1st Lecture

Comparison, Equality and Invariance of Hölder Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

In the first lecture, we recall the notion of Hölder mean (power mean)[i] and we describe the solution of the comparison and equality problems.

Given a real parameter $p \in \mathbb{R}$, the *pth Hölder mean (pth power mean)* $H_p : \bigcup_{n=1}^{\infty} \mathbb{R}_+^n \to \mathbb{R}_+$ is defined by

$$H_p(x_1, \dots, x_n) := \begin{cases} \left(\dfrac{x_1^p + \dots + x_n^p}{n} \right)^{\frac{1}{p}} & \text{if } p \neq 0, \\ \sqrt[n]{x_1 \cdots x_n} & \text{if } p = 0. \end{cases}$$

As particular cases, we have that H_1 is the arithmetic mean, H_0 is the geometric mean and H_{-1} is the harmonic mean.

The following result describes the behavior of Hölder means, as a function of the parameter p.

Theorem. *Hölder means are continuous, symmetric, homogeneous and strict means. Furthermore, for every fixed* $n \in \mathbb{N}$, $x_1, \dots, x_n > 0$ *with* $\min(x_1, \dots, x_n) < \max(x_1, \dots, x_n)$*, the mapping*

$$p \mapsto H_p(x_1, \dots, x_n)$$

is continuous (in particular, at $p = 0$*), strictly increasing and*

$$\lim_{p \to -\infty} H_p(x_1, \dots, x_n) = \min(x_1, \dots, x_n), \qquad \lim_{p \to \infty} H_p(x_1, \dots, x_n) = \max(x_1, \dots, x_n).$$

As a consequence, for the comparison of Hölder means, the following theorem holds:

Theorem. *Let* $p, q \in \mathbb{R}$*. Then the following conditions are equivalent:*

(a) For all $n \in \mathbb{N}$, $x_1, \dots, x_n > 0$,

$$H_p(x_1, \dots, x_n) \leq H_q(x_1, \dots, x_n);$$

(b) For all $x, y > 0$,

$$H_p(x, y) \leq H_q(x, y);$$

(c) $p \leq q$.

For the invariance equation, the following result holds[ii].

Theorem. *Let* $p, q, r \in \mathbb{R}$*. Then the invariance equation*

$$H_r\big(H_p(x, y), H_q(x, y)\big) = H_r(x, y) \qquad (x, y \in \mathbb{R}_+)$$

is satisfied if and only if either $p = q = r$*, i.e., all all the three means are equal to each other, or* $p + q = r = 0$*, i.e.,* H_r *is the geometric mean and* $H_p = H_{-q}$.

[i] O. Hölder. Über einen Mittelwerthsatz. *Nachr. Ges. Wiss. Göttingen*, page 38–47, 1889.

[ii] Z. Daróczy and Zs. Páles. Gauss-composition of means and the solution of the Matkowski–Sutô problem. *Publ. Math. Debrecen*, 61(1-2):157–218, 2002.

2nd Lecture

Comparison, Equality and Homogeneity of Quasi-arithmetic Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

In this second lecture, we recall the notion of quasi-arithmetic mean[iii] and we describe the solution of the comparison, equality and homogeneity problems.

Given a continuous strictly monotonic function $f : I \to \mathbb{R}$, the *quasi-arithmetic mean* $A_f : \bigcup_{n=1}^{\infty} I^n \to I$ is defined by

$$A_f(x_1, \dots, x_n) := f^{-1}\left(\frac{f(x_1) + \cdots + f(x_n)}{n}\right) \qquad (n \in \mathbb{N},\ x_1, \dots, x_n \in I).$$

In the particular case when, for a real parameter p, the function f is given by $f(x) = x^p$ for $p \neq 0$ and f is the logarithmic function for $p = 0$, then the quasi-arithmetic mean A_f so obtained is equal to the *Hölder mean* of parameter p.

For the comparison of quasi-arithmetic means we have the following result.

Theorem. *Let $f, g : I \to \mathbb{R}$ be continuous strictly monotone functions. Then the following conditions are equivalent:*

(i) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$,
$$A_f(x_1, \dots, x_n) \leq A_g(x_1, \dots, x_n);$$
(ii) For all $x, y \in I$,
$$A_f(x, y) \leq A_g(x, y);$$
(iii) If g is increasing (decreasing), then $g \circ f^{-1}$ is convex (concave);

(iv) There exits a function $h : I \to \mathbb{R}$ such that, for all $x, y \in I$,
$$h(y)(f(x) - f(y)) \leq g(x) - g(y).$$

As a consequence, the equality problem admits the following solution.

Theorem. *Let $f, g : I \to \mathbb{R}$ be continuous strictly monotone functions. Then the following conditions are equivalent:*

(i) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$,
$$A_f(x_1, \dots, x_n) = A_g(x_1, \dots, x_n);$$
(ii) For all $x, y \in I$,
$$A_f(x, y) = A_g(x, y);$$
(iii) $g \circ f^{-1}$ is affine;

(iv) There exist constants $a, b \in \mathbb{R}$ with $a \neq 0$ such that $g = af + b$.

An important feature here is that the necessary and sufficient conditions of comparability and equality of quasi-arithmetic means are independent of the number of variables.

For the investigation of homogeneity, one can use the solution of the equality problem and arrives at the result that homogeneous quasi-arithmetic mean are exactly the Hölder means. Another approach is possible by using the separation theorem by Hölder means proved by Páles[iv].

[iii] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge University Press, Cambridge, 1934.

