Theory of fractals

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1 Introduction

This is a course of the fractals' theory from dynamical point of view. I start from material from books of Mattila [24] and of Falconer [7, 8, 9] and then go in direction of Przytycki and Urbański's book [33]. Those books contain much greater material and are much better written than this text, however I managed to include some results that were only done after those books were written. In particular, the multifractal formalism (maybe the fastest developing part of this theory in the last ten years) was only minimally mentioned in the last Falconer's book (but there is a some starting reference in the book by Pesin [30] and in [33]).

1.1 Measures

Let X be a separable metric space.

Definition 1.1. The set function $\mu : 2^X \to [0, \infty]$ is an (outer) measure if

- $\mu(\emptyset) = 0$,
- $\mu(A) \leq \mu(B)$ when $A \subset B$,
- $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

The common definition of measure corresponds to the restriction of the outer measure to its σ-algebra of measureable sets:

Definition 1.2. Given measure μ, the set $E \subset X$ is μ-measureable if for every $A \subset X$
$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E).$$

Fact 1.3. *Let $\mathcal{M}$ be the family of all μ-measureable sets.*

- *$\mathcal{M}$ is a σ-algebra,*
- *if $\mu(A) = 0$ then $A \in \mathcal{M}$,*
- *if $A_1, A_2, \ldots \in \mathcal{M}$ are pairwise disjoint then $\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$,*
- *if $A_1 \subset A_2 \subset \ldots \in \mathcal{M}$ then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_i \mu(A_i)$,*
- *if $A_1 \supset A_2 \supset \ldots \in \mathcal{M}$ and $\mu(A_1) < \infty$ then $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_i \mu(A_i)$.*

Some useful properties for measures to have:

Definition 1.4. Let μ be a measure on X.

- μ is locally finite if $\forall x \in X \exists r > 0 \mu(B(x, r)) < \infty$,
- μ is Borel if Borel sets are μ-measureable,
- μ is Borel regular if it is Borel and for every $A \subset X$ there exists Borel set $B \supset A$ such that $\mu(A) = \mu(B)$,
- μ is Radon if it is Borel and

– $\mu(K) < \infty$ for compact $K \subset X$,

– $\mu(V) = \sup\{\mu(K); K \subset V, K \text{compact}\}$ for open $V \subset X$,

– $\mu(A) = \inf\{\mu(V); V \supset A, V \text{open}\}$.

Fact 1.5. *If X is complete separable metric space and μ is Borel regular locally finite measure then μ is Radon.*

1.2 Dimensions

Given a separable metric space X (in this text X will almost always be $\mathbb{R}^n$, so I will only present the dimension theory in Euclidean spaces), the dimension is a function $\dim : 2^X \to [0, \infty]$ satisfying several conditions:

- monotonicity: $A \subset B$ implies $\dim A \leq \dim B$,
- for Lipschitz map f, $\dim f(A) \leq \dim A$,

- stability under finite or countable sums:

– $\dim(A \cup B) = \max(\dim A, \dim B)$,

– $\dim \bigcup_{i=1}^{\infty} A_i = \sup \dim A_i$,

- $\dim[0,1]^d = d$.

1.2.1 Box-counting dimension

Given set $A \subset X$ and $r > 0$, let $N(A, r)$ be the minimal number of balls of diameter r needed to cover A. We denote

$$\overline{\dim}_B(A) = \limsup \frac{\log N(A,r)}{-\log r},$$

$$\underline{\dim}_B(A) = \liminf \frac{\log N(A,r)}{-\log r}$$

and call them *upper* and *lower box-counting dimension*. If they are equal, the common value is called *box-counting dimension* and denoted $\dim_B(A)$.

If $X = \mathbb{R}^d$ then we can choose any grid of size r and replace $N(A, r)$ by the number $\tilde{N}(A, r)$ of grid cubes intersecting A. $N(A, r)$ and $\tilde{N}(A, r)$ differ at most by a multiplicative constant (depending on d, but not on A), hence the dimension will not change. The name of box-counting dimension comes from this definition.

Important note: the box counting dimension is not stable under infinite sums; lower box counting dimension is not even stable under finite sums.

Given a sequence of positive real numbers $\{a_i\}$, we define *converge exponent* of $\{a_i\}$ as $\inf\{t; \sum a_i^t < \infty\}$. Interesting fact from [37]:

Lemma 1.6. *Let $A \subset \mathbb{R}$ be a compact set of Lebesgue measure zero. Let $\{a_i\}$ be the sequence of lengths of finite components of the complement of A. Then the convergence exponent of $\{a_i\}$ equals $\overline{\dim}_B A$.*

This result is a special case of more general statement:

Proposition 1.7. *Let $A \subset \mathbb{R}^n$ be a compact set of Lebesgue measure zero. Let $\{E_i\}$ be the maximal dyadical partition of the complement of A. Then the convergence exponent of $\{\operatorname{diam} E_i; \operatorname{diam} E_i < 1\}$ equals $\overline{\dim}_B A$.*

Proof. Denote by d_k the number of cubes of size 2^{-k} in the maximal dyadical partition of the complement of A and by e_k the number of dyadic cubes of size 2^{-k} intersecting A. We have

$$d_k + e_k = 2^n e_{k-1}.$$

Let

$$s = \overline{\dim}_B A = \limsup \frac{1}{k} \log_2 e_k.$$

We have to prove two statements.

Claim 1: for $t > s$ the series $\sum_k d_k 2^{-kt}$ converges. Proof: for any $\varepsilon < t - s$ we have $e_k \le c\varepsilon) \cdot 2^{-k(s+\varepsilon)}$, the claim follows.

Claim 2: for $t < s$ the series $\sum_k d_k 2^{-kt}$ diverges. Proof: denote

$$g_r(k) = e_k \cdot 2^{-rk}.$$

We have a sequence k_i for which $g_t(k_i) > 1$. At the same time, $g_n(k)$ converges to 0. By passing to a subsequence if necessary, we can assume that

$$g_n(k_{i+1}) < \frac{1}{2} g_n(k_i).$$

Denote

$$\sum_{k=1}^{\infty} d_k 2^{-tk} = \sum_{i=1}^{\infty} h_i,$$

where

$$h_i = \sum_{k=k_i+1}^{k_{i+1}} d_k 2^{-tk} = \sum_{k=k_i+1}^{k_{i+1}} (2^{n-t} g_t(k-1) - g_t(k)).$$

We have

$$g_d(k_i) - g_d(k_{i+1}) = \sum_{k=k_i+1}^{k_{i+1}} 2^{-kn} d_k \le 2^{-(k_i+1)(n-t)} h_i$$

and

$$g_d(k_i) - g_d(k_{i+1}) \ge \frac{1}{2} g_d(k_i) = 2^{-k_i(n-t)-1} g_t(k_i).$$

Finally,

$$h_i \ge 2^{n-t-1} g_t(k_i) \ge 2^{n-t-1}.$$

□

1.2.2 Hausdorff dimension

Given $A \subset X$ and $r > 0$, consider the family $\mathcal{A}_r$ of all finite or countable covers of A with open sets of diameter smaller than r. For any $s \geq 0$ we denote

$$H_r^s(A) = \inf_{\{E_i\} \in \mathcal{A}_r} \sum_{\{E_i\}} \operatorname{diam}(E_i)^s.$$

$H_r^s(A)$ is a monotone function of r, hence we can consider the limit

$$H^s(A) = \lim_{r \searrow 0} H_r^s(A).$$

It is called *s-dimensional Hausdorff measure* of A.

Fact 1.8. *H^s is a Borel regular measure.*

Lemma 1.9. *For every $A \subset X$ there exists unique $s \in [0, \infty]$ such that for all $t < s$ $H^t(A) = \infty$ and for all $t > s$ $H^t(A) = 0$.*

Proof. We will prove two claims, from which the assertion follows.

Claim 1: if $H^s(A) > 0$ for some s then $H^t(A) = \infty$ for all $t < s$.

Proof: for any $\varepsilon > 0$, for sufficiently small r, for all $\{E_i\} \in \mathcal{A}_r$ $\sum \operatorname{diam}(E_i)^s > H^s(A) - \varepsilon$. Hence, for all $\{E_i\} \in \mathcal{A}_r$

$$\sum \operatorname{diam}(E_i)^t \geq r^{t-s} \sum \operatorname{diam}(E_i)^s \geq r^{t-s}(H^s(A) - \varepsilon)$$

and the right hand side escapes to infinity as $r \to 0$.

Claim 2: if $H^s(A) < \infty$ for some s then $H^t(A) = 0$ for all $t > s$.

Proof: this claim is just a restatement of the first one. □

The number s given by Lemma 1.9 is called *Hausdorff dimension* of A and denoted $\dim_H A$.

1.2.3 Packing measure

Given $A \subset X$ and $r > 0$, consider the family $\mathcal{B}_r$ of all finite or countable families of disjoint balls with centers in A and diameters smaller than r. For any $s \geq 0$ we denote

$$P_r^s(A) = \sup_{\{E_i\} \in \mathcal{B}_r} \sum_{\{E_i\}} \operatorname{diam}(E_i)^s.$$

Like H_r^s, P_r^s is monotone as a function of r. We denote

$$\tilde{P}^s(A) = \lim_{r \searrow 0} P^s_r(A)$$

and

$$P^s(A) = \inf \left\{ \sum_{i=1}^{\infty} \tilde{P}^s(A_i); A = \bigcup_{i=1}^{\infty} A_i \right\}.$$

Fact 1.10. *P^s is a Borel regular measure.*

The analogue of Lemma 1.9 holds for P^s. We call P^s *s-dimensional packing measure* and the unique s such that $P^t(A) = 0$ for $t > s$ and $P^t(A) = \infty$ for $t < s$ is called *packing dimension* of A and denoted $\dim_P A$.

Lemma 1.11. *For $A \subset \mathbb{R}^n$*

$$\dim_P A = \inf\{\sup_i \overline{\dim}_B(A_i); A = \bigcup_i A_i\}. \tag{1.1}$$

Proof. We have

$$\tilde{P}^s_r(B) \geq 2^{-n} \cdot N(B, r) \cdot r^s.$$

Hence,

$$\dim_P A \geq \inf\{\sup_i \overline{\dim}_B(A_i); A = \bigcup_i A_i\}.$$

To obtain the inequality in the other direction, assume it is not true. Let d be the right hand side of (1.1) and assume $d < \dim_P A$. Let $d < t_1 < t_2 < \dim_P A$. We can find a partition $A = \bigcup A_i$ for which $\overline{\dim}_B(A_i) < t_1$ for all i.

At the same time, $P^{t_2}(A) > 0$, hence $\tilde{P}^{t_2}(A_j) > 0$ for some A_j. That is, for any $r > 0$ we can find a packing of disjoint balls $\{E_i\}$ centered at A_j, of diameters smaller than r, such that $\sum \operatorname{diam}(E_i)^{t_2}$ is uniformly bounded away from zero.

