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Renormalized and entropy solutions of partial differential equations

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Note on lecturer

Professor Piotr Gwiazda is a recognized expert in the fields of partial differential equations, applied mathematics, and functional analysis. He has enriched his scientific experiences visiting universities in Darmstadt, Heidelberg, Berlin, Paris, Prague, Brescia and, as a consequence, he collaborates with best mathematicians from France, Germany, Italy, Czech Republic.

Prof. Gwiazda defended his Ph.D. in Faculty Mathematics, Informatics and Mechanics of Warsaw University in 1999 and in 2006 got his habilitation. Currently, he is a professor in Faculty Mathematics, Informatics and Mechanics of Warsaw University. Actually, prof. Piotr Gwiazda is a coordinator of the Ph.D. Programme Mathematical Methods in Natural Sciences financed by European Regional Development Funds via the Foundation for Polish Sciences http://mmns.mimuw.edu.pl/

Piotr Gwiazda (Uniwersytet Warszawski)

Renormalized and entropy solutions of partial differential equations

For most of equations, that are significant from the point of view of mathematical physics, it is impossible to show the existence of classical solutions, ie, solutions which are continuously differentiable as many times as the order derivatives in equations under consideration. On the other hand, the concept of distribution solutions (or weak solutions) appears to be too poor and does not allow us to pose certain problems correctly.

As an example, we can point out the lack of the uniqueness in the class distribution solutions to the simplest Burgers equation or other pathologies which appear in the case of distribution solutions to many partial differential equations.

Hence, as a certain more general idea, we can use the notion of the entropy or the renormalization, that is to postulate that (in addition to the weak formulation of the problem) the equation satisfies certain additional weak formulation in entropic or renormalized sense (for a sufficiently rich family of entipies/renormalizations).

This program (derived essentially from hyperbolic conservation laws) has found applications in many other equations: elliptic equations, parabolic equations, the Boltzmann equation, the transport equation, the Navier-Stokes equation for compressible fluid or with variable density. As another application of the entropy method, one should mention the method of relative entropy, used to analyze the long time behavior of solutions to partial differential equations. This program was successfully applied in the case of equations such as the Fokker-Planck equation, the nonlinear diffusion equation (the porous medium equation), and other equations from mathematical biology.

Since this is an introductory course, we shall concentrate on the following simplest and selected issues:

- The Kruzkov theory entropy solutions for scalar hyperbolic equations;
- the DiPerny-Lions theory of renormalized solutions to the transport equation;
- theory of renormalized and entropy solutions to elliptic and parabolic equations;
- the method of relative entropy in dynamics of biological populations

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Renormalized and entropy solutions for partial differential equations

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Example 1 Cauchy problem for HYPERBOLIC CONSERVATION LAW :

$$\partial_t u(t,x) + \mathrm{div}_x F(u(t,x)) = 0 \quad x \in \mathbb{R}^n,\ t > 0.$$

with initial condition $u(0,x) = u^0(x)$. Function $F : \mathbb{R}^m \to \mathbb{R}^n$ is the flux of the unknown.

Problem:

On one side even in the simpliest case, i.e. Burgers equation ($n = m = 1$, $F(\cdot) = (\cdot)^2$) the solutions with initial condition $u^0 \in C_c^\infty$ in the finite time become discontinuous functions, on the other side distributional (discontinuous) solutions are not unique.

Smooth initial data and discontinuous solutions

$$u_t + f(u)_x = 0, \qquad u(x,0) = u^0(x).$$

Consider the characteristics

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = f'(u).$$

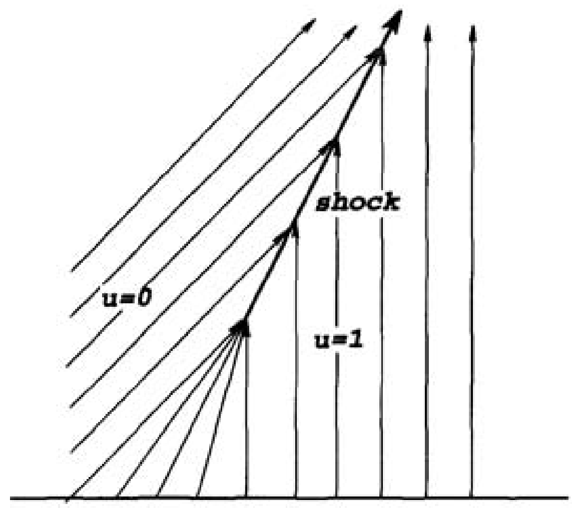
The solution is constant on the characteristics, and their slope is equal to $1/f'(u)$. If there exist $x_1 < x_2$ such that

$$1/f'(u^0(x_1)) < 1/f'(u^0(x_2)),$$

then the characteristics from the points $(x_1, 0)$ i $(x_2, 0)$ will intersect at time $t > 0$.