[iv] Zs. Páles. Nonconvex functions and separation by power means. *Math. Inequal. Appl.*, 3(2):169–176, 2000.

3rd Lecture

Characterization and Invariance of Quasi-arithmetic Means

by Zsolt Páles,

Institute of Mathematics, University of Debrecen

In the third lecture, we present the characterization of quasi-arithmetic means with fixed and also with non-fixed number of variables.

The characterization of quasi-arithmetic means discovered by Kolmogorov is as follows[v].

Theorem. *Let $M : \bigcup_{n=1}^{\infty} I^n \to I$ be a function. Then M is a quasi-arithmetic mean, i.e., $M = A_f$ for some continuous strictly monotone function $f : I \to \mathbb{R}$ if and only if*

(a) M is a continuous symmetric mean;

(b) For all $n \in \mathbb{N}$, $M|_{I^n}$ strictly increasing *in each of its variables;*

(c) M is associative*, i.e., for all $m, n \in \mathbb{N}$ and for all $x_1, \dots, x_n, y_1, \dots, y_m \in I$,*

$$M(x_1, \dots, x_n, y_1, \dots, y_m) = M(m_1, \dots, m_n, y_1, \dots, y_m),$$

where $m_1 = \cdots = m_n = M(x_1, \dots, x_n)$.

The associativity involves the given quasi-arithmetic mean with different number of variables. Therefore, it is unsuitable for characterizing n-variable means only. The characterization of n-variable quasi-arithmetic means using the notion of *bisymmetry* was invented by Aczél[vi].

Theorem. *Let $n \geq 2$ be fixed and let $M : I^n \to I$. Then M is a n-variable quasi-arithmetic mean, i.e., $M = A_f|_{I^n}$ for some continuous strictly monotone function $f : I \to \mathbb{R}$ if and only if*

(a) M is a continuous symmetric n-variable mean;

(b) M strictly increasing in each of its variables;

(c) M is bisymmetric*, i.e., for all $x_{i,j} \in I$ ($i, j \in \{1, \dots, n\}$),*

$$M\big(M(x_{1,1}, \dots, x_{1,n}), \dots, M(x_{n,1}, \dots, x_{n,n})\big) = M\big(M(x_{1,1}, \dots, x_{n,1}), \dots, M(x_{1,n}, \dots, x_{n,n})\big).$$

For the invariance problem of two-variable quasi-arithmetic means, we have the following solution due to Matkowski[vii]and to Daróczy and Páles[viii].

Theorem. *Let $f, g, h : I \to \mathbb{R}$ be continuous strictly monotone functions. Then the invariance equation*

$$A_f\big(A_g(x, y), A_h(x, y)\big) = A_f(x, y) \qquad (x, y \in I)$$

holds if and only if there exist $a, b, c, d, p \in \mathbb{R}$ with $ac \neq 0$ such that

$$g = aE_p \circ f + b, \qquad h = cE_{-p} \circ f + d,$$

where

$$E_p(t) := \begin{cases} \exp(pt) & \text{if } p \neq 0, \\ t & \text{if } p = 0. \end{cases}$$

[v] A. N. Kolmogorov. Sur la notion de la moyenne. *Rend. Accad. dei Lincei (6)*, 12:388–391, 1930.

[vi] J. Aczél. The notion of mean values. *Norske Vid. Selsk. Forh., Trondhjem*, 19(23):83–86, 1947.

[vii] J. Matkowski. Invariant and complementary quasi-arithmetic means. *Aequationes Math.*, 57(1):87–107, 1999.

[viii] Z. Daróczy and Zs. Páles. Gauss-composition of means and the solution of the Matkowski–Sutô problem. *Publ. Math. Debrecen*, 61(1-2):157–218, 2002.

4th Lecture

Equality and Homogeneity of Matkowski Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

In the fourth lecture, we deal with a nonsymmetric generalization of the notion of quasi-arithmetic mean which was invented by Matkowski[ix].

Given the strictly increasing continuous functions $f_1, \dots, f_k : I \to \mathbb{R}$ the Matkowski mean $A_{f_1,\dots,f_k} : I^k \to I$ is defined by

$$A_{f_1,\dots,f_k}(x_1, \dots, x_k) := (f_1 + \cdots + f_k)^{-1}\big(f_1(x_1) + \cdots + f_k(x_k)\big) \qquad (x_1, \dots, x_k \in I).$$

The equality problem of these means leads to a Pexider type functional equation which can easily be solved yielding the following result.

Theorem. *Let I be an open interval, $k \geq 2$ be a fixed integer, $f_1, \dots, f_k$ and $g_1, \dots, g_k$ be strictly increasing continuous functions. Then the functional equation*

$$A_{f_1,\dots,f_k}(x_1, \dots, x_k) = A_{g_1,\dots,g_k}(x_1, \dots, x_k)$$

holds for all $x_1, \dots, x_k \in I$ if and only if there exist constants $a, b_1, \dots, b_k \in \mathbb{R}$ with $a > 0$ such that

$$g_i = af_i + b_i \qquad (i = 1, \dots, k).$$

Using the above solution of the equality problem, also the homogeneous Matkowski means can be determined. They turn out to be the weighted Hölder means.

