Topological methods in model theory summary

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Abstract

In this series of lectures I present selected topics — examples of topological approach to model theory.

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Model theory has its own language, mostly unknown to common mathematicians. We begin with introducing the basic notions. There are many excellent general textbooks on model theory, both introductory [CK, Sa] and advanced [Ho, Pi, Ba, Ma]. In these lectures we follow the notation established there. The reader may consult them for details.

Throughout, L is a countable language and T is a complete theory in L with infinite models. L denotes also the set of formulas of language L.

If A is a set of parameters in a model of T, $L(A)$ denotes the set of formulas of L with parameters in A. If x is a variable (or a tuple of variables), then $L_x(A)$ denotes the set of formulas in $L(A)$ in variable x. $L_x(A)$ may be viewed as a Boolean algebra (Tarski-Lindenbaum algebra). By a type in T over A in variable x we mean any consistent (with T) set $p(x)$ of formulas in $L_x(A)$. A type is called complete if it is maximal with respect to inclusion. (Complete) types in variable x correspond to (ultra-)filters in the Boolean algebra $L_x(A)$. $S_x(A)$ denotes the set of complete types in T in variable x, over A. $S_x(A)$, considered as the space of ultrafilters, is equipped with the Stone space topology.

Fact 1.1 *If B is a Boolean algebra, then its Stone space is compact and Hausdorff. If moreover B is countable, then $S(B)$ is homeomorphic with a closed subset of the Cantor set 2^ω.*

A model M of T is κ-saturated if for every $A \subseteq M$ of power $< \kappa$, every type in $S_x(A)$ is realized in M.

M is κ-universal if for every model N of T of power $\leq \kappa$, there is an elementary embedding $N \overset{\equiv}{\to} M$.

M is κ-homogeneous if for every partial elementary mapping f between subsets of M of power $< \kappa$, for every $a \in M$, f extends to a partial elementary maping g of M whose domain contains a.

M is strongly κ-homogeneous if every partial elementary mapping f between subsets of M of power $< \kappa$ extends to an automorphism of M.

For $A \subseteq M$, $Aut(M/A)$ denotes the group of automorphisms of M fixing A pointwise. There is a natural topology on $Aut(M)$, where the basis of neighbourhoods of the identity consists of the sets of pointwise stabilizers of finite subsets of M.

Fact 1.2 *$Aut(M)$ is a topological group. If M is countable, $Aut(M)$ is a Polish group.*

A **definable** set in M is any subset of M^n of the form

$$\varphi(M, \overline{a}) = \{\overline{b} \in M^n : M \models \varphi(\overline{b}, \overline{a})\},$$

where $\varphi(\overline{x}, \overline{y})$ is a formula of L and $\overline{a}$ is a tuple of elements of M (called parameters in this context). If $\overline{a} \subseteq A \subseteq M$, we say that $\varphi(M, \overline{a})$ is A-definable. A-definable subsets of M^n form a Boolean algebra of sets $Def^n_A(M)$.

Lemma 1.3 *$Def^n_A(M) \cong L_n(A)$, as Boolean algebras.*

Let $\overline{\kappa}$ be a "large" cardinal (that is, a cardinal larger than any cardinal we really would want to deal with). We say that a model $\mathfrak{C}$ of T is a **monster model**, if $\mathfrak{C}$ is $\overline{\kappa}$-saturated and strongly $\overline{\kappa}$-homogeneous.

Any theory T has a monster model. Any subset of $\mathfrak{C}$ of power $< \overline{\kappa}$ is called small, and we tacitly assume that all sets of parameters we consider and all models (except $\mathfrak{C}$) are small.

$\mathfrak{C}$ is useful because of the following properties.

- Every (small) model of T is isomorphic to an elementary submodel of $\mathfrak{C}$.

- For every (small) $A \subseteq \mathfrak{C}$, the types in $S_x(A)$ correspond precisely to the orbits of $Aut(\mathfrak{C}/A)$.

So if we want to study small submodels of T, we can restrict ourselves to elementary submodels of $\mathfrak{C}$, which in model theory and from now on in these lectures we do.

Lemma 1.4 (Tarski-Vaught Test) *Assume $A \subseteq M$ and*

> $(*)$ *for every formula $\varphi(x) \in L_x(A)$, if $M \models \varphi(a)$ for some $a \in M$, then this holds also for some $a \in A$.*

Then A is the universe of an elementary substructure of M.

