SOME COMBINATORICS AND TOPOLOGY RELATED TO THE BANACH SPACES c_0 AND l_1

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Abstract. This is an outline of lectures for PhD students to be delivered in November 2013 at *Instytut Matematyki Uniwersytetu Śląskiego* within the programme of *Środowiskowe Studia Doktoranckie z Nauk Matematycznych.*

The presentation widely uses several chapters from Albiac & Kalton [AK], some elements are taken from the most enjoyable book on Banach space, that is [Di] by Joe Diestel. The proof of Rosenthal theorem follows Todorcevic [To]. Some arguments and examples are taken from the folklore.

The results we discuss in details are (nearly) classical but we also mention some more recent extensions.

Contents

1. Introduction

We consider real Banach spaces, often denoted by $X, Y, \ldots$ and, unless stated otherwise, assume that every Banach space is infinite dimensional. Saying that Y is a subspace of X we always mean a **closed** subspace.

Recall that c_0 is the space of sequences of the reals converging to 0 with the norm $x = \sup_n |x_n|$, while l_1 is the space of all absolutely summing sequences with the norm $||x|| = \sum_n |x(n)|$. In either space e_n denotes the n-th unit vector ($e_n(k) = 1$ if $n = k$ and $e_n(k) = 0$ otherwise). Recall that $c_0^* = l_1$ and $l_1^* = l_\infty$, where l_∞ is the space of all bounded sequences equipped with the supremum norm. Writing $e_n^* \in l_1$ we mean that the unit vector is treated as a functional on c_0, $e_n^*(x) = x(n)$ for $x \in c_0$.

If Γ is any set then $l_\infty(\Gamma)$ is the space of all bounded functions $\Gamma \to \mathbb{R}$, equipped with the supremum norm. We shall also consider $C(K)$ spaces, of continuous functions on a given compactum.

We use a typical jargon of functional analysis: for instance the phrase X *contains a copy of* c_0 means that there is an isomorphic embedding $T : c_0 \to X$ so that c_0 is isomorphic to a subspace of X. Recall that $T : X \to Y$ is an isomorphism onto its image if

$$m \cdot ||x| \le ||Tx|| \le M \cdot ||x||,$$

for every $x \in X$, where $m, M > 0$ are some constants.

Definition 1.1. *If Y is a subspace of a Banach space X then Y is said to be* **complemented** *in X if there is a subspace $Z \subseteq X$ such that $X = Y \oplus Z$.*

Definition 1.2. *A bounded linear operator $P : X \to X$ is called a* projection *if $P \circ P = P$.*

Lemma 1.3. *If $P : X \to X$ is a projection then $Y = P[X]$ is a complemented subspace of X.*

Proof. Clearly Y is a closed subspace of X since $x \in Y$ iff $x = Px$. For every $x \in X$ we can write $x = Px + z$ where $z = (x - Px)$ belongs to $\ker P$ since $Pz = P(x-Px) = Px-P^2x = 0$. Therefore $X = Y \oplus \ker P$ (if $x \in Y \cap \ker P$ then $x = Px'$ and $Px = 0$ so $x = Px' = PPx' = Px = 0$; hence $Y \cap \ker P = \{0\}$). $\square$

Lemma 1.4. *If Y is complemented in X then there is a projection P from X onto Y.*

Proof. Writing $X = Y \oplus Z$ we can define $P : X \to X$ by $P(y + z) = y$. Then P is well-defined linear operator; we check that P is bounded.

Let $|||x||| = |y|| + ||z||$ whenever $x = y + z$, $y \in Y$, $z \in Z$. Then $||| \cdot |||$ is a norm on X and $(X, ||| \cdot |||)$ is a Banach space (!, because Y, Z are closed). Since $||x|| \le |||x|||$ for every $x \in X$, it follows (!) that $|||x||| \le C \cdot ||x||$ for some constant $C > 0$. Then

$$||P(y + z)|| = ||y|| \le |||y + z||| \le C \cdot ||y + z||,$$

so P is bounded. $\square$

Example 1.5. *c_0 is a subspace of the space c, of all convering sequences of the reals. Let*

$$Px = (x_1 - \lim_n x_n, x_2 - \lim_n x_n, \ldots);$$

then P is a projection onto c_0.

Note that if P is any projection from c onto c_0 then $||P|| \geq 2$. Indeed, if $x = (1, 1, \ldots)$ then $x - 2e_n$ has norm 1 and $||P(x - 2e_n)|| = ||Px - 2e_n|| \to 2$ as $n \to \infty$.

Exercise 1.6. *If Y has a finite codimension in X (i.e. X/Y is finite dimensional) then Y is complemented in X.*

Recall that for every Banach space X, the space $(B_{X^*}, weak^*)$ is compact (Banach-Alaoglu). We shall use the following standard fact.

