} = x$;

(L3) if $x \neq \lim_{n\to\infty} x_n$, then there exists a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ such that no subsequence of $\{x_{n_k}\}_{k=0}^{\infty}$ has limit x.

If $A \subseteq X$, then **sequential closure** of A is

$$\mathrm{scl}(A) = \{x \in X; x = \lim_{n\to\infty} x_n \text{ for some } x_n \in A\}.$$

The sequential closure "scl" satisfies the axioms (C1) – (C3). Any topological space is an $\mathcal{L}^*$-space and $\mathrm{scl}(A) \subseteq \overline{A}$. X is **Fréchet** if $\mathrm{scl}(A) = \overline{A}$.

We can define by transfinite induction:

$$\mathrm{scl}_0(A) = a, \qquad \mathrm{scl}_\xi(A) = \mathrm{scl}(\bigcup_{\eta<\xi} \mathrm{scl}_\eta(A)) \text{ for } \xi > 0.$$

Note that

$$\mathrm{scl}_\xi(A) = \mathrm{scl}(\omega_1)A \text{ for any } \xi > \omega_1.$$

Theorem 2. *Assume that X is an $\mathcal{L}^*$-space. Then the following conditions are equivalent:*

a) *If $\lim_{n\to\infty} x_n = x$, $\lim_{m\to\infty} x_{n,m} = x_n$ for every n, then there exist sequences $\{n_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} x_{n_k,m_k} = x$.*

b) *If $\lim_{n\to\infty} x_n = x$, $\lim_{m\to\infty} x_{n,m} = x_n \neq x$ for every n, then there exist sequences $\{n_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{\infty}$ such that $\{n_k\}_{k=0}^{\infty}$ is increasing and $\lim_{k\to\infty} x_{n_k,m_k} = x$.*

c) *If $\lim_{n\to\infty} x_n = x$, $\lim_{m\to\infty} x_{n,m} = x_n \neq x$ for every n, then there exist increasing sequences $\{n_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} x_{n_k,m_k} = x$.*

d) *$\mathrm{scl}_1(A) = \mathrm{scl}_2(A)$ for any $A \subseteq X$.*

X possesses the **sequence selection property**, shortly **SSP**, if X possesses any of conditions a) – d).

The **Fremlin number** $\Sigma(X)$ of an $\mathcal{L}^*$-space X is defined as

$$\sigma(A, X) = \min\{\xi; \mathrm{scl}_\xi(A) = \mathrm{scl}_{\omega_1}(A)\}, \qquad \Sigma(X) = \sup\{\sigma(A, X); A \subseteq X\}.$$

An $\mathcal{L}^*$-space X has the **property** (α_i), $i = 1, 2, 3, 4$, if for any $x \in X$ and for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging to x, there exists a sequence $\{y_m\}_{m=0}^{\infty}$ such that $\lim_{m\to\infty} y_m = x$ and

(α_1) $\{x_{n,m} : m \in \omega\} \subseteq^* \{y_m : m \in \omega\}$ for each n,

(α_2) $\{x_{n,m} : m \in \omega\} \cap \{y_m : m \in \omega\}$ is infinite for each n,

(α_3) $\{x_{n,m} : m \in \omega\} \cap \{y_m : m \in \omega\}$ is infinite for infinitely many n,

(α_4) $\{x_{n,m} : m \in \omega\} \cap \{y_m : m \in \omega\} \neq \emptyset$ for infinitely many n.

It is easy to see that

(1) $$(\alpha_1) \to (\alpha_2) \to (\alpha_3) \to (\alpha_4), \qquad \text{SSP} \equiv (\alpha_4).$$

Metatheorem 1 (A. Dow)**.** *$(\alpha_1) \equiv (\alpha_2)$ in the Laver's model of the set theory* **ZFC**.

Metatheorem 2. *There exist models of* **ZFC***, in which $(\alpha_2) \nrightarrow (\alpha_1)$.*

2 Pointwise Convergence of Real Functions

In what follows, f_n, f are functions from a topological space X into $\mathbb{R}$.

"$f_n \to f$ on X" means that $\{f_n\}_{n=0}^{\infty}$ pointwise converges to f on X, i.e., $(\forall x \in X) \lim_{n\to\infty} x_n(x) = f(x)$. $\mathrm{C}_p(X)$ is the set of all continuous functions from X into $\mathbb{R}$. Topology of $\mathrm{C}_p(X)$ is the subspace topology $\mathrm{C}_p(X) \subseteq {}^X\mathbb{R}$. A set $U \subseteq \mathrm{C}_p(X)$ is a neighbourhood of an $f \in \mathrm{C}_p(X)$ if and only if there exist $x_1, \cdots, x_n \in X$ and $\varepsilon > 0$ such that

$$\{g \in \mathrm{C}_p(X) : |g(x_1) - f(x_1)| < \varepsilon \wedge \cdots \wedge |g(x_n) - f(x_n)| < \varepsilon\} \subseteq U.$$

Theorem 3. *$f_n \to f$ in the product topology on ${}^X\mathbb{R} \equiv f_n \to f$ on X.*

Theorem 4. *If $\mathrm{C}_p(X)$ is a Fréchet space, $X \subseteq \mathbb{R}$, then $\lambda(X) = 0$ and X is meager.*

For a real $x \in \mathbb{R}$ we let $\|x\|$ to be the distance of x to the nearest integer.

$$2\|x\| \le |\sin \pi x| \le \pi\|x\|.$$

Theorem 5 (P. G. Lejeune Dirichlet)**.** *For any $\varepsilon > 0$ and for any reals $x_1, \ldots, x_k \in \mathbb{R}$, there exists arbitrarily large n such that*

(2) $$\|nx_l\| < \varepsilon \qquad \text{for } l = 1, 2, \ldots, k.$$

Corollary 6. *For any non-empty $X \subseteq \mathbb{R}$ we have $0 \in \overline{\{\|nx\| : n > 0\}}$.*

Proof of Theorem 4. Assume that $\|n_i x\| \to 0$ on X. We can assume that $X = \{x \in \mathbb{R} : \|n_i x\| \to 0$ and therefore X is Borel. Then $\|n_i x\| \to 0$ on $X - X = \{x - y : x, y \in X\}$. By Lebesgue Dominated Convergence Theorem

$$\int_{(X-X)} \|n_i x\| \, d\lambda \to 0.$$

If $\lambda(X) > 0$ (or if X is not meager), then by a Steinhaus Theorem there exists reals $a < b$ such that $(a, b) \subseteq X - X$. For any sufficiently large positive integer n there exist positive integers k, m such that

$$\frac{k-1}{n} < a \le \frac{k}{n} < \frac{m}{n} \le b < \frac{m+1}{n}.$$

Then

$$\frac{m-k}{4n} \le \int_{[a,b]} \|nx\| \, d\lambda < \frac{m-k+2}{4n}$$

and therefore

$$\lim_{n\to\infty}\int_{[a,b]} \|nx\|\, d\lambda = \frac{b-a}{4} > 0.$$

q.e.d.

The **Baire Hierarchy** of subsets of ${}^X\mathbb{R}$ with the product topology

$$\mathrm{C}_p(X) \subseteq \mathrm{scl}_1(\mathrm{C}_p(X)) \subseteq \cdots \subseteq \mathrm{scl}_\xi(\mathrm{C}_p(X)) \subseteq \cdots.$$

If $X = [0,1]$ then the hierarchy is proper of length ω_1.

Theorem 7 (D.H. Fremlin). *If X is a topological space, then $\Sigma(\mathrm{C}_p(X))$ is either 1 or ω_1.*

If $\Sigma(\mathrm{C}_p(X)) = 1$, we say that X is an s_1-space.

Theorem 8 (Essentially D. Fremlin). *X is an s_1-space if and only if $\mathrm{C}_p(X)$ possesses* SSP.