Fix r. We can divide the family $\{E_i\}$ into subfamilies, containing the balls of diameter between 2^{-k-1} and 2^{-k} (for all $k > -\log_2 r$). Denoting the size of k-th subfamily by $n_k(r)$, we get that $\sum_k n_k(r) 2^{-kt_2}$ is uniformly (in r) bounded away from zero.

On the other hand, $\overline{\dim}_B(A_j) < t_1$, hence

$$n_k(r) < c \cdot 2^{kt_1},$$

which together with $n_k(r) = 0$ for $k < -\log_2 r$ gives a contradiction. $\square$

1.3 Densities

Let $s \geq 0$, μ a Radon measure on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. We define *upper* and *lower* *s-densities* of μ at a by

$$\Theta^{*s}(\mu, x) = \limsup_{r \to 0} (2r)^{-s} \mu(B(x, r)),$$

$$\Theta^{s}_{*}(\mu, x) = \liminf_{r \to 0} (2r)^{-s} \mu(B(x, r)).$$

Theorem 1.12. *Let $A \subset \mathbb{R}^n$, $H^s(A) < \infty$. Let μ be the restriction of s-dimensional Hausdorff measure to A. Then*

$$2^{-s} \leq \Theta^{*s}(\mu, x) \leq 1 \text{ for } H^s - \text{almost all } x \in A.$$

If A is H^s-measurable then $\Theta^{*s}(\mu, x) = 0$ for H^s-almost every $x \in \mathbb{R}^n \setminus A$.

Theorem 1.13. *Let μ be a Radon measure on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$, $\lambda \in (0, \infty)$. If $\Theta^{*s}(\mu, x) \leq \lambda$ for $x \in A$ then $\mu(A) \leq 2^s \lambda H^s(A)$. If $\Theta^{*s}(\mu, x) \geq \lambda$ for $x \in A$ then $\mu(A) \geq \lambda H^s(A)$.*

Proof. The first statement: by Egorov Theorem, for any $\varepsilon > 0$ we can find $R > 0$ and a set $B \subset A, \mu(B) > (1 - \varepsilon)\mu(A)$ such that for every $x \in B$ and $r < R$

$$\mu(B(x, r)) \leq (1 + \varepsilon)\lambda r^s.$$

Let $\{E_i\}$ be any cover in $\mathcal{A}_R$. Each E_i intersecting B is contained in some $F_i = B(x_i, \operatorname{diam} E_i)$ for $x_i \in B$. Hence,

$$\sum_i \operatorname{diam}(E_i)^s \geq 2^{-s} \sum_i \operatorname{diam}(F_i)^s \geq 2^{-s}(1 + \varepsilon)^{-1}\lambda^{-1} \sum \mu(F_i) \geq$$

$$2^{-s}(1 + \varepsilon)^{-1}\lambda^{-1}\mu(B) \geq \frac{1 - \varepsilon}{2^s \lambda (1 + \varepsilon)} \mu(A).$$

The second statement is immediate consequence of the Vitali Covering Lemma:

Lemma 1.14. *Let μ be a Radon measure on $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$. Let $\mathcal{E}$ be a family of closed balls such that each point in A is the center of arbitrarily small balls in $\mathcal{E}$. Then we can choose from $\mathcal{E}$ a subfamily of disjoint balls $\{B_i\}$ such that*

$$\mu(A \setminus \bigcup B_i) = 0.$$

□

Similar results hold for the packing dimension:

Theorem 1.15. *Let $A \subset \mathbb{R}^n$, $P^s(A) < \infty$. Let μ be the restriction of s-dimensional packing measure to A. Then*

$$\Theta^s_*(\mu, x) = 1 \text{ for } P^s - \text{almost all } x \in A.$$

Theorem 1.16. *Let μ be a Radon measure on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$, $\lambda \in (0, \infty)$. If $\Theta^s_*(\mu, x) \leq \lambda$ for $x \in A$ then $\mu(A) \leq \lambda P^s(A)$. If $\Theta^s_*(\mu, x) \geq \lambda$ for $x \in A$ then $\mu(A) \geq \lambda H^s(A)$.*

Usually our goal is not to calculate Hausdorff/packing measure but only Hausdorff/packing dimension. In such situation it is enough to use the following, weaker version of Theorems 1.13 and 1.16. Given a measure μ and point $x \in \mathbb{R}^n$, we define the upper and lower local dimension of μ at x by

$$\overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r},$$
$$\underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Theorem 1.17. *Let $A \subset \mathbb{R}^n$ be a Borel set. Let μ be a finite measure.*

i) if $\underline{d}_\mu(x) \geq s$ for μ-almost all $x \in A$ and $\mu(A) > 0$ then $\dim_H A \geq s$,

ii) if $\underline{d}_\mu(x) \leq s$ for all $x \in A$ then $\dim_H A \leq s$,

iii) if $\overline{d}_\mu(x) \geq s$ for μ-almost all $x \in A$ and $\mu(A) > 0$ then $\dim_P A \geq s$,

iv) if $\overline{d}_\mu(x) \leq s$ for all $x \in A$ then $\dim_P A \leq s$,

This result is usually called Mass Distribution Principle or Frostman's Lemma.

2 Conformal Iterated Function Systems satisfying Open Set Condition

The main geometric objects of our study will be limit sets (attractors) of contracting iterated function systems and repellers of expanding (or non-uniformly expanding) dynamical systems. The two classes are almost equivalent: for any dynamical system consider the inverse branches of the maps and you will get an IFS (or, if the domains of the inverse branches are smaller than the whole repeller, a more general object called graph-directed system); if the dynamical system was expanding, the resulting IFS is contracting.

In fact, there are some differences (for example, the IFS obtained by the inverse branches of a dynamical system always satisfies the Open Set condition). However, the differences are sufficiently minor that the geometric theory I am about to present is the same for both presentations. In this and following section I will use the IFS version, in the fourth section I will use the dynamical systems version.

2.1 Linear IFS

A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *contraction* if there is $\lambda < 1$ such that for any $x, y \in \mathbb{R}^n$

$$|f(x) - f(y)| \leq \lambda |x - y|.$$

An *iterated function system* is a finite family of contractions.

Lemma 2.1. *For any iterated function system $\{f_1, \ldots, f_k\}$ there exists a unique nonempty compact set Λ satisfying*

$$\Lambda = \bigcup_{i=1}^{k} f_i(\Lambda). \tag{2.1}$$

Proof. The existence of Λ: consider sufficiently big R that $f_i(B(0,R)) \subset B(0,R)$ (such R always exists for any finite family of contractions). Consider the sequence $A_0 = B(0,R), A_{j+1} = \bigcup_i f_i(A_j)$. It is a decreasing sequence of nonempty compact sets. The set $\Lambda = \bigcap_j A_j$ satisfies (2.1).

The uniqueness of Λ: consider the space X of nonempty compact subsets of $\mathbb{R}^n$ with Hausdorff metric (i.e. the distance between sets A and B is the infimum of $r > 0$ such that A is contained in r-neighbourhood of B and vice versa). Then the map

$$F(A) = \bigcup_i f_i(A)$$

acts on X as a contraction, with contraction coefficient $\lambda = \max_i \lambda_i$. Hence, F cannot have more than one fixed point. □

This set Λ is called *limit set* of the iterated function system.

Let $\Sigma = \{1, \ldots, k\}^{\mathbb{N}}$ be the symbolic space of one-sided infinite sequences of symbols $1, \ldots, k$. There exists a natural projection from Σ to Λ:

$$\pi(\omega) = \lim_{m \to \infty} f_{\omega_1} \circ \ldots \circ f_{\omega_m}(0). \tag{2.2}$$

Obviously, in place of 0 we can put any other $x \in \mathbb{R}^n$.

In Σ, let $C[\omega_1, \ldots, \omega_m]$ be the *cylinder*, that is the set of all infinite sequences that start from the word $\omega_1 \ldots \omega_m$.

We say that the iterated function system satisfies *open set condition* (OSC) if there exist a nonempty open set U such that all $f_i(U)$ are contained in U but pairwise disjoint. If the closures of $f_i(U)$ are also disjoint, the iterated function system satisfies *strong separation condition* (SSC), in this case π is a bijection.

We usually use the product (Tychonoff) topology on Σ. In case a metric is needed, we usually use the metric

$$d(\omega, \tau) = \operatorname{diam} f_{\omega_1} \circ \ldots \circ f_{\omega^m}(\Lambda),$$

where m is such that $\omega_i = \tau_i$ for $i \le m$ but $\omega_{m+1} \neq \tau_{m+1}$. In case of OSC, π is Lipschitz (with Lipschitz constant 1) in this metric. Ni case of SSC, π is bi-Lipschitz.

Proposition 2.2. *Let Λ be the limit set of the iterated function system $\{f_i\}$, consisting of contracting similitudes with contraction ratios λ_i and satisfying OSC. Let s be the unique solution of equation*

$$\sum_i \lambda_i^s = 1. \tag{2.3}$$

Then

$$\dim_B \Lambda = \dim_H \Lambda = \dim_P \Lambda = s$$

Moreover, s-dimensional Hausdorff and packing measures of Λ are positive and finite.

Proof. Distribute on Σ the Bernoulli measure ν given by the probabilistic vector $(\lambda_1^s, \ldots, \lambda_k^s)$. Let $\mu = \pi_* \nu$ be the projection of ν onto Λ. We will check that the local density of μ exists and equals s at every point in Λ.

Note first that we know the measure of each cylinder in Σ:

$$\nu(C[\omega_1 \ldots \omega_m]) = \lambda_1^s \cdot \ldots \cdot \lambda_m^s.$$

Also, the image under π of $C[\omega_1 \ldots \omega_m]$ is $f_{\omega_1} \circ \ldots \circ f_{\omega_m}(\Lambda)$, hence

$$\operatorname{diam}(\pi(C[\omega_1 \ldots \omega_m])) = \lambda_{\omega_1} \cdot \ldots \cdot \lambda_{\omega_m} \cdot \operatorname{diam} \Lambda.$$

For any $r > 0$, for any sequence $\omega \in \Sigma$ let $m(r)$ be the smallest natural number for which

$$\lambda_{\omega_1} \cdot \ldots \cdot \lambda_{\omega_{m(r)}} < r.$$

The cylinders $C[\omega_1, \dots, \omega_{m(r)}]$ form a disjoint cover $\mathcal{M}_r$ of Σ (*Moran cover*) with disjoint cylinders such that

$$r \min_i \lambda_i \le \lambda_{\omega_1} \cdot \ldots \cdot \lambda_{\omega_{m(r)}} < r.$$

This cover contains between r^{-s} and $r^{-s} \max_i \lambda_i^{-s}$ sets of diameter between $r \min_i \lambda_i \operatorname{diam} \Lambda$ and $r \operatorname{diam} \Lambda$, which already gives the box counting dimension.