Proof. First we show that A is the universe of a substructure N of M. Then, by induction on formulas we prove that for any sentence φ over N, $N \models \varphi \iff M \models \varphi$.

This lemma may be used to construct elementary submodels of M containing a given set of parameters $B \subseteq M$, in a process of gradually adjoining some elements of M to B, so as to satisfy condition $(*)$ from the Tarski-Vaught test. Since any model of T may be redarded an elementary submodel of $\mathfrak{C}$, we see that essentially every model of T arises in this construction.

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If $M = \{a_\alpha : \alpha < \mu\}$, then the isomorphism type of M is determined by the sequence of types $tp(a_\alpha/a_{<\alpha}), \alpha < \mu$, where $a_{<\alpha} = \{a_\beta : \beta < \alpha\}$. So complete types are the real building bricks of any model. The more types there are in a theory, the more freedom there is to construct models of T, and the more complicated these models can be.

Definition 2.1 *(Stability hierarchy) Let $\kappa \geq \aleph_0$.*
(1) We say that T is κ-stable if for every $A \subseteq \mathfrak{C}$ of power $\leq \kappa$, $|S_x(A)| \leq \kappa$.
(2) We say that T is superstable, if for some κ, T is stable in each power $\geq \kappa$.
(3) We say that T is stable if T is stable in some power κ. Otherwise we say T is unstable.

Definition 2.2 *Assume $p(x)$ is a type over $\mathfrak{C}$ and $A \subseteq \mathfrak{C}$.*
(1) We say that $p(x)$ is realized in M if there is some $a \in M$ with $M \models p(a)$. Otherwise we say that p is omitted in M.
(2) We say that $p(x)$ is isolated over A if there is a consistent formula $\varphi(x) \in L_x(A)$ with $\varphi(x) \vdash p(x)$. Otherwise we say that $p(x)$ is non-isolated over A.

Lemma 2.3 *Assume $p(x)$ is a type over M. Then $p(x)$ is realized in some elementary extension N of M.*

Lemma 2.4 *Assume $A \subseteq \mathfrak{C}$ and $p(x)$ is isolated over A. Then $p(x)$ is realized in every model $M \prec \mathfrak{C}$ containing A.*

Fact 2.5 *Assume $p(x) \in S_x(A)$. Then $p(x)$ is isolated over A iff p is an isolated point in the space $S_x(A)$.*

Definition 2.6 *Assume $A \subseteq M \prec \mathfrak{C}$.*
(1) We say that M is atomic over A iff every tuple in M realizes over A an isolated type.
(2) We say that M is prime over A iff for every $N \prec \mathfrak{C}$ containing A, there is an elementary embedding of M to N, fixing A pointwise.

Theorem 2.7 (Omitting Types Theorem) *(1) Assume $A \subseteq \mathfrak{C}$ is countable and $p(x)$ is a non-isolated type over A. Then there is a model $M \preceq \mathfrak{C}$ containing A and omitting $p(x)$.*
(2) Assume $A \subseteq \mathfrak{C}$ is countable, $\kappa < covK$ and $p_\alpha, \alpha < \kappa$, is a family of non-isolated types over A. Then there is a model $M \prec \mathfrak{C}$ containing A and omitting all types p_α.

Lemma 2.8 *Assume A is countable. T has a prime model over A iff for every $n < \omega$, the isolated types are dense in $S_n(A)$. In this case, the prime model over A is unique up to isomorphism. Also a countable model M containing A is prime over A iff M is atomic over A.*

Definition 2.9 *T is κ-categorical (or: categorical in power κ), if up to isomorphism T has just one model of power κ.*

An early result in model theory is the following characterization of $\aleph_0$-categoricity, due (independently) to C.Ryll-Nardzewski, Svenonius and Engeler. The group $Aut(M)$ acts naturally on M, and also on every Cartesian power M^n.

Theorem 2.10 *The following conditions are equivalent.*

1. *T is $\aleph_0$-categorical.*
2. *For every n, L_n (the algebra of formulas of L in n-many variables $x_1, \ldots, x_n$) is finite.*
3. *For every n, $S_n(\emptyset)$ (the space of complete n-types of T, in n-many variables $x_1, \ldots, x_n$) is finite.*
4. *If $M \models T$, then for every $n < \omega$ on M^n there are finitely many orbits of $Aut(M)$.*

The main examples of $\aleph_0$-categorical structures are an atomless Boolean algebra, a dense linear ordering without endpoints and an infinite vector space over a finite field. Another interesting example is a dense circular order.