Theorem 1.7. *If X is separable then $(B_{X^*}, weak^*)$ is a metrizable compactum*

Proof. Let x_n form a dense set in B_X. Examine the metric ρ, where

$$\rho(x^*, y^*) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x^*(x_n) - y^*(x_n)|.$$

$\square$

2. Basic properties of c_0 and l_1

We first show that every separable Banach space X is a quotient of l_1.

Theorem 2.1. *For every separable Banach space X there is a bounded linear onto operator $T : l_1 \to X$.*

Proof. Let the set $\{x_n : n \in \mathbb{N}\}$ be dense in the unit ball B_X. We define $T : l_1 \to X$ by $T\alpha = \sum_n \alpha(n) x_n$. Then $||T|| = 1$ and $T[B_{l_1}]$ is dense in B_X.

Fix $x \in X$ with $||x|| < 1$ and $\varepsilon > 0$. There is $\alpha_1 \in B_{l_1}$ such that $||x - T\alpha_1|| < \varepsilon$. Next choose α_2' such that $||(1/\varepsilon)(x - T\alpha_1) - T\alpha_2'|| < \varepsilon$. For $\alpha_2 = \varepsilon \cdot \alpha_2'$ this gives $||x - T(\alpha_1 + \alpha_2)|| < \varepsilon^2$.

Continue; finally for $\alpha = \sum_n \alpha_n$ we get $T\alpha = x$. $\square$

Theorem 2.2. *The space l_1 has the Schur property, that is every weakly convergent sequence in l_1 is norm convergent.*

Proof. Assume that $x_n \to 0$ weakly while $1 \geq ||x_n|| \geq \delta > 0$. We shall see (on the blackboard) that there is an infinite set $N \subseteq \mathbb{N}$ and a sequence of pairwise disjoint finite sets $I_n \subseteq \mathbb{N}$ for $n \in N$ such that $\sum_{k \notin I_n} |x_n(k)| \leq \delta/4$ for $n \in N$. This enables to define $x^* \in l_\infty$ such that $|x^*(x_n)| \geq \delta/2$ for $n \in N$, a contradiction. □

Observe that $L_1[0,1]$ does not have the Schur property — try $f_n \in L_1[0,1]$, $f_n(t) = \sin(2\pi t n)$. Therefore $L_1[0,1]$ is not isomorphic to l_1. Note that $L_2[0,1]$ and l_2 are isometric because they are Hilbert.

Clearly in c_0 the sequence $(e_n)_n$ converges to 0 weakly while $||e_n|| = 1$; hence

Corollary 2.3. *l_1 does not contain a copy of c_0.*

Definition 2.4. *A sequence $(x_n)_n$ in a Banach space is said to be* **weakly Cauchy** *if $x^*(x_n)$ converges for every $x^* \in X^*$.*

Exercise 2.5. *Check that in c_0 every bounded sequence has a weakly Cauchy subsequence.*

Note that c_0 does not contain a copy of l_1: in l_1 the sequence (e_n) does not have a subsequence which would be weakly Cauchy.

Exercise 2.6. *Note that if a Banach space X has the Schur property then*

(i) *every weakly compact set $A \subseteq X$ is norm compact;*
(ii) *X is weakly sequentially complete (i.e. every weakly Cauchy sequence has a limit).*

Corollary 2.7. *There is an uncomplemented subspace X of l_1.*

Proof. By Theorem 2.1 there is a surjective operator $T : l_1 \to c_0$. Then $\ker T$ cannot be complemented in l_1; otherwise, if $l_1 = \ker T \oplus Y$ then Y is isomorphic to c_0, a contradiction with Corollary 2.3 □

Remarks.

(i) Note that if X is (isomorphic to) a Hilbert space then every subspace of X is complemented. Johnson and Lindenstrauss showed that this can be reversed; in other words every X which is not isomorphic to a Hilbert space contains an uncomplemented subspace.
(ii) If we embedd l_1 into $C[0,1]$ (which we can do !) then its copy cannot be complemented because one can check that $C[0,1]^*$ is weakly sequentially complete while $l_\infty = l_1^*$ is not.

(iii) If X is reflexive that X cannot be complement in $L_1[0,1]$ or $C[0,1]$ (see [AK], Theorem 5.4.7).
(iv) X^* is always complemented in X^{***} because $X^{***} = X^* \oplus X^\perp$, where $X^\perp = \{x^{***} : x^{***}|X \equiv 0\}$. For $X = c_0$ this gives $l_\infty^* = l_1 \oplus c_0^\perp$.
(v) If $\varphi : 2^{\mathbb{N}} \to [0,1]$ is the usual surjection, $\varphi(x) = \sum_n x(n)/2^n$, then $g \to g \circ \varphi$ is an isometry between $C[0,1]$ and some $X \subseteq C(2^{\mathbb{N}})$. The space X is not complemented in $C(2^{\mathbb{N}})$, see e.g. Rosenthal survey [Ro].