For the Hausdorff and packing dimensions we need one more ingredient, the following geometric lemma:

Lemma 2.3. *There exist a constant K such that for all $r > 0$ and any $x \in \mathbb{R}^n$ the ball $B(x, r \operatorname{diam} \Lambda)$ will intersect at most K sets $\pi(C[\omega_1, \dots, \omega_m]); C[\omega_1, \dots, \omega_m] \in \mathcal{M}_r$.*

Proof. Each set $\pi(C[\omega_1, \dots, \omega_m])$ is contained in $f_{\omega_1} \circ \ldots \circ f_{\omega_m}(\overline{U})$. For different $C[\omega_1, \dots, \omega_m] \in \mathcal{M}_r$, the sets $f_{\omega_1} \circ \ldots \circ f_{\omega_m}(U)$ are disjoint. Those sets are rescaled copies of U, hence the diameter of each is at most $r \operatorname{diam} U$ and the volume of each is at least $(r \min_i \lambda_i)^n \operatorname{vol}(U)$. If the ball $B(x, r)$ intersects one of those sets, the ball $B(x, r(1 + \operatorname{diam} U)$ must contain it. But as those sets are disjoint, the number of them cannot be greater than

$$K = \frac{\operatorname{vol}(B(x, 1 + \operatorname{diam} U))}{\min_i \lambda_i^n \cdot \operatorname{vol}(U)}.$$

□

Now consider, for any $x \in \Lambda$, the ball $B(x, r \operatorname{diam} \Lambda)$. On the one hand, it contains at least one element from $\pi(\mathcal{M}_r)$: the one containing x. Hence,

$$\mu(B(x, r \operatorname{diam} \Lambda)) \ge \min_i \lambda_i^s r^s.$$

On the other hand, it intersects at most K elements from $\pi(\mathcal{M}_r)$. Hence,

$$\mu(B(x, r \operatorname{diam} \Lambda)) \le K r^s.$$

□

The equation (2.3) is called Moran's Equation.

2.2 Nonlinear IFS - equality of dimensions

Let $\{f_i\}_1^k$ be an iterated function system consisting of contracting conformal $C^{1+\alpha}$ diffeomorphisms (for some $\alpha > 0$), satisfying the open set condition. Let Λ be its limit set. Let us make a useful observation: as the maps are uniformly contracting and the contraction ratios are Hölder, the system has *bounded distortion*: there exists a constant K_0 such that for any m, any $\omega_1, \ldots, \omega_m$, and any $x, y \in \mathbb{R}^n$

$$\frac{(f_{\omega_1} \circ \ldots \circ f_{\omega_m})'(x)}{(f_{\omega_1} \circ \ldots \circ f_{\omega_m})'(y)} < K_0 |x - y|^\alpha, \tag{2.4}$$

where $f' = |\det Df|^{1/n}$ is the local contraction coefficient. Proof: apply the chain rule, note that $|f_{\omega_\ell} \circ ... \circ f_{\omega_m}(x) - f_{\omega_\ell} \circ ... \circ f_{\omega_m}(y)|$ is exponentially small as a function of $m - \ell$, apply the Hölder continuity of f'.

Big part of the asserion of Proposition 2.2 is true for nonlinear (but smooth and conformal) systems, even without assumption of OSC, as shows the following result of Falconer.

Theorem 2.4. *Let $s = \dim_H \Lambda$. Then*

$$\dim_P \Lambda = \dim_B \Lambda = s$$

and the s-dimensional Hausdorff and packing measures are finite.

Proof. We can freely assume that $\operatorname{diam} \Lambda = 1$ (the case when Λ is just one point is trivial).

Claim 1: there exist constants $a > 0$ and $r_0 > 0$ such that for every closet ball B with center in Λ and radius $r < r_0$ there exists a map $g : \Lambda \to \Lambda \cap B$ satisfying

$$|g(x) - g(y)| \geq ar|x - y|.$$

Proof: Choose B and let $z \in \Lambda$ be its center. As $\Lambda = \bigcup_1^k f_i(\Lambda)$, $z = f_{\omega_1}(z_1)$ for some $\omega_1 \in \{1, \ldots, k\}$ and some $z_1 \in \Lambda$. Similarly, $z_1 = f_{\omega_2}(z_2)$, hence $z = f_{\omega_1} \circ f_{\omega_2}(z_2)$. Repeating this process, we get an infinite family of maps $g_j = f_{\omega_1} \circ \ldots \circ f_{\omega_j}$ with domain Λ and range contained in smaller and smaller neighbourhoods of z. Moreover,

$$c_1 \leq \frac{|g'_{j+1}|(g_{j+1}^{-1}(z))}{|g'_j|(g_j^{-1}(z))} \leq c_2,$$

where c_1 and c_2 are minimum and maximum of contraction ratios of all maps f_i over Λ. Hence, we can choose one of those maps with contraction ratio close but slightly smaller than r and the claim will follow from bounded distortion.

Claim 2: Let $\{B_{j,r}\}$ be a finite family of disjoint balls of radii $r_{j,r} < r$ and centers in Λ. For small enough r we have

$$\sum_j r_{j,r}^s \leq a^{-s}. \tag{2.5}$$

Proof: assume that we can find arbitrarily small r and $\{B_{r,j}\}$ for which (2.5) does not hold. We can then write

$$\sum_j r_{j,r}^t > a^{-t}$$

for some $t > s$. Let $\{g_j\}$ be the corresponding maps found in claim 1. Consider the iterated function system $\{g_j\}$ and let Λ' be its limit set. We can get the lower bound for $\dim_H \Lambda'$ much like in the proof of Proposition 2.2: we distribute a measure on Λ' (using $ar_{j,r}$) instead of λ_j in the Moran's Equation) and estimate

$$\mu(B(x,r)) \leq c \cdot r^t$$

for all $x \in \Lambda'$. It follows that Λ' (a subset of Λ) has Hausdorff dimension at least t, hence greater than $s = \dim_H \Lambda$ – contradiction. □

We now observe that if we take a maximal set of disjoint balls of a given size, and increase their radii by constant 2, then they will cover Λ (otherwise they would not be the maximal set). Hence, (2.5), which stays true for the special case of all the balls having the same size, implies $\dim_B \Lambda \leq s$.

To prove that $P^s(\Lambda) < \infty$ note that $P_r^s \leq a^{-s}$, and hence $\tilde{P}^s < \infty$, follows immediately from (2.5). □

2.3 Bowen's Theorem

The one element of Proposition 2.2 missing in Theorem 2.4 is the exact value of the Hausdorff, packing, box counting dimensions. To obtain it, to generalize the Moran's Equation, we will need to assume OSC and introduce the following result from thermodynamical formalism.

Let σ be a symbolic space $\{1, \ldots, k\}^{\mathbb{N}}$, as above. We choose on Σ the exponential metric: we choose some constant $\gamma < 1$ and set the distance between sequences ω and τ as $\gamma^{\omega \wedge \tau - 1}$, where $\omega \wedge \tau = \min\{m; \omega_m \neq \tau_m\}$.

Let $\phi : \Sigma \to \mathbb{R}$ be a Hölder function. That is, we denote by $\operatorname{var}_m \phi$ the maximum of $|\phi(\omega) - \phi(\tau)|$ for $\omega_i = \tau_i, i = 1, \ldots, m$ and assume that $\operatorname{var}_m \phi < b\alpha^m$ for some $b > 0$ and $\alpha < 1$.

We will denote the space of probabilistic measures supported on σ by $M(\Sigma)$. The shift-invariant probabilistic measure μ supported on Σ is called *Gibbs measure* if there exist constants K, P such that for every cylinder $C[\omega_1, \ldots, \omega_m]$ and every $\tau \in C[\omega_1, \ldots, \omega_m]$ we have

$$K^{-1} \leq \frac{\mu(C[\omega_1, \ldots, \omega_m])}{\exp(-Pm + \sum_{\ell=0}^{m-1} \phi(\sigma^\ell \tau))} \leq K \tag{2.6}$$

Theorem 2.5. *For every Hölder potential ϕ the Gibbs measure exists and is unique.*

Proof. Consider the following *transfer operator* $\mathcal{L}$, acting on $C(\Sigma)$:

$$(\mathcal{L}f)(\omega) = \sum_{\tau \in \sigma^{-1}\omega} e^{\phi(\tau)} f(\tau).$$

The main step in the proof is the following Ruelle's Perron-Frobenius Theorem:

Theorem 2.6. *Under assumptions as above, there exists $\lambda > 0$, $h \in C(\Sigma), h > 0$, and $\nu \in M(\Sigma)$ such that $\mathcal{L}h = \lambda h$, $\mathcal{L}^*\nu = \lambda\nu$, $\nu(h) = 1$, and for all $g \in C(\Sigma)$ we have*

$$\lim_{m \to \infty} ||\lambda^{-m}\mathcal{L}^m g - \nu(g)h|| = 0.$$

Proof. As $\mathcal{L}$ is positive, the dual operator $\mathcal{L}^*$ moves finite measures to finite measures. Hence, we can define operator

$$G(\mu) = \frac{\mathcal{L}^*\mu}{(\mathcal{L}^*\mu)(1)}$$

acting from $M(\Sigma)$ to $M(\Sigma)$. By Krylov-Bogolyubov Theorem $M(\Sigma)$ is compact in the weak* topology, hence by Schauder-Tychonoff Theorem G has a fixed point ν. We denote $\lambda = (\mathcal{L}^*\mu)(1)$.

We will investigate the action of the operator $\lambda^{-1}\mathcal{L}$. Note that

$$\nu(\lambda^{-1}\mathcal{L}f) = \lambda^{-1}\mathcal{L}^*\nu(f) = \nu(f). \tag{2.7}$$

Set $B_m = \exp(\sum_{\ell=m+1}^{\infty} 2b\alpha^\ell)$ and consider the space

$$H = \{f \in C(\Sigma); f \geq 0, \nu(f) = 1, f(\tau) \leq B_m f(\omega) \text{ when } \tau_i = \omega_i \forall i \leq m\}.$$

Claim 1: there exists $h \in H$ such that $\lambda^{-1}\mathcal{L}h = h$. Proof: we first check that $\lambda^{-1}\mathcal{L}(H) \subset H$. Next we check that H is equicontinuous, hence (by Arzela-Ascoli Theorem) compact. Then the claim will follow from Schauder-Tychonoff Theorem.

Claim 2: there exists $\eta \in (0,1)$ such that for all $f \in H$ $\lambda^{-1}\mathcal{L}f = \eta h + (1-\eta)g$ for some $g \in H$. Proof: calculation. Based on the fact that $\log B_m$ decrease so fast that for $f \in H$ $\lambda^{-1}\mathcal{L}f$ satisfies stronger assumptions about variations.

Claim 3: There are constants $c > 0$ and $\beta < 1$ such that for all $f \in H$ and for all m

$$||\lambda^{-m}\mathcal{L}f - h|| \le c\beta^m.$$

Proof: immediate from Claim 2.

Claim 4: Let $f \in H$ and $f' \in C(\Sigma), f' \ge 0$ with $\mathrm{var}_m f' = 0$. Then $\lambda^{-m}\mathcal{L}^m(ff') \in H$. Proof: Enough to check for g being the characteristic function of a m-th level cylinder.

Claim 5: Let $f \in H$ and $f' \in C(\Sigma)$ with $\mathrm{var}_m f' = 0$. Then

$$||\lambda^{-m-\ell}\mathcal{L}^{m+\ell}(ff') - \nu(ff')h|| \le c\nu(|ff'|)\beta^\ell.$$

Proof: We apply claims 3 and 4 to the positive and negative part of g, separately.