Lemma 2.11 *Assume T is $\aleph_0$-stable. Then T is κ-stable for every infinite κ.*

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Assume X is a compact Hausdorff topological space. The Cantor-Bendixson derivative X' of X is simply the set of non-isolated points of X. We can iterate this derivation process getting a decreasing sequence $X^{(\alpha)}$ of closed subsets of X defined by:
(1) $X^{(\alpha+1)} = (X^{(\alpha)})'$.
(2) For limit δ, $X^{(\delta)} = \bigcap_{\alpha<\delta} X^{(\alpha)}$.

There is an $\alpha < wt(X)^+$ such that $X^{(\alpha)} = X^{(\alpha+1)} = X^{(\beta)}$ for any $\beta > \alpha$. If the set $X^{(\alpha)}$ is non-empty, it is perfect. It is called the perfect core of X.

Related to the Cantor-Bendixson derivative is the Cantor-Bendixson rank $CB : X \to Ord \cup \{\infty\}$, defined by:

$$CB(p) = min\{\alpha : p \notin X^{(\alpha+1)}\}.$$

For an open non-empty set $U \subseteq X$ we define $CB(U) = min\{\alpha : U \cap X^{(\alpha+1)} = \emptyset\}$. We put $CB(\emptyset) = -1$.

Fact 3.1 $CB(p) = \min\{CB(U) : p \in U \subseteq X\}$.

This fact may be used to define the Morley rank RM of formulas and types, a model-theoretic variant of the CB-rank.

Definition 3.2 *(1) Assume $\varphi(x)$ is a formula over $\mathfrak{C}$. We define $RM(\varphi)$ as $CB([\varphi(x)] \cap S(\mathfrak{C}))$. We put $CB(\emptyset) = -1$.*
(2) Assume $A \subseteq \mathfrak{C}$ and $p(x) \in S(A)$. We define $RM(p)$ as $\min\{RM(\varphi(x)) : \varphi(x) \in p(x)\}$.

Lemma 3.3 *T is $\aleph_0$-stable iff $RM(x = x) < \infty$.*

In fact, in a countable theory the range of RM is always a countable initial segment of the class of ordinals, possibly augmented by infinity.

Applications of Morley rank in an $\aleph_0$-stable theory :

- Definition of forking independence.
- The main tool in geometric model theory.
- The main tool in the proof of Morley categoricity theorem and desctiption of uncountably categorical theories.

In 1964 M.Morley proved the following categoricity theorem, answering a question of Jerzy Los.

Theorem 3.4 *If T is categorical in some uncountable categoricity, then T is categorical in every uncountable categoricity.*

Examples of uncountably categorical structures are: an algebraically closed field, a infinite vector space over a division ring, a set with no structure, any algebraic group.

Here we explain the model-theoretic tools, with which every uncountably categorical structure may be described.

Definition 3.5 *(1) Assume $(I, <)$ is a linearly ordered set of indices. We say that an indexed set $\{a_i : i \in I\} \subseteq \mathfrak{C}$ is order-indiscernible, if for every formula $\varphi(x_1, \ldots, x_n)$ of L and every increasing sequences $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$ of indices from I, we have that*

$$\mathfrak{C} \models \varphi(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow \varphi(a_{j_1}, \ldots, a_{j_n}).$$

Equivalently, $tp(a_{i_1}, \ldots, a_{i_n}) = tp(a_{j_1}, \ldots, a_{j_n})$.

(2) Assume I is a set of indices. We say that an indexed set $\{a_i : i \in I\} \subseteq \mathfrak{C}$ is indiscernible, if for every formula $\varphi(x_1, \ldots, x_n)$ of L and every tuples of distinct indices $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ from I, we have that

$$\mathfrak{C} \models \varphi(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow \varphi(a_{j_1}, \ldots, a_{j_n}).$$

Similarly we define the notion of a set that is (order-)indiscernible over a given set of parameters from $\mathfrak{C}$. The notion of order-indiscernible set was introduced in set theory by Silver, then it was adapted and widely used in model theory. An order indiscernible set indexed by ω (with its natural ordering) is called an indiscernible sequence.

Lemma 3.6 *There is an infinite indiscernible sequence of elements of $\mathfrak{C}$.*

Proof. The proof relies on Ramsey theorem.

