RELATIVE RANKS OF SEMIGROUPS OF MAPPINGS;

GENERATING CONTINUOUS MAPS WITH LIPSCHITZ FUNCTIONS

MICHAŁ MORAYNE

INSTYTUT MATEMATYKI I INFORMATYKI

POLITECHNIKA WROCŁAWSKA

WYKŁADY SSDNM

INSTYTUT MATEMATYKI

UNIWERSYTET ŚLĄSKI

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LECTURE 1 (3h)

SIERPIŃSKI'S THEOREM

A *semigroup* is an algebraic structure $(S, *)$ where $*$ is an associative binary operation on elements of S. A *monoid* is a semigroup with an identity element. If $A \subseteq S$, by $\langle A \rangle$ we denote the subsemigroup generated by A. In these series of lectures we will consider only monoids of mappings where the monoid operation will always be the composition of mappings. The starting point for the theory of monoids of mappings was the following Sierpiński's theorem. It is still one of the most important facts of this theory.

Theorem 1.1 (Sierpiński) *Let A be an infinite set. Let $f_i : A \to A$, $i = 1, 2, \ldots$. There exist two mappings $\varphi, \xi : A \to A$ such that*

$$\{f_i : i \in \mathbb{N}\} \subseteq \langle \{\varphi, \xi\} \rangle .$$

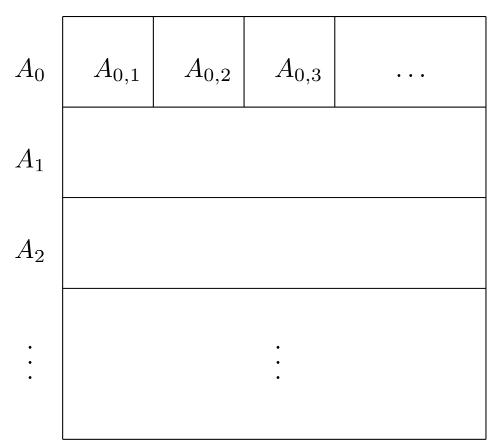
Proof (Banach). Let

$$A = \bigcup_{i=0}^{\infty} A_i$$

be a partition of A into sets of the same cardinality. Let

$$A_0 = \bigcup_{i=1}^{\infty} A_{0,i}$$

be a partition of A_0 into sets of the same cardinality.



First let us define φ. Namely, let it be any mapping that is a bijection from A_n onto A_{n+1}, for every n.

Now let us define ξ. First we define ξ on $A \setminus A_0 = A_1 \cup A_2 \cup \ldots$. Namely, let $\xi|A_n$, $n > 0$, be any mapping that is a bijection from A_n onto $A_{0,n}$.

Let us notice that so far none of the two mappings φ and ξ depends in any way on the mappings $f_1, f_2, \ldots$. Thus the only place where it can happen is the remaining part of the definition of ξ, namely $\xi|A_0$. Before we complete this definition let us notice that the mapping

$$\gamma_n = \xi \circ \varphi^n \circ \xi \circ \varphi$$

is a bijection of A onto $A_{0,n}$.

We want to encode the function f_n on $A_{0,n}$ to ensure that $\xi\gamma_n = f_n$. We can do this defining

$$\xi|A_{0,n} = f_n \circ \gamma_n^{-1}.$$

(Note that we can do this because γ_n is a bijection.) Finally,

$$f_n = \xi^2 \circ \varphi^n \circ \xi \circ \varphi$$

for each $n = 1, 2, \ldots$. This completes the proof. □

Remark In general, the conclusion of Sierpiński's theorem cannot be strengthened to get all mappings f_n, $n \in \mathbb{N}$, as iterations of a single mapping $\psi : A \to A$. Indeed, just let us consider two mappings f_1 and f_2 where f_1 is a bijection and f_2 is not. Assume that $f_1 = \psi^n$ and $f_2 = \psi^m$. $f_1 = \psi^n$ implies that ψ is a bijection and $f_2 = \psi^m$ implies that ψ is not a bijection.

Let S be a semigroup and $A \subseteq S$. The *relative rank of S with respect to A* is the cardinal number

$$\operatorname{rank}(S : A) = \min\{|B| : \langle A \cup B\rangle = S\}.$$

Corollary 1.2 *Let A be any infinite set. Let $\mathcal{V} \subseteq A^A$. Then* $\operatorname{rank}(A^A : \mathcal{V}) = 0, 1, 2$ *or is uncountable.*

Proof. Assume that $\operatorname{rank}(A^A : \mathcal{V})$ is countable. It means that there exists a countable family $\mathcal{F} = \{f_1, f_2, \ldots\}$ of mappings from A^A such that $\langle \mathcal{V} \cup \mathcal{F}\rangle = A^A$. By Sierpiński's theorem we can find two mappings $\varphi, \xi \in A^A$ such that $\mathcal{F} \subseteq \langle\{\varphi, \xi\}\rangle$, whence $\langle \mathcal{V} \cup \{\varphi, \xi\}\rangle = A^A$. □

There are two groups of problems that arise naturally here. Once we know the rank is finite we may ask which one of these three possibilities $0, 1, 2$ holds. It is sometimes a difficult and challenging problem. The whole new realm of problems start with trying to make precise the word *uncountable* in the above conclusion. We will discuss several such problems for various monoids of mappings, also going beyond the monoid A^A.

The following two propositions provide examples of submonoids of $\mathbb{N}^{\mathbb{N}}$ of cardinality continuum with very different relative ranks.

Let us define a submonoid $\mathcal{O}_{\mathbb{N}}$ to consist of all those mappings $g \in \mathbb{N}^{\mathbb{N}}$ that are weakly order preserving i.e. $m \leq n$ implies $g(m) \leq g(n)$.

Proposition 1.3 *We have*

$$\text{rank}(\mathbb{N}^{\mathbb{N}} : \mathcal{O}_{\mathbb{N}}) = 1.$$

Proof. Let $\phi : \mathbb{N} \to \mathbb{N}$ be such a mapping that the set $\varphi^{-1}[\{i\}]$ is infinite for each $i \in \mathbb{N}$. Let now f be any mapping from $\mathbb{N}^{\mathbb{N}}$. Let us define $g \in \mathcal{O}_{\mathbb{N}}$. We do it inductively. Let $g(1) \in \varphi^{-1}[\{f(1)\}]$. Let $g(2) \in \varphi^{-1}[\{f(2)\}]$ and $g(2) \geq g(1)$. It is possible to find such a $g(2)$ because the set $\varphi^{-1}[\{f(2)\}]$ is infinite. Once $g(1) < g(2) < \dots g(n-1)$ have been defined we define $g(n) \in \varphi^{-1}[\{f(n)\}]$ and $g(n) \geq g(n-1)$. It is possible because the set $\varphi^{-1}[\{f(n)\}]$ is infinite. Now we have

$$\varphi \circ g(n) = f(n).$$

□

Before we state and prove the next proposition let us recall the notion of an almost disjoint family of sets. We say that a family of sets $\mathcal{A}$ is almost disjoint if any two of its elements have finite intersection. We will need the following two lemmas.