3. From c_0 to injective Banach spaces

Theorem 3.1 (Sobczyk). *The space c_0 is complemented in every separable superspace X.*

Proof. (the sketch following [Di])

Exend every $e_n^* \in l_1 = c_0^*$ to a norm one functional $x_n^* \in X$. Let

$$F = \{x^* \in B_{X^*} : x^*|c_0 \equiv 0\}.$$

Note that any *weak** limit point of x_n^* belongs to F. Since X is separable $(B_{X^*}, weak^*)$ is a metrizable compactum; say ρ is a metric on the ball. Then $\lim_n \operatorname{dist}_\rho(x_n, F) = 0$.

Choose $z_n^* \in F$ so that $\lim_n \rho(x_n^*, z^*) = 0$. Define P on X by

$$Px = (x_n^*(x) - z_n^*(x))_n$$

; then P is the required projection. □

By a result due to Zippin, c_0 is (up to isomorphism) the only separable space satisfying 3.1.

Theorem 3.2 (Phillips-Sobczyk). *The space c_0 is not complemented in l_∞.*

Proof. We check that if $T : l_\infty \to l_\infty$ is any bounded operator such that $c_0 \subseteq \ker T$ then there is infinite $A \subseteq \mathbb{N}$ such that T is zero on the space $l_\infty(A)$ (of all $x \in l_\infty$ with support contained in A). Once we know this, it is immediate that there is no projection $P : l_\infty \to c_0$: try $T = Id - P$.

We argue by contradiction. Let $\{A_\gamma : \gamma \in \Gamma\}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$, where Γ is uncountable (so $A_\gamma \cap A_\eta$ is finite whenever $\gamma \neq \eta$ are in Γ). For every γ there is $x_\gamma \in l_\infty(A_\gamma)$ such that $Tx_\gamma \neq 0$. There is $n \in \mathbb{N}$ such that $\Gamma_n = \{\gamma \in \Gamma : x_\gamma(n) \neq 0\}$ is uncountable, and so there is

k such that $\Gamma_{n,k} = \{\gamma \in \Gamma : |x_\gamma(n)| \geq 1/k\}$ is uncountable. Choose r_γ so that $r_\gamma \cdot x_\gamma(n) = |x_\gamma(n)|$.

Take any finite $I \subseteq \Gamma_{n,k}$ and consider $y_I = \sum_{\gamma \in I} r_\gamma x_\gamma$. Then $||Ty_I|| \leq ||T||$ (!) so $|I|/k = e_n^* T y_I \leq ||T||$, which is a contradiction if we take a set I with $|I| \geq k \cdot ||T||$. □

Exercise 3.3. *Modify the above argument to check that $C[0,1]$ is not complemented on $L_\infty[0,1]$.*

Note that Theorem 3.2 says only that a concrete copy of c_0 inside l_∞ is not complemented. This can be generalized to saying that whenever $T : c_0 \to l_\infty$ is an isomorphic embedding then $T[c_0]$ is not complemented in l_∞. Indeed, and possible complement would be isomorphic to l_∞/c_0, and it is enough to know that l_∞/c_0 cannot be embedded into l_∞. There are several reasons for this - see an interesting discussion presented in [CCY].

Definition 3.4. *A Banach space X is* **injective** *if for every Banach spaces $E \subseteq F$ every bounded operator $T : E \to X$ has an extension to a bounded operator $\widetilde{T} : F \to X$. If, moreover, we can demand that $||\widetilde{T}|| = ||T||$ then X is called* **1-injective** *(or isometrically injective).*

Theorem 3.5. *The space l_∞ is 1-injective. More generally, the space*

$$l_\infty(\Gamma) = \{x : \Gamma \to \mathbb{R} : ||x|| = \sup_\gamma |x(\gamma)| < \infty\},$$

is 1-injective.

Proof. Given $E \subseteq F$ and $T : E \to l_\infty$, we can write $Tx = (e_n^*(x))_n$ and $||T|| = \sup_n ||e_n^*||$, where $e_n^* \in E^*$ is defined as $e_n^*(x) = Tx(n)$. Extend every e_n^* to $f_n^* \in F^*$ with the same norm using the Hahn-Banach theorem. Then $\widetilde{T} : F \to X$, $\widetilde{T}x = (f_n^*(x))_n$ is as required.