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad u^0(x) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 \le x \le 1, \\ 0, & x \ge 1. \end{cases}$$

Solution



$$u(t,x) = \begin{cases} 1, & x \le t, 0 \le t \le 1, \\ \frac{1-x}{1-t}, & t \le x \le 1, 0 \le t \le 1 \\ 0, & x \ge 1, 0 \le t \le 1. \end{cases}$$

For $t \ge 1$ the characteristics intersect.

Example of non-uniqueness Riemanna problem for Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad u^0(x) = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases}$$

For each $\alpha \ge 1$ the problem possesses solution u_α defined as follows

$$u_\alpha(x) = \begin{cases} 1 & \text{dla} \quad 2x < (1-\alpha)t, \\ -\alpha & \text{dla} \quad (1-\alpha)t < 2x < 0, \\ \alpha & \text{dla} \quad 0 < 2x < (\alpha-1)t, \\ -1 & \text{dla} \quad (\alpha-1)t < 2x. \end{cases}$$

Observation: If (η, q) are such that $\nabla_u \eta \cdot \nabla_u F = \nabla_u q$ then the classical solution of

$$\partial_t u(t,x) + \mathrm{div}_x F(u(t,x)) = 0 \quad x \in \mathbb{R}^n, \; t > 0.$$

also satisfies

$$\partial_t \eta(u(t,x)) + \mathrm{div}_x q(u(t,x)) = 0 \quad x \in \mathbb{R}^n, \; t > 0.$$

Remark:
(η, q) satisfying $\nabla_u \eta \cdot \nabla_u F = \nabla_u q$ are called entropy pair, and if additionally η is a convex function, then we call it convex entropy pair.

Definition:
We say that u is a weak entropy solution to hyperbolic conservation law if it is a distributional solution and additionally

$$\partial_t \eta(u(t,x)) + \mathrm{div}_x q(u(t,x)) \leq 0 \quad \text{in} \quad \mathcal{D}'$$

for each convex entropy pair (η, q).
Remark:
If $m = 1$, then there are a lot of such pairs. They are used for showing uniqueness, compactness in strong topology L^p, regularity in fractional Sobolew spaces, etc. Unfortunately for $m > 1$ the situation becomes essentially more complicated and the reach enough family of pairs we have only in the case when $m = 2$ and $n = 1$.
Motivation for the definition (Parabolic approximation)

$$\partial_t u^\epsilon(t,x) + \mathrm{div}_x F(u^\epsilon(t,x)) = \epsilon \Delta_x u^\epsilon(t,x) \quad x \in \mathbb{R}^n,\ t > 0.$$

then

$$\begin{aligned}
&\partial_t \eta(u^\epsilon(t,x)) + \mathrm{div}_x q(u^\epsilon(t,x)) = \\
&\quad = \epsilon[-\nabla_x u^\epsilon \cdot \nabla_u^2 \eta(u^\epsilon) \cdot \nabla_x u^\epsilon + \Delta_x \eta(u^\epsilon(t,x))] \quad x \in \mathbb{R}^n,\ t > 0.
\end{aligned}$$

Observe that $\nabla_x u^\epsilon \cdot \nabla_u^2 \eta(u^\epsilon) \cdot \nabla_x u^\epsilon \geq 0$ since η is convex.
The maximum principle for parabolic equation provides that $\|u^\epsilon\|_\infty \leq \|u^0\|_\infty$, so $\epsilon \Delta_x \eta(u^\epsilon(t,x)) \to 0$ in $\mathcal{D}'$ for $\epsilon \to 0$. Finally we obtain in the limit nonnegative distribution, namely the Radon measure.
Example 2. Transport equation with a non-Lipschitz coefficient.
Consider the initial value problem for the transport equation

$$\frac{\partial}{\partial t} u(t,x) + b(x) \cdot \nabla_x u(t,x) = 0, \quad x \in \mathbb{R}^n,\ t > 0.$$

Question:
what regularity of the coefficient $b(\cdot)$ is needed to obtain the existence and uniqueness of classical solutions (i.e., differentiable functions)?

Answer:
$b(\cdot)$ should be Lipschitz functions ($\nabla_x b \in L^\infty$).
Explanation:
Solving the Cauchy problem for transport equation reduces to solving the ordinary differential equation backwards for the characteristics:

$$\frac{d}{dt} X(t) = b(X(t)) \text{ with an end condition } X(t) = x$$

(Thm. Picard-Lindeloef)!
Recall that $u(t, X(t))$ solution to the transport equation is constant along characteristics. Hence the solution to transport equation can be computed from the initial condition according to the formula:

$$u(t, x) := u(0, X(0)).$$

Renormalized solutions – DiPerna, Lions '89:

1. assume only that $b \in W^{1,1}_{loc}$ and $\operatorname{div} b \in L^\infty$,
2. we require that the solution satisfies for each $\beta \in C^1$, $\beta' \in L^\infty$

   $$\tfrac{\partial}{\partial t}\beta(u(t,x)) + b(x) \cdot \nabla_x \beta(u(t,x)) = 0 \qquad \textsc{renormalization property}$$

   in the distributional sense. In particular, for $\beta(\xi) = \xi$ we obtain the original equation

   $$\tfrac{\partial}{\partial t}u(t,x) + b(x) \cdot \nabla_x u(t,x) = 0$$

3. let additionally assume, that the set of solutions is a linear space (natural demand for the linear equation)

Result:
In this class we obtain existence and uniqueness of solutions

Question:
Why the renormalization property implies the uniqueness of solutions?