Claim 6: For all $g \in C(\Sigma)$ we have

$$\lim_{m\to\infty} ||\lambda^{-m}\mathcal{L}^m g - \nu(g)h|| = 0.$$

Proof: for any $\varepsilon > 0$ we can find f and f' as in claim 5 so that $0 \le g - ff' \le \varepsilon h$. Then

$$\limsup_e ll||\lambda^{-\ell}\mathcal{L}^\ell g - \nu(g)h|| \le \varepsilon(1 + ||h||).$$

□

The measure we seek is $\mu_\phi = h\nu$. Let us first check that it is σ-invariant.

Let us start from a useful equality

$$((\mathcal{L}f)\cdot g)(\omega) = \sum_{\tau\in\sigma^{-1}(\omega)} e^{\phi(\tau)}f(\tau)g(\omega) = \sum_{\tau\in\sigma^{-1}(\omega)} e^{\phi(\tau)}f(\tau)g(\sigma\tau) = \mathcal{L}(f\cdot(g\circ\sigma))(\omega).$$

Now we get

$$\mu(f) = \nu(hf) = \nu(\lambda^{-1}\mathcal{L}h \cdot f) = \lambda^{-1}\nu(\mathcal{L}(h\cdot(f\circ\sigma))) =$$
$$\lambda^{-1}(\mathcal{L}^*\nu)(h\cdot(f\circ\sigma)) = \nu(h\cdot(f\circ\sigma)) = \mu(f\circ\sigma).$$

□

Denote $a = \sum_{m=0}^{\infty} \mathrm{var}_m \phi$. Denote

$$S_m\phi(\omega) = \sum_{\ell=0}^{m-1} \phi(\sigma^\ell\omega).$$

Important property: for ω, τ with $\omega_i = \tau_i \forall i \leq m$,

$$|S_m\phi(\omega) - S_m\phi(\tau)| < a.$$

We can now check the Gibbs property. Denote by C the m-th level cylinder $C[\omega_1, \ldots, \omega_m]$. For every point $\tau \in \Sigma$ there exists precisely one point in $E \cap \sigma^{-m}(\tau)$. Hence,

$$\mathcal{L}^m(h\chi_C)(\tau) = \sum_{\omega \in \sigma^{-m}\tau} e^{S_m\phi(\omega)} h(\omega)\chi_C(\omega) \in e^{S_m\phi(\omega)} \cdot (e^{-a}||h^{-1}||, e^a||h||).$$

As

$$\mu(C) = \nu(h\chi_C) = \lambda^{-m}\nu(\mathcal{L}^m(h\chi_C)),$$

the Gibbs property (2.6) follows, with $P = \log \lambda$.

Last thing to check is the uniqueness of the Gibbs measure. First let us check that P is uniquely defined. Indeed, for any Gibbs measure, it is probabilistic, hence the measures of all m-th level cylinders must sum up to 1. Hence, for any $\omega \in \Sigma$

$$K^{-1}e^{-a} \leq e^{-Pm} \sum_{\tau \in \sigma^{-m}\omega} e^{S_m\phi(\tau)} \leq Ke^a.$$

Passing with m to infinity we get a formula for P

$$P = \lim_{m \to \infty} \frac{1}{m} \log \sum_{\tau \in \sigma^{-m}\omega} e^{S_m\phi(\tau)} \tag{2.8}$$

independent of the choice of the Gibbs measure (and ω).

Let us assume that we have two Gibbs measures, μ and μ', for the same P. From the Gibbs property (2.6) applied to both μ and μ' we obtain the existence of a constant $\tilde{K}$ such that for any cylinder C

$$\tilde{K}^{-1} \leq \frac{\mu(C)}{\mu'(C)} \leq \tilde{K}.$$

Hence, the two measures are equivalent, that is there exists a function $h', \nu(h') = 1$ such that $\mu' = h'\nu$. Moreover, as μ' is σ-invariant, $\lambda^{-1}\mathcal{L}h' = h'$. We can approximate h' with continuous functions in $L_1(\nu)$, that is we find a sequence $h_i \in C(\Sigma)$ such that $\nu(|h' - h_i|) \to 0$. We apply then claim 6 together with (2.7) to prove that h' must be equal to h. $\square$

The constant P is called *pressure*. It is usually considered as a function of potential ϕ. The equation (2.8) is often used as the definition of pressure (because it lets us define the pressure for continuous potentials, while (2.6) cannot usually be satisfied for non-Hölder potentials).

Important remark: Gibbs measures are ergodic, mixing, they are equilibrium states, they are probably the most important class of measures from dynamical point of view. From the geometric point of view one can often do with some simpler (and easier to obtain) measures.

Indeed, it is often enough to find a *conformal measure*, that is a (in general not invariant) probabilistic measure such that for every Borel set $A \subset \Sigma$ on which σ is 1-1 we have

$$\mu(\sigma(A)) = \int_A e^{P-\phi(\omega)} d\mu(\omega).$$

Such measures always exist for continuous potentials, see for example [6]. They are usually constructed using the *Patterson-Sullivan construction*: we take a point ω, consider all the m-th preimages of ω, and distribute on each of them a Dirac mass:

$$\mu_m = \sum_{\tau \in \sigma^{-m}(\omega)} \rho_m \cdot e^{-Pm + \sum_{\ell=0}^{m-1} \phi(\sigma^\ell \tau)},$$

where ρ_m is some special correction term. Choosing ρ_m properly one can guarantee that the sequence μ_m will converge in the weak topology, and that the limit measure will be conformal.

2.4 Nonlinear IFS with OSC

We are coming back to the conformal iterated function systems. Consider the symbolic space $\Sigma = \{1, \ldots, k\}^{\mathbb{N}}$ and the natural projection defined by (2.2). We define on Σ a potential

$$\phi(\omega) = \log f'_{\omega_1}(\pi(\sigma\omega)). \tag{2.9}$$

For any $t \in \mathbb{R}$ consider the pressure of the potential $t \cdot \phi$.

Lemma 2.7. *The function $t \to P(t\phi)$ has unique zero.*

Proof. We will apply (2.8). There exist constants $c_1, c_2 < 0$ such that $c_1 < \phi < c_2$. Hence, for any t and any $\delta > 0$

$$c_1\delta < P((t+\delta)\phi) - P(t\phi) < c_2\delta.$$

□

The equation

$$P(t\phi) = 0 \tag{2.10}$$

is called *Bowen's Equation.*

Proposition 2.8. *Let s be the solution of* (2.10)*. Then*

$$\dim_B \Lambda = \dim_H \Lambda = \dim_P \Lambda = s.$$

Moreover, s-dimensional Hausdorff and packing measures of Λ are positive and finite.

Proof. Let ν be the Gibbs measure on Σ for the potential $s\phi$. Let $\mu = \pi^*\nu$ be the projection of ν onto Σ.

For any cylinder $C[\omega_1, \dots, \omega_m]$ we can write

$$\pi(C[\omega_1, \dots, \omega_m]) = f_{\omega_1} \circ \dots \circ f_{\omega_m}(\Lambda).$$

By the bounded distortion, for any $\tau \in \Sigma$ we can write

$$\operatorname{diam} \pi(C[\omega_1, \dots, \omega_m]) \approx \operatorname{diam} \Lambda \cdot (f_{\omega_1} \circ \dots \circ f_{\omega_m})'(\pi(\tau)) = \operatorname{diam} \Lambda \cdot e^{S_m\phi(\omega_1 \dots \omega_m \tau)}.$$

At the same time,

$$\nu(C[\omega_1, \dots, \omega_m]) \approx e^{s \cdot S_m\phi(\omega_1 \dots \omega_m \tau)}.$$

Hence, for any cylinder $C[\omega_1, \dots, \omega_m]$ we have

$$\nu(C[\omega_1, \dots, \omega_m]) \approx \operatorname{diam}(\pi(C[\omega_1, \dots, \omega_m]))^s.$$

We can now repeat the proof of Proposition 2.2: construct the Moran cover and calculate the s-density of μ. The only difference will be in the proof of Lemma 2.3, where we cannot say that $f_{\omega_1} \circ \dots \circ f_{\omega_m}(U)$ is a exact smaller copy of U. However, bounded distortion gives us a lower bound for

$$\frac{\operatorname{vol} f_{\omega_1} \circ \dots \circ f_{\omega_m}(U)}{(\operatorname{diam} f_{\omega_1} \circ \dots \circ f_{\omega_m}(U))^n}$$

which is sufficient for the proof to follow. $\square$

2.5 Open Set Condition

We now know that OSC implies positivity of s-dimensional Hausdorff measure of the limit set, where s is the solution to the Bowen's Equation (2.10). Here we will prove that the two statements are actually equivalent.

Let us improve the notation. For a sequence $\omega^m = \omega_1 \dots \omega_m$ we will denote

$$f_{\omega^m} = f_{\omega_1} \circ \dots \circ f_{\omega_m}$$

and

$$d(\omega^m) = \operatorname{diam}(f_{\omega^m}(\Lambda)).$$

For two finite words ω^m and τ^ℓ we say that the maps f_{ω^m} and f_{τ^ℓ} are *ε-relatively close* if for all $x \in \Lambda$

$$|f_{\omega^m}(x) - f_{\tau^\ell}(x)| \le \varepsilon \min\{d(\omega^m), d(\tau^\ell)\}.$$

We say the iterated function system satisfies *Bandt-Graf condition* if there exists $\varepsilon > 0$ such that no two maps $f_{\omega^m}, f_{\tau^\ell}$, $\omega^m \neq \tau^\ell$ are ε-relatively close. The following result is from [29] (version for linear IFS was done before in [1] and [34])

Theorem 2.9. *The following are equivalent:*

i) OSC,

ii) $H^s(\Lambda) > 0$,

iii) Bandt-Graf condition.

Proof. We only need to consider the case when Λ is not a single point − without weakening of assumptions we can assume $\operatorname{diam} \Lambda = 1$. We will set a new metric ρ on Σ: for $\omega, \tau \in \Sigma$ such that $\omega_i = \tau_i$ for $i = 1, \dots, m$ but not for $i = m + 1$ we set $\rho(\omega, \tau) = d(\omega^m)$. We denote by μ the Gibbs measure for potential ϕ and by $\nu = \pi_*(\mu)$ its projection to Λ. The following lemma is another formulation of Proposition 2.8.

Lemma 2.10. *The set Σ with metric ρ has Hausdorff dimension s. The measure μ is equivalent to the s-dimensional Hausdorff measure on Σ.*

Fix some small δ_0 and let V' be the δ_0-neighbourhood of Λ and V'' be the $2\delta_0$-neghbourhood of Λ. As $f_i(\Lambda) \subset \Lambda$ and f_i are contractions, $f_i(V') \subset V'$ and $f_i(V'') \subset V''$. We will denote

$$||f'_{\omega^m}|| = \sup_{x \in V''} |f'_{\omega^m}(x)|.$$

Let now formulate several version of bounded distortion (the last one follows from Gibbs property).