Lemma 1.4 *If $|A| = \aleph_0$ then there is an almost disjoint family of subsets of A of cardinality continuum $\mathfrak{c} = 2^{\aleph_0}$.*

Proof. Of course, it is enough to provide a proof for any A satisfying the hypothesis. We choose

$$A = \bigcup_{n=1}^{\infty} \{0,1\}^{\{1,2,\dots,n\}},$$

(i.e. the set of all vertices of the infinite complete binary tree). Now our family is the family of all branches of the tree, i.e. the family whose elements are sequences of functions $(\sigma_n)_n$, $\sigma_n : \{1,2,\dots,n\} \to \{0,1\}$, where $\sigma_n|\{1,2,\dots,m\} = \sigma_m$ for any $m < n$. □

Lemma 1.5 *Let $k \in \mathbb{N}$. Let $A_1, A_2, \dots$ be subsets of $\mathbb{N}$ each of cardinality not exceeding k. Then there is no uncountable family $\mathcal{F}$ of functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(i) \in A_i$ for each $i \in \mathbb{N}$, such that for any two functions $f, g \in \mathcal{F}$ the set $\{i : f(i) = g(i)\}$ is finite.*

Proof. The proof will be an induction with respect to k. For $k = 1$ the conclusion is obvious. Assume that the conclusion is true for some $k \in \mathbb{N}$ and that sets

$A_1, A_2, \ldots$ are of cardinalities not exceeding $k+1$. Assume for contradiction that such an uncountable family $\mathcal{F}$ exists. Let us take one function $f \in \mathcal{F}$. Each function g from $\mathcal{F}$ has now diiferent values than f from some point on, say n_g. Thus

$$\mathcal{F} = \bigcup_{i=1}^{\infty} \{g \in \mathcal{F} : n_g = i\},$$

whence there must be $i_0 \in \mathbb{N}$ such that the set G=$\{g \in \mathcal{F} : n_g = i_0\}$ is uncountable. Of course, the family $\{g | \{i_0+1, i_0+2, \ldots\} : g \in \mathcal{G}\}$ satisfies now the induction hypothesis for k and thus it cannot be uncountable. $\square$

Let $\mathcal{S}_2 \subseteq \mathbb{N}^{\mathbb{N}}$ consist of all the functions f such that

$$f[\{2i-1, 2i\}] \subseteq \{2i-1, 2i\}$$

for all $i \in \mathbb{N}$.

Proposition 1.6 *We have*

$$\mathrm{rank}(\mathbb{N}^{\mathbb{N}} : \mathcal{S}_2) = \mathfrak{c}.$$

Proof. For contradiction assume that there exists $\mathcal{F} \in \mathbb{N}^{\mathbb{N}}$ such that $\langle \mathcal{S}_2 \cup \mathcal{F} \rangle = \mathbb{N}^{\mathbb{N}}$ and $|\mathcal{F}| = \kappa < \mathfrak{c}$. Let $f_1, f_2, \ldots, f_n$ be any functions from $\mathcal{F}$. Let

$$\mathcal{F}_{(f_1, f_2, \ldots, f_n)} = \{\varphi_{n+1} \circ f_n \circ \varphi_n \cdots \circ f_1 \circ \varphi_1 : \varphi_1, \ldots, \varphi_{n+1} \in \mathcal{S}_2\}.$$

Now we are going to prove via induction with respect to n that the family $\mathcal{F}_{(f_1, f_2, \ldots, f_n)}$ satisfies the hypothesis of Lemma 1.5. Let $n = 0$. Then the claim is satisfied with $k = 2$ because we deal only with $\varphi_1 \in \mathcal{S}_2$. Assume now that the claim is satisfied for $n \in \mathbb{N} \cup \{0\}$. Thus

$$\varphi_n \circ f_{n-1} \cdots \circ f_1 \circ \varphi_1(i) \in A'_i$$

for some $|A_i| \leq k$. Then

$$f_n \circ \varphi_n \cdots \circ f_1 \circ \varphi_1 : \varphi_1(i) \in f_n[A'_i]$$

and $|f_n[A'_i]| \leq |A'_i|$. By the properties of mappings from $\mathcal{S}_2$

$$\varphi_{n+1}[f_n[A'_i]] \subseteq A_i,$$

where

$$|A_i| \leq 2|f_n[A'_i]| \leq 2|A'_i| \leq 2k.$$

and the claim is proved. By Lemma 1.4 there exists an almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$. Let $A \in \mathcal{A}$. Let $f_A : \mathbb{N} \to A$ be any strictly increasing function. The set $\{i : f_A(i) = f_B(i)\}$ is finite for any sets $A, B \in \mathcal{A}$ and any

$i \in \mathbb{N}$, for otherwise the sets A and B would have infinite intersection which would contradict almost disjointness of $\mathcal{A}$. Thus, by Lemma 2, only countably many functions f_A, $A \in \mathcal{A}$ may belong to $\mathcal{F}_{(f_1,f_2,\ldots,f_n)}$. As there are only κ families $\mathcal{F}_{(f_1,f_2,\ldots,f_n)}$ and $\mathfrak{c}$ functions f_A some functions f_A do not belong to any family $\mathcal{F}_{(f_1,f_2,\ldots,f_n)}$ and thus to $\langle \mathcal{S}_2 \cup \mathcal{F} \rangle = \mathbb{N}^{\mathbb{N}}$. $\square$

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LECTURE 2 (6h)

UNIVERSAL SETS AND FUNCTIONS

The Baire space is the set $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ endowed with the metric d defined as follows:

$$d((m_i)_{i\in\mathbb{N}}, (n_i)_{i\in\mathbb{N}}) = \frac{1}{\min\{i : m_i \neq n_i\}}.$$

For considering the relative rank of the monoid of continuous mappings on the Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ with respect to the monoid of Lipschitz mappings on $\mathcal{N}$ we need several facts about the existence of so called universal functions (some of them standard and some not quite so standard).

Let us first introduce some notation.

Let X be any set. By $\mathbf{P}(A)$ we denote the power set of X i.e. the family of all subsets of X.

Let X, Y be any sets. For $A \subseteq Y \times X$ and $y \in Y$ by A_y we denote the *vertical section of A given by y* i.e. the set $\{x \in X : (y, x) \in A\}$. Let $\mathcal{A} \subseteq \mathbf{P}(\mathrm{A})$. We say that a set $A \subseteq Y \times X$ is *universal* for $\mathcal{A}$ if $\mathcal{A} \subseteq \{A_y : y \in Y\}$.

Let now X be a topological space. By $\mathbf{\Sigma}^0_1(X)$ we denote the family of all open subsets of X (the topology on X) and by $\mathbf{\Pi}^0_1(X)$ the family of all closed subsets of X.

First we prove the following (classical and fundamental) lemma.

Lemma 2.1 *Let X be a second countable topological space and Y be a topological space containing a topological copy C of a Cantor set $\{0,1\}^{\mathbb{N}}$. Then there exists $A \in \mathbf{\Sigma}^0_1(Y \times X)$ universal for $\mathbf{\Sigma}^0_1(X)$.*

Proof. By the hypothesis there exists a homeomorphism $\varphi : \{0,1\}^{\mathbb{N}} \to C \subseteq Y$. Let $\{U_1, U_2, \ldots\}$ be a base for the topology on X. We define now $A \subseteq Y \times X$ as follows:

$$(y, x) \in A \quad \Leftrightarrow \quad (y \notin C) \text{ or } \Big(x \in \bigcup\{U_i : y = \varphi(t) \text{ and } t_i = 1\}\Big).$$

We claim that A is open in $Y \times X$. Indeed, if $y \notin C$ then $(y, x) \in (Y \setminus C) \times X \subseteq A$ and thus we have found an open neighbourhood of (y, x), namely $(Y \setminus C) \times X$, that is in A. Now assume that $(y, x) \in A$ and $y \in C$. Thus $\varphi(t) = y$, and $x \in U_k$ and $t_k = 1$ for some $k \in \mathbb{N}$. But then also $s_k = 1$ for all $s \in V$, where V is a certain open neighbourhood of t. Then $\varphi[V]$ is an open neighbourhood of y in C and thus $\varphi[V] = C \cap W$ for some set W open in Y. As $(u, r) \in A$ for all $u \in \phi[V]$ and $r \in U_k$ and $(u, r) \in A$ for all $u \in Y \setminus C$ yield $(y, x) \in W \times U_k \subseteq A$ we again found an open neighbourhood of (y, x) in A. □

Let X, Y, Z be any sets. Let $\mathcal{F} \subseteq Z^X$. We say that a mapping $\Phi : Y \times X \to$ is *universal* for $\mathcal{F}$ if $\mathcal{F} \subseteq \{\Phi(y, \cdot) : y \in Y\}$.