Explanation:
Consider $w = u_1 - u_2$ the difference of two solutions with the same initial condition. Integrating the equation w.r.t. x we obtain:

$$\frac{d}{dt}\int_{\mathbb{R}^n} \beta(w(t,x))dx + \int_{\mathbb{R}^n} b(x) \cdot \nabla_x \beta(w(t,x))dx = 0.$$

After applying the formula for integrating per parts and Höldera inequality we obtain:

$$\frac{d}{dt}\int_{\mathbb{R}^n} \beta(w(t,x))dx \le \|\operatorname{div} b\|_{L^\infty} \int_{\mathbb{R}^n} \beta(w(t,x))dx.$$

Due to applying $\beta(\cdot) = |\cdot|$ (small technical problem) and Gronwalla inequality we obtain

$$\|w(t)\|_{L^1} \leq e^{(t\|\operatorname{div} b\|_{L^\infty})}\|w(0)\|_{L^1}$$

and hence uniqueness of solutions.
The idea of the proof of existnce of renormalized solutions

1. for regular solutions (of class C^1) of the transport equation the renormalization property is a consequence of the chain rule for the derivative of a composition (we multiply the transport equation with $\beta'(u)$),

2. it is easy to show the existence of distributional solutions:

   - approximation of b with a sequence $b_n \in W^{1,\infty}$,
   - a priori estimates independent of n for the sequence u_n (e.g. in the space L^∞),
   - Banach-Alaoglu theorem and the property of weak star continuity of linear equations $\Rightarrow$ existence of distributional solutions

3. unfortunately they do not have to satisfy a priori the renormalization property (the problemem of the lack of weak star continuity of nonlinear mapping $u \mapsto \beta(u)$)

4. fact: if $b \in W^{1,1}_{loc}$, $u \in L^\infty$ then

   $$\varphi_\epsilon * (b \cdot \nabla_x u) - b \cdot (\varphi_\epsilon * \nabla_u) \xrightarrow{\epsilon \to 0^+} 0 \text{ w } L^1_{loc},$$

   where $\varphi_\epsilon(x) := \varphi\left(x/\epsilon\right)$ $(\varphi \in C^\infty_c)$.

5. we obtain:

   $$\frac{\partial}{\partial t}(\varphi_\epsilon * u) + b \cdot \nabla_x(\varphi_\epsilon * u) = r_\epsilon \xrightarrow{\epsilon \to 0^+} 0 \text{ w } L^1_{loc},$$

   multiplying the above equation with $\beta'(\varphi_\epsilon * u)$ and using the chain rule we obtain

   $$\frac{\partial}{\partial t}\beta(\varphi_\epsilon * u) + b \cdot \nabla_x \beta(\varphi_\epsilon * u) = r_\epsilon \beta'(\varphi_\epsilon * u) \xrightarrow{\epsilon \to 0^+} 0 \text{ w } L^1_{loc}$$

6. recall that if $u \in L^1_{loc}$ then

   $$\varphi_\epsilon * u \xrightarrow{\epsilon \to 0^+} u \text{ w } L^1_{loc},$$

due to linear growth conditions β (tj. $|\beta(\xi)| \leq C(1+|\xi|)$)

$$\beta(\varphi_\epsilon * u) \xrightarrow{\epsilon\to 0^+} \beta(u) \ \text{ w } \ L^1_{loc},$$

Example 3. Renormalized solutions to SCALAR ELLIPTIC AND PARABOLIC consider the elliptic equation

$$\mathrm{div}_x(|\nabla_x u(x)|^{p-2}\nabla_x u(x)) + \mathrm{div}_x F(u(x)) = f(x) \quad x \in \Omega$$

with the boundary condition $u_{|\partial\Omega} = 0$.
Difficulties:
$F \in W^{1,\infty}_{loc}(\mathbb{R}, \mathbb{R}^n)$, and $f \in L^1(\Omega)$ for $p \in (1, n)$. In this situation Sobolew embedding theorem does not provide the embedding of $W^{1,p}_0(\Omega)$ into $L^\infty(\Omega)$. Hence the problem is the lack of proper growth conditions for F and the fact that f is not an element of the dual space to $W^{1,p}_0(\Omega)$!
Solution:
Let us introduce the truncation operator T_n, then $\|T_n(u)\|_\infty \leq n$. Using $T_n(u)$ as a test function in a weak formulation we obtain apriori estimate