The set ${}^\omega\mathbb{R}$ is preordered as

$$f \le^* g \equiv \{n \in \omega : \neg f(n) \le g(n)\} \text{ is finite.}$$

W. Hurewicz introduced properties of a topological space X:

H* for any sequence $\langle f_n \in \mathrm{C}_p(X) : n \in \omega\rangle$ the family of sequences of reals $\{\{f_n(x)\}_{n=0}^\infty; x \in X\}$ is not **dominating**, i.e.

$$(\exists g \in {}^\omega\omega)(\forall x \in X)(\forall n_0)(\exists n \ge n_0)\, f_n(x) < g(n).$$

H** for any sequence $\langle f_n \in \mathrm{C}_p(X) : n \in \omega\rangle$ the family of sequences of reals $\{\{f_n(x)\}_{n=0}^\infty; x \in X\}$ is **eventually bounded**, i.e.

$$(\exists g \in {}^\omega\omega)(\forall x \in X)(\exists n_0)(\forall n \ge n_0)\, f_n(x) \le g(n).$$

Evidently

$$\mathrm{H}^{**} \to \mathrm{H}^*.$$

Fact:

Any σ-compact topological space possesses the property H**.

Theorem 9. *Both properties are preserved by passing to a closed subset and continuous image.*

3 Quasi-normal Convergence

Let $\langle Y, \rho\rangle$ be a metric space, $f_n, f : X \longrightarrow Y$, $n \in \omega$. A sequence $\{f_n\}_{n=0}^\infty$ **converges quasi-normally** to f on X if there exists a sequence $\{\varepsilon_n\}_{n=0}^\infty$ of positive reals converging to zero and such that

(3) $$(\forall x \in X)(\exists k)(\forall n \ge k)\, \rho(f_n(x), f(x)) < \varepsilon_n.$$

$\{\varepsilon_n\}_{n=0}^\infty$ is a **control sequence** or $\{\varepsilon_n\}_{n=0}^\infty$ **witnesses** the quasi-normal convergence. We shall write "$f_n \xrightarrow{\mathrm{QN}} f$ on X".

A sequence $\{f_n\}_{n=0}^{\infty}$ **converges discretely** to f on X if

$$(\forall x \in X)(\exists k)(\forall n \geq k)\, f_n(x) = f(x) \tag{4}$$

We shall write "$f_n \xrightarrow{\text{DS}} f$ on X".

Theorem 10. *Let $f_n \xrightarrow{\text{QN}} f$ on X. For any sequence $\varepsilon_n \to 0$ of positive reals there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $f_{n_k} \xrightarrow{\text{QN}} f$ on X with the control $\{\varepsilon_k\}_{k=0}^{\infty}$.*

Theorem 11. *Let $\{f_n\}_{n=0}^{\infty}$ and f be functions from X into a metric space Y. Then the following conditions are equivalent:*

a) *$f_n \xrightarrow{\text{QN}} f$ on X.*

b) *There exists a sequence $\{X_k\}_{k=0}^{\infty}$ of subsets of X such that $X = \bigcup_k X_k$ and $f_n \rightrightarrows f$ on X_k for each k.*

c) *There exists a non-decreasing sequence $\{X_k\}_{k=0}^{\infty}$ of subsets of X such that $X = \bigcup_k X_k$ and $f_n \rightrightarrows f$ on X_k for each k.*

Moreover, if X is a topological space and $\{f_n\}_{n=0}^{\infty}$ are continuous, then conditions a)–c) *are equivalent with*

d) *There exists a non-decreasing sequence $\{X_k\}_{k=0}^{\infty}$ of closed subsets of X such that $X = \bigcup_k X_k$ and $f_n \rightrightarrows f$ on X_k for each k.*

Theorem 12. *Let $\{f_n\}_{n=0}^{\infty}$ and f be functions from X into a metric space Y. Then the following conditions are equivalent:*

a) *$f_n \xrightarrow{\text{DS}} f$ on X.*

b) *There exists a sequence $\{X_k\}_{k=0}^{\infty}$ of subsets of X such that $X = \bigcup_k X_k$ and $f_n(x) = f(x)$ for each $x \in X_k$ for each $n \geq k$.*

c) *There exists a non-decreasing sequence $\{X_k\}_{k=0}^{\infty}$ of subsets of X such that $X = \bigcup_k X_k$ and $f_n(x) = f(x)$ for each $x \in X_k$ for each $n \geq k$.*

Moreover, if X is a topological space and $\{f_n\}_{n=0}^{\infty}$ are continuous, then conditions a)–c) *are equivalent with*

d) *There exists a non-decreasing sequence $\{X_k\}_{k=0}^{\infty}$ of closed subsets of X such that $X = \bigcup_k X_k$ and $f_n(x) = f(x)$ for each $x \in X_k$ for each $n \geq k$.*

X is called a **QN-space** if every sequence of continuous real functions converging pointwise to 0 on X converges quasi-normally to 0 on X as well.

X is called a **wQN-space** if for every sequence $\{f_n\}_{n=0}^{\infty}$ of continuous real functions converging pointwise to 0 on X there exists an increasing sequence of integers $\{n_k\}_{k=0}^{\infty}$ such that $f_{n_k} \xrightarrow{\text{QN}} 0$ on X.

"$f_n \searrow 0$ on X" means that $f_n \to f$ on X and $f_n(x) \geq f_{n+1}(x)$ for each $x \in X$ and each n.

X is called an **mQN-space** if for every sequence $\{f_n\}_{n=0}^{\infty}$ of continuous functions such that $f_n \searrow 0$ on X also $f_n \xrightarrow{\text{QN}} 0$ on X.

Every compact space is an mQN*-space.*

Theorem 13. *Each of introduced properties is preserved by passing to an* F_σ *subset and by passing to an image by a function, that is a quasi-normal limit of continuous functions.*

Theorem 14. *There exists a sequence* $\{h_n\}_{n=0}^\infty$ *of continuous real functions defined on Cantor middle-third set* $\mathbb{C}$ *such that* $h_n \to 0$ *on* $\mathbb{C}$ *and for any increasing sequence* $\{n_k\}_{k=0}^\infty$ *of natural numbers there exists a* $z \in \mathbb{C}$ *such that*

$$\sum_{k=0}^{\infty} h_{n_k}(z) = +\infty. \tag{5}$$

Proof. For $x = \sum_{k=0}^\infty x_k 3^{-k-1} \in \mathbb{C}$, $x_k \in \{0, 2\}$, we set

$$h_n(x) = \begin{cases} 0 & \text{if } x_n = 0, \\ 1/m_n & \text{where } m_n = |\{k < n : x_k = 2\}| + 1, \text{ otherwise.} \end{cases}$$

If $\{n_k\}_{k=0}^\infty$ is increasing, we set $z = \sum_{k=0}^\infty z_k 3^{-k}$, where $z_{n_k} = 2$ and $z_n = 0$ otherwise. Then $h_{n_k}(z) = 1/(k+1)$.

q.e.d.

Corollary 15. *If* X *contains topologically* $\mathbb{C}$*, then* X *is not a* wQN*-space.*

Theorem 16. *Let* X *be a perfect separable metric space. Then there exists a sequence* $\{f_n\}_{n=0}^\infty$ *of continuous real-valued functions such that*

i) $f_n \to 0$ *on* X*;*

ii) *if* $A \subseteq X$*,* $\{n_k\}_{k=0}^\infty$ *is an increasing sequence such that* $f_{n_k} \rightrightarrows 0$ *on* A*, then* A *is nowhere dense.*

Proof. Let $Q = \{r_i : i \in \omega\}$ be a countable dense subset of X. Since no point r_i is isolated there exists a sequence $x_{i,n} \longrightarrow r_i$, such that $x_{i,n} \notin Q$ for each $n \in \omega$. Let $h_{i,n} : X \longrightarrow [0, 2^{-i}]$ be continuous and such that $h_{i,n}(x_{i,n}) = 2^{-i}$ and $h_{i,n}(x) = 0$ if $\rho(x, x_{i,n}) \geq 1/2\rho(r_i, x_{i,n})$. Denote

$$f_n(x) = \sum_{i=0}^{\infty} h_{i,n}(x) \text{ for } x \in X,\ n \in \omega.$$

Each f_n is continuous function from X into $[0, 2]$. Evidently $h_{i,n} \to 0$ on X for every fixed i. If $x \in X$ and $\varepsilon > 0$, one can find an i_0 such that $\sum_{i>i_0} 2^{-i} < \varepsilon/2$ and such n_0 that $\sum_{i\leq i_0} h_{i,n}(x) < \varepsilon/2$ for every $n \geq n_0$. Thus $f_n \to 0$ on X.