Theorem 3.7 *In a stable theory, every infinite order-indiscernible set is an indiscernible set.*

Definition 3.8 *Assume $\varphi(x)$ is a formula (with parameters) such that $RM(\varphi) < \infty$. We define the Morley multiplicity of φ (denoted by $Mlt(\varphi)$) as the number of types in $S(\mathfrak{C}) \cap [\varphi]$ of maximal Morley rank in $S(\mathfrak{C}) \cap [\varphi]$. Similarly we define the multiplicity of any (partial) type.*

It is not hard to see that for any type $p(x)$, $Mlt(p)$ is the minimum of $Mlt(\varphi(x)$, for $\varphi(x)$ such that $p(x) \vdash \varphi(x)$ and $RM(p) = RM(\varphi)$. Also, $Mlt(\varphi)$ is finite.

Alternatively, Morley rank may be defined combinatorially (and this was the approach of Shelah). Namely, we may define RM as the minimal function on formulas with values in $\{-1\} \cup Ord \cup \{\infty\}$ with the following properties.

1. $RM(\varphi(x)) = -1 \iff \varphi(x)$ is inconsistent (with T).
2. For a limit ordinal δ, $RM(\varphi(x)) \geq \delta \iff RM(\varphi(x) \geq \alpha$ for every $\alpha < \delta$.
3. For an ordinal α, $RM(\varphi(x)) \geq \alpha + 1 \iff$ for every $n < \omega$ there are pairwise contradictory consistent formulas $\varphi_i(x), i < n$ implying $\varphi(x)$, with $RM(\varphi_i) \geq \alpha$.

Remark 3.9 *If $p \in S_x(M)$ and $RM(p) < \infty$, then $Mlt(p) = 1$.*

Proof. When M is $\aleph_0$-saturated, the proof is easy. In the general case, the proof is a bit harder.

A type or a formula of Morley rank 1 and Morley multiplicity 1 is called strongly minimal. We have that a formula $\varphi(x)$ is strongly minimal if $\varphi(\mathfrak{C})$ is infinite and for every formula $\psi(x)$ implying $\varphi(x)$, one of the sets $\varphi(\mathfrak{C}) \setminus \psi(\mathfrak{C})$ and $\psi(\mathfrak{C})$ is finite. For example, the formula $x = x$ is strongly minimal in the theory of algebraically closed fields (of a fixed characteristic).

Definition 3.10 *(1) A consistent formula $\varphi(x)$ is called algebraic, if $\varphi(\mathfrak{C})$ is finite.*
(2) For $A \subseteq \mathfrak{C}$ the algebraic closure of A is the set

$$acl(A) = \bigcup\{\varphi(\mathfrak{C}) : \varphi(x) \in L_x(A) \text{ is algebraic }\}.$$

(3) For $A \subseteq \mathfrak{C}$ the definable closure of A is the set

$$dcl(A) = \bigcup\{\varphi(\mathfrak{C}) : \varphi(x) \in L_x(A) \text{ and } |\varphi(\mathfrak{C}) = 1\}.$$

. If $A, B \subseteq \mathfrak{C}$, we put $acl_A(B) = acl(A \cup B)$. Hence, acl_A is a function $\mathcal{P}(\mathfrak{C}) \to \mathcal{P}(\mathfrak{C})$.

Definition 3.11 *Assume $d : \mathcal{P}(X) \to \mathcal{P}(X)$. We say that d is a (combinatorial) pregeometry (or a matroid) on the set X if the following conditions hold.*

1. *If $A \subseteq B$ then $d(a) \subseteq d(B)$.*
2. *$d(d(A)) = d(A)$.*
3. *(finite character) $d(A) = \bigcup\{d(A_0) : A_0 \subseteq_{fin} A\}$.*
4. *(Steinitz exchange principle) If $a \in d(A \cup \{b\}) \setminus acl(A)$, then $b \in d(A \cup \{a\})$.*

Examples of pregeometries are the linear closure in a vector space, the algebraic closure in an algebraically closed field. In every pregeometry (X, d) we have notions of a d-independent set, a basis of X and the dimension of X. In a vector space these are linear independence, linear basis and linear dimension, in an algebraically closed field these are algebraic independence, transcendental basis and transcendental dimension.