Let now X be a topological space. A function $f : X \to \mathbb{R}$ is *lower* (*upper*) *semicontinuous* if for very $\alpha \in \mathbb{R}$ the set $f^{-1}[(\alpha, \infty)]$ ($f^{-1}[(-\infty, \alpha)]$) is open in X. Let $LSC(X)$ ($USC(X)$) denote the class of all lower (upper) semicontinous functions from X into the interval $[0, 1]$.

The following four lemmas will play an important technical role.

Lemma 2.2 *Let X, Y be topological spaces, $\varphi : X \to Y$ be a continuous mapping, and $f : Y \to \mathbb{R}$ be a lower semicontinous function. Then $\varphi \circ f : X \to \mathbb{R}$ is also a lower semicontinuous function.*

Proof. Let $\alpha \in \mathbb{R}$. We have

$$(\varphi \circ f)^{-1}[(\alpha, \infty)] = f^{-1}[\varphi^{-1}[(\alpha, \infty)]].$$

Because φ is lower semicontinuous the set $\varphi^{-1}[(\alpha, \infty)]$ is open in Y, and because f is continuous the set $f^{-1}[\varphi^{-1}[(\alpha, \infty)]]$ is open in X. □

Lemma 2.3 *Let X be a topological space and $\mathcal{F} \subseteq LSC(X)$. Then* $\sup \mathcal{F} \in LSC(X)$.

Proof. Let $\alpha \in [0, 1]$. We have

$$(\sup \mathcal{F})^{-1}[(\alpha, 1]] = \bigcup_{f \in \mathcal{F}} f^{-1}[(\alpha, 1]],$$

and this set is open in X because it is a union of open sets. □

Lemma 2.4 *Let X be a topological space. Let $f \in LSC(X)$. Then $f = \sup \mathcal{F}$, where $\mathcal{F}$ is a family of functions of the form $q_n \chi_{U_n}$ where $\{q_1, q_2, \ldots\}$ is a certain enumeration of $\mathbb{Q} \cap [0, 1]$ and U_n are open sets.*

Proof. It is easy to check that $f = \sup\{q_n \chi_{f^{-1}[(q_n,1]]}\}$. □

Lemma 2.5 *Let X be a topological space and $A \in X$. Let $f \in LSC(A)$, where A is equipped with the relative topology inherited from X. Then f can be extended to $f^* \in LSC(X)$.*

Proof. Let U be an open set in A. Then $U = A \cap U^*$, where U^* is an open set in X. By Lemma 2.4 the function f can be expressed as $f = \sup \mathcal{F}$, where $\mathcal{F}$ is a countable family of functions of the form $q\chi_U$ where $q \in \mathbb{Q} \cap [0, 1]$ and U is open in A. Let

$$f^* = \sup\{q\chi_{U^*} : q\chi_B \in \mathcal{F}\}.$$

Lemma 2.3 implies $f^* \in LSC(X)$. Simple checking shows that $f^*|A = f$. □

Theorem 2.6 *Let X be a second countable topological space and Y be any topological space containing a topological copy of the Cantor set $\{0, 1\}^{\mathbb{N}}$. Then there exists a function $F \in LSC(Y \times X)$ universal for $LSC(X)$.*

Proof. The set $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ is homeomorphic to the Cantor set. To avoid technicalities assume simply that $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \subseteq Y$. By Theorem 2.1 there exists a set $V \in \mathbf{\Sigma}^0_1(\{0,1\}^{\mathbb{N}} \times X)$ which is universal for $\mathbf{\Sigma}^0_1(X)$. Let $q_1, q_2, \ldots$ be an enumeration of the elements of $\mathbb{Q} \cap [0,1]$. For $t = (t^{(n)})_n \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ and $x \in X$, let $\Phi : (\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \times X :\rightarrow \mathbb{R}$ be defined as follows:

$$\Phi(t,x) = \sup\{q_n \chi_{V_{t^{(n)}}}(x) : n \in \mathbb{N}\}.$$

The function Φ is lower semicontinuous. To show this it is enough, by Lemma 2.3, to show that $(t,x) \longmapsto \chi_{V_{t^{(n)}}}$ is lower semicontinuous. We have

$$\chi_{V_{t^{(n)}}}(x) = \chi_V(\pi_n(t), x) = \chi_V(\Theta(t,x)),$$

where $\Theta(t,x) = (\pi_n(t), x)$. Obviously, $\Theta : (\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \times X \to \{0,1\}^{\mathbb{N}} \times X$ is a continuous mapping. Thus by Lemma 2.2 $\chi_{U_{t^{(n)}}}$ is lower semicontinuous, and therefore so is Φ.

Now we will show that Φ is universal for $LSC(X)$. Let $f \in LSC(X)$. By Lemma 2.4 we can express f as

$$f = \sup\{q_n \chi_{U_n} : n \in \mathbb{N}\},$$

where U_n's are some open sets in X. Let $t = (t^{(n)})_n \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ satisfy

$$V_{t^{(n)}} = U_n.$$

Thus we have for this t:

$$f(\cdot) = \Phi(t, \cdot),$$

and this shows that Φ is indeed universal for $LSC(X)$. Finally, by Lemma 2.5 Φ can be extended to $F \in LSC(Y \times X)$ which is, of course, also universal for $LSC(X)$. □

Let (X,d) be a metric space. Let $\mathcal{C}(X)$ denote the monoid of all continuous mappings from X to X, where the monoid operation is the composition of functions. Let $\mathcal{L}(X)$ denote the submonoid of $\mathcal{C}(X)$ consisting of all Lipschitz mappings from X to X, i.e. the mappings f satisfying $d(f(x), f(y)) \le Cd(x,y)$, where C is a constant depending on f.

Proposition 2.7 *There is no continuous mapping from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$ that is universal for $C(\mathcal{N})$.*

Proof. Assume that there is such a mapping, call it F. Then the diagonal mapping $f(x) = F(x,x)$ is continuous. For each mapping $g \in C(\mathcal{N})$ there exists $x \in \mathcal{N}$ such that $g(\cdot) = F(x, \cdot)$, and thus $g(x) = F(x,x) = f(x)$. Consider now the function $g \in C(\mathcal{N})$ defined as $g(x) = ((f(x))_1 + 1, (f(x))_2, \ldots)$. it is obvious that $g(x) \neq f(x)$ for any $x \in \mathcal{N}$. It is a contradiction. □

Let X, Y, Z be any sets. Let $\mathcal{F} \subseteq Z^X$. We say that a mapping $F : A \to Z$, $A \subseteq Y \times X$, *is universal for $\mathcal{F}$ in a generalized sense* if $\mathcal{F} \subseteq \{F(y, \cdot) : y \in Y\}$.

The word "generalized" used here refers to the fact that the domain of F is not the whole $Y \times X$. Note, however, that the domain of functions which we want to realize as sections of F is the whole X.