The proof for $l_\infty(\Gamma)$ is virtually the same. □

Theorem 3.6. *A Banach space is injective if and only if X is complemented in any superspace.*

Proof. Forward implication: if $X \subseteq Y$ then apply the definition of injectivity to $E = X, F = Y$ to see that $Id : X \to X$ extends to $P : Y \to X$, which is clearly a projection.

To check the backward implication note that every X can be isometrically embedded into some $l_\infty(\Gamma)$ (!!). Then we have a projection $P : l_\infty(\Gamma) \to X$. If $E \subseteq F$, $T : E \to X$ are given then T extends to $\widetilde{T} : F \to l_\infty(\Gamma)$ by Theorem 3.5, and $P \circ \widetilde{T} : F \to X$ extends T. □

Looking at Theorem 3.1, we can say that c_0 is separably injective. The space c_0 can be complemented also in some nonseparable spaces:

Example 3.7. *The space c_0 is complemented in every space $C(K)$, where a compact space K contains a nontrivial converging sequence.*

Proof. Let $t_n \to t$ in K, where t_n are all different. We can find pairwise disjoint open sets $U_n \subseteq K$ such that $t_n \in U_n$, and continuous functions $\varphi_n : K \to [0,1]$ such that $\varphi_n(t_n) = 1$, $\varphi_n|(K \setminus U_n) \equiv 0$.

The operator $T : c_0 \to C(K)$, $Tx = \sum_n x(n)\varphi_n$ is an isometric embedding of c_0 into $C(K)$ (!). The space $X = T[c_0]$ is complemented in $C(K)$ because

$$Pg = \sum_n (g(t_n) - g(t))\varphi_n,$$

is a projection. □

Definition 3.8. *A Banach space is said to be a* **Grothendieck space** *if every sequence $x_n^* \in X^*$ which is weak* convergent is necessarily weakly convergent.*

Theorem 3.9. *A Grothendieck space X cannot contain a complemented copy of c_0.*

Proof. Suppose that c_0 lies inside X and that $P : X \to c_0$ is a projection onto c_0. Let e_n^* be a functional on c_0, $e_n^*(x) = x(n)$. Then $e_n^* \circ P \in X^*$ is a sequence *weak** converging but not converging weakly (!). □

The above can be reversed for $C(K)$ spaces:

Theorem 3.10. *$C(K)$ is Grothendieck if and only if $C(K)$ does not contain a complemented copy of c_0.*

Proof. The condition is necessary by Theorem 3.9. If $C(K)$ is not Grothendieck then there is a bounded sequence of signed measures μ_n on K converging in the *weak** topology but not weakly. In particular $\{\mu_n : n \in \mathbb{N}\}$ is not relatively weakly compact and by the Dieudonne-Grothendieck Theorem, see [Di], Thm.

14 on page 98, there is $\varepsilon > 0$ and a pairwise disjoint sequence of open sets U_n such that $|\mu_n(U_n)| \geq \varepsilon$. We can continue as in Example 3.7. □

Corollary 3.11. *If $C(K)$ is injective then $C(K)$ is Grothendieck and, consequently, K contains no nontrivial converging sequences.*

Proof. If $C(K)$ is not Grothendieck then it contains a complemented copy of c_0. If $C(K)$ were injective we could conclude that c_0 is complemented in $l_\infty(K)$. □

4. A characterization of 1-injective spaces

A topological space K is said to be **extremally disconnected** if $\overline{U}$ is open for every open $U \subseteq K$; equivalently, in K the following very strong separation axiom is satisfied:

if $U, V \subseteq K$ are open and $U \cap V = \emptyset$ then $\overline{U} \cap \overline{V} = \emptyset$.

Natural examples of compact extremally disconnected spaces come from the Stone representation of Boolean algebras, which we now outline.

Let $\mathfrak{A}$ be a Boolean algebra with operations $\vee$ and $\wedge$ (if you are not sure about the axioms think of any algebra of sets with the union and intersection). A family $\mathcal{F} \subseteq \mathfrak{A}$ is an ultrafilter in $\mathfrak{A}$ if

(i) $1 \in \mathcal{F}$, $0 \notin \mathcal{F}$;
(ii) if $a, b \in \mathcal{F}$ then $a \wedge b \in \mathcal{F}$;
(iii) if $a \in \mathcal{F}$, $b \geq a$ then $b \in \mathcal{F}$;
(iv) for every $a \in \mathfrak{A}$, either $a \in \mathcal{F}$ of $-a \in \mathcal{F}$.