Lemma 3.12 *Assume $U \subseteq M$ is a strongly minimal definable set. Then acl is a pregeometry on U.*

Now we can describe models of an uncountably categorical theory T. T is $\aleph_0$-stable (of finite RM-rank). It has a prime model M_0. There is a stongly minimal formula $\varphi(x)$ in T, with parameters in M_0. Every model M is assigned its dimension $dim(M)$, namely the dimension of the acl-pregeometry on the strongyl minimal set $\varphi(M)$. Two models of T are isomorphic iff they have the same dimension.. Moreover, the class of models of T forms an elementary chain, where models with smaller dimension precede models with larger dimension.

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Besides definable sets in M, in model theory we consider a generalization of this notion, namely type-definable sets. Assume $p(x)$ is a collection of formulas in variable x, with parameters in a set $A \subseteq M$. Then

$$p(M) = \{b \in M : M \models \varphi(b, \overline{a}) \text{ for every } \varphi(x, \overline{a}) \in p\} = \bigcap_{\varphi \in p} \varphi(M).$$

We call the set $p(M)$ type-definable (over A). If this set is non-empty, then $p(x)$ is a type. We say that any element of $p(M)$ realizes the type $p(x)$.

Lemma 4.1 *(1) If $U, V \subseteq \mathfrak{C}$ are type-definable, then $U \cup V$ and $U \cap V$ are type-definable.*
(2) If $U \subseteq \mathfrak{C} \times \mathfrak{C}$ is type-definable, then the sets
$\pi[U] = \{a \in U : (\exists b \in \mathfrak{C})\langle a, b\rangle \in U\}$ *and*
$\nu[U] = \{a \in U : (\forall b \in \mathfrak{C})\langle a, b\rangle \in U\}$
are type-definable.

Let $A \subseteq \mathfrak{C}$. We define a trace function $\pi_A : \mathfrak{C} \to S_x(A)$ by:

$$\pi_A(a) = tp(a/A).$$

Lemma 4.2 *The image under π_A of any type-definable set is closed in $S_x(A)$.*

Lemma 4.3 *Assume $A \subseteq \mathfrak{C}$ is small and $U \subseteq \mathfrak{C}$ is A-invariant (that is, $Aut(\mathfrak{C}/A)$-invariant).*
(1) If U is definable, then U is definable over A.
(2) If U is type-definable, then U is type-definable over A.

Proof. The proof uses the trace function.

So A-definable subsets of $\mathfrak{C}$ correspond to clopen subsets of $S_x(A)$, while A-type-definable subsets of $\mathfrak{C}$ correspond to closed subsets of $S_x(A)$.

In model theory, given a model M of T, we adjoin to it its imaginary elements. We describe the procedure now. We call a formula $E(x, y)$ of L an equivalence relation if it defines an equivalence relation in some (equivalently: every) model of T. Here x, y are two tuples of variables of the same length n and given $M \models T$, E^M is an equivalence relation on M^n. Let Eq be the set of all equivalence relations in L.

Fix an $M \models T$. For an equivalence relation $E(x, y)$ in L let M_E be the set of E-classes of elements of M_n and $f_E^M : M^n \to M_E$ be the canonical function mapping a to its E-class a_E.

We work in a many sorted logic. For every equivalence relation E we consider a sort S_E. For the formula $x = y$ (equality) the sort $S_=$ is called the home sort, or the standard sort. In a many sorted logic every variable and every function symbol is assigned a fixed sort.

We define three objects: L^{eq}, M^{eq} and T^{eq}. L^{eq} is a many sorted language, with sorts $S_E, E \in Eq$. We assign the variables and function symbols of L to the standard sort of L^{eq}. Besides that, for every $E(x, y)$ in Eq we include into L^{eq} a function symbol $f_E(x)$ of the sort S_E, where the variables x are of the standard sort.

M^{eq} is a many-sorted structure for L^{eq}. We specify the interpretations of the sorts and symbols of L^{eq} in M^{eq}.

$S_=^{M^{eq}} = M$, and the symbols of L are interpreted in $S_=^{M^{eq}}$ in a natural way.

$S_E^{M^{eq}} = M_E$ and $f_E^{M^{eq}} : M^n \to M_E$ is the natural function.

$T^{eq} = Th(M^{eq})$.

This construction is functorial. T^{eq} does not depend on the choice of $M \models T$. $\mathfrak{C}^{eq}$ is a monster model of T^{eq}. Elements of the non-standard sorts of M^{eq} are called imaginary elements (or: imaginaries), while elements of the standard sort of M^{eq} are called real, or standard. The standard sort of M^{eq} is identified with M. $M^{eq} = dcl(M)$ (via the functions $f_E^{M^{eq}}$), so $Aut(M)$ acts also on M^{eq}. In fact we identify $Aut(M)$ with $Aut(M^{eq})$.