Theorem 2.8 *There exists a Borel subset $\mathfrak{B}$ of $\mathcal{N}\times\mathcal{N}$ and a continuous mapping $F : \mathfrak{B} \to \mathcal{N}$ universal for $\mathcal{C}(\mathcal{N})$ in a generalized sense.*

Proof. Let $\psi : \mathcal{N} \to \mathcal{N}\times\mathcal{N}$ and $\xi : [0,1]\setminus\mathbb{Q} \to \mathcal{N}$ be homeomorphisms. Let $L \in LSC(\mathcal{N}\times\mathcal{N})$ and $U \in USC(\mathcal{N}\times\mathcal{N})$ be universal for $LSC(\mathcal{N})$ and $USC(\mathcal{N})$, respectively. Let $(y,x) \in \mathfrak{B}$ if $L((\psi(y))_1, x) = U((\psi(y))_2, x) \in [0,1]\setminus\mathbb{Q}$. In other words

$$\mathfrak{B} = G^{-1}(\{0\}) \cap \bar{L}^{-1}[[0,1]\setminus\mathbb{Q}],$$

where $\bar{L}(y,x) = L((\psi(y))_1, x)$ and $G = \bar{L} - \bar{U}$, for $\bar{U}(y,x) = U((\psi(y))_2, x)$. Thus it is obvious that $\mathfrak{B}$ is a Borel subset of $\mathcal{N}\times\mathcal{N}$. Finally, define

$$F(y,x) = \xi \circ \bar{L}(y,x)$$

for $(y,x) \in \mathfrak{B}$. It is obvious by construction that F is universal for $\mathcal{C}(\mathcal{N})$ in a generalized sense. □

Theorem 2.9 *There exists a coanalytic set $D \in \mathbf{\Pi}^1_1(\mathcal{N})$ and a continuous mapping $G : D\times\mathcal{N} \to \mathcal{N}$ universal for $\mathcal{C}(\mathcal{N})$.*

Proof. Let π_1 denote the projection onto the first axis. Let $D = \mathcal{N}\setminus\pi_1(\mathcal{N}^2\setminus\mathfrak{B})$, where $\mathfrak{B}$ is the set from Theorem 2.8. Note that $D\times\mathcal{N} \subseteq \mathfrak{B}$. By the properties of $\mathfrak{B}$ and F from Theorem 2.9, we can define G as $F|D\times\mathcal{N}$. □

Theorem 2.10 *There exists a continuous mapping $L : \mathcal{N}\times\mathcal{N} \to \mathcal{N}$ universal for $\mathcal{L}(\mathcal{N})$.*

Proof. Let $\mathcal{L}(\mathcal{N}, k)$ denote the class of Lipschitz mappings with constant k Let $\tau_k : \mathbb{N}^{k-1} \to \mathbb{N}$ be a bijection. For a sequence $x = (x_1, x_2, \ldots) \in \mathcal{N}$ let $x(i) = x_i$.

We will define a mapping $\Phi_k : \mathcal{N}^{\mathbb{N}}\times\mathcal{N} \to \mathcal{N}$ that is universal for $\mathcal{L}(\mathcal{N}, k)$. Let

$$(\Phi_k((c^{(i)})_{i\in\mathbb{N}}, x))(n) = c^{(n)}(\tau_{k(n+1)}(x_1, x_2, \ldots, x_{k(n+1)-1})).$$

It is easy to see that Φ_k is continuous.

Now we will show that Φ_k is universal for $\mathcal{L}(\mathcal{N}, k)$. Let $f \in \mathcal{L}(\mathcal{N}, k)$. Let us define $c_f^{(i)} \in \mathcal{N}$ as follows:

$$c_f^{(i)}(\tau_{k(i+1)}(x_1, x_2, \ldots, x_{k(i+1)-1})) = (f(x_1, x_2, \ldots, x_{k(i+1)-1}, 1, 1, \ldots))(i),$$

for $(x_1, x_2, \ldots, x_{k(i+1)-1}) \in \mathbb{N}^{k(i+1)-1}$.

We claim that $f(x) = \Phi_k((c_f^{(i)})_{i\in\mathbb{N}}, x)$ for each $x \in \mathcal{N}$. Fix $x \in \mathcal{N}$ and $n \in \mathbb{N}$. Then

$$d(x, (x_1, \ldots, x_{k(n+1)-1}, 1, 1, \ldots)) \leq 1/k(n+1),$$

and so

$$d(f(x), f(x_1, \ldots, x_{k(n+1)-1}, 1, 1, \ldots)) \leq 1/(n+1).$$

Thus

$$(f(x))(n) = (f(x_1, \ldots, x_{k(n+1)-1}, 1, 1, \ldots))(n) =$$

$$c_f^{(n)}(\tau_{k(n+1)}(x_1, \ldots, x_{k(n+1)-1}) = (\Phi_k((c_f^{(i)})_{i \in \mathbb{N}}, x))(n).$$

Next define $\Psi : (\mathbb{N} \times \mathcal{N}^{\mathbb{N}}) \times \mathcal{N} \to \mathcal{N}$ by

$$\Psi(k, ((c_f^{(i)})_{i \in \mathbb{N}}, x)) = \Phi_k((c_f^{(i)})_{i \in \mathbb{N}}, x).$$

Note that the spaces $\mathbb{N} \times \mathcal{N}^{\mathbb{N}}$ and $\mathcal{N}$ are homeomorphic. Let $\theta : \mathcal{N} \to \mathbb{N} \times \mathcal{N}^{\mathbb{N}}$ be such a homeomorphism. Finally define

$$F(y, x) = \Psi(\theta(y), x).$$

$F : \mathcal{N}^2 \to \mathcal{N}$ is the desired continuous mapping universal for $\mathcal{L}(\mathcal{N})$. □

Remark In fact, any section $\Phi_k(c, x)$ is Lipschitz with constant $k+1$. Assume that $1/(k+1)n < d(x, y) \leq 1/kn$. Thus $x_i = y_i$ for all $i \leq kn$. Hence for $j < n$

$$(\Phi_k((c^{(i)})_{i \in \mathbb{N}}, x))(j) =$$

$$c^{(j)}(\tau_{k(j+1)}(x_1, x_2, \ldots, x_{k(j+1)-1})) = c^{(j)}(\tau_{k(j+1)}(y_1, x_2, \ldots, y_{k(j+1)-1})) =$$

$$(\Phi_k((c^{(i)})_{i \in \mathbb{N}}, y))(j).$$

Thus

$$d(\Phi_k((c^{(i)})_{i \in \mathbb{N}}, x), \Phi_k((c^{(i)})_{i \in \mathbb{N}}, y) \leq 1/n = (k+1)\frac{1}{(k+1)n} \leq (k+1)d(x, y).$$

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LECTURE 3 (3h)

CONTINUOUS FUNCTIONS vs LIPSCHITZ FUNCTIONS ON SOME CLASSICAL METRIC SPACES

We are now prepared to find $\mathrm{rank}(\mathcal{C}(\mathcal{N}) : \mathcal{L}(\mathcal{N}))$.

Theorem 3.1 $\mathrm{rank}(\mathcal{C}(\mathcal{N}) : \mathcal{L}(\mathcal{N})) = \aleph_1$.

Proof. We divide the proof into two parts. In the first part we prove that $\mathrm{rank}(\mathcal{C}(\mathcal{N}) : \mathcal{L}(\mathcal{N})) \geq \aleph_1$. In the second part we prove that $\mathrm{rank}(\mathcal{C}(\mathcal{N}) : \mathcal{L}(\mathcal{N})) \leq \aleph_1$.

Part 1. Assume for contradiction that the rank $\mathrm{rank}(\mathcal{C}(\mathcal{N}) : \mathcal{L}(\mathcal{N}))$ is countable. Thus there exists a countable collection $\{f_1, f_2, \ldots\}$ such that

$$\mathcal{C}(\mathcal{N}) = \langle \mathcal{L}(\mathcal{N}) \cup \{f_1, f_2, \ldots\} \rangle .$$

Hence every mapping $g \in \mathcal{C}(\mathcal{N})$ is a composition

$$g = \lambda_{k+1} \circ f_{n_k} \circ \lambda_k \circ \ldots \circ \lambda_2 \circ f_{n_1} \circ \lambda_1,$$

for some $\lambda_1, \ldots, \lambda_{k+1} \in \mathcal{L}(\mathcal{N})$ and $\mathbf{n} = (n_1, \ldots, n_k)$.