Theorem. *Let $k \geq 2$ be a fixed integer, $f_1, \dots, f_k$ be strictly increasing continuous functions on $\mathbb{R}_+$. Then the following conditions are equivalent:*

(a) $A_{f_1,\dots,f_k}$ is homogeneous;

(b) There exist functions $a : \mathbb{R}_+ \to \mathbb{R}_+$ and $b_1, \dots, b_k : \mathbb{R}_+ \to \mathbb{R}$ such that, for all $i \in \{1, \dots, k\}$ and $t, x > 0$

$$f_i(tx) = a(t)f_i(x) + b_i(t);$$

(c) There exist constants $p \in \mathbb{R}$, $a_1, \dots, a_k \in \mathbb{R}_+$, $b_1, \dots, b_k \in \mathbb{R}$ such that

$$f_i = a_i\chi_p + b_i \qquad (i = 1, \dots, k);$$

(d) There exist constants $p \in \mathbb{R}$ and $a_1, \dots, a_k \in \mathbb{R}_+$ such that $A_{f_1,\dots,f_k}$ is equal to the pth weighted Hölder mean with weights $a_1, \dots, a_k$.

[ix] J. Matkowski. Generalized weighted and quasi-arithmetic means. *Aequationes Math.*, 79(3):203–212, 2010.

5th Lecture

Comparison of Matkowski Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

The problem of comparison is a more delicate one. To attack it, one should observe that a k-variable Matkowski mean, for all $2 \le m \le k-1$, uniquely determines all the m-variable Matkowski means whose generating functions form a subsequence of the generating functions of the given k-variable Matkowski mean. It is even more surprising, and it is much more difficult to prove that a k-variable Matkowski mean also determines all the k (standard) quasi-arithmetic means, whose generating function is one of the k generating functions of the k-variable Matkowski mean. We call these means the *derived means* of the given k-variable Matkowski mean. The following result holds[x].

Theorem. *Let I be an open interval, $k \ge 2$ be a fixed integer, $f_1, \dots, f_k$ and $g_1, \dots, g_k$ be strictly increasing continuous functions. Then the following assertions are equivalent:*

(a) For all $x_1, \dots, x_k$ in I,

$$A_{f_1,\dots,f_k}(x_1, \dots, x_k) \le A_{g_1,\dots,g_k}(x_1, \dots, x_k);$$

(b) For all $i, j \in \{1, \dots, k\}$ and for all x, y in I,

$$A_{f_i,f_j}(x, y) \le A_{g_i,g_j}(x, y);$$

(c) For all $i, j \in \{1, \dots, k\}$ and for all $x < u < y$ in I,

$$\frac{g_i(u) - g_i(x)}{f_i(u) - f_i(x)} \le \frac{g_j(y) - g_j(u)}{f_j(y) - f_j(u)};$$

(d) There exists a function $h : I \to \mathbb{R}$ such that, for all $i = 1, \dots, k$ and x, u in I,

$$h(u)(f_i(x) - f_i(u)) \le g_i(x) - g_i(u);$$

In the proof of the latter statement, an implicit Gauss type iteration should be constructed. For the existence of this iteration, the Brouwer Fixed Point theorem will be applied. Interesting byproducts of these constructions are that whenever two k-variable Matkowski means are comparable then all their corresponding derived means are comparable. This finally leads to the complete solution of the comparison problem of Matkowski means.

The separation problem of two Matkowski means by weighted Hölder means (which are the homogeneous Matkowski means) will be formulated as an open problem.

To find a characterization theorem for the Matkowski means, is also a challenging open question.

[x]Zs. Páles. Unpublished manuscript.

6th Lecture

Invariance Equations for Matkowski Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

The following result of Baják and Páles[xi] describes all the pairs of Matkowski means that leave the arithmetic mean invariant.

Theorem. *Let $f_1, f_2, g_1, g_2 : I \to \mathbb{R}$ be four times continuously differentiable strictly increasing functions. Then the invariance equation*

$$A_{f_1,f_2}(x,y) + A_{g_1,g_2}(x,y) = x + y$$

holds for all $x, y \in I$ if and only if

(a) either there exist real constants $p, a_1, a_2, c_1, c_2, b_1, b_2, d_1, d_2$ with $p \neq 0$, $a_1a_2 > 0$, $c_1c_2 > 0$ and $a_1c_1 = a_2c_2$ such that, for $x \in I$,

$$f_1(x) = a_1e^{px} + b_1, \qquad f_2(x) = a_2e^{px} + b_2,$$
$$g_1(x) = c_1e^{-px} + d_1, \qquad g_2(x) = c_2e^{-px} + d_2;$$

(b) or there exist real constants a, b, c, d_1, d_2 with $ac \neq 0$ such that, for $x \in I$,

$$f_1(x) + f_2(x) = ax + b,$$
$$g_1(x) = cf_2(x) + d_1, \qquad g_2(x) = cf_1(x) + d_2.$$

One can mention two open problems here. First, if the conclusion of the theorem remains valid without requiring four times continuously differentiability. The second problem is to find the solution (if possible, without strong regularity assumptions) of the following more general invariance equation:

$$A_{h_1,h_2}\big(A_{f_1,f_2}(x,y), A_{g_1,g_2}(x,y)\big) = A_{h_1,h_2}(x,y) \qquad (x, y \in I).$$

Concerning a simpler invariance equation Matkowski and Volkmann[xii] were able to remove the differentiability assumptions.

Theorem. *Let $f_1, f_2 : I \to \mathbb{R}$ be continuous strictly increasing functions. Then the invariance equation*

$$A_{f_1,f_2}(x,y) + A_{f_2,f_1}(x,y) = x + y$$

holds for all $x, y \in I$ if and only if there exist real constants a, b with $a \neq 0$ such that, for $x \in I$,

$$f_1(x) + f_2(x) = ax + b.$$

[xi]Sz. Baják and Zs. Páles. Invariance equation for generalized quasi-arithmetic means. *Aequationes Math.*, 77:133–145, 2009.