Lemma 4.4 *Assume $U \subseteq \mathfrak{C}$. Then U is definable in $\mathfrak{C}$ iff U is definable in $\mathfrak{C}^{eq}$.*

Proof. Description of formulas of L^{eq}.

Also, inside $(\mathfrak{C}^{eq})^{eq}$, every element of $(\mathfrak{C}^{eq})^{eq}$ is interdefinable with an element of $\mathfrak{C}^{eq}$. So essentially $(\mathfrak{C}^{eq})^{eq} = \mathfrak{C}^{eq}$. From now on we work in $\mathfrak{C} = \mathfrak{C}^{eq}$.

Definition 4.5 *Assume $U \subseteq \mathfrak{C}$ is definable. $c \in \mathfrak{C}$ is a name of U iff $Aut(\mathfrak{C}/c) = Aut(\mathfrak{C}/\{U\})$.*

Notice that if c is a name of U, then U is definable by a formula with parameter c. The main reason for considering the imaginary extension of $\mathfrak{C}$ is the following lemma.

Lemma 4.6 *Every definable set $U \subseteq \mathfrak{C}^{eq}$ has a name in $\mathfrak{C}^{eq}$.*

Proof. For simplicity, consider a set U definable in $\mathfrak{C}$ by a formula $\varphi(x, a)$. The formula

$$E(y, z) = \forall x(\varphi(x, y) \leftrightarrow \varphi(x, z))$$

defines an equivalence relation in $\mathfrak{C}$. $c = a^E$ is a name of U.

$\mathfrak{C}^{eq}$ is extremely useful in stable model theory. We consider also some generalizations of imaginaries. These are hyperimaginaries and ultraimaginaries.

Let us consider a 0-type-definable equivalence relation $E(x, y)$. Here we allow x, y to be tuples of infinite length. We say that E is countable if it is defined by a countable type and x, y are countable tuples of variables.

Assume $E(x, y)$ is an equivalence relation defined by a type $\pi(x, y)$. We can assume $\pi(x, y)$ is closed under conjunction and every formula in it is symmetric.

Lemma 4.7 *(1) For every $\varphi(x, y) \in \pi(x, y)$ there is $\varphi'(x, y) \in \pi(x, y)$ with*

$$\varphi'(x, y) \wedge \varphi'(y, z) \vdash \varphi(x, z).$$

(2) $\pi(x, y)$ may be presented as a union of countable types $\pi_i(x_i, y_i), i \in I$, where each $\pi_i(x_i, y_i)$ defines a equivalence relation $E_i(x_i, y_i)$.

Given a 0-type-definable equivalence relation $E(x, y)$ we call a_E a hyperimaginary element of type E. Let $\mathfrak{C}_E = \{a_E : a \in \mathfrak{C}\}$. $\mathfrak{C}' = \mathfrak{C} \cup \bigcup_E \mathfrak{C}_E$. By Lemma 4.7 we can restrict to countable E here, and then we call a_E a countable hyperimaginary.

Let $\mathfrak{C}^{heq} = \mathfrak{C} \cup \bigcup \{\mathfrak{C}_E : E \text{ is countable}\}$. $Aut(\mathfrak{C})$ acts on $\mathfrak{C}^{heq}$. For $a \in \mathfrak{C}^{heq}$ and $A \subseteq \mathfrak{C}^{heq}$ let $o(a/A)$ be the orbit of a under the action of $Aut(\mathfrak{C}/A)$. We extend the definitions of algebraic and definable closure to $\mathfrak{C}^{heq}$, and also introduce a new kind of closure, the bounded closure $bdd(A)$.

Definition 4.8 *Let $A \subseteq \mathfrak{C}^{heq}$.*

$$acl(A) = \{a \in \mathfrak{C}^{heq} : |o(a/A)| < \aleph_0\}$$

$$dcl(A) = \{a \in \mathfrak{C}^{heq} : |o(a/A)| = 1\}$$

$$bdd(A) = \{a \in \mathfrak{C}^{heq} : |o(a/A)| < \overline{\kappa}\},$$

where $\overline{\kappa}$ is the cardinal such that $\mathfrak{C}$ is $\overline{\kappa}$-saturated and strongly homogeneous.