Lemma 2.11. *There exist* $C_1, C_2, C_3, C_4, C_5 \geq 1$ *such that for all* ω^m, τ^ℓ *for all* $x, y \in V''$, $w, z \in V'$

$$|f'_{\omega^m}(x)| \leq C_1 |f'_{\omega^m}(y)|,$$

$$C_2^{-1} ||f'_{\omega^m}|| \cdot |z - w| \leq |f_{\omega^m}(z) - f_{\omega^m}(w)| \leq C_2 ||f'_{\omega^m}|| \cdot |z - w|,$$

$$C_3^{-1} ||f'_{\omega^m}|| \leq d(\omega^m) \leq C_3 ||f'_{\omega^m}||,$$

$$C_4^{-1} \max\{||f'_{\omega^m}|| d(\tau^\ell), ||f'_{\tau^\ell}|| d(\omega^m)\} \leq d(\omega^m \tau^\ell) \leq C_4 \max\{||f'_{\omega^m}|| d(\tau^\ell), ||f'_{\tau^\ell}|| d(\omega^m)\},$$

$$C_5^{-1} ||f'_{\omega^m}|| \leq \mu(C[\omega^m]) \leq C_5 ||f'_{\omega^m}||.$$

We start the proof. The implication i)→ ii) we already have, there are two implications left to prove.

Implication ii) → iii). We will prove that if for every $\varepsilon > 0$ one can find $\omega^m \neq \tau^\ell$ with $f_{\omega^m}, f_{\tau^\ell}$ ε-relatively close then $H^s(\Lambda) > 0$.

First thing to do: consider the impact of bounded distortion on the pairs of relatively close maps.

Lemma 2.12. *Let* $f_{\omega^m}, f_{\tau^\ell}$ *be* ε*-relatively close. Let* η^r *be arbitrary. We have*

- $f_{\eta^r \omega^m}, f_{\eta^r \tau^\ell}$ *are* $C_2 C_4 \varepsilon$*-relatively close,*
- $f_{\omega^m \eta^r}, f_{\tau^\ell \eta^r}$ *are* $C_4 ||f'_{\eta^r}||^{-1} \varepsilon$*-relatively close,*
- $d(\tau^\ell) \leq (1 + 2\varepsilon) d(\omega^m)$,
- *if* f_{ω^m}, f_{η^r} *are also* ε*-relatively close then* $f_{\eta^r}, f_{\tau^\ell}$ *are* $2\varepsilon(1+2\varepsilon)$*-relatively close.*

The rest of the proof is in three steps.

Claim 1: For any ε and any $N > 0$ we can find N maps pairwise ε-relatively close.

Proof: by induction. If we can find N such maps $f_{\omega^{(i)}}$, we find a pair of maps $f_{\tau^\ell}, f_{\eta^r}$ $||f_{\omega^{(1)}}||\varepsilon$-relatively close and the $2N$ maps $f_{\tau^\ell \omega^{(i)}}, f_{\eta^r \omega^{(i)}}$ are pairwise $C_2 C_4 \varepsilon(1 + C_2 C_4 \varepsilon)$-relatively close.

Claim 2: Upper s-density of ν is greater than cN ν-almost everywhere.

Proof: We have N blocks $\omega^{(i)}$ such that the corresponding maps $f_{\omega^{(i)}}$ are ε-relatively close. The μ-typical sequence τ contains the block $\omega^{(1)}$ infinitely often. For each m such that the sequence $\omega^{(1)}$ appears in τ at positions $m+1, \ldots, m+r$ the maps $f_{\tau^m \omega^{(i)}}$ are pairwise $C_2 C_4 \varepsilon$-relatively close. Hence, the ball centered at $\pi(\tau)$ of radius $cd(\tau^{m+r})$ will contain each $\pi(C[\tau^m \omega^{(i)}])$.

Claim 3: $H^s(\Lambda) \leq cN^{-1}$
Proof: Follows from Theorem 1.12.
Implication iii) $\to$ i).
For $t \geq 1$, $a \geq 0$, and ω^m we define

$$W_{a,T}(\omega^m) = \left\{ \tau^\ell; \frac{1}{T} \leq \frac{d(\tau^\ell)}{d(\omega^m)} \leq T, \mathrm{dist}(f_{\omega^n}(\Lambda), f_{\tau^\ell}(\Lambda)) \leq ad(\omega^m) \right\}$$

Claim 1: Bandt-Graf condition implies that for each a and T $W_{a,T}(\omega^m)$ has at most $C(a,T)$ elements, with $C(a,T)$ not depending on ω^m.

Proof: Bounded distortion implies compactness of the space of maps with approximately the same $||f'||$.

Claim 2: Let $T_0 \geq 1$ and $\varepsilon > 0$. There exists $\delta(T_0, \varepsilon) > 0$ such that for all ω^m with $d(\omega^m) \leq \delta$, for all $a \in [0,1]$ and $T \in [T_0, 2T_0]$ we have

$$\tau^\ell \in W_{a,T}(\omega^m) \implies \eta^r \tau^\ell \in W_{a(1+\varepsilon),T(1+\varepsilon)}(\eta^r \omega^m) \forall \eta^r.$$

Proof: Calculation based on bounded distortion.

Main construction: We fix T_0 sufficiently large that $d(\omega^m j) \geq T_0^{-1} d(\omega^m)$. We will use notation $W_a(\omega^m) = W_{a,(1+a)T_0}(\omega^m)$ and $M_a(\omega^m) = \sharp W_a(\omega^m)$. We know that $M_a(\cdot)$ is bounded above by a constant C. Fix $\varepsilon = 1/2C$ and $\delta = \delta(T_0, \varepsilon)$. We denote by $\widetilde{M}_a(r)$ the maximum of $M_a(\omega^m)$ over all ω^m with $d(\omega^m) < r$.

The function $a \to \widetilde{M}_a(\delta)$ is nondecreasing, integer valued and bounded by C. Hence, there is an interval $[a_1, (1+\varepsilon)a_1]$ on which $\widetilde{M}_{a_1}(\delta)$ is constant.

Let ω^m be one word on which $\widetilde{M}_{a_1}(\delta)$ is achieved. Then for any $\tau^\ell \in W_{a_1}(\omega^m)$ for all η^r we have

$$\eta^r \tau^\ell \in W_{a_1(1+\varepsilon)}(\eta^r \omega^m).$$

Hence,

$$M_{a_1(1+\varepsilon)}(\eta^r \omega^m) \geq M_{a_1}(\omega^m) = \widetilde{M}_{a_1} = \widetilde{M}_{a_1(1+\varepsilon)} \geq M_{a_1(1+\varepsilon)}(\eta^r \omega^m).$$

This means that

$$W_{a_1(1+\varepsilon)}(\eta^r \omega^m) = \eta^r W_{a_1(1+\varepsilon)}(\omega^m)$$

for all η^r. Hence, the set

$$U = \bigcup_{\eta^r} f_{\eta^r}(B(f_{\omega^m}(\Lambda), \varepsilon'))$$

will satisfy OSC for ε' sufficiently small. □

Corollary 2.13. *If $s = n$ and $H^s(\Lambda) > 0$ then Λ is a closure of an open set.*

Proof. By the above proof we have OSC and can choose a set U contained in some small neighbourhood of Λ satisfying it. Let $W = U \setminus \bigcup_i f_i(U)$.

Claim 1: $\mathrm{Leb}(W) = 0$.

Proof: the forward images of W are all disjoint and all contained in U, which has finite volume. Plus bounded distortion.

It implies

$$U \subset \overline{\bigcup_i f_i(U)},$$

hence $\overline{U} = \bigcup f_i(\overline{U})$. □

3 Iterated Function Systems without Open Set Condition

Important remark: one of the goals of this section is to provide the basics of the transversality method, first introduced in [32], of studying the iterated function systems with overlaps. The material is current for the year 2011. But in 2012 Mike Hochman wrote a paper [13] which quite possibly will make this whole approach obsolete...

3.1 Differentiation of measures

Let us add one more tool we will use soon.

Given two Radon measures μ, ν we define the *upper and lower derivative* of μ with respect to ν at a point x by

$$\overline{D}(\mu, \nu, x) = \limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))},$$

$$\underline{D}(\mu, \nu, x) = \liminf_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

(where 0/0=0). If the limits coincide, we denote the common value by $D(\mu, \nu, x)$.

Fact 3.1. *For any pair of Radon measures μ, ν*

- *$D(\mu, \nu, x)$ exists and is finite ν-almost everywhere,*

- *For all Borel $B \subset \mathbb{R}^n$,*

$$\int_B D(\mu, \nu, x) d\nu(x) \le \mu(B),$$

with equality if μ is absolutely continuous with respect to ν,

- *μ is absolutely continuous with respect to ν if and only if $\underline{D}(\mu, \nu, x) < \infty$ for μ-almost every $x \in \mathbb{R}^n$.*

3.2 Capacity dimension

Given a finite measure μ on $\mathbb{R}^n$, we define *t-energy* as

$$I_t \mu) = \int \int |x - y|^{-t} d\mu(x) d\mu(y).$$

Lemma 3.2. *If $\mu(B(x, r)) \le cr^s$ for some $c > 0$, all $x \in \mathbb{R}^n$, and all $r > 0$ then*

$$I_t(\mu) < \infty$$

for all $t < s$. If $I_s(\mu) < \infty$ then there exists a set E of positive measure μ such that if ν is the retriction of μ to E then

$$\mu(B(x, r)) \le cr^s$$

for some $c > 0$, all $x \in \mathbb{R}^n$, and all $r > 0$.

Proof. The first part of the assertion:

$$\int |x - y|^{-t} d\mu(y) = \int_0^\infty \mu(\{y; |x - y|^{-t} \ge u\}) du =$$

$$\int_0^\infty \mu(B(x, u^{-1/t})) du = t \int_0^\infty r^{-t-1} \mu(B(x, r)) dr.$$

The second part of assertion: the set

$$E = \left\{ x; \int |x - y|^{-s} d\mu(y) \le M \right\}$$

has positive measure μ for sufficiently lage M. Fixing such E we get for all $x \in E$, $r > 0$

$$\nu(B(x, r)) \le r^s \cdot \int_{B(x,r)} |x - y|^{-s} d\nu(y) \le Mr^s.$$

For $x \notin A$, the ball $B(x, r)$ either intersects E (and is thus contained in a ball $B(z, 2r)$ for some $z \in E$) or does not intersect E (and hence has measure ν 0). □

Hence, for any $A \subset \mathbb{R}^n$ the supremum of numbers $t \in \mathbb{R}$ for which there exists a finite Radon measure μ supported on A such that $I_t(\mu) < \infty$ and the supremum of numbers $t \in \mathbb{R}$ for which there exists a finite Radon measure μ supported on A such that $\mu(B(x,r)) \leq r^t$ for all x, r are the same number. We denote it $\dim_C A$ and call it *capacity dimension* of A.