[xii]J. Matkowski and P. Volkmann. A functional equation with two unknown functions. *Seminar LV (http://www.mathematik.uni-karlsruhe.de/∼semlv/)*, (30):6 pp., 2008.

7th Lecture

Comparison and Equality of Gini Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

As a generalization of Hölder means, Gini[xiii] introduced a new class of means. For $p, q \in \mathbb{R}$, the *Gini mean* $G_{p,q} : \bigcup_{n=1}^{\infty} \mathbb{R}_+^n \to \mathbb{R}_+$ is defined by

$$G_{p,q}(x_1, \dots, x_n) := \begin{cases} \left(\dfrac{x_1^p + \dots + x_n^p}{x_1^q + \dots + x_n^q} \right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ \exp\left(\dfrac{x_1^p \ln(x_1) + \dots + x_n^p \ln(x_n)}{x_1^p + \dots + x_n^p} \right) & \text{if } p = q. \end{cases}$$

It is immediate to see that If $p \neq 0 = q$ or $p = 0 \neq q$, then $G_{p,q} = H_p$ and if $p = q = 0$, then $G_{0,0} = H_0$, therefore Hölder means are indeed Gini means. For the comparison of Gini means, the following theorem of Daróczy and Losonczi[xiv] holds.

Theorem. *Let $p, q, r, s \in \mathbb{R}$. Then the following conditions are equivalent:*

(a) For all $n \in \mathbb{N}$, $x_1, \dots, x_n > 0$,

$$G_{p,q}(x_1, \dots, x_n) \leq G_{r,s}(x_1, \dots, x_n);$$

(b) For all $n \in \mathbb{N}$ and $x, y > 0$,

$$G_{p,q}(x, \underbrace{y, \dots, y}_{n-\text{times}}) \leq G_{r,s}(x, \underbrace{y, \dots, y}_{n-\text{times}});$$

(c) For all $t > 0$,

$$\chi_{p,q}(t) \leq \chi_{r,s}(t), \qquad \textit{where} \quad \chi_{p,q}(t) := \begin{cases} \dfrac{t^p - t^q}{p - q} & \textit{if } p \neq q, \\ t^p \log(t) & \textit{if } p = q; \end{cases}$$

(d) $\min(p, q) \leq \min(r, s)$ and $\max(p, q) \leq \max(r, s)$.

The comparison of Gini means on compact subintervals of $\mathbb{R}_+$ was obtained by Losonczi[xv].

Theorem. *Let $p, q, r, s \in \mathbb{R}$ and $0 < a < b$. Then the following conditions are equivalent:*

(a) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in [a, b]$,

$$G_{p,q}(x_1, \dots, x_n) \leq G_{r,s}(x_1, \dots, x_n);$$

(b) For all $n \in \mathbb{N}$ and $x, y \in [a, b]$,

$$G_{p,q}(x, \underbrace{y, \dots, y}_{n-\text{times}}) \leq G_{r,s}(x, \underbrace{y, \dots, y}_{n-\text{times}});$$

(c)

$$\chi_{p,q}\left(\frac{a}{b}\right) \leq \chi_{r,s}\left(\frac{a}{b}\right) \qquad \textit{and} \qquad \chi_{p,q}\left(\frac{b}{a}\right) \leq \chi_{r,s}\left(\frac{b}{a}\right).$$

[xiii] C. Gini. Di una formula compressiva delle medie. *Metron*, 13:3–22, 1938.

[xiv] Z. Daróczy and L. Losonczi. Über den Vergleich von Mittelwerten. *Publ. Math. Debrecen*, 17:289–297, 1970.

[xv] L. Losonczi. Über den Vergleich von Mittelwerten die mit Gewichtsfunktionen gebildet sind. *Publ. Math. Debrecen*, 17:203–208, 1970.

8th Lecture

Comparison of Bajraktarević Means

by Zsolt Páles,

Institute of Mathematics, University of Debrecen

In 1958, Bajraktarević[xvi] introduced the following generalization of quasi-arithmetic means. Given two continuous functions $f, g : I \to \mathbb{R}$, such that $g > 0$ and f/g is strictly monotone, the *Bajraktarević mean* $B_{f,g} : \bigcup_{n=1}^{\infty} I^n \to I$ is defined by

$$B_{f,g}(x_1, \dots, x_n) = \left(\frac{f}{g}\right)^{-1} \left(\frac{f(x_1) + \cdots + f(x_n)}{g(x_1) + \cdots + g(x_n)}\right) \qquad (n \in \mathbb{N},\ x_1, \dots, x_n \in I).$$

The first comparison problem of the Bajraktarević means was established by Daróczy and Losonczi[xvii] under differentiability assumptions on the generating functions.

Theorem. *Let $f, g, h, k : I \to \mathbb{R}$ be differentiable functions such that $g, k > 0$ and $(f/g)' \neq 0$, $(h/k)' \neq 0$. Then the following conditions are equivalent:*

(a) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$,
$$B_{f,g}(x_1, \dots, x_n) \leq B_{h,k}(x_1, \dots, x_n);$$

(b) For all $n \in \mathbb{N}$ and $x, y \in I$,
$$B_{f,g}(x, \underbrace{y, \dots, y}_{n-\text{times}}) \leq B_{h,k}(x, \underbrace{y, \dots, y}_{n-\text{times}});$$

(c) For all $x, y \in I$,
$$\frac{\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix}}{\begin{vmatrix} f'(y) & f(y) \\ g'(y) & g(y) \end{vmatrix}} \leq \frac{\begin{vmatrix} h(x) & h(y) \\ k(x) & k(y) \end{vmatrix}}{\begin{vmatrix} h'(y) & h(y) \\ k'(y) & k(y) \end{vmatrix}}.$$

The necessary and sufficient conditions which does not involve differentiability conditions was found 11 years later by Daróczy and Páles[xviii].