A further generalization of imaginaries are ultraimaginaries. An ultraimaginary is of the form a_E, where $E(x, y)$ is an equivalence relation invariant under $Aut(\mathfrak{C})$.

Remark 4.9 *A set $U \subseteq \mathfrak{C}$ is invariant over A iff $U = \bigcup_{p \in P} p(\mathfrak{C})$ for some $P \subseteq S(A)$.*

In particular, the number of invariant equivalence relations is bounded.

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In the stable case of particular importance are definable equivalence relations with finitely many classes, called finite equivalence relations. Let $FE(\emptyset)$ denote the family of all finite 0-definable equivalence relations.

Lemma 5.1 *(in $\mathfrak{C}^{eq}$) $acl(\emptyset) = dcl(\bigcup\{\mathfrak{C}_E : E \in FE(\emptyset)\}$.*

Types may be regarded as orbits of $Aut(\mathfrak{C})$. A generalization of these notions are various notions of strong types. These are the orbits of some special subgroups of $Aut(\mathfrak{C})$.

Definition 5.2 *$Autf_{Sh}(\mathfrak{C})$ is the group of automorphisms of $\mathfrak{C}$ preserving pointise $\mathfrak{C}_E$ for every $E \in FE(\emptyset)$. Its elements are called strong (Shelah) automorphisms of $\mathfrak{C}$ and its orbits are called strong (Shelah) types.*

Lemma 5.3 *(1)* $Autf_{Sh}(\mathfrak{C}) = Aut(\mathfrak{C}/acl^{eq}(\emptyset))$.
(2) a and b have the same strong (Shelah) type iff $tp(a/acl^{eq}(\emptyset)) = tp(b/acl^{eq}(\emptyset))$.

Hence $tp(a/acl^{eq}(\emptyset))$ is denoted by $stp_{Sh}(a)$. We see that there is a natural bijection between the quotient group $Aut(\mathfrak{C})/Autf_{Sh}(\mathfrak{C})$ and the group of elementary permutations of $acl^{eq}(\emptyset)$. The group $Aut(\mathfrak{C})/Autf_{Sh}(\mathfrak{C})$ is called the Galois group of T for Shelah strong types and is denoted by $Gal_{Sh}(T)$. It does not depend on the choice of the monster model.

In the case of type-definable or invariant equivalence relations strong Shelah types correspond to strong Kim-Pillay types and strong Lascar types, respectively. We shall describe them now.

First we deal with the type-definable equivalence relations. We say that a type-definable equivalence relation is bounded if it has $< \overline{\kappa}$-many classes.

Lemma 5.4 *A type-definable equivalence relation is bounded iff it has* $\leq 2^{\aleph_0}$*-many classes.*

Assume E is a bounded type-definable equivalence relation. Consider the natural function $\pi_E : \mathfrak{C} \to \mathfrak{C}_E$. We define a logic topology on $\mathfrak{C}_E$ by declaring a set $X \subseteq \mathfrak{C}_E$ closed iff $\pi_E^{-1}[X]$ is type-definable in $\mathfrak{C}$.

Lemma 5.5 *(1)* $\mathfrak{C}_E$ *is a compact Hausdorff space.*
(2) $f_E[U]$ *is closed for any type-definable set* $U \subseteq \mathfrak{C}$.

However, unlike in the case of Shelah strong types or usual types, the topologies that arise this way need not be 0-dimensional. Every compact Polish space any occur here. For example, a unit circle.

There is a unique finest bounded type-definable equivalence relation. It is denoted by E_{KP}.

Lemma 5.6 *Assume* $f \in Aut(\mathfrak{C})$. *Then* f *preserves* E*-classes of every bounded type-definable equivalence relation* E *iff* f *preserves every* E_{KP}*-class.*

Definition 5.7 $Autf_{KP}(\mathfrak{C})$ *is the group of those* $f \in Aut(\mathfrak{C})$ *that preserve every* E_{KP}*-class. It is called the group of strong Kim-Pillay automorphisms of* $\mathfrak{C}$.