$$\|\nabla_x T_n(u)\|_{L^p} \leq n\|f\|_{L^1}$$

Definition:
The function u is called the renormalized solution if $T_n(u) \in W^{1,p}_0(\Omega)$ for each $n \in \mathbb{N}$, and moreover in the distributional sense the following equation is satisfied

$$\mathrm{div}_x(|\nabla_x u|^{p-2}\nabla_x u)\beta(u) + \mathrm{div}_x F(u)\beta(u) = f\beta(u)$$

for each function $\beta \in C^\infty_c(\mathbb{R}, \mathbb{R}^n)$.
Question:
Why all the terms appearing in the equation in the definition are well defined?
Answer:
Observe that if $\mathrm{supp}\beta \in (-n, n)$ then it holds:

$$\mathrm{div}_x(|\nabla_x u|^{p-2}\nabla_x u)\beta(u) = \mathrm{div}_x(|\nabla_x T_n(u)|^{p-2}\nabla_x T_n(u))\beta(T_n(u))$$

and

$$\mathrm{div}_x F(u)\beta(u) = \mathrm{div}_x F(T_n(u))\beta(T_n(u))$$

so

$$\mathrm{div}_x(|\nabla_x u|^{p-2}\nabla_x u)\beta(u) \in L^{p'}(\Omega), \qquad \mathrm{div}_x F(u)\beta(u) \in L^p(\Omega)$$

Example 4 Renormalized solutions for BOLTZMANN EQUATION

$$\partial_t f(t,x,v) + v\nabla_x f(t,x,v) = Q[f(t,x\cdot), f(t,x,\cdot)]$$

where $f(t,x,v)$- density of particles of gas in a given time, space and velocity vector .

Problem:
It is possible to show only the estimates in L^1 for the bilinear collision term $Q[f,f]$.
Solution:
R. DiPerna i P.-L. Lions '89, existence of renormalized solutions for Boltzmann equation

Example 5 Renormalizations in the conservation law of mass in the NAVIERA-STOKES system for the fluid of a NONHOMOGENEOUS DENSITY

$$\partial_t \rho(t,x) + \mathrm{div}_x[\rho(t,x)v(t,x)] = 0$$

$$\partial_t[\rho(t,x)v(t,x)] + \mathrm{div}_x[\rho(t,x)v(t,x)\otimes v(t,x)] + \nabla_x p(t,x) = \Delta_x u(t,x)$$

$$\mathrm{div}_x v(t,x) = 0$$

with an initial condition and boundary conditions (e.g. Dirichlet).
Application: In a natural way $\mathrm{div}_x v \in L^\infty$, and from the total energy estimate $v \in W_0^{1,2}$, but it is not possible to show that $v \in W^{1,\infty}$, hence using the method of renormalizations we obtain the compactness properties for ρ in a strong topology L^p. More detailly, since $\frac{d}{dt}\|\rho\|_{L^p} = 0$ hence showing the weak and strong convergence is the same.

Example 6 Renormalizations in a mass conservation law in the NAVIERA-STOKES system for the COMPRESSIBLE fluid

$$\partial_t \rho(t,x) + \mathrm{div}_x[\rho(t,x)v(t,x)] = 0$$

$$\partial_t[\rho(t,x)v(t,x)] + \mathrm{div}_x[\rho(t,x)v(t,x)\otimes v(t,x)] + \nabla_x \rho^\gamma(t,x) = \Delta_x u(t,x)$$

with an initial condition and boundary conditions (e.g. Dirichlet). The constant $\gamma > 1$ and it is dependent on the number of degrees of freedom. For the one atomic gas it takes the discrete set of values.

Hyperbolic conservation laws - Luc Tartare
Compansated compactness and Young measures

Example (L.C. Young)

Minimize

$$\int_0^1 (u_x^2 - 1)^2 + u^2 dx$$

subject to

$$u(0) = u(1) = 0$$

The infimum of the functional is zero, but it cannot be attained since there is no function that satisfies $u \equiv 0$ and $u_x = \pm 1$ almost everywhere. Minimizing sequence must oscillate and converge weakly, but not strongly, to zero.

Notation

$C_0(\mathbb{R}^d)$ – closure of the space of continuous functions on $\mathbb{R}^d$ with compact support w.r.t. the $\|\cdot\|_\infty$-norm.
$(C_0(\mathbb{R}^d))^* = \mathbb{M}(\mathbb{R}^d)$ – the space of signed Radon measures with finite mass. The duality pairing is given by

$$\langle \mu, f \rangle = \int_{\mathbb{R}^d} f(\lambda)\, d\mu(\lambda).$$

Definicja 1 *A map $\mu : \Omega \to \mathcal{M}(\mathbb{R}^d)$ is called weakly* measurable if the functions $x \to \langle \mu(x), f \rangle$ are measurable for all $f \in C_0(\mathbb{R}^d)$.*

