



# An elementary harmonic analysis of arithmetic functions

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## Note on lecturer

Profesor Jörg Brüdern jest wybitnym teoretykiem liczb zajmującym się analitycznymi aspektami arytmetyki i geometrii algebraicznej, autorem ponad 90 prac naukowych oraz monografii *Einführung in die analytische Zahlentheorie* (Springer 1995). Jest absolwentem Uniwersytetu w Getyndze. Studiował również w Oxfordzie pod kierunkiem Rogera Heath-Browna. Promotorem jego rozprawy doktorskiej (Getynga 1988) był Samuel J. Patterson. Habilitację uzyskał w 1991 roku. W latach 1994-2010 był profesorem zwyczajnym na Uniwersytecie w Stuttgarcie, gdzie pełnił wiele odpowiedzialnych funkcji: dziekana Wydziału Matematyki, prorektora i członka senatu (Advisory Board). Na początku 2010 roku powrócił do Getyngi by objąć prestiżowe stanowisko Gauss-Professur. W 1993 otrzymał nagrodę Oberwolfach prize Algebra-Zahlentheorie. Oprócz intensywnej pracy naukowej prowadzi zakrojoną na szeroką skalę działalność wspierającą badania naukowe. Między innymi jest członkiem komitetu sterującego (Steering Committee) Oberwolfach-Foundation oraz członkiem komitetu redakcyjnego *Functiones et Approximatio*. Jego zainteresowania naukowe dotyczą głównie addytywnych problemów teorii liczb, równań i nierówności diofantycznych, problemu Waringa oraz zastosowań metody kołowej Hardy-Littlewooda.

## 1 Background

In this chain of lectures we intend to demonstrate how most mathematical research works: from concrete examples to the development of a theory, and backwards to new examples that would have been out of reach without the insight provided by the abstract material. We introduce a harmonic analysis of arithmetic functions that is motivated by the Hardy-Littlewood method. The latter is known as a powerful tool in additive number theory, but in the new context it can be applied to study the distribution of multiplicative functions and sheds new light on limit-periodic functions. Thus, the topics covered range from classical analytic number theory to functional analysis. It is hoped that participants are familiar with elementary number theory and basic concepts in functional analysis. An introductory course in analytic number theory (prime number theorem, Dirichlet series) will be very helpful, but is not really needed to follow the course. The Hardy-Littlewood circle method will be developed from scratch, and we do not expect the audience to be familiar with this method.

The course is based to a large extent on unpublished work of the lecturer. Related material and part of the course work can be found in the following papers.

[1] J. Brüdern, A. Granville, A. Perelli, R.C. Vaughan and T.D. Wooley. *On the exponential sum over  $k$ -free numbers*. R. Soc. Lond. Philos. Trans. A356 (1998), 739–761.

[2] J. Brüdern. *Binary additive problems and the circle method, multiplicative sequences and convergent sieves*. Analytic number theory, Essays in honour of K.F. Roth. pp. 91–132, Cambridge Univ. Press, Cambridge, 2009.

For background reading on analytic number theory, arithmetical functions and the circle method, the following books are recommended.

H. Davenport. *Multiplicative number theory* 2nd edn. Springer 1980.

W. Schwarz and J. Spilker. *Arithmetical functions*. Cambridge University Press 1994

R.C. Vaughan. *The Hardy-Littlewood method*. 2nd edn. Cambridge University Press 1997.

## 2 Examples. The principal spaces

Analytic number theory is a fertile tool to study counting problems. Most prominent among the many interesting examples where the method of Dirichlet series has delivered an asymptotic formula for the counting problem at hand, is the prime number theorem. This asserts that

$$\sum_{p \leq x} \log p = x + o(x) \quad (x \rightarrow \infty),$$

with  $p$  reserved for primes. Similarly one may count square-free numbers. If  $\mu$  is Möbius' function, then  $\mu(n)^2$  is the indicator function of the square-free numbers, and one has

$$\sum_{n \leq x} \mu(n)^2 = \frac{6}{\pi^2} x + o(x).$$

One may also study twins of square-free numbers. There are several, rather different proofs of the asymptotic formula

$$\sum_{n \leq x} \mu(n)^2 \mu(n+1)^2 = x \prod_p \left(1 - \frac{2}{p^2}\right) + o(x), \quad (1)$$

two of which will be presented in the course. The analogous problem for the primes is famous and open: count the *twin primes*, that is, primes  $p$  with  $p+2$  prime. The number theorist has a simple answer why twins of square-free numbers are “easy” to detect, and prime twins are “hard”: the sieve for square-free numbers converges, and the sieve (of Eratosthenes) for primes does not. We now set up a more analytical model to describe these problems from a different perspective.