Lemma 5.8 *(1)* $bdd(\emptyset) = dcl(\mathfrak{C}_{E_{KP}})$.
(2) The orbits of $Autf_{KP}(\mathfrak{C})$ *are exactly the classes of* E_{KP}*. They are called strong Kim-Pillay types.*

The set $\mathfrak{C}_{E_{KP}}$ is a compact Hausdorff space. The quotient group $Aut(\mathfrak{C})/Autf_{KP}(\mathfrak{C})$ is called the Galois group of T for Kim-Pillay strong types and is denoted by $Gal_{KP}(T)$. The group $Gal_{KP}(T)$ may be identified with the group of all elementary permutations of $\mathfrak{C}_{E_{KP}}$. All such permutations are continuous and the group of such permutations is a compact topological group.

In the case of a stable T, strong Shelah types and strong Kim-Pillay types coincide and $Autf_{Sh}(\mathfrak{C}) = Autf_{KP}(\mathfrak{C})$. In the unstable case it is no longer so, for example see the random graph.

Now we turn to the invariant equivalence relations and Lascar strong types. Just like in the case of type-definable equivalence relations we have that there is a finest invariant equivalence relation with boundedly many classes. It is denoted by E_{Ls}, its classes are called Lascar strong types. We can describe E_{Ls} more constructively.

We define two type-definable relations θ and Φ on $\mathfrak{C}$ by putting

> $\theta(a,b)$ iff $a = a_0$ and $b = a_1$ for some indiscernible sequence $a_0, a_1, a_2, \ldots$ in $\mathfrak{C}$.
>
> $\Phi(a,b)$ iff $tp(a/M) = tp(b/M)$ for some $M \prec \mathfrak{C}$.

Theorem 5.9 *(1) θ and Φ are type-definable relations.*
(2) E_{Ls} is the transitive closure of each of θ and Φ

Fix a countable model $M \prec \mathfrak{C}$. We see that E_{Ls} is finer than the equality of types over M. Hence there is an equivalence relation E on the space $S(M)$ such that E_{Ls} is the lifting of E. The space $S(M)$ is Polish and E is an F_σ-relation. This brings descriptive set theory into the subject and the questions on Borel cardinality of the set of Lascar strong types.

Let $Autf_{Ls}(\mathfrak{C})$ be the group of those $f \in Aut(\mathfrak{C})$ that preserve every Lascar strong type. Its elements are called strong Lascar automorphisms of $\mathfrak{C}$. Just as in the case of Kim-Pillay strong types we have the following lemma.

Lemma 5.10 *(1) Strong Lascar types are exactly the orbits of $Autf_{Ls}(\mathfrak{C})$.*
(2) $Autf_{Ls}(\mathfrak{C})$ is generated (as a group) by $\bigcup\{Aut(\mathfrak{C}/M) : M \prec \mathfrak{C}\}$.

Also we define $Gal_{Ls}(T)$, the Galois group of T for Lascar strong types, as the quotient group $Aut(\mathfrak{C})/Autf_{Ls}(\mathfrak{C})$. It may be endowed with a compact topology, which need not be Hausdorff, however.

A theory where $E_{KP} = E_{Ls}$ is called G-compact. Stable theories and simple theories are G-compact. For a long time no example was known of a theory that is not G-cpmpact. The first example was given by Martin Ziegler in 2001. Still, the structure of Lascar strong types remained mysterious. This was explained by L.Newelski [Ne].

Namely, for $a, b \in \mathfrak{C}$ let $d(a, b)$ be the minimal n such that there ate $a_0 = a, a_1, \ldots, a_n = b$ such that $\theta(a_i, a_{i+1})$ for all i. If no such n exists, we put $d(a, b) = \infty$. We define the diameter of the Lascar strong type of a as

$$diam(a_{E_{Ls}}) = \sup\{d(a, b) : b \in a_{E_{Ls}}\}.$$

Theorem 5.11 *(1) $a_{E_{Ls}} = a_{E_{KP}}$ iff $diam(a_{Ls})$ is finite.*
(2) T is G-compact iff every Lascar strong type has finite diameter. In this case $Autf_{Ls}(\mathfrak{C})$ is generated by $\bigcup\{Aut(\mathfrak{C}/M) : M \prec \mathfrak{C}\}$ in finitely many steps.
(3) If $a_{E_{Ls}} \neq a_{E_{KP}}$, then $a_{E_{Ls}}$ is the union of at least $2^{\aleph_0}$ many Kim-Pillay strong types. and $[Autf_{KP}(\mathfrak{C}) : Autf_{Ls}(\mathfrak{C})] \geq 2^{\aleph_0}$.

The new methods used in the proof opened new directions of research and led to applications of topological dynamics in model theory.

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