Let

$$\mathcal{A}_{\mathbf{n}} = \{\lambda_{k+1} \circ f_{n_k} \circ \lambda_k \circ \ldots \circ \lambda_2 \circ f_{n_1} \circ \lambda_1 : \lambda_1, \ldots \lambda_{k+1} \in \mathcal{L}(\mathcal{N})\}.$$

First we are going to construct a continuous mapping $H_{\mathbf{n}} : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ universal for $\mathcal{A}_{\mathbf{n}}$. Let $\tau : \mathcal{N} \to \mathcal{N}^{k+1}$ be a homomorphism whose coordinate functions are denoted by τ_i,$i = 1, 2, \ldots, k+1$. Let $F : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ be a continuous mapping universal for $\mathcal{L}(\mathcal{N})$, which exists by Theorem 2.10. Let us define $H_{\mathbf{n}}$ as follows:

$$H_{\mathbf{n}}(y, x) = F(\tau_{k+1}(y), \cdot) \circ f_{n_k} \circ F(\tau_k(y), \cdot) \circ \ldots \circ F(\tau_2(y), \cdot) \circ f_{n_1} \circ F(\tau_1(y), \cdot).$$

Let us enumerate all $\mathbf{n}$'s as $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \ldots$. Let us now define a mapping $H : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ as:

$$H((y_1, y_2, \ldots), x) = H_{\mathbf{n}^{(y_1)}}((y_2, y_3, \ldots), x).$$

It is easy to check that H is universal for $C(\mathcal{N})$ but this contradicts Proposition 2.7. And thus we have proved $\mathrm{rank}(\mathcal{C}(\mathcal{N}) : \mathcal{L}(\mathcal{N})) \geq \aleph_1$.

Part 2. Let $D \in \mathbf{\Pi}^1_1(\mathcal{N})$ and $G : D \times \mathcal{N} \to \mathcal{N}$ be a continuous mapping universal for $\mathcal{C}(\mathcal{N})$. They exist by Theorem 2.7. As every nonempty coanalytic set is a union of $\aleph_1$ nonempty Borel sets, we can write

$$D = \bigcup_{\alpha < \aleph_1} B_\alpha,$$

where B_α's are Borel subsets of $\mathcal{N}$. It is also known that every Borel subset of a Polish space, in particular of $\mathcal{N}$, is a continuous image of $\mathcal{N}$. Let then $\xi_\alpha : \mathcal{N} \to B_\alpha$ be a continuous 'onto' mapping.

Let us consider the following homeomorphism $\phi : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$:

$$\phi((y_1, y_2, \ldots), (x_1, x_2, \ldots)) = (y_1, x_1, y_2, x_2, \ldots).$$

The metric $\rho((y,x),(y',x')) = d(y,y') + d(x,x')$ coincides with the product topology on $\mathcal{N} \times \mathcal{N}$ and it is easy to see that ϕ is a two ways Lipschitz homeomorphism. It is also easy to notice that the mapping $x \longmapsto (y,x)$ from $\mathcal{N}$ into $\mathcal{N}^2$ is Lipschitz.

Now let $f \in \mathcal{C}(\mathcal{N})$. Then there exist $\alpha < \aleph_1$ and $z \in B_\alpha$ such that $f(\cdot) = G(y, \cdot)$. Let $z = \xi_\alpha(y)$. Let us consider the following series of compositions which give f:

$$x \longmapsto (x,y) \longmapsto \phi(y,x) \longmapsto (y,x) \longmapsto (\xi_\alpha(y), x) \longmapsto G(\xi_\alpha(y), x) = f(x).$$

Notice that the mapping λ defined by

$$x \longmapsto (x,y) \longmapsto \phi(y,x)$$

is Lipschitz, and the mapping h_α defined by

$$\phi(y,x) \longmapsto (y,x) \longmapsto (\xi_\alpha(y), x) \longmapsto G(\xi_\alpha(y), x)$$

is continuos and depends exclusively on α. Thus $f = h_\alpha \circ \lambda$. Hence we can express every $f \in C(\mathcal{N})$ as a composition of a Lipschitz mapping and a continouos mapping that belongs to the family of cardinality not exceeding $\aleph_1$. Thus we have shown that $\mathrm{rank}(\mathcal{C}(\mathcal{N}) : \mathcal{L}(\mathcal{N})) \leq \aleph_1$. $\square$

Remark It turns out that if x is any point of $\mathcal{N}$ then

$$\mathrm{rank}(\mathcal{C}(\mathcal{N} \setminus \{x\}) : \mathcal{L}(\mathcal{N} \setminus \{x\})) = 1.$$

Because the spaces $\mathcal{N}$ and $\mathcal{N} \setminus \{x\}$ are homeomorphic this shows very strongly that it is the metric not topological structure of X that decides about $\mathrm{rank}(\mathcal{C}(X) : \mathcal{L}(C))$. We will see further examples later.

A metric space X is *concentric* if it is unbounded and it is the union of countably many compact balls. Such spaces are common, for example every euclidean space $\mathbb{R}^n$ is concentric. The induced topology of a concentric metric space is noncompact, locally compact and second countable. On the other hand, for a topological space X which is noncompact, locally compact and second countable there exists a metric d such that (X,d) is concentric.

Let σ be a sequence in a metric space X and let Y be the set of elements that occur in σ, and its limit l, if it exists. Then σ is said to be *extendible* if Y is infinite and every continuous map from Y to Y that fixes l, if it exists, can be extended to an element of $\mathcal{C}(X)$.

$\mathbb{N}^{<\mathbb{N}}$ denotes the set of all finite sequences of natural numbers. An element of $\mathbb{N}^{<\mathbb{N}}$ is denoted by $\mathbf{i}$ and $|\mathbf{i}|$ denotes the length of the sequence.

Thoerem 3.2 *Let X be a concentric metric space which contains an extendible sequence with no convergent subsequence. Then*

$$\operatorname{rank}(\mathcal{C}(X) : \mathcal{L}(X)) > \aleph_0.$$

Proof. Let $\mathcal{L}(X, n)$ denote the family of all Lipschitz mappings from X to X with constant n.

It suffices to prove that for every countable subset of $\mathcal{C}(X)$ there exists a continuous mapping that is not generated by the union of the Lipschitz mappings and this subset. We construct such a mapping γ recursively.

To this end, let $\mu_1, \mu_2, \ldots \in \mathcal{C}(X)$ be arbitrary, let $\rho = (x_0, x_1, x_2, \ldots)$ be an extendible sequence with no convergent subsequence, and let $B(p, 1) \subseteq B(p, 2) \subseteq \ldots$ be compact balls that comprise X for some p. Since all the balls we consider in this proof are centered on p, for brevity we write $B(n)$ instead of $B(p, n)$. The elements of $\langle \mathcal{L}(X) \cup \{\mu_1, \mu_2, \ldots\} \rangle$ are finite compositions of the form

$$\Phi_{m+1} \circ \mu_{i_m} \cdots \circ \Phi_2 \circ \mu_{i_1} \circ \Phi_1,$$

for some $i_1, i_2, \ldots, i_m \in \mathbb{N}$ and $\Phi_1, \Phi_2, \ldots, \Phi_{m+1} \in \mathcal{L}(X)$. We represent the family $\langle \mathcal{L}(X) \cup \{\mu_1, \mu_2, \ldots\} \rangle$ as a countable union of families of such compositions. These families are determined, roughly speaking, according to the elements of $\{\mu_1, \mu_2, \ldots\}$ that appear, the Lipschitz constants involved, and the balls that contain images of x_0. The mapping γ is initially defined on the sequence ρ. Thus there are countably many steps in the definition; at each of which, we ensure that γ does not belong to one of the given families. This process exhausts every possibility.