Assume that $\{n_k\}_{k=0}^\infty$ is increasing and $f_{n_k} \rightrightarrows 0$ on a set A. We can assume that A is closed. If $\mathrm{Int}(A) \neq \emptyset$ then there exist an $r_i \in \mathrm{Int}(A)$. Hence there exists an m such that $x_{i,n} \in \mathrm{Int}(A)$ for $n \geq m$. Since for $n \geq m$ we have

$$\sup\{f_n(x) : x \in A\} \geq f(x_{i,n}) \geq h_{i,n}(x_{i,n}) = 2^{-i},$$

we get a contradiction.

q.e.d.

Corollary 17. *Let* $\{f_n\}_{n=0}^\infty$ *be the sequence of the Theorem. If* $f_{n_k} \xrightarrow{\mathrm{QN}} 0$ *on* A*, then* A *is meager.*

A subset $A \subseteq X$ of a topological space X is **perfectly meager** if for any perfect set $P \subseteq X$ the intersection $A \cap P$ is meager in the subspace P, i.e. if for any perfect set P there are closed sets F_n such that $A \cap P \subseteq \bigcup_n F_n$ and the inclusion $P \cap U \subseteq F_n$ implies $P \cap U = \emptyset$ for any $n \in \omega$ and any open set U. If X is a perfect Polish space, then any perfectly meager subset of X is meager in X. Moreover,the family $\mathcal{PM}(X)$ of perfectly meager subsets of X is a σ-ideal.

Theorem 18. *Any* wQN-*subset A of a separable metric space X is perfectly meager.*

Theorem 19. *X is a normal topological space, $\{f_n\}_{n=0}^{\infty}$, f are real functions. If $f_n \xrightarrow{\text{QN}} f$ on X and all f_n are continuous, then there exist $\{g_n\}_{n=0}^{\infty}$ continuous functions such that $g_n \xrightarrow{\text{DS}} f$ on X. Thus f is $\mathbf{\Delta}^0_2$-measurable.*

Proof. If $\{U_n : n \in \omega\}$ is a countable base of the topology on $\mathbb{R}$, then for any open set $U \subseteq \mathbb{R}$ we have

$$f(x) \in U \equiv (\exists n)(\exists k)\,(\overline{U_n} \subseteq U \wedge (\forall m \geq k)\, f_m(x) \in \overline{U_n}.$$

Thus, f is F_σ-measurable.

Take closed subsets $\langle X_k : k \in \omega\rangle$ such that $X = \bigcup_k X_k$ and $f_n \rightrightarrows f$ on X_k for each k. Thus $f|X_k$ is continuous. By Tietze-Urysohn Theorem there exist real continuous functions $\{g_k\}_{k=0}^{\infty}$ defined on X such that $g_k|X_k = f|X_k$ for every k. Then $g_n \xrightarrow{\text{DS}} f$ on X.

Since for any open U we have

$$f(x) \in U \equiv (\forall n)(\exists m > n)\, g_m(x) \in U,$$

the function f is G_δ-measurable.

q.e.d.

Theorem 20 (M. Scheepers – D.H.Fremlin)**.** *For a topological space X the following are equivalent:*

a) *X is a* wQN-*space.*

b) *The space $\mathrm{C}_p(X)$ possesses the sequence selection property.*

c) *The space $\mathrm{C}_p(X)$ possesses the property (α_2).*

d) *The space $\mathrm{C}_p(X)$ possesses the property (α_3).*

e) *The space $\mathrm{C}_p(X)$ possesses the property (α_4).*

Proof.

e) $\rightarrow$ a). Assume that $\langle f_n : n \in \omega\rangle$ are continuous, $f_n \rightarrow 0$ on X and $f_n > 0$ for every n. We denote $f_{n,m} = 2^n \cdot f_{n+m}$. Then $f_{n,m} \rightarrow 0$ on X for every n. By (α_4), there exist increasing sequences $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=0}^{\infty}$ such that $f_{n_k,m_k} \rightarrow 0$ on X. We claim that $f_{m_k+n_k} \xrightarrow{\text{QN}} 0$ on X with control $\{2^{-n_k}\}_{k=0}^{\infty}$.

Indeed, if $f_{n_k,m_k} \rightarrow 0$ on X, then for any $x \in X$ there exists an n_0 such that $f_{n_k,m_k}(x) < 1$ for $n \geq n_0$. Hence $f_{m_k}(x) < 2^{-n_k}$ for any $n \geq n_0$.

a) → b). Let $f_{n,m} \to 0$ on X for every n. We set

$$g_m(x) = \sum_{n=0}^{\infty} \min\{2^{-n}, f_{n,m}(x)\}, \quad x \in X. \tag{6}$$

Let $x \in X$, $\varepsilon > 0$ being a real. Then there exists an n_0 such that $2^{-n_0+2} < \varepsilon$. For every $n < n_0$ there exists an m_n such that $f_{n,m}(x) < \varepsilon/(2n_0)$ for every $m \geq m_n$. We denote $k = \max\{m_n : n < n_0\}$. For $m \geq k$ we have

$$g_m(x) \leq \sum_{n<n_0} \frac{\varepsilon}{2n_0} + \sum_{n \geq n_0} 2^{-n} < \varepsilon.$$

Thus $\lim_{m\to\infty} g_m(x) = 0$. Since X is a wQN-space there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such that $g_{m_n} \xrightarrow{\mathrm{QN}} 0$ on X with the control $\{2^{-n}\}_{n=0}^{\infty}$. However by (6), if $g_{m_n}(x) < 2^{-n}$, then also $f_{n,m_n}(x) < 2^{-n}$. Thus $f_{n,m_n} \xrightarrow{\mathrm{QN}} 0$ on X.

b) → c). Let $f_{n,m} \to 0$ on X for every n. We set $h_{n,m} = f_{\lambda(n),m}$, where λ is the left inverse to a pairing function[1]. Therefore the sequence $\{h_{n,m}\}_{n=0}^{\infty}$ contains every $f_{i,m}$ infinitely many times. Let $g_{n,m} = \max\{h_{i,m} : i \leq n\}$. Evidently $g_{n,m} \to 0$ for every n. Then there exist increasing sequences $\{n_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} g_{n_k,m_k} = 0$. We set $f_i = h_{i,m_k}$ for $n_{k-1} < i \leq n_k$. Then $f_i \leq g_{n_k,m_k}$ for $n_{k-1} < i \leq n_k$ and therefore $f_i \to 0$ on X. Every sequence $\{f_{n,m}\}_{m=0}^{\infty}$ contains infinitely many members of $\{f_i\}_{i=0}^{\infty}$.

q.e.d.

Since a Fréchet space possesses the sequence selection property we obtain:

Corollary 21. *If $\mathrm{C}_p(X)$ is a Fréchet space, then X is a* wQN*-space.*

There is a similar result for a QN-space.

Theorem 22 (M. Scheepers)**.** *If $\mathrm{C}_p(X)$ has the property (α_1), then the topological space X is a* QN*-space.*

Proof. If $\{f_m\}_{m=0}^{\infty}$ is a sequence of continuous functions converging pointwise to 0 on X, we set $f_{n,m}(x) = 2^n \cdot |f_m(x)|$. Then $\{f_{n,m}\}_{m=0}^{\infty}$ converges pointwise to 0 on X for each n. By (α_1) there exists a sequence $\{h_k\}_{k=0}^{\infty}$ converging to 0 on X and such that the sequence $\{h_k\}_{k=0}^{\infty}$ contains all but finitely many members of the sequence $\{f_{n,m}\}_{m=0}^{\infty}$ for every n. Thus, there exists an increasing sequence $\{m_n\}_{n=0}^{\infty}$ such that $\{f_{n,m}\}_{m=m_n}^{\infty}$ is a subsequence of $\{h_k\}_{k=0}^{\infty}$ for each n. Moreover, we can assume that

$$(\forall m \geq m_n)(\forall i)\,(h_i = f_{n,m} \to i \geq n)$$

for every n. Let

$$\varepsilon_m = 2^{-k} \text{ for } m_k \leq m < m_{k+1}, \quad \varepsilon_m = 1 \text{ for } m < m_0.$$

Let $x \in X$, then there exists a k_0 such that $|h_k(x)| < 1$ for $k \geq k_0$. For any $m \geq m_{k_0}$ there exists a $k \geq k_0$ such that $m_k \leq m < m_{k+1}$. Then $f_{k,m} = h_i$ for some $i \geq k \geq k_0$. Hence $f_{k,m}(x) < 1$. Since $\varepsilon_m = 2^{-k}$ we obtain $|f_m(x)| < \varepsilon_m$.

q.e.d.