Conditions (i)—(iii) defines a proper filer in $\mathfrak{A}$, while (iv) says that $\mathcal{F}$ is maximal proper filer in $\mathfrak{A}$. Let $\mathrm{ult}(\mathcal{F})$ denote the family of all ultrafilters in $\mathfrak{A}$. The following summarizes the so-called Stone duality between Boolean algebras and compact totally disconnected compact spaces

Theorem 4.1 (Stone). *If $\mathfrak{A}$ is a Boolean algebra then $\mathrm{ult}(\mathfrak{A})$ is a compact totally disconnected space in the topology generated by the base of all $\widehat{a}$, $a \in \mathfrak{A}$, where*

$$\widehat{a} = \{\mathcal{F} \in \mathrm{ult}(\mathfrak{A}) : \mathrm{a} \in \mathcal{F}\}.$$

The mapping $\mathfrak{A} \ni a \to \widehat{a}$ is a Boolean isomorphisms between $\mathfrak{A}$ and the algebra of clopen sets in $\mathrm{ult}(\mathfrak{A})$.

Theorem 4.2. *The algebra $\mathfrak{A}$ is complete (every subfamily has a least upper bound) if and only if* $\mathrm{ult}(\mathfrak{A})$ *is extremally disconnected.*

Proof. Forward implication: let $U \subseteq \mathrm{ult}(\mathfrak{A})$ be open; write $U = \bigcup_{t\in T} \widehat{a_t}$. Check that $\overline{U} = \widehat{\bigvee_{t\in T} a_t}$.

Backward implication: if $\{a_t : t \in T\} \subseteq \mathfrak{A}$ then $U = \overline{\bigcup_{t\in T} \widehat{a_t}}$ is open so $\overline{U} = \widehat{a}$ for some $a \in \mathfrak{A}$. Check that a is the least upper bound of our family. □

Example 4.3. *If we take $\mathfrak{A} = P(\mathbb{N})$ then* $\mathrm{ult}(\mathfrak{A})$ *is $\beta\omega$ - the maximal compactification of $\mathbb{N}$. l_∞ is isometric to $C(\beta\omega)$.*

Example 4.4. *Let $\mathfrak{A}$ be the the measure algebra, i.e. the family of all equivalence classes $[B]$, $B \in Bor[0,1]$. Here $[B] = \{A \in Bor[0,1] : \lambda(A \triangle B) = 0\}$.*

One can check that $L_\infty[0,1]$ is isometric to $C(\mathrm{ult}(\mathfrak{A}))$.

Theorem 4.5. *If K is an extremally disconnected compact space then $C(K)$ is an order complete Banach lattice, i.e. for every bounded from above family $\Phi \subseteq C(K)$ there is an least upper bound in $C(K)$.*

Proof. One reduces the task to considering a family $\Phi \subseteq C(K)$ of functions $K \to [0,1]$. We shall check that there is $g \in C(K)$ which is the least upper bound, that is: $g \geq f$ for $f \in \Phi$ and for every $g' \in C(K)$ if $g' \geq f$ for $f \in \Phi$ then $g' \geq g$.

Let $\varphi(x) = \sup_{f\in\Phi} f(x)$. Then $\varphi : K \to \mathbb{R}$ is not necessarily continuous but is semicontinuous: the sets of the form $\{x \in K : \varphi(x) > r\}$ are open. Take any $n \in \mathbb{N}$, define $U_n^i = \{\varphi > i/2^n\}$; then $A_{n,i} = \overline{A_{n,i}}$ are open. Let

$$\varphi_n = \frac{1}{2^n}\sum_{i=1}^{n} \chi_{U_{n,i}} \quad g_n = \frac{1}{2^n}\sum_{i=1}^{n} \chi_{A_{n,i}}.$$

Then $\varphi_n \leq \varphi \leq \varphi_n + 1/2^n$ and $\varphi_n \leq \varphi_{n+1}$. It follows (!) that g_n is uniformly convergent. Check that $g = \lim g_n$ is as required. □

Exercise 4.6. *Prove the converse: if $C(K)$ is order complete then K is extremally disconnected. If $U \subseteq K$ is open then consider the least upper bound g of*

$$\{f \in C(K) : 0 \leq f \leq \chi_U\}.$$

Check that $g = \chi_{\overline{U}}$ which implies that $\overline{U}$ is open.

Theorem 4.7 (Goodner, Nachbin, see [AK], Theorem 4.3.6)**.** *The space $C(K)$ is 1-injective if and only if K is extremally disconnected.*

Proof. Assume that K is extremally disconnected, so $C(K)$ is order complete by 4.5. Examine the proof of the classical Hahn-Banach theorem.