Theorem. *Let $f, g, h, k : I \to \mathbb{R}$ be continuous functions such that $g, k > 0$ and f/g, h/k are strictly increasing. Then the following conditions are equivalent:*

(a) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$,
$$B_{f,g}(x_1, \dots, x_n) \leq B_{h,k}(x_1, \dots, x_n);$$

(b) For all $n, m \in \mathbb{N}$ and $x, y \in I$,
$$B_{f,g}(\underbrace{x, \dots, x}_{n-\text{times}}, \underbrace{y, \dots, y}_{m-\text{times}}) \leq B_{h,k}(\underbrace{x, \dots, x}_{n-\text{times}}, \underbrace{y, \dots, y}_{m-\text{times}});$$

(c) There exists a function $\ell : I \to \mathbb{R}$ such that, for all $x, y \in I$,
$$\ell(y) \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} \leq \begin{vmatrix} h(x) & h(y) \\ k(x) & k(y) \end{vmatrix}.$$

[xvi] M. Bajraktarević. Sur une généralisation des moyennes quasilinéaires. *Publ. Inst. Math. (Beograd) (N.S.)*, 3 (17):69–76, 1963.

[xvii] Z. Daróczy and L. Losonczi. Über den Vergleich von Mittelwerten. *Publ. Math. Debrecen*, 17:289–297 (1971), 1970.

[xviii] Z. Daróczy and Zs. Páles. On comparison of mean values. *Publ. Math. Debrecen*, 29(1-2):107–115, 1982.

9th Lecture

Equality and Homogeneity of Bajraktarević Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

The equality problem of Bajraktarević means (with arbitrary many variables) was solved by Aczél and Daróczy[xix].

Theorem. *Let $f, g, h, k : I \to \mathbb{R}$ be continuous functions such that $g, k > 0$ and f/g, h/k are strictly monotone. Then the following conditions are equivalent:*

(a) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$,

$$B_{f,g}(x_1, \dots, x_n) = B_{h,k}(x_1, \dots, x_n);$$

(b) For all $n, m \in \mathbb{N}$ and $x, y \in I$,

$$B_{f,g}(\underbrace{x, \dots, x}_{n-times}, \underbrace{y, \dots, y}_{m-times}) = B_{h,k}(\underbrace{x, \dots, x}_{n-times}, \underbrace{y, \dots, y}_{m-times});$$

(c) There exists a constant $C \neq 0$ such that, for all $x, y \in I$,

$$C \begin{vmatrix} h(x) & h(y) \\ k(x) & k(y) \end{vmatrix} = \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix};$$

(d) There exist four constants a, b, c, d with $ad \neq bc$ such that

$$h = af + bg, \qquad k = cf + dg.$$

Using this characterization, Aczél and Daróczy also found the homogeneous means among Bajraktarević means. In the case when $I = \mathbb{R}_+$, these means are exactly the Gini means.

Theorem. *Let $f, g, h, k : \mathbb{R}_+ \to \mathbb{R}$ be continuous functions such that $g, k > 0$ and f/g, h/k are strictly monotone. Then the following are equivalent:*

(a) $B_{f,g}$ is homogeneous;

(b) There exist four functions $a, b, c, d : \mathbb{R}_+\mathbb{R}$ with $ad \neq bc$ such that

$$f(tx) = a(t)f(x) + b(t)g(x), \qquad g(tx) = c(t)f(x) + d(t)g(x);$$

(c) There exist real constants a, b, c, d, p, q with $ad \neq bc$ such that, for $p \neq q$,

$$f(x) = ax^p + bx^q, \qquad g(x) = cx^p + dx^q \qquad (x > 0)$$

and, for $p = q$,

$$f(x) = ax^p \ln(x) + bx^p, \qquad g(x) = cx^p \ln(x) + dx^p \qquad (x > 0);$$

(d) There exist real constants p, q such that $B_{f,g} = G_{p,q}$.

The equality problem of two variable Bajraktarević means was treated by Losonczi under 6 times continuous differentiability assumptions in 1999. He found 32 other cases of the equality. The comparison problem of two variable Bajraktarević means is still untouched.

[xix] J. Aczél and Z. Daróczy. Über verallgemeinerte quasilineare Mittelwerte, die mit Gewichtsfunktionen gebildet sind. *Publ. Math. Debrecen*, 10:171–190, 1963.

10th Lecture

Characterization of Bajraktarević Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

The characterization of the Bajraktarević means was found by Páles in 1987[xx].