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[1]A bijection $\pi : \omega \times \omega \xrightarrow[\text{onto}]{1-1} \omega$ is a **pairing function**, λ, ρ are the left and the right inverse to π. Thus $\pi(\lambda(n), \rho(n)) = n$, $\lambda(\pi(n,m)) = n$ and $\rho(\pi(n,m)) = m$ for any n, m.

Theorem 23 (L. Bukovský – J. Haleš – M. Sakai)**.** *If X is a* QN*-space, then* $\mathrm{C}_p(X)$ *possesses the property* (α_1)*.*

Proof. Let $\{\{f_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ be a sequence of sequences converging to 0 on X. We can assume that values of each $f_{n,m}$ are in $[0,1]$. We define the functions g_m by (6). Then g_m are continuous and $g_m \to 0$ on X.

Since X is a QN-space, there exist positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$, $\varepsilon_n \to 0$ such that

$$(\forall x)(\exists l_x)(\forall m \geq l_x)\, g_m(x) < \varepsilon_m. \tag{7}$$

There are also natural numbers m_k such that

$$(\forall k)(\forall m \geq m_k)\, \varepsilon_m < 2^{-k}.$$

We can assume that $m_k < m_{k+1}$ for any k. We claim that the sequence

$$\{f_{n,m} : n \in \omega \wedge m \geq m_n\} \tag{8}$$

converges to 0 on X.

q.e.d.

Corollary 24. *A topological space X is a* QN*-space if and only if* $\mathrm{C}_p(X)$ *possesses the property* (α_1)*.*

X has the **quasi-normal sequence selection property**, **QSSP**, if for any functions $f, f_n, f_m^n : X \longrightarrow \mathbb{R}$, $n, m \in \omega$, such that

(1) $f_n \xrightarrow{\mathrm{QN}} f$ on X,

(2) $f_m^n \xrightarrow{\mathrm{QN}} f_n$ on X for every $n \in \omega$,

(3) every f_m^n is continuous,

there exists an increasing $\beta \in {}^{\omega}\omega$ such that $f_{\beta(n)}^n \xrightarrow{\mathrm{QN}} f$ on X.

Theorem 25 (L. Bukovský – J. Šupina)**.** *Any* QN*-space possesses* QSSP*.*

Proof. We assume that the control of the quasi-normal convergences in (2) is $\{2^{-2m-1}\}_{m=0}^{\infty}$ and the control of that in (1) is $\{\varepsilon_n\}_{n=0}^{\infty}$.

We set

$$g_m^n(x) = \min\{|f_m^n(x) - f_{m+1}^n(x)| \cdot 2^m, 1\}.$$

Evidently for a fixed $n \in \omega$ we have $g_m^n \to 0$ on X. Since the space $\mathrm{C}_p(X)$ satisfies (α_1), there exists an increasing function $\beta \in {}^{\omega}\omega$ such that the set $\{g_m^n : m \geq \beta(n) \wedge n \in \omega\}$ converges to 0.

Then $f_{\beta(n)}^n \xrightarrow{\mathrm{QN}} f$ with the control $\{2^{-\beta(n)+1} + \varepsilon_n\}_{n=0}^{\infty}$.

q.e.d.

A topological space is a **σ-space**, if every F_σ subset is also a G_δ subset. Then every Borel subset is an F_σ set.

Theorem 26 (I. Recław)**.** *If a perfectly normal topological space X has the property* QSSP*, then X is a σ-space. Therefore every perfectly normal topological* QN*-space is a σ-space.*

Proof. Let $F = \bigcup_n F_n$, F_n being closed, $F_n \subseteq F_{n+1}$ for any $n \in \omega$. We show that the characteristic function χ_F is G_δ-measurable.

Since X is perfectly normal, there exist closed sets $\langle F_{n,m} : n, m \in \omega\rangle$ such that $F_{n,m} \subseteq F_{n,m+1}$ and $X \setminus F_n = \bigcup_k F_{n,k}$ for any n and m. For any n and m, there exists a continuous function $f_{n,m} : X \longrightarrow [0,1]$ such that $f_{n,m}(x) = 1$ for $x \in F_n$ and $f_{n,m}(x) = 0$ for $x \in F_{n,m}$. Evidently $f_{n,m} \xrightarrow{\mathrm{DS}} \chi_{F_n}$ on X. Moreover, $\chi_{F_n} \xrightarrow{\mathrm{DS}} \chi_F$ on X.

By QSSP there exists a β such that $f_{n,\beta(n)} \xrightarrow{\mathrm{QN}} \chi_F$. Since $f_{n,\beta(n)}$ are continuous, by Theorem 19 the function χ_F is G_δ-measurable.

q.e.d.

Corollary 27. *Every subset of a metric separable* QN-*space is a* QN-*space.*

Lemma 28. *If* $\mathrm{Ind}(X) = 0$, *then every simple* $\mathbf{\Delta}^0_2$-*measurable function* $g : X \longrightarrow [0,1]$ *is a discrete limit of a sequence* $\{g_n\}_{n=0}^\infty$ *of simple continuous functions.*

Proof. Assume that $g = \sum_{i=0}^k a_i \chi_{A_i}$, where $A_i \in \mathbf{\Delta}^0_2$ are pairwise disjoint, $\bigcup_{i=0}^k A_i = X$ and $0 = a_0 < a_1 < \cdots < a_k \leq 1$. Then for every $i = 0, \ldots, k$ there exist non-decreasing and non-increasing sequences $\{F_n^i\}_{n=0}^\infty$ and $\{G_n^i\}_{n=0}^\infty$ of F_σ and G_δ sets, respectively, such that

$$A_i = \bigcup_n F_n^i = \bigcap_n G_n^i \text{ for } i = 0, \ldots, k.$$

Since $\mathrm{Ind}(X) = 0$, there exists clopen sets C_n^i such that $F_n^i \subseteq C_n^i \subseteq G_n^i$ for every $i \leq k$ and every $n \in \omega$. Replacing eventually C_n^0 by $X \setminus \bigcup_{0<i\leq k} C_n^i = X$ we can assume that $\bigcup_{i\leq k} C_n^i = X$. Let $D_n^i = C_n^i \setminus \bigcup_{j<i} C_n^j$. Then D_n^i are pairwise disjoint and $\bigcup_{i\leq k} D_n^i = X$. Set $g_n = \sum_{i=0}^k a_i \chi_{D_n^i}$. Since each D_n^i is clopen, g_n is continuous.

q.e.d.

Theorem 29. *If X is a normal topological space possessing property* QSSP, *then any Borel measurable function* $f : X \longrightarrow [0,1]$ *is a quasi-normal limit of a sequence of continuous functions.*

Proof. If $f : X \longrightarrow {}^\omega\omega \subseteq [0,1]$ is Borel measurable, then by Recław's Theorem 26 the function f is $\mathbf{\Delta}^0_2$-measurable. For any n and any $i < 2^n - 1$, we set

$$A_n^i = \{x \in X : \frac{i}{2^n} \leq f(x) < \frac{i+1}{2^n}\}, \quad A_n^{2^n-1} = \{x \in X : \frac{2^n-1}{2^n} \leq f(x)\}.$$

Then the sequence of simple $\mathbf{\Delta}^0_2$-measurable functions

$$f_n = \sum_{i=0}^{2^n-1} \frac{i}{2^n} \chi_{A_n^i}$$

converges uniformly to f with control 2^{-n}. By the Lemma, for every n there exists a sequence $\{g_m^n\}_{m=0}^\infty$ of simple continuous functions such that $g_m^n \xrightarrow{\mathrm{DS}} f_n$ on X. Thus, by Theorem 25 there exists an increasing $\alpha \in {}^\omega\omega$ such that $g_{\alpha(n)}^n \xrightarrow{\mathrm{QN}} f$.

q.e.d.