Suppose that $C(K)$ is 1-injective. Then there is a projection of norm one $P : l_\infty(K) \to K$. Let $\Phi \subseteq C(K)$ be bounded from above; then $\xi(x) = \sup_{f\in\Phi} f(x)$ defines $\xi \in l_\infty(K)$. Check that $g = P\xi \in C(K)$ is the least upper bound of Φ (note first that P is a positive operator). □

Theorem 4.8 (Kelley, see [AK], Theorem 4.3.7)**.** *A Banach space X is 1-injective if and only if it is isometric to $C(K)$ space with K extremally disconnected.*

Proof. May be mentioned during the lecture. □

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It is still open if every injective Banach space is isomorphic to some $C(K)$ space; see Wolfe [Wo] for results on < 3-injectivity.

Theorem 4.9. *If $C(K)$ is isometric to a dual Banach space then $C(K)$ is 1-injective.*

In particular $L_\infty(\mu)$ is 1-injective for every σ-finite measure μ.

Proof. Informally, $C(K) = X^*$; check that the positie cone

$$C = \{f \in C(K) : f \geq 0\},$$

is *weak** closed – by the Banach-Dieudonne theorem it is sufficient to check that $r \cdot B_{C(K)} \cap C$ is *weak**-closed.

Conclude that $C(K)$ is order complete and apply Theorem 4.4.

For the second statement, note that $L_\infty(\mu) = L_1(\mu)^*$ and $L_\infty(\mu)$ is isometric to some $C(K)$ space. □

Theorem 4.10 (Pełczyński)**.** *The spaces $L_\infty[0,1]$ and l_∞ are isomorphic*

Proof. The proof uses the so-called Pełyński decomposition method: If X and Y are Banach space such that $X \sim X \oplus X$, $Y \sim Y \oplus Y$ and each space can be embedded as a complemented subspace of the other then $X \sim Y$. The proof is ingenious but simple: write $X \sim Y \oplus X_1$, $Y \sim X \oplus Y_1$. Then

$$X \sim X_1 \oplus Y \sim X_1 \oplus Y \oplus Y \sim X \oplus Y \sim X \oplus Y_1 \oplus X \sim X \oplus Y_1 \sim Y.$$

We have $l_\infty = l_\infty \oplus l_\infty$ and $L_\infty[0,1] = L_\infty[0,1] \oplus L_\infty[0,1]$. It is not difficult to find embeddings $l_\infty \hookrightarrow L_\infty[0,1]$. and $L_\infty[0,1] \hookrightarrow l_\infty$; for the latter use general remark that if X is separable then X^* can be isometrically embedded into l_∞. Therefore the last assumption is fulfilled since both spaces are injective. □

Note that $l_\infty = C(\beta\omega)$ is not isometric to $L_\infty = C(S)$ (where $S = \mathrm{ult}(\mathfrak{A})$ is a compact space, see Example 4.4) — this follows from the classical Banach-Stone theorem: if $C(L)$ is isometric to $C(L)$ then K is homeomorphic to L (here K and L are any compacta). But S is different from $\beta\omega$:

Exercise 4.11. *Show that the Stone space of the measure algebra is not separable and has no isolated points.*

5. Rosenthal's l_1-theorem

Note that in the unit ball B_X of a Banach space one has

(a) every sequence $(x_n)_n$ in B_X has a norm converging subsequence iff B_x is norm compact iff X is finite dimensional;
(b) every sequence $(x_n)_n$ in B_X has a weakly converging subsequence iff B_x is weakly compact iff X is reflexive.

We shall investigate when every sequence in B_X has a weakly Cauchy subsequence. Recall that (x_n) is weakly Cauchy if the limit $x^*(x_n)$ exists for every $x^* \in X^*$. If $x_n = (1,1,\ldots,1,0,\ldots) \in c_0$ then x_n is weakly Cauchy but it is not weakly convergent.

Example 5.1. *$e_n \in l_1$ is a sequence which has no weakly Cauchy subsequence.*

A deep theorem due to Rosenthal says that the above example is essentially unique.

Definition 5.2. *A bounded sequence (x_n) in a Banach space X is said to be* **equivalent to the standard basis of** l_1*, briefly l_1-sequence if there is $\theta > 0$ such that*
$$\theta \cdot \sum_{i=1}^{n} |a_i| \le \left\| \sum_{i=1}^{n} a_i x_i \right\|,$$
for every n and every scalars a_i.

Note that the condition in the above definition is equivalent to saying that there is an isomorphic embedding $T : l_1 \to X$ such that $Te_n = x_n$.

Theorem 5.3 (Rosenthal l_1-theorem)**.** *If (x_n) is a bounded sequence in a Banach space X then*

1. *either (x_n) has a weakly Cauchy subsequence, or*
2. *(x_n) has a l_1-subsequence.*

In particular, if X does not contain an isomorphic copy of l_1 then every bounded sequence has a weakly Cauchy subsequence.

The Rosenthal theorem will be derived from the abstract result stated below. Let S be any space and $f_n : S \to S$ be any sequence of functions.