Theorem. *Let $M : \bigcup_{n=1}^{\infty} I^n \to I$ be a function. Then M is a Bajraktarević mean, i.e., $M = B_{f,g}$ for some continuous functions $f, g : I \to \mathbb{R}$ with $g > 0$ and f/g strictly monotone if and only if*

(a) M is a continuous and symmetric mean;

(b) For all $x < u < v < y$ in I, there exists $n, m \in \mathbb{N}$ such that

$$u < M(\underbrace{x, \ldots, x}_{n-\text{times}}, \underbrace{y, \ldots, y}_{m-\text{times}}) < v;$$

(c) For all $n, m \in \mathbb{N}$, $x_1, \ldots, x_n, y_1, \ldots, y_m$,

$$\lim_{k\to\infty} M(\underbrace{x_1, \ldots, x_1}_{k-\text{times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{k-\text{times}}, y_1, \ldots, y_m) = M(x_1, \ldots, x_n);$$

(d) M satisfies the following linking condition*: If, for some $x_1, \ldots, x_n, y_1, \ldots, y_m$, $u_1, \ldots, u_k$, $v_1, \ldots, v_\ell \in I$ the inequalities*

$$M(x_1, \ldots, x_n, u_1, \ldots, u_k) \le M(x_1, \ldots, x_n, v_1, \ldots, v_\ell),$$
$$M(y_1, \ldots, y_m, u_1, \ldots, u_k) \le M(y_1, \ldots, y_m, v_1, \ldots, v_\ell)$$

hold, then

$$M(x_1, \ldots, x_n, u_1, \ldots, u_k, y_1, \ldots, y_m, u_1, \ldots, u_k)$$
$$\le M(x_1, \ldots, x_n, v_1, \ldots, v_\ell, y_1, \ldots, y_m, v_1, \ldots, v_\ell)$$

is also valid.

This characterization theorem, compared to the characterization of quasi-arithmetic means by Kolmogorov, does not contain functional equations for the Bajraktarević means, instead functional inequalities are used.

The invariance equation problems in terms of Bajraktarević means have not been solved yet, they form a challenging problem.

[xx]Zs. Páles. On the characterization of quasi-arithmetic means with weight function. *Aequationes Math.*, 32(2-3):171–194, 1987.

11th Lecture

Comparison, Equality and Homogeneity of Daróczy Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

In this lecture we deal with the theory of deviation means (called Daróczy means in the sequel) which were introduced by Daróczy[xxi].

A map $D : I \times I \to \mathbb{R}$ is called a *deviation* if

— for each fixed $x \in I$, the function $y \mapsto D(x, y)$ is continuous and strictly decreasing.

— for all $x \in I$, $D(x, x) = 0$.

Given a deviation D on I, the *deviation mean* or *Daróczy mean* $M_D : \bigcup_{n=1}^{\infty} I^n \to I$ is defined in the following way: $y = M_D(x_1, \ldots, x_n)$ is the unique solution of the equation

$$D(x_1, y) + \cdots + D(x_n, y) = 0.$$

The following particular cases are of great importance.

1) If $f : I \to \mathbb{R}$ is strictly increasing and continuous and $D(x, y) := f(x) - f(y)$, then $M_D = A_f$, i.e., *quasi-arithmetic means are deviation means*.

2) If $f, g : I \to \mathbb{R}$ are continuous functions such that $g > 0$ and f/g is strictly increasing, and $D(x, y) := g(y)\left(\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right)$, then $M_D = B_{f,g}$, i.e., *Bajraktarević means are deviation means*.

The comparison problem of these means was solved under differentiability assumptions by Daróczy in 1971. In 1982, Daróczy and Páles[xxii] found the necessary and sufficient condition of the comparability Daróczy means without involving and assuming differentiability.

Theorem. *Let $D, E : I^2 \to \mathbb{R}$ be deviations. Then the following conditions are equivalent:*

(a) For all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in I$,

$$M_D(x_1, \ldots, x_n) \le M_E(x_1, \ldots, x_n);$$

(b) For all $k, n \in \mathbb{N}$ and $x, y \in I$,

$$M_D(\underbrace{x, \ldots, x}_{k-times}, \underbrace{y, \ldots, y}_{n-times}) \le M_E(\underbrace{x, \ldots, x}_{k-times}, \underbrace{y, \ldots, y}_{n-times});$$

(c) For all $x, y, z \in I$ with $x < y < z$,

$$\frac{E(x, y)}{D(x, y)} \le \frac{E(z, y)}{D(z, y)};$$

(d) There exists a function $h : I \to \mathbb{R}$ such that, for all $x, y \in I$,

$$h(y)D(x, y) \le E(x, y).$$

The above result, obviously directly leads to the solution of the equality and henceforth, of the homogeneity problems. As particular cases of these results, the comparison theorems of quasi-arithmetic and Bajraktarević means can also be deduced.

[xxi] Z. Daróczy. A general inequality for means. *Aequationes Math.*, 7(1):16–21, 1971.

[xxii] Z. Daróczy and Zs. Páles. On comparison of mean values. *Publ. Math. Debrecen*, 29(1-2):107–115, 1982.

12th Lecture

Multiplicativity and Characterization of Daróczy Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

The applications of deviation means to information theory lead to a property, called *multiplicativity*. Recalling the results of the paper by Daróczy and Páles[xxiii], it will be proved that Daróczy means that satisfy this multiplicativity property are exactly the Gini means. Thus, among deviation means, only the Gini means are good candidates to define generalized entropies.

Theorem. *Let $D : \mathbb{R}_+^2 \to \mathbb{R}$ be a deviation. Then the following two conditions are equivalent*

(a) M_D is multiplicative, i.e., for all $n, m \in \mathbb{N}$, $x_1, \dots, x_n, y_1, \dots, y_m > 0$,

$$M(x_1y_1, \dots, x_1y_m, \dots, x_ny_1, \dots, x_ny_m) = M(x_1, \dots, x_n)M(y_1, \dots, y_m)$$

(b) M_D is a Gini mean, i.e., there exist $p, q \in \mathbb{R}$ such that $M_D = G_{p,q}$.