Theorem 30 (B. Tsaban – L. Zdomskyy)**.** *The image of a perfectly normal topological* QN*-space X by a Borel measurable function into ${}^\omega\omega$ is eventually bounded.*

Proof. Assume that $f : X \longrightarrow {}^\omega\omega \subseteq [0,1]$ is Borel measurable. By Theorem 29 there exists a sequence of continuous functions $\langle f_n : X \longrightarrow [0,1] : n \in \omega\rangle$ such that $f_n \xrightarrow{\mathrm{QN}} f$ on X. By Theorem 13 the set $f(X) \subseteq {}^\omega\omega$ is eventually bounded.
q.e.d.

Corollary 31. *For a perfectly normal topological space X the following are equivalent:*

a) *X is a* QN*-space.*

b) *If $\langle f_n : n \in \omega\rangle$ are Borel measurable function from X into $[0,1]$ and $f_n \to f$ on X, then $f_n \xrightarrow{\mathrm{QN}} f$ on X.*

c) *Any Borel measurable image of X into ${}^\omega\omega$ is eventually bounded.*

Proof. a) $\to$ c) follows by Tsaban-Zdomskyy Theorem, the implication b) $\to$ a) is trivial.

We show that c) $\to$ b). Let $\langle f_n : n \in \omega\rangle$ be Borel measurable functions from X into $[0,1]$ and $f_n \to f$ on X. Set $g_n(x) = \sup\{|f_m(x) - f(x)| : m \geq n\}$. Then g_n is Borel measurable and $g_n \searrow 0$. The function $\psi : X \to {}^\omega\omega$ defined as $\psi(x)(m) = \min\{n : g_n(x) < 2^{-m}\}$ is Borel measurable. By c), the set $\psi(X)$ is eventually bounded by a $\beta \in {}^\omega\omega$. Then $g_{\beta(n)} \xrightarrow{\mathrm{QN}} 0$ with the control $\{2^{-n}\}_{n=0}^\infty$. Since $\{g_n\}_{n=0}^\infty$ is non-increasing we obtain $g_n \xrightarrow{\mathrm{QN}} 0$ and also $f_n \xrightarrow{\mathrm{QN}} f$.
q.e.d.

Lemma 32 (J. Haleš)**.** *If $f_n \searrow 0$ on X are continuous then there exists a continuous function $h : X \longrightarrow \mathbb{R}$ such that*

$$(\forall x \in X)(\forall n > h(x))\, f_n(x) < 1.$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \min\{1, f_n(x)\} \cdot 2^{-n}.$$

The function $g : X \longrightarrow [0,2)$ is continuous. Set $h(x) = -\log_2(2 - g(x))$. □

Theorem 33 (L.B. – J. Haleš)**.** *A topological space X is an* mQN*-space if and only if X possesses the property* H**.

Theorem 34. *If A is a subset of a Polish space X with property* H***, then there exists a σ-compact set $B \subseteq X$ such that $A \subseteq B$.*

Proof. Let $\{r_n : n \in \omega\}$ be a countable dense subset of X. Denote

$$f_n(x) = \min\{\rho(x, r_i) : i = 0, \dots, n\}.$$

Then $f_n \searrow 0$ on X and therefore $f_n \xrightarrow{\mathrm{QN}} 0$ on A. If $\{\varepsilon_n\}_{n=0}^\infty$ is a control sequence for the quasi-normal convergence of $\{f_n\}_{n=0}^\infty$ on A, then the set

$$B_n = \{x \in X : (\forall m \geq n)\, f_m(x) \leq \varepsilon_m\}$$

is closed. If $m \geq n$ is such that $\varepsilon_m \leq \varepsilon$, then the set $\{r_0, \ldots, r_m\}$ is an ε net on B_n. Thus, B_n is totally bounded and therefore, B_n is compact. Evidently $A \subseteq \bigcup_n B_n$.

q.e.d.

Corollary 35 (Hurewicz Theorem). *Assume that D is an analytic subset of a Polish space X such that D is not an F_σ set. Then there exists a closed subset of D homeomorphic to ${}^\omega\omega$.*

4 Covering Properties

A **cover** $\mathcal{U}$ of X is a set of subsets of X such that $\bigcup \mathcal{U} = X$ and $X \notin \mathcal{U}$. A cover $\mathcal{U}$ of X is called **essentially infinite** if no finite subset of $\mathcal{U}$ is a cover of X. A cover $\mathcal{V} \subseteq \mathcal{U}$ is said to be a **subcover** of $\mathcal{U}$. An infinite cover $\mathcal{U}$ is a γ**-cover** if every point $x \in X$ is in all but finitely many sets from $\mathcal{U}$. $\Gamma(X)$ or simply Γ is the family of all open γ-covers of X. Γ_ω is the family of all countable open γ-covers. A cover $\mathcal{U}$ is an ω**-cover** if for every finite $A \subseteq X$ there is a $U \in \mathcal{U}$ such that $A \subseteq U$. $\Omega(X)$ or simply Ω is the family of all open ω-covers of X.

A cover $\mathcal{V}$ is a **refinement** of a cover $\mathcal{U}$ if

$$(\forall V \in \mathcal{V})(\exists U \in \mathcal{U})\, V \subseteq U.$$

A countable open γ-cover $\mathcal{U}$ is **shrinkable** if there exists a closed γ-cover that is a refinement of $\mathcal{U}$. $\Gamma^{sh}(X)$ or simply Γ^{sh} is the family of shrinkable γ-covers of X.

A topological space X is a γ-**space** if from every open ω-cover of X one can choose a γ-subcover.

Let $\mathcal{A}(X)$, $\mathcal{B}(X)$ be families of covers of a topological space X. X is said to be an $\mathrm{S}_1(\mathcal{A}, \mathcal{B})$**-space** if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of covers from $\mathcal{A}(X)$ there exist sets $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega\}$ is a cover belonging to $\mathcal{B}(X)$.

Theorem 36. *Assume that the families $\mathcal{A}(X)$ and $\mathcal{B}(X)$ of open covers have the following property:*

i) *if $\mathcal{V} \in \mathcal{B}(X)$ is a refinement of an open cover $\mathcal{U}$, then there exists a subcover of $\mathcal{U}$ which belongs to $\mathcal{B}(X)$,*

ii) *every two covers of $\mathcal{A}(X)$ have a common refinement which belongs to $\mathcal{A}(X)$.*

Then X is an $\mathrm{S}_1(\mathcal{A}, \mathcal{B})$-space if and only if for every sequence $\{\mathcal{U}_n\}_{n=0}^\infty$ of covers from $\mathcal{A}(X)$ such that $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for every n, there exist sets $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega\}$ is a cover belonging to $\mathcal{B}(X)$.

X is an $\mathrm{U}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$**-space** if for any sequence $\{\mathcal{U}_n\}_{n=0}^\infty$ of essentially infinite $\mathcal{A}$-covers of X there exist finite $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a $\mathcal{B}$-cover.

Theorem 37 (J. Gerlits – Z. Nagy). *A topological space X is a γ-space if and only if X is an $\mathrm{S}_1(\Omega, \Gamma)$-space.*

Proof. Let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of ω-covers such that $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for every n. Choose distinct $\langle x_n \in X : n \in \omega \rangle$. Then

$$\mathcal{U} = \{U \setminus \{x_n\} : U \in \mathcal{U}_n \wedge n \in \omega\}$$

is an ω-cover. Then there exists a γ-subcover $\{V_k : k \in \omega\} \subseteq \mathcal{U}$. Let n_k be such that $V_k = U \setminus \{x_{n_k}\}$, where $U \in \mathcal{U}_{n_k}$. If $\{x_0, \dots, x_n\} \subseteq V_k$, then $n_k > n$. Thus the set $\{n_k : k \in \omega\}$ is infinite and therefore we can assume that the sequence $\{n_k\}_{k=0}^{\infty}$ is increasing and $n_0 = 0$. For any $m < n_k$, $m \geq n_{k-1}$, $k > 0$ take $U_m \in \mathcal{U}_m$ such that $V_k \subseteq U_m \setminus \{x_{n_k}\}$. One can easily see that $\{U_m : m \in \omega\}$ is a γ-cover.

q.e.d.