Precisely speaking, we consider the set

$$\Sigma = \{[\mathbf{i}, n] : \mathbf{i} \in \mathbb{N}^{<\mathbb{N}}, n \in \mathbb{N}\} \tag{1}$$

and for $[\mathbf{i}, n] \in \Sigma$, with $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ and $n \in \mathbb{N}$, the family of compositions

$$\mathcal{F}_{[\mathbf{i},n]} = \{\Phi_{k+1} \circ \mu_{i_k} \cdots \circ \Phi_2 \circ \mu_{i_1} \Phi_1 : \Phi_{j+1} \circ \mu_{i_j} \circ \cdots \circ \mu_{i_1} \circ \Phi_1(x_0) \in B(n)\,, \\ \Phi_{j+1} \in \mathcal{L}(X, n) \text{ for each } 0 \leq j \leq k\}. \tag{2}$$

Obviously,

$$\bigcup_{[\mathbf{i},n] \in \Sigma} \mathcal{F}_{[\mathbf{i},n]} = \langle \mathcal{L}(X) \cup \{\mu_1, \mu_2, \ldots\} \rangle\,.$$

Since Σ is countable we may enumerate its elements as $\sigma_1, \sigma_2, \ldots$.

Step 0 in the definition of γ is made by setting $\gamma(x_0) = x_0$.

At step $r > 0$ we define γ on x_r in such a way that when the definition is complete γ will not be an element of $\mathcal{F}_{\sigma_r}$. Assume that γ was defined on

$x_1, x_2, \ldots, x_{r-1}$ in the previous steps such that any element of $\mathcal{C}(X)$ agreeing with γ on these points does not lie in $\mathcal{F}_{\sigma_i}$ for $i < r$. Also assume that $\sigma_r = [\mathbf{i}, n]$ with $\mathbf{i} = (i_1, \ldots, i_k)$ and $n \in \mathbb{N}$.

This step is completed by proving that every element of $\mathcal{F}_{\sigma_r}$ maps x_r into the same ball $B(m)$ for some m. In fact, it suffices to prove that if for each $0 \leq j \leq k-1$ and Φ_{j+1} as in (2) we have

$$\Phi_{j+1} \circ \mu_{i_j} \circ \cdots \Phi_2 \circ \mu_{i_1} \circ \Phi_1(x_r) \in B(m_{j+1}),$$

for some m_{j+1}, then every

$$\Phi_{j+2} \circ \mu_{i_{j+1}} \circ \cdots \circ \Phi_2 \circ \mu_{i_1} \circ \Phi_1(x_r) \in B(m_{j+2}),$$

for some m_{j+2}. Thus within step r we perform a finite induction.

For the base case, $\Phi_1(x_0) \in B(n)$ and $d(\Phi_1(x_0), \Phi_1(x_r)) \leq nd(x_0, x_r)$. Therefore $d(\Phi_1(x_r), p) \leq n + nd(x_0, x_r)$, and so $\Phi_1(x_r) \in B(m_1)$ where $m_1 = n + nd(x_0, x_r)$. Note that the choice of m_1 does not depend on Φ_1, but on its Lipschitz constant.

The inductive hypothesis states that for every

$$\Phi_{j+1} \circ \mu_{i_j} \circ \cdots \circ \Phi_2 \circ \mu_{i_1} \circ \Phi_1 \in \mathcal{F}_{[(i_1, \ldots, i_j), n]}$$

we have

$$u = \Phi_{j+1} \circ \mu_{i_j} \circ \cdots \circ \Phi_2 \circ \mu_{i_1} \circ \Phi_1(x_r) \in B(m_{j+1}),$$

for some m_{j+1}. By the definition of $\mathcal{F}_{[\mathbf{i},n]}$

$$v = \Phi_{j+1} \circ \mu_{i_j} \circ \cdots \circ \Phi_2 \circ \mu_{i_1} \circ \Phi_1(x_0) \in B(n).$$

If M is the maximum of m_{j+1} and n then $\mu_{i_{j+1}}(u), \mu_{i_{j+1}}(v) \in \mu_{i_{j+1}}(B(M))$. Since the continuous image of a compact set is compact there exists M' such that $\mu_{i_{j+1}}(B(M)) \subseteq B(M')$. Thus $d(\mu_{i_{j+1}}(u), \mu_{i_{j+1}}(v)) \leq 2M'$. But, again by the definition of $\mathcal{F}_{[\mathbf{i},n]}$, $\Phi_{j+2} \circ \mu_{i_{j+1}}(v) \in B(n)$. Therefore

$$d(p, \Phi_{j+2} \circ \mu_{i_{j+1}}(u)) \leq n + 2nM'.$$

We deduce that $\Phi_{j+2} \circ \mu_{i_{j+1}} \circ \cdots \circ \Phi_2 \circ \mu_{i_1} \circ \Phi_1(x_r) \in B(m_{j+2})$ where $m_{j+2} = n + 2nM'$.

Define $\gamma(x_r)$ in such a way that $\gamma(x_r) \notin B(m_{k+1})$.

When the recursion is complete, the extension of γ to an element of $\mathcal{C}(X)$ is not contained in $\mathcal{F}_\sigma$ for any $\sigma \in \Sigma$. This implies that $\gamma \notin \langle \mathcal{L}(X) \cup \{\mu_1, \mu_2, \ldots\} \rangle$. $\square$

Now we can derive the following corollary about euclidean spaces and their subsets.

Corollary 3.3 *Let $X = \mathbb{N}$, $\mathbb{Z}$ or $\mathbb{R}$ with the usual euclidean metric. Then*

$$\operatorname{rank}(\mathcal{C}(X^n) : \mathcal{L}(X^n)) > \aleph_0$$

for any $n \in \mathbb{N}$.

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LECTURE 4 (3h)

CONTINUOUS FUNCTIONS vs LIPSCHITZ FUNCTIONS ON COUNTABLE DISCRETE METRIC SPACES

In this lecture we return to monoids X^X for infinite countable X's. Namely, we consider countable discrete metric spaces (X, d). Because X is discrete we have $\mathcal{C}(X) = X^X$. We will be interested in the relative rank $\mathrm{rank}(X^X : \mathcal{L}(X))$. The monoid $\mathcal{L}(X)$ is the next example of a submonoid of X^X. By Sierpiński's theorem $\mathrm{rank}(X^X : \mathcal{L}(X)) = 0, 1, 2$ or $\mathrm{rank}(X^X : \mathcal{L}(X)) > \aleph_0$. The full description of these ranks for all countable discrete metric spaces is not known. In fact, it is not known even for every $X \subseteq \mathbb{R}$ with the euclidean metric. Nevertheless for the most classical examples of (X, d) the values of $\mathrm{rank}(X^X : \mathcal{L}(X))$ have been found. In this last lecture we will give an account of this research.

A *dominating family* is any subfamily $\mathcal{A}$ of $\mathbb{N}^{\mathbb{N}}$ such that for any $f \in \mathbb{N}^{\mathbb{N}}$ there exists $g \in \mathcal{A}$ such that $g(n) \geq f(n)$ for all n's greater than some N. We say that g *eventually dominates* f. The cardinal $\mathfrak{d}$ is the minimal possible cardinality of a dominating family. It is known that both statements $\mathfrak{d} < \mathfrak{c}$ and $\mathfrak{d} = \mathfrak{c}$ are consistent with ZFC.