Actually, it is not a new dimension:

Proposition 3.3. *For any Borel $A \subset \mathbb{R}^n$,*

$$\dim_C A = \dim_H A$$

Proof. I will only present the proof for A such that $0 < H^s(A) < \infty$ for some s. The general statement can then be proven using result of Davies [5]

Fact 3.4. *For any Borel set $A \in \mathbb{R}^n$ for all s there exists a sequence of closed sets $B_i \subset A$ such that*

$$H^s(A) = \lim H^s(B_i).$$

It is also true for the packing measure, [19].

For H^s-measureable set of finite s-dimensional Hausdorff measure its upper s-density is bounded by 1 almost everywhere. Restricting the measure to a subset we get $\mu(B(x,r)) \leq r^s$ for all x, r. Hence, $\dim_C A \geq s$.

To get the other direction, assume that $I_s(\mu) < \infty$ for some probabilistic Radon measure supported on A. We have $\int |x-y|^{-s} d\mu(y) < \infty$ for μ-almost every $x \in \mathbb{R}^n$, hence for those x

$$\lim_{r \searrow 0} \int_{B(x,r)} |x-y|^{-s} d\mu(y) = 0.$$

For given $\varepsilon > 0$ we can choose $B \subset A$ and $r_0 > 0$ such that $\mu(B) > 1/2$ and that for all $r \leq r_0$, $x \in B$

$$\mu(B(x,r)) \leq r^s \int_{B(x,r)} |x-y|^{-s} d\mu(y) \leq \varepsilon r^s.$$

Let $\{E_i\}$ be a cover of B of diameter smaller than r_0, such that

$$\sum \operatorname{diam}(E_i)^s < H^s(B) + 1$$

Choosing some $x_i \in B \cap E_i$, we get

$$\frac{1}{2} < \mu(B) \leq \sum_i \mu(B(x_i, \operatorname{diam} E_i)) \leq \varepsilon \sum_i \operatorname{diam}(E_i)^s \leq \varepsilon(H^s(B) + 1).$$

□

3.3 Marstrand Theorem

For $A \subset \mathbb{R}^n$ denote

$$C_s(A) = \sup\{I_s(\mu)^{-1}\},$$

where the supremum is taken over probabilistic Radon measures, supported on A and of compact support. This number is called *s-capacity* of A.

Consider the Grassmanian manifold $G(n,m)$ of m-dimensional linear subspaces of $\mathbb{R}^n$, with the natural measure $\gamma_{n,m}$ (the case we will really need is $n=2$, $m=1$). For a subspace V let P_V be the orthogonal projection to V.

Proposition 3.5. *Let $0 < s < m$. There exists $c = c(n,m,s)$ such that for all $A \subset \mathbb{R}^n$*

$$\int C_s(P_V(A))^{-1} d\gamma_{n,m}(V) \leq cC_s(A)^{-1}.$$

Proof. For any μ Radon, supported on A

$$\int C_s(P_V(A))^{-1} d\gamma_{n,m}(V) \leq \int I_s(V_*\mu) d\gamma_{n,m}(V) =$$

$$\int\int\int |P_V(x-y)|^{-s} d\mu(x)d\mu(y)d\gamma_{n,m}(V) =$$

$$\int\int\int |P_V(x-y)|^{-s} d\gamma_{n,m}(V)d\mu(x)d\mu(y) \leq cI_s(\mu)$$

and we just take infimum over all μ. □

We will use the notation: for a measure ν supported on V let $D(\nu,x)$ be the derivative of ν with respect to the m-dimensional Hausdorff measure on V, at point $x \in V$. Similarly, let $\underline{D}(\nu,x)$ stand for the lower derivative.

Proposition 3.6. *Let μ be a Radon measure on $\mathbb{R}^n$ with compact support. Assume $I_m(\mu) < \infty$. Then for $\gamma_{n,m}$-almost every $V \in G(n,m)$ we have $P_{V*}\mu << H^m$ and*

$$\int\int_V D(P_{V*}\mu, x)^2 dH^m(x)d\gamma_{n,m}(V) \leq cI_m(\mu)$$

for some $c = c(n,m)$.

Proof. We have

$$\int\int \underline{D}(P_{V*}\mu, x) dP_{V*}\mu(x) d\gamma_{n,m}(V) \leq$$

$$\liminf_{r\searrow 0}(2r)^{-m} \int\int P_{V*}\mu(B(x,r))dP_{V*}\mu(x)d\gamma_{n,m}(V) =$$

$$\liminf_{r\searrow 0}(2r)^{-m}\int\int \mu(\{y;|P_V(y-z)|\leq r\})d\mu(z)d\gamma_{n,m}(V)=$$

$$\liminf_{r\searrow 0}(2r)^{-m}\int\int \gamma_{n,m}(\{V;|P_V(y-z)|\leq r\})d\mu(y)d\mu(z)\leq$$

$$cI_m(\mu).$$

That is, $\int \underline{D}(P_{V*}\mu, x) < \infty$ (and hence $P_{V*}\mu$ is absolutely continuous wrt to H^m) for almost every V. In particular, $D(P_{V*}\mu, x)$ exists for $P_{V*}\mu$-almost every $x \in V$. As

$$\int D(\mu,\nu,x)^2 d\nu(x) = \int D(\mu,\nu,x)d\mu(x),$$

the assertion follows. □

As a corollary, we get the Marstrand Theorem:

Theorem 3.7. *Let $A \in \mathbb{R}^n$ be a Borel set. If $\dim_H A \leq m$ then $\dim_H P_V(A) = \dim_H A$ for $\gamma_{n,m}$-almost all $V \in G(n,m)$. If $\dim_H A > m$ then $H^m(P_V(A)) > 0$ for $\gamma_{n,m}$-almost all $V \in G(n,m)$.*

Proof. The first part follows immediately from Proposition 3.5. The second part follows from Proposition 3.6, because for any Radon measure μ supported on A with $I_m(\mu) < \infty$ we have on $\gamma_{n,m}$-almost all V both $P_{V*}\mu(P_V(A)) = 1$ and

$$\int d(P_{V*}\mu, x)dP_{V*}\mu(x) < \infty.$$

□

The corresponding results for packing and box counting dimensions are more complicated. The dimension of the projected set is still the same for almost all projections, but the common value does not need to be equal to the dimension of the original set (it can be strictly smaller). See [10] and [14].

3.4 Transversality method

We will now apply the same method to iterated function systems without OSC. For a $C^{1+\alpha}$ conformal iterated function system $\{f_i\}_1^k : \mathbb{R} \to \mathbb{R}$ we define the symbolic space Σ, construct the projection π by (2.2), we set the potential ϕ by (2.9) and let s be the solution of Bowen's Equation (2.10). Let μ_ϕ be the Gibbs measure on Σ and let $\nu = \pi_*\mu_\phi$. We can choose a metric ρ on Σ in which $\dim_H \Sigma = s$ and π is Lipschitz.

We will consider not one such system but a one-parameter family of them, with parameter t in an open interval I. We will assume that the dependance of f_i on t is $C^{1+\alpha}$ and that the following *transversality condition* is satisfied: there exists $\delta > 0$ such that for any pair $\omega, \tau \in \Sigma$, $\omega_1 \neq \tau_1$ for all $t \in I$ we have either

$$|\pi_t(\omega) - \pi_t(\tau)| > \delta$$

or

$$\left| \frac{d\pi_t(\omega)}{dt} - \frac{d\pi_t(\tau)}{dt} \right| > \delta.$$

Theorem 3.8. *Let $\{f_i^{(t)}\}_1^k$ be a transversal one-parameter family of iterated function systems in $\mathbb{R}$ of the form*

$$f_i^{(t)}(x) = \lambda_i x + a_i^{(t)}.$$

Then for Leb-almost every $t \in I$, the following is true:

- *If $s < 1$ then $\dim_H \Lambda_t = s$,*
- *If $s > 1$ then $\mathrm{Leb}(\Lambda) > 0$.*

Proof. Let μ be the Bernoulli measure for the vector $\lambda_1^s, \ldots, \lambda_k^s$. In the metric ρ, this measure has dimension s. We want to prove that its projection to $\mathbb{R}$ by the map π has dimension s (if $s < 1$) or is absolutely continuous with respect to the Lebesgue measure (if $s > 1$).

Case 1: $s < 1$

Let us start from two lemmas, showing the use of transversality. We will denote

$$g_t(\omega, \tau) = |\pi_t(\omega) - \pi_t(\tau)|.$$

Claim 1: There exists a constant $K(s) < \infty$ such that for any $\omega, \tau \in \Sigma$, $\omega_1 \neq \tau_1$

$$\int g_t^{-s}(\omega, \tau) dt \leq K(s).$$

Proof: We can divide the integral into integral over $I_1 = \{t; g_t(\omega, \tau) > \delta\}$ and $I_2 = \{t; |d/dt g_t(\omega, \tau)| > \delta\} \setminus I_1$. We have

$$\int_{I_1} g_t^{-s}(\omega, \tau) dt \leq \delta^{-s} |I|.$$

Let J be one of components of I_2. We have

$$\int_J g_t^{-s}(\omega, \tau) dt \leq \int_{-\delta}^{\delta} t^{-s} dt = 2\delta^{1-s}.$$

As $g_t(\omega,\tau)$ is $C^{1+\alpha}$ (as a function of t) with Hölder constant not depending on ω and τ and between any two consecutive components of I_2 the derivative d/dtg_t has to change at least by 2δ, the components of I_2 are in bounded from below distance from each other. Hence, their number is uniformly bounded by some $M < \infty$.

Claim 2: There exists a constant $L(s) < \infty$ such that for any $\omega, \tau \in \Sigma$,

$$\int_\Sigma \int_\Sigma \int_I g_t^{-s}(\omega,\tau) dt d\mu(\omega) d\mu(\tau) \leq L(s). \tag{3.1}$$

Proof: Let us divide $\Sigma \times \Sigma$ into a countable family of sets. For any finite word η^ℓ let

$$A_{\eta^\ell} = \{(\omega,\tau) \in \Sigma^2; \omega_j = \tau_j = \eta^j \forall j \leq \ell, \omega_{\ell+1} \neq \tau_{\ell+1}\},$$
$$A_\infty = \{(\omega,\tau) \in \Sigma^2; \omega = \tau\}.$$

Denote the integral in (3.1) by W and the similar integrals with (ω,τ) restricted to A_{η^ℓ} by W_{η^ℓ}. We have

$$W = \sum_{\eta^\ell} W_{\eta^\ell} + W_\infty.$$

W_∞ is zero because $\mu \times \mu(A_\infty) = 0$. For $(\omega,\tau) \in A_{\eta^\ell}$ we have $(\sigma^\ell\omega, \sigma^\ell\tau) \in A_\emptyset$,

$$g_t(\omega,\tau) = \lambda_{\eta^\ell}^{-1} g_t(\sigma^\ell\omega, \sigma^\ell\tau),$$

and

$$d\mu \times d\mu(\omega,\tau) = \lambda_{\eta^\ell}^{2s} d\mu \times d\mu(\sigma^\ell\omega, \sigma^\ell\tau).$$

Hence,

$$W_{\eta^\ell} = \lambda_{\eta^\ell}^s W_\emptyset$$

and

$$W = \sum_{\eta^\ell} W_{\eta^\ell} = \sum_{\eta^\ell} \lambda_{\eta^\ell}^s W_\emptyset < \infty$$

by the first claim.