Corollary 38. *A topological γ-space X is an $\mathrm{S}_1(\Gamma, \Gamma)$-space.*

A topological $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$-space is said to have the **Menger Property**. A topological space $\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma)$-space is said to have the **Hurewicz Property**. A topological $\mathrm{S}_1(\mathcal{O}, \mathcal{O})$-space is said to have the **Rothbereger Property** or to be a **C''-space**.

$$\text{Rothberger} \rightarrow \text{Menger}, \qquad \text{Hurewicz} \rightarrow \text{Menger}.$$

Theorem 39 (D.H. Fremlin – A.W. Miller)**.** *If $\langle X, \mathcal{O} \rangle$ is a separable metrizable topological space, then X has the Rothberger Property if and only if X has strong measure zero with respect to any metric compatible with the topology $\mathcal{O}$.*

Theorem 40 (W. Hurewicz)**.** *Let X be a perfectly normal space. Then*

a) *X has the property H^* if and only if X has the Menger Property,*

b) *X has the property H^{**} if and only if X has the Hurewicz Property.*

Proof. If $\langle \{U_{n,k} : k \in \omega\} : n \in \omega \rangle$ is a sequence of countable open covers of X we take continuous $f_{n,k} : X \longrightarrow [0,1]$ such that $\mathrm{Z}(f_{n,k}) = X \setminus U_{n,k} \neq \emptyset$. Set

$$f_n = \sum_{k=0}^{\infty} 2^{-k} \cdot f_{n,k}.$$

$f_n > 0$ is continuous.

If X has the property H^*, there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of positive reals such that

$$(\forall x \in X)(\forall m)(\exists n > m)\, 1/f_n(x) < a_n.$$

Let k_n be such that

$$\sum_{i > k_n} 2^{-i} < 1/a_n. \tag{9}$$

We denote $\mathcal{V}_n = \{U_{n,i} : i \leq k_n\}$. Then $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a cover of X.

If X has the property H^{**} then there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of positive reals such that $\{1/f_n(x)\}_{n=0}^{\infty} \leq^* \{a_n\}_{n=0}^{\infty}$. Let k_n and $\mathcal{V}_n$ be as above. If $f_n(x) \geq 1/a_n$, then $\sum_{i=0}^{k_n} 2^{-i} \cdot f_{n,i}(x) > 0$ and therefore $x \in \bigcup \mathcal{V}_n$. Then $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a γ-cover of X.

Now, let $\{f_n : X \longrightarrow \mathbb{R}\}_{n=0}^{\infty}$ be a sequence of continuous real functions. Set $U_{n,m} = \{x \in X : |f_n(x)| < m\}$. Then for every n, $\mathcal{U}_n = \{U_{n,m} : m \in \omega\}$ is an open cover of X and $U_{n,m} \subseteq U_{n,m+1}$. Also $\mathcal{U}_{n,k} = \{U_{n,m} : m \geq k\}$ is a cover.

If X has the Menger Property, then there exist finite sets $\mathcal{V}_n \subseteq \mathcal{W}_n = \mathcal{U}_{\rho(n),\lambda(n)}$ (λ, ρ are the inverse functions to the pairing function π), such that $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a cover of X. Since $\bigcup \mathcal{V}_n = U_{\rho(n),m_n}$ for some $m_n \geq \lambda(n)$, we obtain for every $x \in X$ that $|f_n(x)| < m_n$ for infinitely many n.

If X has the Hurewicz Property, then there exists finite subsets $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a γ-cover of X. Then for every $x \in X$ we have $|f_n(x)| < m_n$ for all but finitely many n.

q.e.d.

Corollary 41. *A γ-space has the property* H^{**}.

Theorem 42. *If X has a countable base of the topology and $|X| < \mathfrak{p}$, then X is a γ-space.*

Proof. Let $\mathcal{U}$ be a countable open ω-cover of X. Let

$$\mathcal{U}_x = \{U \in \mathcal{U} : x \in U\}.$$

The family $\{\mathcal{U}_x : x \in X\}$ possesses the finite intersection property. Thus, there exists an infinite set $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \setminus \mathcal{U}_x$ is finite for every $x \in X$. Then $\mathcal{V}$ is desired γ-cover.

q.e.d.

Theorem 43. *There exists a subset A of ${}^{\omega}2$ of cardinality $\mathfrak{p}$ that is not a γ-space.*

Proof. Let $\mathcal{F}$ be a family of subsets of ω of cardinality $\mathfrak{p}$ such that $\mathcal{F}$ has f.i.p. and has no pseudointersection. Let $A \subseteq {}^{\omega}2$ be the set of all characteristic functions of sets from the family $\mathcal{F}$. We claim that A is not a γ-space.

q.e.d.

Theorem 44 (F. Galvin – A.W.Miller)**.** *If $\mathfrak{p} = \mathfrak{c}$, then there exists a γ-space A of cardinality $\mathfrak{c}$.*

Theorem 45 (J. Gerlits – Z. Nagy)**.** *A completely regular topological space X is a γ-space if and only if the topological space $\mathrm{C}_p(X)$ is Fréchet.*

Proof. If X is an infinite completely regular topological space, we fix mutually distinct elements $\langle x_n \in X : n \in \omega\rangle$. For $h \in \mathrm{C}_p(X)$ and $n \in \omega$ we let

$$U_{h,n} = \{x \in X : |h(x)| < 2^{-n} \wedge x \neq x_n\}.$$

Evidently $U_{h,n}$ is an open set.

Assume that X is a γ-space. Let $A \subseteq \mathrm{C}_p(X)$, $f \in \overline{A} \setminus \{f\}$.

The family

$$\mathcal{U} = \{U_{f-g,n} : n \in \omega, g \in A\}$$

is an open ω-cover of X.

If $\mathcal{G} = \{G_k : k \in \omega\}$ is a countable γ-subcover of $\mathcal{U}$, then there are $g_k \in A$ and $n_k \in \omega$ such that

$$G_k = U_{f-g_k,n_k} \text{ for each } k \in \omega.$$

We can assume that $\{n_k\}_{k=0}^{\infty}$ is increasing (otherwise some x_{n_k} does not belong to infinitely many G_k).

We claim that $f = \lim_{k\to\infty} g_k$ on X: if $x \in X$, then $|f(x) - g_k(x)| < 2^{-n_k}$ for all but finitely many k.

Assume now that $\mathrm{C}_p(X)$ is Fréchet space. Let $\mathcal{U}$ be an open ω-cover of X. We set

$$A = \{f \in \mathrm{C}_p(X) : (\exists U \in \mathcal{U})\, \{x \in X : |f(x)| < 1\} \subseteq U\}.$$

Then $0 \in \overline{A}$.

Since $\mathrm{C}_p(X)$ is a Fréchet space, there exists a sequence $\langle f_n \in A : n \in \omega\rangle$ such that $\lim_{n\to\infty} f_n = 0$. By definition of the set A, for every n there exists a set $U_n \in \mathcal{U}$ such that $\{x \in X : |f_n(x)| < 1\} \subseteq U_n$. Then $\{U_n : n \in \omega\}$ is an open γ-cover of X.

q.e.d.

Theorem 46 (L. Bukovský – J. Haleš)**.** *Every* $\mathrm{S}_1(\Gamma^{sh}, \Gamma)$*-space is a* wQN*-space.*

Proof. If $\langle f_n : n \in \omega\rangle$ are continuous, $f_n \to 0$ on X, $f_n) > 0$, we set

$$U_{n,m} = \{x \in X : f_m(x) < 2^{-n}\}, \quad \mathcal{U}_n = \{U_{n,m} : m \in \omega\}. \tag{10}$$

Let $L = \{n \in \omega : X \notin \mathcal{U}_n\}$.