Note that if in the definition above we require that $g(n) \geq f(n)$ for each n (that g *dominate* f) this gives an equivalent definition of $\mathfrak{d}$. It is also not important whether we use the sharp or weak inequality.

We start with the following theorem where we find $\mathrm{rank}(X^X : \mathcal{L}(X))$ for a vast family of countable discrete metric spaces (X, d).

Theorem 4.1 *Let (X, d) be a countable discrete metric space containing a Cauchy sequence. Then*

$$\mathrm{rank}(X^X : \mathcal{L}(X)) = 1.$$

Proof. Let $y_1, y_2, \ldots$ be a Cauchy sequence of different points of X. Let $f \in X^X$ be any function such that the set $f^{-1}[\{x\}] \cap \{y_1, y_2, \ldots\}$ is infinite for every $x \in X$. Let $g \in X^X$. Let $B(x_i, r_i) = \{x_i\}$, for $r_i > 0$. Let $N(1)$ be chosen in such a way that $d(y_m, y_n) < r_1$ for $m, n \geq N(1)$. Let $\lambda(x_1) \in \{y_{N(1)}, y_{N(1)+1}, \ldots\}$ and $f(\lambda(x_1)) = g(x_1)$. By the assumption about f we can find such $\lambda(x_1)$. Let $N(2)$ be chosen in such a way that $d(y_m, y_n) < r_2$ for $m, n \geq N(2)$. Let $\lambda(x_2) \in \{y_{N(2)}, y_{N(2)+1}, \ldots\}$ and $f(\lambda(x_2)) = g(x_2)$. By the assumption about f we can find such $\lambda(x_2)$. We continue this process of choosing $\lambda(x_i)$'s. Finally we obtain that $d(y_m, y_n) < r_i$ for $m, n \geq N(i)$ and and $\lambda(x_i) \in \{y_{N(i)}, y_{N(i)+1}, \ldots\}$ for each i and $f(\lambda(x_i)) = g(x_i)$. The last property means

$$f \circ \lambda = g.$$

Let us show that $\lambda : X \to X$ is Lipschitz with constant 1. Let $i < j$. We have

$$d(\lambda(x_i), \lambda(x_j)) < \max\{r_i, r_j\} \leq d(x_i, x_j).$$

Thus

$$\langle \mathcal{L}(X) \cup \{f\} \rangle = X^X,$$

which gives

$$\operatorname{rank}(X^X : \mathcal{L}(X)) = 1.$$

□

We have proved (Corollary 3.3) that $\operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathcal{L}(\mathbb{N})) > \aleph_0$. Here we will make this result more precise. Namely, we will prove the following theorem.

Theorem 4.2 $\operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathcal{L}(\mathbb{N})) = \mathfrak{d}$.

Proof. The proof consists of two parts. First we will prove that $\operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathcal{L}(\mathbb{N})) \geq \mathfrak{d}$. Next, we will prove the opposite inequality.

Part 1. Let us assume that

$$\langle \mathcal{L}(\mathbb{N}) \cup \mathcal{F} \rangle = \mathbb{N}^{\mathbb{N}} \tag{3}$$

and $|\mathcal{F}| = \operatorname{rank}(\mathbb{N}^{\mathbb{N}} : \mathcal{L}(\mathbb{N}))$.

Let $f_1, f_2, \dots, f_n \in \mathcal{F}$. Recall that $\mathcal{L}_N(\mathbb{N}, N)$ denotes the family of functions from $\mathbb{N}^{\mathbb{N}}$ satisfying the Lipschitz condition with constant N. Let

$$\mathcal{A}^{(N)}_{(f_1,\dots,f_n)} = \{\lambda_{n+1} \circ f_n \circ \lambda_n \ldots \circ f_1 \circ \lambda_1 : \lambda_1, \dots, \lambda_{n+1} \in \mathcal{L}(\mathbb{N}, N).$$

Of course,

$$\langle \mathcal{L}(\mathbb{N}) \cup \mathcal{F} \rangle = \bigcup_{(f_1,\dots,f_n) \in \mathcal{F}^n, N} \mathcal{A}^{(N)}_{(f_1,\dots,f_n)}. \tag{4}$$

We will use the following notation. Let $\varphi_M(x) = Mx$. For $f \in \mathbb{N}^{\mathbb{N}}$, the function $\bar{f}$ is defined by

$$\bar{f}(m) = \max\{f(1), f(2), \dots, f(m)\}.$$

Obviously $\bar{f}(m)$ is nondecreasing. We define $F(f_1, \dots, f_n; M) \in \mathbb{N}^{\mathbb{N}}$ by

$$F(f_1, \dots, f_n; M) = \varphi_M \circ \bar{f}_n \circ \varphi_M \circ \bar{f}_{n-1} \circ \bar{f}_1 \circ \varphi_M.$$

Let now g be any function from $\mathbb{N}^{\mathbb{N}}$. It follows from (3) and (4) that $g \in \mathcal{A}^{(N)}_{(f_1,\dots,f_n)}$ for some $f_1, f_2, \dots, f_n \in \mathcal{F}$ and $N \in \mathbb{N}$ which means that

$$g = \lambda_{n+1} \circ f_n \circ \lambda_n \ldots \circ f_1 \circ \lambda_1,$$

$\lambda_1, \dots, \lambda_{n+1} \in \mathcal{L}(\mathbb{N}, N)$.

Let $M = \max\{\lambda_1(1), \dots \lambda_{n+1}(1), N\}$. It is easy to check that $g \leq F(f_1, \dots, f_n; M)$. Thus the family $\{F(f_1, \dots, f_n; M) : f_1, \dots, f_n \in \mathcal{F}, M \in \mathbb{M}\}$ is dominating for $\mathbb{N}^{\mathbb{N}}$. Of course,

$$|\{F(f_1, \dots, f_n; M) : f_1, \dots, f_n \in \mathcal{F}, M \in \mathbb{M}\}| = |\mathcal{F}|.$$

Hence $|\mathcal{F}| \geq \mathfrak{d}$ and this ends the first part of the proof.

Part 2. Let $\mathcal{G}$ be any dominating family for $\mathbb{N}^{\mathbb{N}}$ and $|\mathcal{G}| = \mathfrak{d}$. For $g \in \mathbb{N}^{\mathbb{N}}$ we define a new function g^*: $g^*(1) = g(1)$, $g^*(2) = 2g^*(1) + g(2)$, and, recursively, $g^*(n+1) = 2g^*(n) + g(n+1)$. Of course, $g^* \geq g$, whence the family

$$\mathcal{G}^* = \{g^* : g \in \mathcal{G}\}$$

is also dominating for $\mathbb{N}^{\mathbb{N}}$ and $|\mathcal{G}^*| = \mathfrak{d}$.

For $f \in \mathbb{N}^{\mathbb{N}}$ let $\hat{f}(n) = f(n+1)$.