Twierdzenie 1 (Fundamental theorem on Young measures) *Let $\Omega \subset \mathbb{R}^d$ and let $z^j : \Omega \to \mathbb{R}^d$ be a sequence of measurable functions. Then there exists (z^{j_k}) and a weakly* measurable map $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$ such that:*

i) $\nu_x \geq 0, \quad \|\nu_x\|_{\mathbb{M}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} d\nu_x \leq 1$ *for a.a.* $x \in \Omega$.

ii) For all $f \in C_0(\mathbb{R}^d)$

$$f(z^{j_k}) \overset{*}{\rightharpoonup} \overline{f}, \text{ where } \overline{f}(x) = \langle \nu_x, f \rangle \text{ in } L^\infty(\Omega)$$

(iii) Let $K \subset \mathbb{R}^d$ *be compact. Then*

$$\operatorname{supp}\nu_x \subset K \text{ if } \operatorname{dist}(z^{j_k}, K) \to 0 \text{ in measure.}$$

iv) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1$ *for a.a.* $x \in \Omega \iff$ *the 'tightness condition' is satisfied, i.e.*

$$\lim_{M\to\infty} \sup_k |\{|z^{j_k}| \geq M\}| = 0.$$

v) Tightness condition + $A \subset \Omega$ *is measurable,* $f \in C(\mathbb{R}^d)$ *and* $f(z^{j_k})$ *is relatively weakly compact in* $L^1(A)$*, then*

$$f(z^{j_k}) \rightharpoonup \overline{f} \text{ where } \overline{f}(x) = \langle \nu_x, f\rangle \quad \text{in } L^1(A). \tag{0.1}$$

Remark *The map* $\nu : \Omega \to \mathbb{M}(\mathbb{R}^d)$ *is called the Young measure generated by the sequence* (z^{j_k})*. Every (weakly* measurable map)* $\nu : \Omega \to \mathbb{M}(\mathbb{R}^d)$ *that satisfies (i) is generated by some sequence* (z^k)*.*

Proof: Let

$$\nu_x^j = \delta_{\{z^j(x)\}}.$$

Then

$$\|\nu_x^j\|_{\mathbb{M}(\mathbb{R}^d)} = 1 \quad \text{and} \quad \langle \nu_x^j, f\rangle = f(z^j(x))$$

$\Rightarrow \nu^j$ is bounded in $L^\infty_w(\Omega; \mathbb{M}(\mathbb{R}^d))$ - the space of weakly* measurable maps $\mu : \Omega \to \mathbb{M}(\mathbb{R}^d)$ that are essentially bounded. Note that

$$L^\infty_w(\Omega; \mathbb{M}(\mathbb{R}^d)) \cong \left(L^1(\Omega; C_0(\mathbb{R}^d))\right)^*$$

with the duality pairing

$$\langle \mu, g\rangle = \int_\Omega \langle \mu_x, g\rangle dx.$$

$\overset{\text{Banach}-\text{Alaoglu}}{\Rightarrow}$

$$\nu^{j_k} \overset{*}{\rightharpoonup} \nu \quad \text{in} \quad L^\infty_w(\Omega; \mathbb{M}(\mathbb{R}^d)) \tag{0.2}$$

Lower semicontinuity of the norm provides $\|\nu_x\|_{\mathbb{M}(\mathbb{R}^d)} \leq 1$.
(0.2) means that for all $\psi \in L^1(\Omega; C_0(\mathbb{R}^d))$

$$\lim_{k\to\infty} \int_\Omega \psi(x, z^{j_k}(x))dx = \int_\Omega \langle \nu_x, \psi(x,\cdot)\rangle dx.$$

Choose

$$\psi(x, \lambda) = \varphi(x) f(\lambda)$$

where $\varphi \in L^1(\Omega)$ and $f \in C_0(\mathbb{R}^d)$. Thus

$$f(z^{j_k}) \overset{*}{\rightharpoonup} \langle \nu_x, f\rangle \quad \text{in} \quad L^\infty(\Omega).$$

Let

$$T^M(\lambda) = \begin{cases} 1 & |\lambda| \le M \\ 1 + M - |\lambda| & M \le |\lambda| \le M+1 \\ 0 & |\lambda| \ge M+1 \end{cases}$$

Then

$$\lim_{k\to\infty} \int_\Omega T^M(z^{j_k}(x))dx = \lim_{k\to\infty} \int_\Omega \int_{\mathbb{R}^d} T^M(\lambda) d\delta_{\{z^{j_k}(x)\}} dx$$
$$= \int_\Omega \langle \nu_x, T^M\rangle dx \le \int_\Omega \langle \nu_x, 1\rangle dx = \int_\Omega \|\nu_x\|_{\mathbb{M}(\mathbb{R}^d)} dx$$

Observe that

$$\int_\Omega (1 - T^M(z^{j_k}(x)))dx \le |\{|z^{j_k}| \ge M\}|$$

which means that

$$|\Omega| - |\{|z^{j_k}| \ge M\}| \le \int_\Omega T^M(z^{j_k}(x))dx$$

so

$$|\Omega| - \sup_{k\in\mathbb{N}} |\{|z^{j_k}| \ge M\}| \le \int_\Omega \|\nu_x\|_{\mathbb{M}(\mathbb{R}^d)} dx.$$

Let $M \to \infty$

$$|\Omega| \le \int_\Omega \|\nu_x\|_{\mathbb{M}(\mathbb{R}^d)} dx.$$

Hence ν_x is a probability measure a.e.