Definition 5.4. *A sequence $(A_n^0, A_n^1)_n$ of disjoint pairs of subsets of S is said to be* **independent** *if for every n and every $\varepsilon : \{1, 2, \ldots, n\} \to \{0, 1\}$ we have*

$$\bigcap_{i \le n} A_n^{\varepsilon(n)} \neq \emptyset.$$

A sequence $(f_n)_n$ is **independent** *if there are $p < q$ such that putting*

$$A_n^0 = \{s : f_n(s) < p\}, \quad A_n^1 = \{s : f_n(s) > q\},$$

we get an independent sequence of disjoint pairs.

A sequence $(A_n^0, A_n^1)_n$ of disjoint pairs is **convergent** *if no $s \in S$ belong sto infinitely many A_n^0 and infinitely many A_n^1.*

Example 5.5. *Let $X = C(\{-1, 1\}^{\mathbb{N}})$, $f_n : \{-1, 1\}^{\mathbb{N}} \to \{-1, 1\}$, $f_n(x) = x(n)$, be the sequence of projections. Clearly no subsequence of (f_n) is pointwise convergent. Note that (f_n) is an l_1-sequence.*

Lemma 5.6. *If (f_n) is an independent sequence then it is l_1-sequence in the sup norm of $l_\infty(S)$.*

Proof. Suppose that $(p + q) \sum_{i \le n} c_i \ge 0$ (the other case is symmetric). Consider scalars a_i, let $I = \{i \le n : a_i \ge 0\}$. Choose $s \in \bigcap_{i \in I} A_i^1 \cap \bigcap_{i \notin I} A_i^0$. Then

$$\sum_{i \le n} a_i f_i(s) = \sum_{i \in I} + \sum_{i \notin I} \ge q \cdot \sum_{i \in I} a_i + p \cdot \sum_{i \notin I} a_i =$$

$$= \frac{q}{2} \sum_{i \le n} (a_i + |a_i|) + \frac{p}{2} \sum_{i \le n} (a_i - |a_i|) =$$

$$\frac{p+q}{2} \sum_{i \le n} a_i + \frac{q-p}{2} \sum_{i \le n} |a_i| \ge \frac{q-p}{2} \sum_{i \le n} |a_i|.$$

Hence (f_n) is l_1-sequence with $\theta = (q - p)/2$. ☐

We shall need the following Ramsey-type result:

Theorem 5.7 (Galvin, Nash-Williams, see [To]). *If we colour all finite subsets of $\mathbb{N}$ into white or black then*

- *either there is an infinite set $B \subseteq \mathbb{N}$ whose all finite subsets are black; or*
- *there is a infinite set B such that every infinite set $C \subseteq B$ has a white initial segment.*

Theorem 5.8 (Rosenthal's lemma). *Every sequence of disjoint pairs $((A_n^0, A_n^1))_n$ either contains a convering subsequence or contains an independent subsequence.*

Proof. For every finite $a \subseteq \mathbb{N}$ let $\varepsilon_a : a \to \{0,1\}$ be defined so that $\varepsilon_a(n) = 0$ iff n is an **even element** of a. Say that a is white if $\bigcap_{n\in a} A_n^{\varepsilon_a(n)} = \emptyset$.

We apply Theorem 5.7; suppose that B is infinite and all finite subsets of B are black. Let B^* be the subsets of all even elements of B. If $b \subseteq B^*$ then any $\varphi : b \to 2$ then there is (!!) $b^* \supseteq b$ such that ε_{b^*} extends φ. Then

$$\bigcap_{n\in b} A_n^{\varphi(n)} \supseteq \bigcap_{n\in b^*} A_n^{\varepsilon_{b^*}(n)} \neq \emptyset.$$

Therefore B^* enumerates an independent subsequence.

Let the other possibility holds: there is B such that every infinite $C \subseteq B$ has a white initial segment. Then B enumerates a converging subsequence: indeed, otherwise, find $s \in S$ and $n_1 < m_1 < n_2 < m_2 < \ldots$ such that $s \in A_{n_k}^0$ and $s \in A_{m_k}^1$. But then no initial segment of $C = \{n_1, m_1, n_2, m_2, \ldots\}$ is white, a contradiction. $\square$

Theorem 5.9 (Rosenthal l_1-theorem in abstract form). *Every sequence $f_n : S \to [-1,1]$ either contains a pointwise converging subsequence or contains a subsequence which is l_1-sequence for the sup norm.*

Proof. Let $((p_j, q_j))_j$ be an enumeration of all pairs $p < q$ of rational numbers. Given j, apply 5.8 to pairs $A_n^0 = \{f_n < p_j\}$, $A_n^1 = \{f_n > q_j\}$. If there is an independent subsequence then stop: 5.6 says that (f_n) has an l_1-subsequence. Otherwise, pass to a convergent subsequence of pairs and apply (p_{j+1}, q_{j+1}).