Let now f be any function from $\mathbb{N}^{\mathbb{N}}$. Let $g \in \mathcal{G}$ dominate $\max\{f, \hat{f}\}$. We have

$$g^*(n+1) - g^*(n) \geq g^*(n) \geq g(n)$$

and

$$g(n) \geq f(n) > f(n) - f(n+1)$$

and

$$g(n) \geq \hat{f}(n) = f(n+1) > f(n+1) - f(n).$$

Hence

$$g^*(n+1) - g^*(n) > |f(n+1) - f(n)|. \tag{5}$$

Let $G = g^*[\mathbb{N}]$. We define $\lambda_f : G \to \mathbb{N}$ by $\lambda_f(g^*(i)) = f(i)$. By (5) λ_f satisfies the Lipschitz condition with constant 1. We extend λ_f to a function $\Lambda_f \in \mathcal{L}(\mathbb{N}, 2)$. Let $r \in G$, $s \in G$, $r < s$ and $(r, s) \cap G = \emptyset$. Let $v \in (r, s)$. We define

$$\Lambda_f(v) = \left\lfloor \frac{\lambda_f(s) - \lambda_f(r)}{s - t}(v - r) + \lambda_f(r) \right\rfloor.$$

To check that Λ_f satisfies the Lipschitz condititio n with constant 2 it is enough to check it for two consecutive points $v, v+1 \in [r, s]$. Consider the case $\lambda_f(s) - \lambda_f(r) \geq 0$; the other case is simillar. We have

$$\Lambda_f(v+1) - \Lambda_f(v) = \left\lfloor \frac{\lambda_f(s) - \lambda_f(r)}{s - t}(v + 1 - r) \right\rfloor - \left\lfloor \frac{\lambda_f(s) - \lambda_f(r)}{s - t}(v - r) \right\rfloor \leq$$

$$\frac{\lambda_f(s) - \lambda_f(r)}{s - t}(v + 1 - r) - \frac{\lambda_f(s) - \lambda_f(r)}{s - t}(v - r) + 1 = \frac{\lambda_f(s) - \lambda_f(r)}{s - t} + 1 \leq 2.$$

We also have $f = \Lambda_f \circ g^*$. Hence $\mathbb{N}^{\mathbb{N}} = \langle \mathcal{L}(\mathbb{N}) \cup \mathcal{G}^* \rangle$. $\square$

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APPENDIX

Let $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$. Let ρ be the the usual euclidean metric of the real line. We will prove the following theorem.

Theorem *The spaces* $(\mathbb{N}^{\mathbb{N}}, \mathrm{d})$ *and* $(\mathbb{I}, \rho)$ *are homeomorphic.*

Proof. It is easy to notice that the space $(\mathbb{I}, \rho)$ is homeomorhic to the space $((0,1) \cap \mathbb{I}, \rho)$.

Let

$$\mathbb{Q} \cap (0,1) = \{s_j : j = 1, 2, \ldots\}$$

and

$$I_\emptyset = (0,1).$$

Let now

$$0 = q_0 < q_1 < q_2 < \ldots \to 1,$$

where all q_i are rational numbers.

Let us make sure that $s_1 \in \{q_i : i \in \mathbb{N}\}$.

Let

$$I_i = (q_{i-1}, q_i),$$

for $i \in \mathbb{N}$. We make sure that $|I_i| < 1/2$ for each $i \in \mathbb{N}$.

Now we divide the intervals $I_i = (q_{i-1}, q_i)$ into subintervals $I_{i,j} = (q_{i,j}, q_{i,j-1})$, $j \in \mathbb{N}$, where

$$q_{i-1} \leftarrow \ldots q_{i,2} < q_{i,1} < q_{i,0} = q_i.$$

At this second stage of the construction we make sure that $|I_{i,j}| < 1/3$ for each interval $I_{i,j}$ and that

$$s_2 \in \{q_i : i \in \mathbb{N}\} \cup \{q_{i,j} : i, j \in \mathbb{N}\}.$$

We continue this construction and at the stage k, for k odd and $i_{k-1} \in \mathbb{N}$, we have

$$q_{i_1,\ldots,i_{k-1}} = q_{i_1,\ldots,i_{k-1},0} < q_{i_1,\ldots,i_{k-1},1} < \ldots \to q_{i_1,\ldots,i_{k-1}-1}$$

and for k even and $i_{k-1} \in \mathbb{N}$, we have

$$q_{i_1,\ldots,i_{k-1}-1} \leftarrow \cdots < q_{i_1,\ldots,i_{k-1},1} < q_{i_1,\ldots,i_{k-1},0} = q_{i_1,\ldots,i_{k-1}}.$$

We also make sure that s_k is of the form $q_{i_1,\ldots,i_r}$, $r \leq k$, $i_1, \ldots i_r \in \mathbb{N}$, and that for $I_{i_1,\ldots,i_k}$ being the open interval of the endpoints $q_{i_1,\ldots,i_k-1}, q_{i_1,\ldots,i_k}$,

$$|I_{i_1,\ldots,i_k}| < \frac{1}{k+1}. \tag{6}$$

We have

$$I_{i_1,\ldots,i_k,i_{k+1}} \subset I_{i_1,\ldots,i_k} \tag{7}$$

and

$$I_{i_1,\dots,i_k} \cap I_{j_1,\dots,j_k} = \emptyset \tag{8}$$

unless $i_1 = j_1, \dots, i_k = j_k$. We also have

$$q_{i_1,\dots,i_k} \notin \bar{I}_{j_1,\dots,j_{k+1},j_{k+2}} \tag{9}$$

for any sequence $j_1, \dots, j_{k+1}, j_{k+2}$. It follows now from (6) and (7) that for each fixed sequence $(I_{i_1,\dots,i_k})_k$ the intersection of the closures of all the intervals from this sequence is a one-point set

$$\bigcap_{k=1}^{\infty} \bar{I}_{i_1,\dots,i_k} = \{x_{i_1,i_2,\dots}\}.$$

We claim now that $x_{i_1,i_2,\dots}$ is an irrational number. Indeed, if it were rational, then it would be s_k for some $k \in \mathbb{N}$. Then, by the rules of the construction, it would be of the form $q_{i_1,\dots,i_r}$, $r \le k$. Then, however, by (9) it could not be an element of $\bar{I}_{i_1,\dots,i_{k+1},i_{k+2}}$.

The argument above also shows that

$$\bigcap_{k=1}^{\infty} \bar{I}_{i_1,\dots,i_k} = \bigcap_{k=1}^{\infty} I_{i_1,\dots,i_k}.$$

Now we will define our homeomorphism $f : \mathcal{N} \to \mathbb{I} \cap (0,1)$. Let

$$\{x_{i_1,i_2,\dots}\} = \bigcap_{k=1}^{\infty} I_{i_1,\dots,i_k},$$

and

$$f((i_k)_k) = x_{i_1,i_2,\dots}.$$

By (8) the mapping f is an injection.

Let now $x \in \mathbb{I} \cap (0,1)$. Then let $x \in I_{i_1}, x \in I_{i_1,i_2}, \dots$. We have

$$\{x\} = \bigcap_{k=1}^{\infty} I_{i_1,\dots,i_k}$$

and thus

$$f((i_k)_k) = x.$$

Hence f is also a surjection.

Now we are going to prove that f is continuous. Actually, we will show more, namely, that f is a contraction. Let $(i_k)_k$, $(j_k)_k$ be two different elements of $\mathcal{N}$. Let $\mathrm{d}((i_k)_k, (j_k)_k)) < 1/k_0$. Then the finite sequences $(i_1, \dots, i_{k_0})$ and $(j_1, \dots, j_{k_0})$ are identical and $f((i_k)_k), f((j_k)_k) \in I_{i_1,\dots,i_{k_0}}$. By (6) we obtain $|f((i_k)_k) - f((j_k)_k)| < 1/k_0$.

In the last step we will show that the mapping f^{-1}, inverse to f, is continuous as well.

Let us assume that $y_n \in (0,1) \cap \mathbb{I}$ for $n \in \mathbb{N}$ and $y_n \to y \in (0,1) \cap \mathbb{I}$.

Let $y \in I_{i_1,\dots,i_k}$ for some fixed k. Then for certain $n_0 \in \mathbb{N}$ and all $n \ge n_0$ we have $y_n \in I_{i_1,\dots,i_k}$. This implies that $\mathrm{d}(f^{-1}(y), f^{-1}(y_n)) \le 1/k$ for $n \ge n_0$. This concludes the proof. □