Generalized Jensen's inequality

Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be strictly convex, μ - a probability measure on $\mathbb{R}^d$ with compact support. Then

$$\langle \mu, g\rangle \ge g(\langle \mu, \mathrm{Id}\rangle), \tag{0.3}$$

with equality occurring if and only if μ is a Dirac measure.

Lemat 2 *$z^j : \Omega \to \mathbb{R}^d$ generates the Young measure ν. Then*

$$z^j \to z \quad \textit{in measure} \iff \nu_x = \delta_{z(x)} \quad \textit{a.e.}$$

Proof:
($\Rightarrow$) If $z^j \to z$ in measure, then for all $f \in C_0(\mathbb{R}^d)$

$$f(z^j) \to f(z) \quad \text{in measure.}$$

By Theorem 1 (ii) $\langle \nu_x, f \rangle = f(z(x)) = \langle \delta_{\{z(x)\}}, f \rangle$ for all $f \in C_0(\mathbb{R}^d)$ and thus $\nu_x = \delta_{\{z(x)\}}$.

Example: Scalar conservation law

$$\partial_t u + \partial_x f(u) = 0$$

- Let (η, q) be the entropy-entropy flux pair
- $\{u^k(x,t)\}$ a bounded sequence in $L^\infty(\Omega)$
- $\partial_t \eta(u^k) + \partial_x q(u^k) \subset$ compact set in $W^{-1,2}_{loc}(\Omega)$.

Then

$$u^k \overset{*}{\rightharpoonup} \overline{u} \quad \text{and} \quad f(u^k) \overset{*}{\rightharpoonup} \overline{f(u)} \quad \text{in} \quad L^\infty$$

$$\overline{f(u)} \overset{?}{=} f(\overline{u})$$

Div-curl lemma

Let $\Omega \subset \mathbb{R}^d$ and

$$\begin{aligned} U^j &\rightharpoonup U \quad \text{in} \quad L^2(\Omega), \\ V^j &\rightharpoonup V \quad \text{in} \quad L^2(\Omega) \end{aligned}$$

$\operatorname{div} U^j$ and $\operatorname{curl} V^j \subset$ compact set in $W^{-1,2}(\Omega)$. Then

$$U^j \cdot V^j \to U \cdot V \quad \text{in} \quad \mathcal{D}'(\Omega).$$

Consider two entropy-entropy flux pairs

$$(\eta_1, q_1), \qquad (\eta_2, q_2).$$

Notice that

$$\operatorname{div}(q_2(u^k), \eta_2(u^k)), \operatorname{curl}(\eta_1(u^k), -q_1(u^k)) \subset \text{ compact set of } W^{-1,2}$$

Use div-curl lemma and characterization of the limit to conclude

$$\langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle = \langle \nu, \eta_1 q_2 - \eta_2 q_1 \rangle$$

Applying some algebraic inequalities we obtain $[\langle \nu, f \rangle - f(\overline{u})]^2 \leq 0$ hence $\langle \nu, f \rangle = f(\overline{u})$. If f is strictly convex, then generalized Jensen's inequality (0.3) provides that ν is a Dirac measure.

Method of doubling the variables

We consider the equation

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} F(u) = 0, \tag{0.4}$$

We will introduce the class of weak entropy solutions, which is the appropriate one for the above system.

Definicja 2 *Suppose that $\eta = \eta(u)$, $q = q(u)$ are scalar C^1-functions and $F(u)$ is a C^1-function, satisfying*

$$\nabla_u \eta(u) \cdot \nabla_u F(u) = \nabla_u q(u).$$

Such functions (η, q) are called entropy-entropy flux pair for the system (0.4). If η is convex, then (η, q) is called a convex entropy-entropy flux pair.

Definicja 3 *We call $u \in \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R})$ a weak entropy solution to*

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial t} F(u) = 0$$

with the initial data $u^0 \in \mathbb{L}^\infty(\mathbb{R}; \mathbb{R})$ iff

1. u is a weak solution, i.e.

$$\int\limits_{[0,T)\times\mathbb{R}} \left[u(t,x) \cdot \frac{\partial}{\partial t} \psi(t,x) + F(u(t,x)) \cdot \frac{\partial}{\partial x} \psi(t,x) \right] dtdx + \int\limits_{\mathbb{R}} u^0(x) \cdot \psi(0,x) dx = 0$$

for all test functions $\psi \in C^1_c([0,T) \times \mathbb{R}; \mathbb{R})$.