If $\omega \setminus L = \{n_k : k \in \omega\}$ is infinite, then there exist increasing sequences $\{n_k\}_{k=0}^{\infty}$, $\{m_k\}_{k=0}^{\infty}$ such that $U_{n_k,m_k} = X$ for every k. Since $f_{m_k}(x) < 2^{-n_k}$ for every $x \in X$ we obtain that $f_{m_k} \rightrightarrows 0$ on X.

Assume that $\omega \setminus L$ is finite. We can assume that $L = \omega$. Then $\mathcal{U}_n$ is a shrinkable γ-cover for every n. Since X is a $\mathrm{S}_1(\Gamma^{sh}, \Gamma)$-space there exist $V_n \in \mathcal{U}_n$ such that $\{V_n : n \in \omega\}$ is a γ-cover. Let m_n be such that $V_n = U_{n,m_n}$. Tthe sequence $\{m_n\}_{n=0}^{\infty}$ is unbounded and we can assume that $\{m_n\}_{n=0}^{\infty}$ is increasing. Then $f_{m_n} \xrightarrow{\mathrm{QN}} 0$ on X with the control $\{2^{-n}\}_{n=0}^{\infty}$.

q.e.d.

Corollary 47 (M. Scheepers)**.** *Every* $\mathrm{S}_1(\Gamma, \Gamma)$*-space is a* wQN*-space.*

Theorem 48 (L. Bukovský – J. Haleš)**.** *A normal topological space* X *is a* wQN*-space if and only if* X *is an* $\mathrm{S}_1(\Gamma^{sh}, \Gamma)$*-space.*

Conjecture 49 (M. Scheepers)**.** *A normal topological space* X *is a* wQN*-space if and only if* X *is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space.*

Metatheorem 3. *The Scheepers Conjecture* 49 *is consistent with* **ZFC***.*

Theorem 50. *If* $\mathfrak{t} = \mathfrak{b}$*, then there exists a set of reals* $X \subseteq {}^{\omega}2$ *of cardinality* $\mathfrak{b}$ *such that* X *is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space, therefore also a* wQN*-space, and* $X \setminus [\omega]^{<\omega}$ *is not a* wQN*-space. Hence* X *is not a* QN*-space.*

Metatheorem 4. *The equivalences* $\mathrm{QN} = \mathrm{S}_1(\Gamma, \Gamma)$ *and* $\mathrm{QN} = \mathrm{wQN}$ *are undecidable in* **ZFC***.*

5 Borel and Analytic Sets

Let $\langle X, \mathcal{O}\rangle$ be a topological space. $\textsc{Borel}(X,\mathcal{O})$ will denote the smallest σ-algebra of subsets of X containing all open subsets of X.

$$\mathbf{\Sigma}^0_0(X,\mathcal{O}) = \mathbf{\Pi}^0_0(X,\mathcal{O}) = \text{ the set of all clopen subsets of } \langle X, \mathcal{O}\rangle,$$
$$\mathbf{\Sigma}^0_1(X,\mathcal{O}) = \mathcal{O} = \text{ the set of all open subsets of } \langle X, \mathcal{O}\rangle,$$
$$\mathbf{\Pi}^0_1(X,\mathcal{O}) = \text{ the set of all closed subsets of } \langle X, \mathcal{O}\rangle,$$
$$\mathbf{\Sigma}^0_\xi(X,\mathcal{O}) = \left\{\bigcup_n A_n : A_n \in \bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(X,\mathcal{O}), n\in\omega\right\},$$
$$\mathbf{\Pi}^0_\xi(X,\mathcal{O}) = \{X\setminus A : A\in \mathbf{\Sigma}^0_\xi(X,\mathcal{O})\} = \left\{\bigcap_n A_n : A_n \in \bigcup_{\eta<\xi} \mathbf{\Sigma}^0_\eta(X,\mathcal{O}), n\in\omega\right\}$$

Any of the inclusions $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_1 \subseteq \mathbf{\Pi}^0_2$ is equivalent to the condition that $\langle X, \mathcal{O}\rangle$ is perfectly normal.

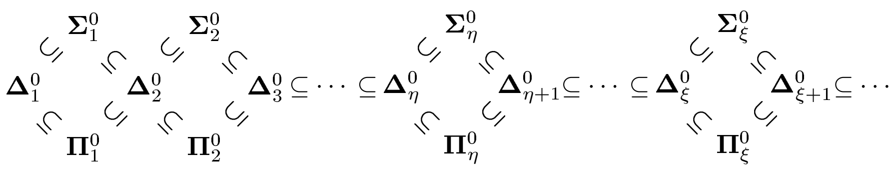
Theorem 51. *If $\langle X, \mathcal{O}\rangle$ is perfectly normal, especially metric, space then*

$$\mathbf{\Sigma}^0_\eta(X,\mathcal{O}) \subseteq \mathbf{\Delta}^0_\xi(X,\mathcal{O}) \subseteq \mathbf{\Sigma}^0_\xi(X,\mathcal{O}), \quad \mathbf{\Pi}^0_\eta(X,\mathcal{O}) \subseteq \mathbf{\Delta}^0_\xi(X,\mathcal{O}) \subseteq \mathbf{\Pi}^0_\xi(X,\mathcal{O})$$

for any $\eta < \xi < \omega_1$. Therefore

$$\textsc{Borel}(X,\mathcal{O}) = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi(X,\mathcal{O}) = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi(X,\mathcal{O}) = \bigcup_{\xi<\omega_1} \mathbf{\Delta}^0_\xi(X,\mathcal{O}).$$

Borel Hierarchy:



For an uncountable Polish space X all inclusions are proper.

Theorem 52. *Any Polish space X is a continuous image of ${}^\omega\omega$.*

Theorem 53. *For any Polish space X there exists a closed set $C \subseteq {}^\omega\omega$ and a continuous bijection $f : C \xrightarrow[\text{onto}]{1-1} X$.*

Theorem 54. *If B is a Borel subset of a Polish space X, then there exist a closed set $C \subseteq {}^\omega\omega$ and a continuous bijection $f : C \xrightarrow[\text{onto}]{1-1} B$.*

Let X be Polish space. $A \subseteq X$ is **analytic** if there exist a Polish space Y, a Borel subset $B \subseteq Y$ and a continuous mapping $f : Y \longrightarrow X$ such that $A = f(B)$. The family of all analytic subsets of the space X is denoted by $\mathbf{\Sigma}^1_1(X)$. A subset $A \subseteq X$ of a Polish space X is called **co-analytic** if $X \setminus A$ is analytic. The family of all co-analytic subsets of the space X is denoted by $\mathbf{\Pi}^1_1(X)$.

Theorem 55. *Let A be a subset of a Polish space X. Then the following are equivalent:*

a) *A is analytic.*

b) *A is sifted by a closed Souslin scheme with vanishing diameter.*

c) *There exists a continuous $f : {}^{\omega}\omega \longrightarrow X$ such that $A = f({}^{\omega}\omega)$.*

d) *There exist a Polish space Y, a Borel subset $B \subseteq Y$ and a continuous surjection $f : B \overset{\text{onto}}{\longrightarrow} A$.*

e) *There exist a Polish space Y and a Borel subset $B \subseteq Y \times X$ such that $A = \text{proj}_2(B)$.*

h) *There exists a closed set $B \subseteq {}^{\omega}\omega \times X$ such that $\text{proj}_2(B) = A$.*

i) *There exists a Borel measurable $f : {}^{\omega}\omega \longrightarrow X$ such that $A = f({}^{\omega}\omega)$.*

Theorem 56. *An uncountable analytic subset of a Polish space contains a perfect subset homeomorphic copy to the Cantor middle-third set. Thus every analytic set is either countable or of cardinality continuum.*

Theorem 57. *There exists a model of* **ZFC***, in which there exists an uncountable co-analytic set that does not contain a perfect subset.*

Theorem 58 (N. N. Luzin)**.** *Assume that $A \subseteq X$ is a Borel subset of a Polish space X. If $f : X \longrightarrow Y$ is continuous and injective on A, then $f(A)$ is Borel.*

We shall call two disjoint subsets $A, B \subseteq X$ **Borel separable** if there exists a Borel set $C \subseteq X$ such that $A \subseteq C$ and $B \subseteq X \setminus C$.