The remaining case is that we get infinite sets $\mathbb{N} \supseteq N_1 \supseteq \ldots$ and we can take infinite N which is almost included in every N_j. Now it is enough to check (!!) that $(f_n)_{n\in N}$ converges pointwise. $\square$

Proof. (of 5.3) Apply 5.9 to $S = (B_{X^*}, weak^*)$ and $f_n(x^*) = x^*(x_n)$. □

Theorem 5.10. *(Haydon, Odell, see* [Di]*, page 215) For a separable Banach space X the following are equivalent*

(i) *X contains no copy of l_1;*
(ii) *every $x^{**} \in B_{X^{**}}$ is a weak* limit of a sequence from B_X;*
(iii) *the cardinality of X^{**} does not exceed $\mathfrak{c}$;*
(iv) *$B_{X^{**}}$ is sequentially compact in its weak* topology.*

Proof. Parts may be mentioned. □

See [AGR] for abundance of other consequences and related results.

6. A few pecular Banach spaces

The Tsirelson space (or its dual).

Let c_{00} denote the (incomplete) linear space of all eventually zero sequences of the reals. Of $x \in c_{00}$ and $E \subseteq \mathbb{N}$ then Ex denotes 'the projection onto E, i.e. $Ex(n) = x(n)$ for $n \in E$ and $Ex(n) = 0$ otherwise.

A finite sequence of pairwise disjoint intervals $I_1, I_2, \ldots, I_n$ in $\mathbb{N}$ will be called **admissible** if $I_i \subseteq [n+1, \infty)$ for every $j \le n$.

Lemma 6.1. *There is a norm $||\cdot||_T$ on c_{00} such that*

$$||x||_T = \max\left\{ ||x||, (1/2)\sum_{j=1}^{n} ||I_j x||_T \right\},$$

where the supremum is taken over all admissible families $I_1, I_2, \ldots, I_n$ of intervals in $\mathbb{N}$.

Proof. Define inductively a sequence $||\cdot||_n$ by

$$||x||_n = \max\left\{ ||x||, (1/2)\sum_{j=1}^{n} ||I_j x||_{n-1} \right\},$$

and check that $||\cdot||_T = \sup_n ||\cdot||_n$ is as required. □

Theorem 6.2. *(see* [AK]*, Theorem 10.3.2) Let T be the completion of c_{00} with respect to $||\cdot||_T$ norm. Then T is a Banach space containing neither c_0 or l_p, $p \ge 1$ (a posteriori, T is reflexive).*

Proof. We shall try to get an idea of the proof. □

Banach's hyperplane problem and beyond

A hyperplane H in a Banach space is a subspace of codimention 1, so that X is isomorphic to $H \oplus \mathbb{R}$.

Exercise 6.3. *Check that all hyperplanes in a given Banach space are isomorphic. Check that a hyperplane H is isomorphic to X iff X is isomorphic to $X \oplus \mathbb{R}$.*

Banach's hyperplane problem was the question if every (infinitely dimensional) X is isomorphic to its hyperplane. It was solved in the negative by Gowers and Maurey in a very strong form: there is a (separable) Banach space X such that

(a) X is isomorphic to no of its proper subspaces;
(b) for all (infinitely dimensional) subspace $Z \subseteq Y \subseteq X$, Z is not complemented in Y (in other words, X is hereditarily indecomposable: no subspace Y can be written as $Y = Y_1 \oplus Y_2$ with Y_i infinitely dimensional.

See Maurey's survey [Ma] for more information. Note that the Gowers-Maurey space cannot contain any classical Banach space you may think of!

Koszmider [Ko] and Plebanek [Pl] constructed a (necessarily) connected compact space K such that $C(K)$ is in particular indecomposable (a $C(K)$ space cannot be hereditarily indecomposable) and not isomorpic to any of its proper subspaces, so in particular $C(K)$ is not isomorphic to $C(K+1)$. Here $K+1$ denotes K with one additional isolated point added, so $C(K+1) = C(K) \oplus \mathbb{R}$.

Note that for a typical compactum K, $C(K+1) \sim C(K)$. Indeed, if for instance K contains a nontrivial converging sequence then by Example 3.7 $C(K)$ contains a complemented copy of c_0 and we have

$$C(K) \sim c_0 \oplus Y \sim \mathbb{R} \oplus c_0 \oplus Y \sim \mathbb{R} \oplus C(K) \sim C(K+1).$$

On the other hand, in $K = \beta\omega$ one can write a similar line using the properties of l_∞.

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