2. The entropy inequality

$$\int\limits_{[0,T)\times\mathbb{R}} \left[\eta(u(t,x))\frac{\partial}{\partial t}\phi(t,x) + q(u(t,x))\frac{\partial}{\partial x}\phi(t,x)\right] dtdx$$
$$+ \int\limits_{\mathbb{R}} \eta(u^0(x))\phi(0,x)dx \quad \geq \quad 0$$

holds for all nonnegative test functions $\phi \in C_c^1([0,T)\times\mathbb{R};\mathbb{R})$ *and all convex entropy-entropy flux pairs* (η, q).

Remark
The above definition is standard in the theory of conservation laws.

Twierdzenie 3 *Let* u, $\overline{u} \in C^0([0,T);\mathbb{L}^1_{loc}(\mathbb{R};\mathbb{R})) \cap \mathbb{L}^\infty([0,T)\times\mathbb{R};\mathbb{R})$ *be two weak entropy solutions to (0.4) with* $(u-\overline{u}) \in \mathbb{L}^\infty([0,T);\mathbb{L}^1(\mathbb{R};\mathbb{R}))$ *and initial data* u^0, $\overline{u}^0 \in \mathbb{L}^\infty(\mathbb{R};\mathbb{R})$.
Then for any $0 < t < T$ *it holds*

$$\|u(t) - \overline{u}(t)\|_{L^1(\mathbb{R})} \leq \|u^0 - \overline{u}^0\|_{L^1(\mathbb{R})}$$

Proof
Henceforth $\eta_\delta(w,\overline{w})$ defined below, becomes an entropy function, $q_\delta(w,\overline{w})$ denotes a corresponding entropy flux, where $\overline{u}$ is a parameter taking values in $\mathbb{R}$, $i = 1,2$

$$\eta_\delta(u,\overline{u}) = \begin{cases} 0 & \text{for} \quad u \leq \overline{u} \\ \frac{(u-\overline{u})^2}{4\delta} & \text{for} \quad \overline{u} < u \leq \overline{u}+2\delta \\ u-\overline{u}-\delta & \text{for} \quad u > \overline{u}+2\delta \end{cases} .$$

Note that

$$\partial_u\eta_\delta(u,\overline{u}) = \begin{cases} 0 & \text{for} \quad u \leq \overline{u} \\ \frac{u-\overline{u}}{2\delta} & \text{for} \quad \overline{u} < u \leq \overline{u}+2\delta \\ 1 & \text{for} \quad u > \overline{u}+2\delta \end{cases} ,$$

and

$$\partial_{\overline{u}}\eta_\delta(u,\overline{u}) = \begin{cases} 0 & \text{for} \quad u \leq \overline{u} \\ -\frac{u-\overline{u}}{2\delta} & \text{for} \quad \overline{u} < u \leq \overline{u}+2\delta \\ -1 & \text{for} \quad u > \overline{u}+2\delta \end{cases} .$$

To the entropy inequality as a test function we can insert nonnegative function $\phi(t,x,\bar{t},\overline{x}) \in C_c^1(((0,T)\times\mathbb{R})^2,\mathbb{R})$. Then for some fixed $(\bar{t},\overline{x})$ the inequality takes the form

$$\int\limits_{[0,T)\times\mathbb{R}} \{\frac{\partial}{\partial t}\phi(t,x,\bar{t},\overline{x})\eta_\delta(u,\overline{u}) + \partial_x\phi(t,x,\bar{t},\overline{x})q_\delta(u,\overline{u})\}dtdx \geq 0$$

In the same manner with (t,x) fixed, the following inequality can be obtained

$$\int\limits_{[0,T)\times\mathbb{R}} \{\frac{\partial}{\partial \bar{t}}\phi(t,x,\bar{t},\overline{x})\eta_\delta(u,\overline{u}) + \frac{\partial}{\partial \overline{x}}\phi(t,x,\bar{t},\overline{x})q_\delta(u,\overline{u})\}d\bar{t}d\overline{x} \geq 0$$

Integrating the first inequality with respect to $(\bar{t},\overline{x})$ and the second with respect to (t,x), then adding them leads to the following result

$$\begin{aligned}\int\limits_{([0,T)\times\mathbb{R})^2} &\{(\frac{\partial}{\partial t}+\frac{\partial}{\partial \bar{t}})\phi(t,x,\bar{t},\overline{x})\eta_\delta(u,\overline{u}) \\ &+(\frac{\partial}{\partial x}+\frac{\partial}{\partial \overline{x}})\phi(t,x,\bar{t},\overline{x})q_\delta(u,\overline{u})\}dtdxd\bar{t}d\overline{x} \geq 0\end{aligned} \tag{0.5}$$