Theorem 59 (Luzin Separation Theorem)**.** *Any two disjoint analytic subsets of a Polish space are Borel separable.*

Corollary 60 (M. J. Souslin)**.** *Let X be a Polish space. If $A \subseteq X$ is both analytic and co-analytic, then A is Borel. Thus*

$$\textsc{Borel} = \mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1. \tag{11}$$

Theorem 61. *Every analytic and therefore every co-analytic set of reals is Lebesgue measurable.*

Lebesgue Decomposition:

A real $x \in [0,1]$ can be identified with a subset $p(x)$ of ω: take an infinite binary expansion $x = \sum_{n=0}^{\infty} x_n 2^{-n=1}$ and let $p(x) = \{n \in \omega : x_n = 1\}$. For any $a \subseteq \omega$, the set $\pi^{-1}(a) \subseteq \omega \times \omega$ is a binary relation on ω. Set

$$L_\xi = \{x \in [0,1] : \langle \omega, \pi^{-1}(p(x)) \rangle \text{ is a well-ordered set of the type } \xi\}.$$

The family

$$\{L_\xi : \omega \leq \xi < \omega_1\} \cup \{[0,1] \setminus \bigcup_{\omega \leq \xi < \omega_1} L_\xi$$

is the the **Lebesgue decomposition** of $[0,1]$.

Theorem 62. *The set $\bigcup_{\omega\le\xi<\omega_1} L_\xi$ is a co-analytic set, that is not analytic. Every L_ξ is a Borel set.*

Moreover, if $A \subseteq \bigcup_{\omega\le\xi<\omega_1} L_\xi$ is analytic, then there is an ordinal $\xi_0 < \omega_1$ such that $A \subseteq \bigcup_{\omega\le\xi<\xi_0} L_\xi$. Thus A is analytic.

The Lebesgue decomposition plays an important role in the investigation of projective sets.

6 Projective Sets

The families of projective sets are defined as follows:

$$\boldsymbol{\Pi}^1_0(X) = \boldsymbol{\Sigma}^1_0(X) = \textsc{Borel}(X), \tag{12}$$

$$\boldsymbol{\Sigma}^1_{n+1}(X) = \exists^{{}^\omega\omega}[\boldsymbol{\Pi}^1_n]({}^\omega\omega \times X), \tag{13}$$

$$\boldsymbol{\Pi}^1_{n+1}(X) = \neg\boldsymbol{\Sigma}^1_{n+1}(X), \tag{14}$$

$$\boldsymbol{\Delta}^1_{n+1}(X) = \boldsymbol{\Sigma}^1_{n+1}(X) \cap \boldsymbol{\Pi}^1_{n+1}(X), \tag{15}$$

for any $0 < n \in \omega$.

Projective Hierarchy

$$\begin{array}{ccccccccccccc}
 & \boldsymbol{\Sigma}^1_1 & & \boldsymbol{\Sigma}^1_2 & & & & & \boldsymbol{\Sigma}^1_n & & & \\
\rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} \;\; \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \\
\boldsymbol{\Delta}^1_1 & & \boldsymbol{\Delta}^1_2 & & \boldsymbol{\Delta}^1_3 & \subseteq \cdots \subseteq & \boldsymbol{\Delta}^1_n & & & & \boldsymbol{\Delta}^1_{n+1} & \subseteq \cdots \\
\rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} \;\; \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \\
 & \boldsymbol{\Pi}^1_1 & & \boldsymbol{\Pi}^1_2 & & & & & \boldsymbol{\Pi}^1_n & & &
\end{array}$$

Theorem 63.

a) *The families $\boldsymbol{\Sigma}^1_n(X)$, $\boldsymbol{\Pi}^1_n(X)$ and $\boldsymbol{\Delta}^1_n(X)$ are closed under Borel measurable inverse images, especially under continuous inverse images.*

b) *Borel measurable image of a $\boldsymbol{\Sigma}^1_n$ set is a $\boldsymbol{\Sigma}^1_n$ set for any $n > 0$.*

c) *The families $\boldsymbol{\Sigma}^1_n(X)$ and $\boldsymbol{\Pi}^1_n(X)$ are closed under countable unions and countable intersections. Thus the family $\boldsymbol{\Delta}^1_n(X)$ is a σ-field of subsets of X.*

d) *If $f : X \xrightarrow[\text{onto}]{1-1} Y$ is a continuous bijection from a Polish space X into a Polish space Y and $A \subseteq X$ is a $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$ or $\boldsymbol{\Delta}^1_n(X)$ set, $n > 0$, then $f(A)$ is a $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$ or $\boldsymbol{\Delta}^1_n(X)$ set, respectively, as well.*

Theorem 64. *There exists a model of* **ZFC**, *in which there exists a $\boldsymbol{\Delta}^1_2(X)$ set, that is not Lebesgue measurable.*

There exists a model of **ZFC**, *in which every $\boldsymbol{\Sigma}^1_2(X)$ set is Lebesgue measurable.*

A family Γ of subsets of X has the **separation property** if for any sets $A_1, A_2 \in \Gamma$, $A_1 \cap A_2 = \emptyset$ there exists a set $B \in \Gamma$ such that $X \setminus B \in \Gamma$, $A_1 \subseteq B$, and $A_2 \cap B = \emptyset$. A family Γ of subsets of X has the **reduction property** if for any sets $A_1, A_2 \in \Gamma$, there exist sets $B_1, B_2 \in \Gamma$ such that $B_1 \cap B_2 = \emptyset$, $B_1 \subseteq A_1$, $B_2 \subseteq A_2$ and $A_1 \cup A_2 = B_1 \cup B_2$.

Luzin Separation Theorem 59 says that $\mathbf{\Sigma}^1_1(X)$ has the separation property.

Theorem 65.

a) *If a family $\Gamma \subseteq \mathcal{P}(X)$ has the reduction property, then the dual family $\neg\Gamma = \{X \setminus A : A \in \Gamma\}$ has the separation property.*

b) *If X is an uncountable Polish space, then for any $n > 0$ neither $\mathbf{\Sigma}^1_n(X)$ nor $\mathbf{\Pi}^1_n(X)$ has both the separation and reduction property.*

Theorem 66. *If X is an uncountable Polish space, then $\mathbf{\Pi}^1_1(X)$ and $\mathbf{\Sigma}^1_2(X)$ possess the reduction property and therefore $\mathbf{\Pi}^1_2(X)$ possesses the separation property.*

What about $\mathbf{\Sigma}^1_3(X)$ and $\mathbf{\Pi}^1_3(X)$?

N.N. Luzin: "*the domain of projective sets is a domain where the principle of excluded third cannot be applied more*".

A boxed family in the next pictures possesses the reduction property.

If every set is constructible (consistent to assume):

$$\begin{array}{cccccccc}
\mathbf{\Sigma}^1_1 & \boxed{\mathbf{\Sigma}^1_2} & \boxed{\mathbf{\Sigma}^1_3} & \cdots & \boxed{\mathbf{\Sigma}^1_{2n-1}} & \boxed{\mathbf{\Sigma}^1_{2n}} & \boxed{\mathbf{\Sigma}^1_{2n+1}} & \cdots \\
\boxed{\mathbf{\Pi}^1_1} & \mathbf{\Pi}^1_2 & \mathbf{\Pi}^1_3 & \cdots & \mathbf{\Pi}^1_{2n-1} & \mathbf{\Pi}^1_{2n} & \mathbf{\Pi}^1_{2n+1} & \cdots
\end{array}$$

Assuming the Axiom of Determinacy:

$$\begin{array}{cccccccc}
\mathbf{\Sigma}^1_1 & \boxed{\mathbf{\Sigma}^1_2} & \mathbf{\Sigma}^1_3 & \cdots & \mathbf{\Sigma}^1_{2n-1} & \boxed{\mathbf{\Sigma}^1_{2n}} & \mathbf{\Sigma}^1_{2n+1} & \cdots \\
\boxed{\mathbf{\Pi}^1_1} & \mathbf{\Pi}^1_2 & \boxed{\mathbf{\Pi}^1_3} & \cdots & \boxed{\mathbf{\Pi}^1_{2n-1}} & \mathbf{\Pi}^1_{2n} & \boxed{\mathbf{\Pi}^1_{2n+1}} & \cdots
\end{array}$$

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