We can now finish the proof much like the proof of Proposition 3.5. As $W < \infty$, the integral

$$\int_\sigma \int_\Sigma g_t(\omega,\tau)^{-s} d\mu(\omega) d\mu(\tau)$$

is finite for almost every t. For each t for which it is finite, the Hausdorff dimension of $\pi_t(\Sigma) = \Lambda_t$ is not smaller than s. At the same time this dimension cannot be greater than s because $\dim \Sigma = s$.

Case 2: $s > 1$
Claim: There exists K such that for every $r > 0$

$$\int_\Sigma \int_\Sigma \int_I \chi(|\pi_t(\omega) - \pi_t(\tau)| < r) dt d\mu(\omega) d\mu(\tau) < Kr$$

Proof: We separate the integral into integrals over the sets A_{η^ℓ}. Denote this integral by Z and the restriction to $(\omega, \tau) \in A_{\eta^\ell}$ by Z_{η^ℓ}. For $\delta\lambda_{\eta^\ell} \leq r$ we have

$$Z_{\eta^\ell} \leq \mu \times \mu(A_{\eta^\ell}) = \lambda_{\eta^\ell}^{2s}.$$

Otherwise,

$$Z_{\eta^\ell} \leq M \cdot \frac{2r}{\delta\lambda_{\eta^\ell}} \cdot \lambda_{\eta^\ell}^{2s}.$$

Note that. as $2s - 1 > s$, we have

$$\sum \lambda_{\eta^\ell}^{2s-1} < \infty. \tag{3.2}$$

We get

$$Z = Z_1 + Z_2,$$

where

$$Z_1 = \sum_{\lambda_{\eta^\ell} \leq r} Z_{\eta^\ell} = \sum_{\lambda_{\eta^\ell} < r} \lambda_{\eta^\ell} \lambda_{\eta^\ell}^{2s-1} \leq r \sum \lambda_{\eta^\ell}^{2s-1}$$

and

$$Z_2 = \sum_{\lambda_{\eta^\ell} > r} Z_{\eta^\ell} \leq \frac{2r}{\delta} \sum \lambda_{\eta^\ell}^{2s-1}.$$

Applying (3.2) we get the claim.
The rest of the proof goes along the same lines as Proposition 3.6. □

4 Multifractal formalism

The dimension of an attractor of a uniformly hyperbolic dynamical system is not the only question we can ask. Another interesting question is: how big is a subset of points with some special geometric or dynamical properties.

Historically, the questions of this type came from two directions. There was a line of work in mathematics (Eggleston, Billingsley) investigating the size of the set of points with given frequencies of digits in their binary or decimal expansion. And there was a line of work in mathematical physics

(Hentschel, Procaccia) investigating the size of set on which a given measure (usually, the Sinai-Ruelle-Bowen measure for some dynamical system) really lives (i.e. not the topological support of the measure, which usually was the whole attractor). The former is an example of what we now call Birkhoff spectrum, the latter is related to the local dimension spectrum. Those two types of spectra are the two main objects investigated by the multifractal formalism.

4.1 Dimensions of measures

Studying fractals we investigate not only sets but also measures living on them. So, it makes sense to try to define dimension of a measure.

The definitions of Hausdorff and packing dimensions are complicated. For this reason, when we try to define a version of those dimensions working on measures, we do not have much of a choice. For a measure μ, the *upper Hausdorff dimension* of μ is defined as the infimum of Hausdorff dimensions of sets of full measure μ. The *lower Hausdorff dimension* of μ is the infimum of Hausdorff dimensions of sets of positive measure μ. The *upper packing dimension* and *lower packing dimension* of μ are defined analogously.

On the other hand, the definition of box counting dimension is simple. For this reason, there are very very many different definitions of dimensions for measures based on the box counting dimension. The definitions are not really fixed in the literature, either. However, main definitions coincide for the Gibbs measures, and those are the only measures for which we will calculate a dimension.

Let μ be a probabilistic measure and fix $q \in [0, \infty) \setminus \{1\}$. For any $r > 0$ consider a grid $\{E_i\}$ of cubes of size r. Let

$$Z(\mu, r, q) = \sum \mu(E_i)^q. \tag{4.1}$$

One can check that changing the grid changes $Z(\mu, r)$ only by a multiplicative constant (depending on n and q, but not on r). The numbers

$$\overline{L}^q(\mu) = \frac{1}{q-1} \limsup_{r \to 0} \frac{\log Z(\mu, r, q)}{-\log r}, \tag{4.2}$$

$$\underline{L}^q(\mu) = \frac{1}{q-1} \liminf_{r \to 0} \frac{\log Z(\mu, r, q)}{-\log r} \tag{4.3}$$

are called *upper and lower Rényi dimensions* of μ. If they coincide, the common value is called Rényi dimension and is denoted $L^q(\mu)$.

The definition can be extended for other values of q. For $q = 1$ instead of $Z(\mu, r)$ we denote

$$Y(\mu, r) = -\sum \mu(E_i) \log \mu(E_i) \tag{4.4}$$

and

$$\overline{L}^1(\mu) = \limsup_{r \to 0} \frac{\log Y(\mu, r)}{-\log r},$$
$$\underline{L}^1(\mu) = \liminf_{r \to 0} \frac{\log Y(\mu, r)}{-\log r}.$$

The case of negative q is more complicated. Formally, (4.2) and (4.3) are still well defined (provided we restrict the sum in (4.1) to E_i with $\mu(E_i) > 0$). Unfortunately, so defined dimension would depend on the choice of grid. Reason: some of E_i's might barely intersect the support of μ, hence have very small measure μ, which taken to the negative power q will give a very big number.

The correct approach is as follows: replace (4.1) by

$$Z'(\mu, r, q) = \sum_{\mu(E_i) > 0} (\mu(3E_i))^q,$$

where $3E_i$ is the cube with the same center and orientation as E_i but three times larger. One can check that substituting Z' instead of Z to (4.2) and (4.3) one gets a definition of dimension which is well defined for all $q \neq 1$ and which coincides with the previous one for $q \geq 0$.

Some properties of the Rényi dimensions:

- For any probabilistic measure μ, both $\overline{L}^1(\mu)$ and $\underline{L}^1(\mu)$ are nonincreasing functions of q,
- $\overline{L}^0(\mu)$ and $\underline{L}^0(\mu)$ are upper and lower box counting dimensions of the topological support of μ. $\underline{L}^2(\mu)$ (*lower correlation dimension* of μ) is a lower bound for the Hausdorff dimension of the support of μ,
- For any Gibbs measure μ the Rényi dimensions can be calculated using in (4.1) or (4.4) cylinders from Moran cover $\mathcal{M}_r$ instead of grid of size r. This works for all $q \in \mathbb{R}$.

4.2 Multifractal formalism for conformal uniformly expanding repellers

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^{1+\varepsilon}$ conformal expanding map. We call a nonempty compact set Λ a *repeller* of T if $T(\Lambda) = \Lambda$ and if there is an open neighbourhood $U \supset \Lambda$ such that the trajectory of any point $x \in U \setminus \Lambda$ will eventually leave U. For simplicity, we will consider expanding maps T topologically conjugate with the k-shift, that is $T : \Lambda \to \Lambda$ is $k - -1$, but the results are valid in greater generality, see for example [33].

Let $\phi : \Lambda \to \mathbb{R}$ be a Hölder potential. We denote the *Birkhoff average* of ϕ at x as

$$\chi(\phi, x) = \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \phi(T^i x)$$

(if the limit exists). Let

$$\Lambda_\phi(\alpha) = \{x \in \Lambda; \chi(\phi, x) = \alpha\}$$

and

$$f_\phi(\alpha) = \dim_H \Lambda_\phi(\alpha).$$

The function f_ϕ is called *Birkhoff spectrum* of the potential ϕ.

Let μ be the Gibbs measure for the potential ϕ. Let

$$\Lambda_\mu(\alpha) = \{x \in \Lambda; d_\mu(x) = \alpha\}$$

and

$$f_\mu(\alpha) = \dim_H \Lambda_\mu(\alpha).$$

The function f_μ is called *local dimension spectrum* of the measure μ.

4.2.1 Local dimension spectrum

The following result was proven independently by Olsen [28] and by Pesin and Weiss [31]. Assumptions as above.

Theorem 4.1. *Let ϕ be a Hölder potential with pressure $P(\phi) = 0$ and Gibbs measure μ. Let $\beta(q)$ satisfy*

$$P(q\phi - \beta(q) \log |T'|) = 0.$$

Let $\psi_q = q\phi - \beta(q) \log |T'|$. Then

- *Function β is real analytic, $\beta' < 0$, $\beta'' \geq 0$,*

- $\beta(0) = \dim_H \Lambda$, $\beta(1) = 0$,
- *The domain of* $\alpha \to f_\mu(\alpha)$ *is a closed interval and coincides with the range of* $-\beta'$,
- f_μ *and* β *form a Legendre pair: for any* q *for* $\alpha = -\beta'(q)$

$$f_\mu(\alpha) = \alpha q + \beta(q) = \min_p(\alpha p + \beta(p)),$$

- $\Lambda_\mu(\alpha)$ *is dense in* Λ,
- *the Gibbs measure* μ_{ψ_q} *for potential* ψ_q *is supported on* $\Lambda_\mu(\alpha)$.

Proof. I won't present the full proof, only the main argument. In particular, the first point of assertion will not be proven. $C_m(x)$ denotes the m-th level cylinder set (in Λ) containing x.

For any measure ν we denote by $\lambda(\nu)$ it's *Lyapunov exponent*

$$\lambda(\nu) = \int_\Lambda \log |T'(x)| d\nu(x).$$

Let

$$\alpha(q) = -\frac{\phi(\mu_{\psi_q})}{\lambda(\mu_{\psi_q})}.$$

Claim 1: μ_{ψ_q}-typical point $x \in \Lambda$ belongs to $\Lambda_\mu(\alpha(q))$. Proof: We start by stating that for any two Gibbs measures μ_1, μ_2 we have

$$d_{\mu_1}(x) = \lim_{m\to\infty} \frac{\log \mu_1(C_m(x))}{\log \operatorname{diam} C_m(x)}$$

for μ_2-almost every point $x \in \Lambda$ (in case of SSC it is obvious, in case of OSC one has to work with ergodic properties of the Gibbs measures, we skip the argument). Hence, it is enough to check the high level cylinder $C_m(x)$ around a μ_{ψ_q}-typical point. We have

$$\operatorname{diam} C_m(x) \approx \exp(-\sum_{i=0}^{m-1} \log |T'(T^i x)|) \tag{4.5}$$

and

$$\mu(C_m(x)) \approx \exp(\sum_{i=0}^{m-1} \phi(T^i x)). \tag{4.6}$$

As the measure μ_{ψ_q} is ergodic, for any continuous potential ψ we have

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \psi(T^i x) = \int_\Lambda \psi(y) d\mu_{\psi_q}(y)$$

for μ_{ψ_q}-typical $x \in \Lambda$. The claim follows.