We fix a smooth, compactly supported function $\xi : \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfying $\int_{\mathbb{R}} \xi(x)dx = 1$ and we choose as a test function in (0.5)

$$\phi(t,x,\bar{t},\overline{x}) = \frac{1}{\epsilon^2}\psi\left(\frac{t+\bar{t}}{2},\frac{x+\overline{x}}{2}\right)\xi\left(\frac{t-\bar{t}}{2\epsilon}\right)\xi\left(\frac{x-\overline{x}}{2\epsilon}\right),$$

where $\psi : \mathbb{R}\times\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a C^1-function with a compact support ($\mathrm{supp}(\psi) \subset ((0,T)\times\mathbb{R})^2$). Note that

$$(\frac{\partial}{\partial t}+\frac{\partial}{\partial \bar{t}})\phi(t,x,\bar{t},\overline{x}) = \frac{1}{\epsilon^2}\frac{\partial}{\partial t}\psi\left(\frac{t+\bar{t}}{2},\frac{x+\overline{x}}{2}\right)\xi\left(\frac{t-\bar{t}}{2\epsilon}\right)\xi\left(\frac{x-\overline{x}}{2\epsilon}\right)$$

and

$$(\frac{\partial}{\partial x}+\frac{\partial}{\partial \overline{x}})\phi(t,x,\bar{t},\overline{x}) = \frac{1}{\epsilon^2}\frac{\partial}{\partial x}\psi\left(\frac{t+\bar{t}}{2},\frac{x+\overline{x}}{2}\right)\xi\left(\frac{t-\bar{t}}{2\epsilon}\right)\xi\left(\frac{x-\overline{x}}{2\epsilon}\right).$$

Letting $\epsilon \searrow 0$ we find that the inequality (0.5) yields to (for more details on this step we refer the reader to [1])

$$\int\limits_{[0,T)\times\mathbb{R}} \{\frac{\partial}{\partial t}\psi(t,x)\eta_\delta(u,\overline{u})+\frac{\partial}{\partial x}\psi(t,x)q_\delta(u,\overline{u})\}dtdx \geq 0 \tag{0.6}$$

for all nonnegative C^1-functions ψ with a compact support in $(0,T)\times\mathbb{R}$. Density of the space $C^1_c((0,T)\times\mathbb{R};\mathbb{R})$ in $\mathbb{W}^{1,1}_0([0,T)\times\mathbb{R};\mathbb{R})$, together with the fact that $u,\,\overline{u}\in\mathbb{L}^\infty([0,T)\times\mathbb{R};\mathbb{R})$ imply the above inequality holds also for $\psi\in\mathbb{W}^{1,1}_0([0,T)\times\mathbb{R};\mathbb{R})$. Therefore we use as a test function $\psi(t,x)=\zeta_r(x)\theta_{\epsilon,s}(t)$ where

$$\zeta_r(x)=\begin{cases} 0 & |x|>r+1\\ r+1-|x| & r<|x|<r+1\\ 1 & |x|<r\end{cases} \tag{0.7}$$

and

$$\theta_{\epsilon,s}(t)=\begin{cases} 0 & 0<t\le s \text{ or } t>\tau+\epsilon\\ 1 & s+\epsilon<t\le\tau\\ \frac{1}{\epsilon}t-\frac{s}{\epsilon} & s<t\le s+\epsilon\\ -\frac{1}{\epsilon}t+1+\frac{\tau}{\epsilon} & \tau<t\le\tau+\epsilon\end{cases} \tag{0.8}$$

for $r>0,\ 0<s<\tau,\ 0<\epsilon<\tau-\epsilon$.

$$\frac{1}{\epsilon}\int_{-r}^{r}\int_{\tau}^{\tau+\epsilon}\eta_\delta(u,\overline{u})dtdx-\frac{1}{\epsilon}\int_{\{r<|x|<r+1\}}\int_{s}^{\tau+\epsilon}\theta_{\epsilon,s}(t)q_\delta(u,\overline{u})dtdx\le\frac{1}{\epsilon}\int_{-r}^{r}\int_{s}^{s+\epsilon}\eta_\delta(u,\overline{u})dtdx$$

Let first $s\searrow 0$, and then $\epsilon\searrow 0$. Using in both cases continuity with respect to t (i.e. $u,\,\overline{u}\in C^0\left([0,T);\mathbb{L}^1_{loc}(\mathbb{R};\mathbb{R})\right)$) we conclude that

$$\int_{-r}^{r}\eta_\delta(u,\overline{u})dx-\int_0^\tau\int_{\{r<|x|<r+1\}}q_\delta(u,\overline{u})dxdt\le\int_{-(r+1)}^{(r+1)}\eta_\delta(u^0,\overline{u}^0)dx$$

for all $t\in[0,T)$. Letting first $\delta\searrow 0$, and then $r\to\infty$ we conclude with standard dominated Lebesgue convergence theorem argument

$$\int_{\mathbb{R}}\{[u(\tau,x)-\overline{u}(\tau,x)]^+\le\int_{\mathbb{R}}\{[u^0(x)-\overline{u}^0(x)]^+$$

Interchanging u with $\overline{u}$ leads to the analogous inequality. Adding both inequalities yields the assertion of the theorem. ∎

References

[1] Kruzkov, S., First-order quasilinear equations with several space variables *Mat. Sbornik* **123** (1970), 228-255. English translation: *Math. USSR Sbornik* **10** (1970), 217-273.