The characterization of deviation means require the slight extension of the notion of deviation and Daróczy means. These more general notions, called *quasi-deviation and quasi-deviation means*, were introduced by Páles[xxiv]. We will present the characterization theorem of quasi-deviation means obtained therein by using the notions of *internality and infinitesimality*. The interesting feature of this characterization, contrary to the characterization of quasi-arithmetic means by Kolmogorov, is that the main properties are expressed in terms of systems of functional inequalities instead of systems of functional equations.

Theorem. *Let $M : \bigcup_{n=1}^{\infty} I^n \to I$ be a function. Then M is a quasi-deviation mean, i.e., $M = M_D$ for some quasi-deviation $D : I^2 \to \mathbb{R}$ if and only if*

(a) M is a symmetric mean;

(b) M is infinitesimal, i.e., for all $x < u < v < y$ in I, there exists $n, m \in \mathbb{N}$ such that

$$u < M(\underbrace{x, \dots, x}_{n-times}, \underbrace{y, \dots, y}_{m-times}) < v;$$

(c) M is internal, i.e., for all $n, m \in \mathbb{N}$, $x_1, \dots, x_n, y_1, \dots, y_m$,

$$\begin{aligned}\min(M(x_1, \dots, x_n), M(y_1, \dots, y_m)) &\le M(x_1, \dots, x_n, y_1, \dots, y_m)\\ &\le \max(M(x_1, \dots, x_n), M(y_1, \dots, y_m))\end{aligned}$$

and both inequalities are strict if $M(x_1, \dots, x_n) \ne M(y_1, \dots, y_m)$.

Combining the above theorems, one can obtain a characterization theorem of Gini means.

[xxiii]Z. Daróczy and Zs. Páles. Multiplicative mean values and entropies. In *Functions, series, operators, Vol. I, II (Budapest, 1980)*, pp. 343–359. North-Holland, Amsterdam, 1983.

[xxiv]Zs. Páles. Characterization of quasideviation means. *Acta Math. Acad. Sci. Hungar.*, 40(3-4):243–260, 1982.

13th Lecture

Comparison of Two-Variable Gini Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

In this lecture, the comparison problem of two-variable Gini means is discussed. Recall, that for $p, q \in \mathbb{R}$, the *two-variable Gini mean* $G_{p,q} : \mathbb{R}^2 \to \mathbb{R}_+$ is defined by

$$G_{p,q}(x,y) := \begin{cases} \left(\dfrac{x^p + y^p}{x^q + y^q}\right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ \exp\left(\dfrac{x^p \ln(x) + y^p \ln(y)}{x^p + y^p}\right) & \text{if } p = q, \end{cases}$$

The following theorem of Páles[xxv] describes the comparison of two-variable Gini means when the domain of the means is $\mathbb{R}_+$.

Theorem. *Let $p, q, r, s \in \mathbb{R}$. Then the following conditions are equivalent:*

(a) For all $x, y > 0$,

$$G_{p,q}(x,y) \leq G_{r,s}(x,y);$$

(b) The following three inequalities hold:

$$p + q \leq r + s, \qquad m(p,q) \leq m(r,s), \qquad \mu(p,q) \leq \mu(r,s),$$

where

$$m(p,q) := \begin{cases} \min(p,q) & \textit{if } p, q \geq 0, \\ 0 & \textit{if } pq < 0, \\ \max(p,q) & \textit{if } p, q \leq 0, \end{cases}$$

$$\mu(p,q) := \begin{cases} \dfrac{|p| - |q|}{p - q} & \textit{if } p \neq q, \\ \textit{sign}(p) & \textit{if } p = q. \end{cases}$$

When the domain is a compact subinterval of $\mathbb{R}_+$, then another theorem of Páles[xxvi] gives an answer to the comparison problem.

Theorem. *Let $p, q, r, s \in \mathbb{R}$ and $0 < a < b$. Then the following conditions are equivalent:*

(a) For all $x, y \in [a,b]$,

$$G_{p,q}(x,y) \leq G_{r,s}(x,y);$$

(b) The following two inequalities hold:

$$p + q \leq r + s, \qquad G_{p,q}(a,b) \leq G_{r,s}(a,b).$$

[xxv] Zs. Páles. Inequalities for sums of powers. *J. Math. Anal. Appl.*, 131(1):265–270, 1988.

[xxvi] Zs. Páles. Comparison of two variable homogeneous means. In W. Walter, editor, *General Inequalities, 6 (Oberwolfach, 1990)*, International Series of Numerical Mathematics, page 59–70. Birkhäuser, Basel, 1992.

14th Lecture

Comparison of Stolarsky Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

For $p, q \in \mathbb{R}$, the *Stolarsky mean*[xxvii] $S_{p,q} : \mathbb{R}^2 \to \mathbb{R}_+$ is defined by

$$S_{p,q}(x,y) := \begin{cases} \left(\dfrac{q(x^p - y^p)}{p(x^q - y^q)}\right)^{\frac{1}{p-q}} & \text{if } pq(p-q)(x-y) \neq 0, \\ \left(\dfrac{x^p - y^p}{p(\ln x - \ln y)}\right)^{\frac{1}{p}} & \text{if } p(x-y) \neq 0, q = 0, \\ \left(\dfrac{q(\ln x - \ln y)}{x^q - y^q}\right)^{-\frac{1}{q}} & \text{if } q(x-y) \neq 0, p = 0, \\ \exp\left(-\dfrac{1}{p} + \dfrac{x^p \ln x - y^p \ln y}{x^p - y^p}\right) & \text{if } p(x-y) \neq 0, p = q, \\ \sqrt{xy} & \text{if } x - y \neq 0, p = q = 0, \\ x & \text{if } x - y = 0. \end{cases}$$

For the comparison of Stolarsky means, the following two theorems of Páles hold[xxviii],[xxix].