Claim 2: μ_{ψ_q}-typical point $x \in \Lambda$ belongs to $\Lambda_{\mu_{\psi_q}}(q\alpha(q) + \beta(q))$. Proof: This calculation is done in a very similar way. The diameter of $C_m(x)$ we get from (4.5), we only need to write

$$\mu_{\psi_q}(C_m(x)) \approx \exp(\sum_{i=0}^{m-1} \psi_q(T^i x)) = \left(\exp(\sum_{i=0}^{m-1} \phi(T^i x))\right)^q \cdot \left(\exp(-\sum_{i=0}^{m-1} \log |T'(T^i x)|)\right)^{\beta(q)} . \tag{4.7}$$

So, by Frostman Lemma (Theorem 1.17) we get

Statement 1:

$$q\alpha(q) + \beta(q) = \dim_H \mu_{\psi_q} \leq \dim_H \Lambda_\mu(\alpha(q)) = f_\mu(\alpha(q)).$$

Let now $x \in \Lambda_\mu(\alpha)$ for some α (not necessarily the $\alpha(q)$ defined above).

Claim 3: for any q $x \in \Lambda_{\mu_{\psi_q}}(\alpha q + \beta(q))$. Proof: For any m we have (4.5). As $x \in \Lambda_\mu(\alpha)$, we must also have

$$\mu(C_m(x)) \approx (\operatorname{diam} C_m(x))^\alpha$$

for m large enough. Substituting (4.6) and (4.7) we prove the claim.

Applying Frostman Lemma again we get

$$f_\mu(\alpha) = \dim_H \Lambda_\mu(\alpha) \leq \alpha q + \beta(q).$$

As q could have been chosen arbitrarily, we get

Statement 2:

$$f_\mu(\alpha) = \inf_q (\alpha q + \beta(q)).$$

Combining Statements 1 and 2 we see that the infimum is achieved when $\alpha = \alpha(q)$ and at this point we have equality. □

Remark. The function $\beta(q)$ has an important geometric interpretation: it is the Rényi spectrum of the measure μ:

$$\beta(q) = L^q(\mu)$$

4.2.2 Birkhoff spectrum

This part was done by Barreira and Saussol [4]. Assumptions as above.

Let $\alpha^- = \inf \mu(\phi)$ and $\alpha^+ = \sup \mu(\phi)$, where the infimum and supremum are over all probabilistic invariant measures. Given $\alpha \in (\alpha^-, \alpha^+)$ let β_α satisfy

$$\inf_q P(q(\phi - \alpha) - \beta_\alpha \log |T'|) = 0.$$

Theorem 4.2. *For $\alpha \notin [\alpha^-, \alpha^+]$ the set $\Lambda_\phi(\alpha)$ is empty. For $\alpha \in (\alpha^-, \alpha^+)$ we have*

$$f_\phi(\alpha) = \beta_\alpha.$$

Proof. I will again provide only the sketch of the proof.

Claim 1: for $\alpha \in (\alpha^-, \alpha^+)$ the pressure $P(q(\phi - \alpha) - \beta \log |T'|)$ escapes to infinity as q goes to plus or minus infinity. Proof: as $\alpha > \alpha^-$, there exist a point $x \in \Lambda$ and a backward branch $y_m \in T^{-1}(y_{m-1}); y_0 = x$ such that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \phi(T^i(y_m)) = \ell > 0.$$

Hence, as $|\beta| \log |T'|$ is bounded by some $k(\beta)$, we have for positive q

$$P(q(\phi - \alpha) - \beta \log |T'|) \geq q\ell - k(\beta).$$

The other part is proven analogously: as $\alpha < \alpha^+$, we can find a point and a backward branch satisfying

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \phi(T^i(y_m)) = \ell < 0,$$

and hence for negative q

$$P(q(\phi - \alpha) - \beta \log |T'|) \geq -q\ell - k(\beta).$$

Claim 2: β_α is well defined. Proof: the function $q \to P(q(\phi - \alpha) - \beta \log |T'|)$ is continuous, by Claim 1 escapes to infinity on both sides, hence it has well defined minimum. Varying β by Δ we change $P(q(\phi-\alpha)-\beta \log |T'|)$ by between $\Delta \inf \log |T'|$ and $\Delta \sup \log |T'|$. Hence, there exists a unique β_α for which this minimum is equal to 0.

Let $q(\alpha)$ be a parameter value, for which this minimum is achieved. That is,

$$\frac{d}{dq} P\left(q(\phi - \alpha) - \beta_\alpha \log |T'|\right)_{|q=q(\alpha)} = 0. \tag{4.8}$$

We will use (without proof) the following fact:

Fact 4.3. *Let ϕ_1, ϕ_2 be two Hölder potentials. Then*

$$\frac{d}{dq} P(\phi_1 + q\phi_2)_{|q=0} = \int \phi_2 d\mu_{\phi_1}.$$

Applying this to (4.8) we get

Claim 3:

$$0 = \int (\phi - \alpha) d\mu_{\psi_{q(\alpha)}},$$

where $\psi_{q(\alpha)} = q(\alpha)(\phi - \alpha) - \beta_\alpha \log |T'|$.

Consider now $\mu_{\psi_{q(\alpha)}}$-typical point $x \in \Lambda$. As this measure is ergodic, we have

$$x \in \Lambda_\phi(\mu_{\psi_{q(\alpha)}}(\phi)).$$

Calculating $d_{\mu_{\psi_{q(\alpha)}}}(x)$ in the usual way, we get

$$d_{\mu_{\psi_{q(\alpha)}}}(x) \approx \lim_{m\to\infty} \frac{q(-\alpha + \frac{1}{m}\sum_{i=0}^{m-1}\phi(T^i x))}{-\frac{1}{m}\sum_{i=0}^{m-1}\log|T'(T^i x)|} + \beta_\alpha \approx -q\frac{\mu_{\psi_{q(\alpha)}}(\phi) - \alpha}{\lambda(\mu_{\psi_{q(\alpha)}})} + \beta_\alpha = \beta_\alpha$$

(where the last equality follows from claim 3). By Frostman Lemma, we get

Statement 1:

$$\beta_\alpha = \dim_H \mu_{\psi_{q(\alpha)}} \le f_\phi(\mu_{\psi_{q(\alpha)}}(\phi)) = f_\phi(\alpha).$$

Consider now any $x \in \Lambda_\phi(\alpha)$. Modifying the proof slightly, we get $d_{\mu_{\psi_{q(\alpha)}}}(x) = \beta_\alpha$, and hence

Statement 2:

$$f_\phi(\alpha) \le \beta_\alpha.$$

We are done. □

Note that the formula for Birkhoff spectrum is significantly more complicated than for the local dimension spectrum (where it was a usual Legendre transform). There exists, however, a special potential for which the Birkhoff spectrum is simple: the potentials of the form $\phi(x) = \log|T'(x)|$. The Birkhoff average of this potentials is the Lyapunov exponent, so the Birkhoff spectrum for this potential is usually called *Lyapunov spectrum*. Checking the formulation of Theorem 4.2 we see that in this case we just need to calculate a usual Legendre transform, as before.

The dynamical explanation of this phenomenon is that the Lyapunov spectrum can be obtained from the local dimension spectrum of the *measure of maximal entropy* (the Gibbs measure for constant potential).

5 Further interesting questions

This is the end of the course, but only the beginning of interesting mathematics. Some of interesting areas of research are the following:

5.1 Infinite iterated function systems

Let $\{f_i\}_{i=1}^{\infty}$ be a family of contractions in $\mathbb{R}^n$. We will assume that the open set condition is satisfied. We define the symbolic space $\Sigma = \mathbb{N}^{\mathbb{N}}$ and define the natural projection π by (2.2). The limit set of infinite iterated function systems is defined as $\pi(\Sigma)$. It is not compact.

The most important differences in comparison with iterated function systems with finitely many branches are as follows:

- The symbolic space is not compact. Hence, the space M_σ of σ-invariant measures supported on Σ is also not compact in the weak* topology.
- The Gibbs measures do not exist.
- We cannot construct Moran covers: as the contraction coefficients of the maps are unbounded from below, for every $r > 0$ there will be some $x \in \pi(\Sigma)$ such that each cylinder x belongs to is either much greater than r or much smaller than r.
- The metric entropy is not upper semicontinuous on Σ.

The Hausdorff dimension of the attractor of an infinite IFS is given by Bowen's formula, [25]. The multifractal spectra many new phenomena, not happening for finite systems, can appear, see [15] for the local dimension spectrum and [11] for Birkhoff spectrum.

5.2 Parabolic maps

We assume the map is topologically expanding but with one (or more) periodic point with derivative ± 1. This kind of maps is closely related to maps with infinitely many branches because one can transform a parabolic system into an infinite one by so-called *inducing*. This operation is very convenient because it lets us to work with a hyperbolic system and it does not change the repeller much (only the countable set is missing). For multifractal questions inducing is very distorting and not always the best approach.

The main problem in investigating parabolic (or more general notion, non-uniformly hyperbolic) systems is, naturally, lack of hyperbolicity. Which implies lack of bounded distortion, lack of Gibbs measures, in fact one can even argue lack of Hölder potentials (for non-hyperbolic systems Hölder potential on $\mathbb{R}^n$ is not, in general, Hölder in symbolic space). There are still some tools we can use (for example, conformal measures and approximating by hyperbolic subsystems). However, we notably lack tools to work with points of Lyapunov exponent 0.

The pressure function for a parabolic system has more than one zero: it is constantly equal to 0 from some moment on. The Hausdorff dimension of the limit set of a parabolic system is given by the minimal zero of the pressure function [26]. For multifractal results, check for example [36], [21], [12], [16], [17].

5.3 Nonconformal maps

Here we assume that the iterated function system does not consist of conformal maps. This leads to change of our whole approach: the high level cylinders are not going to be more-or-less balls, they are going to be very long and thin. We cannot expect them to form an optimal cover, even if OSC holds.

There are two kinds of results here. We can approach 'typical' systems (in some special class). Or we can consider special classes of systems for which we can calculate everything.

The first approach is epitomized by the Falconer-Solomyak theorem [35]: consider a system of maps $f_i : \mathbb{R}^n \to \mathbb{R}^n$ of the form $f_i(x) = A_i x + a_i$, where $\{A_i\}_1^k$ is a fixed family of n by n real-valued matrices and a_i are vectors in $\mathbb{R}^n$. Then, provided that the maximal eigenvalues of all matrices A_i are smaller than $1/2$, for almost every choice of $\{a_i\}$ (with respect to Lebesgue measure on $\mathbb{R}^{nk}$) the Hausdorff dimension of the limit set is given by the zero of so-called *singular value pressure function.*

For the second approach, there are only four known classes of systems we can investigate: the Bedford-McMullen carpets [27] and its generalizations by Lalley and Gatzouras [23], Barański [2], and Kenyon and Peres [20].

There are also some multifractal results: of the first type one could mention [3], of the second type [22] and [18].

5.4 Interval maps and maps of complex variable

This is a gigantic area of dynamical systems. I have no chance to present even a little part of it, so I will not even start.

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