Theorem. *Let $p, q, r, s \in \mathbb{R}$. Then the following conditions are equivalent:*

(a) For all $x, y > 0$,

$$S_{p,q}(x,y) \le S_{r,s}(x,y);$$

(b) The following three inequalities hold:

$$p + q \le r + s, \qquad \ell(p,q) \le \ell(r,s), \qquad \mu(p,q) \le \mu(r,s),$$

where

$$\ell(p,q) := \begin{cases} \dfrac{\ln(p/q)}{p-q} & \textit{if } pq > 0, \\ 0 & \textit{if } pq \le 0, \end{cases}$$

$$\mu(p,q) := \begin{cases} \dfrac{|p| - |q|}{p-q} & \textit{if } p \neq q, \\ \textit{sign}(p) & \textit{if } p = q. \end{cases}$$

Theorem. *Let $p, q, r, s \in \mathbb{R}$ and $0 < a < b$. Then the following conditions are equivalent:*

(a) For all $x, y \in [a,b]$,

$$S_{p,q}(x,y) \le S_{r,s}(x,y);$$

(b) The following two inequalities hold:

$$p + q \le r + s, \qquad S_{p,q}(a,b) \le S_{r,s}(a,b).$$

[xxvii] K. B. Stolarsky. Generalizations of the logarithmic mean. *Math. Mag.*, 48:87–92, 1975.

[xxviii] Zs. Páles. Inequalities for differences of powers. *J. Math. Anal. Appl.*, 131(1):271–281, 1988.

[xxix] Zs. Páles. Comparison of two variable homogeneous means. In W. Walter, editor, *General Inequalities, 6 (Oberwolfach, 1990)*, International Series of Numerical Mathematics, page 59–70. Birkhäuser, Basel, 1992.

15th Lecture

Invariance Equations for Two-Variable Gini and for Stolarsky Means

by ZSOLT PÁLES,

Institute of Mathematics, University of Debrecen

The results presented in this lecture are from two papers of Baják and Páles[xxx],[xxxi], and they completely describe the solutions of the two invariance equations.

The first result is for two-variable Gini means.

Theorem. *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation*

$$G_{p,q}\big(G_{a,b}(x,y), G_{c,d}(x,y)\big) = G_{p,q}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

is satisfied if and only if one of the following 6 possibilities hold:

(a) $a+b=c+d=p+q=0$*, i.e., all the three means are equal to the geometric mean,*
(b) $\{a,b\}=\{c,d\}=\{p,q\}$*, i.e., all the three means are equal to each other,*
(c) $\{a,b\}=\{-c,-d\}$ *and* $p+q=0$*, i.e.,* $G_{p,q}$ *is the geometric mean and* $G_{a,b}=G_{-c,-d}$*,*
(d) there exist $u, v \in \mathbb{R}$ *such that* $\{a,b\}=\{u+v, v\}$*,* $\{c,d\}=\{u-v,-v\}$*, and* $\{p,q\}=\{u,0\}$ *(in this case,* $G_{p,q}$ *is a power mean),*
(e) there exists $w \in \mathbb{R}$ *such that* $\{a,b\}=\{3w,w\}$*,* $c+d=0$*, and* $\{p,q\}=\{2w,0\}$ *(in this case,* $G_{p,q}$ *is a power mean and* $G_{c,d}$ *is the geometric mean),*
(f) there exists $w \in \mathbb{R}$ *such that* $a+b=0$*,* $\{c,d\}=\{3w,w\}$*, and* $\{p,q\}=\{2w,0\}$ *(in this case,* $G_{p,q}$ *is a power mean and* $G_{a,b}$ *is the geometric mean).*

The second result is for Stolarsky means.

Theorem. *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation*

$$S_{p,q}\big(S_{a,b}(x,y), S_{c,d}(x,y)\big) = S_{p,q}(x,y) \qquad (x, y \in \mathbb{R}_+)$$

is satisfied if and only if one of the following 3 possibilities hold:

(a) $a+b=c+d=p+q=0$*, i.e., all the three means are equal to the geometric mean,*
(b) $\{a,b\}=\{c,d\}=\{p,q\}$*, i.e., all the three means are equal to each other,*
(c) $\{a,b\}=\{-c,-d\}$ *and* $p+q=0$*, i.e.,* $S_{p,q}$ *is the geometric mean and* $S_{a,b}=S_{-c,-d}$*.*

Invariance equations that mix up Gini and Stolarsky means can also be considered. The solution to these problems require computer algebraic tools, because the method is to differentiate the above invariance equations up to 12 times and then to solve the systems of so obtained polynomial equations for the 6 unknown parameters.

[xxx]Sz. Baják and Zs. Páles. Computer aided solution of the invariance equation for two-variable Gini means. *Comput. Math. Appl.*, 58:334–340, 2009.

[xxxi]Sz. Baják and Zs. Páles. Computer aided solution of the invariance equation for two-variable Stolarsky means. *Appl. Math. Comput.*, 216(11):3219–3227, 2010.