A mapping  $f: \mathbb{N} \rightarrow \mathbb{C}$  is an *arithmetical function*. Its *Besicovic norm* is defined by

$$\|f\| = \limsup_{N \rightarrow \infty} \left( \frac{1}{N} \sum_{n \leq N} |f(n)|^2 \right)^{1/2}.$$

This is a semi-norm, and the space  $\mathcal{L}$  of all functions with finite norm happens to be a Banach space. Limits of periodic functions form a closed subspace  $\mathcal{B}$ , the space of limit-periodic functions. This is a Hilbert space with inner product

$$\langle f, g \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} f(n) \overline{g(n)}. \quad (2)$$

It is not difficult to see that  $\mu(n)^2$  is limit-periodic. Hence, except for the product formula for the limit, (1) is only a special case of (2), with  $f(n) = \mu(n)^2$  and  $g(n) = \mu(n+1)^2$ . The difficulty for the primes comes from the fact that  $\mu$  is not limit-periodic, it is actually very far from  $\mathcal{B}$ . This will be made precise in the course.

The early part of the course will focus on the analytic problem of extending the sesquilinear form (2) to larger spaces. Analytic number theory often provides access to the limits

$$\eta_f(q, a) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} f(n). \quad (3)$$

If the limits (3) exist for all integers  $a$ , all natural numbers  $q$ , then  $f$  is called *distributed*. The set of all distributed functions in  $\mathcal{L}$  is another Banach space  $\mathcal{D}$  that contains  $\mathcal{B}$ . Note that  $\mu \in \mathcal{D}$  with  $\eta_\mu = 0$ . One of our key theorems is that the limit (2) exists whenever  $f \in \mathcal{B}$ ,  $g \in \mathcal{D}$ , and we will find that  $\langle f, \mu \rangle = 0$ .

### 3 The singular series

Following an extended introduction, the core part of the course will begin with a discussion of the *singular series* of a distributed function, defined formally by

$$\mathfrak{S}(f) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left| \sum_{b=1}^q \eta_f(q, b) e^{2\pi i ab/q} \right|^2.$$

The whole theory then depends on the inequality

$$\mathfrak{S}(f) \leq \|f\|^2 \tag{4}$$

that is established by means of the Hardy-Littlewood method.

Both sides of (4) are quadratic forms on  $\mathcal{D}$ . The singular series encodes “local” information, and the norm is “global”. The following result is therefore very interesting for the number theorist: equality holds in (4) if and only if  $f \in \mathcal{B}$ . Hence, the arithmetic data  $\eta_f$  know whether  $f$  is limit-periodic.

## 4 Extension theorems and convolution formulae

The theory of the singular series can be applied to the analysis of the sesquilinear form (2). In particular, we shall use it to show that (2) is well-defined on  $\mathcal{B} \times \mathcal{D}$ . There is a similar theory for a suitable bilinear form related to the quadratic form defined by the singular series. There is a natural shift operation on arithmetical functions,  $f \rightarrow f^+$  with  $f^+(n) = f(n+1)$ , and consequently a natural operation  $\mathcal{B} \times \mathcal{D} \rightarrow \mathcal{D}$  via

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} f(n+m)g(n).$$

This is of convolution type. One may also study the Fourier series associated with  $f \in \mathcal{D}$ , defined by

$$S_f(\alpha) = \sum_{n \leq N} f(n)e^{2\pi i \alpha n}.$$

Then, the identity

$$S_{fg}(\alpha) = \int_0^1 S_f(\alpha - \beta)S_g(\beta) d\beta$$

relates the products of functions with the convolution on  $\mathbb{R}/\mathbb{Z}$ . Likewise, one can link the local data  $\eta_{fg}$  with convolutions on  $\mathbb{Q}/\mathbb{Z}$ . These interrelations will be made precise. Then, the theory will be applied to arithmetic pattern problems. This is the question of providing an asymptotic formula for the sum

$$\sum_{n \leq x} f(n)f(n+h_1)\dots f(n+h_r)$$

for fixed natural numbers  $h_1, \dots, h_r$ . This can be done for all bounded functions  $f \in \mathcal{B}$ , for example, and there is a simple formula for the limit. This will include (1) as a very special case, and almost all formulae of a similar nature that have been obtained by elementary means.

It should be stressed that the bilinear or sesquilinear structure in (2) is crucial, not as an artefact implied by our approach, but in a very natural way. In most cases where analytic number theory is invoked as an auxiliary in other subjects, it is a limit of the type (2) that is to be examined. Also, closer to the core of number theory, for example in sieve theory or in the

study of automorphic forms, convolution sums of the shape (2) are a natural object of study. Our methods provide easy access to asymptotic formulae, and to a useful interpretation of the leading term. However, our methods are somewhat “soft”, so that for a concrete example one might get better error terms by direct treatment. It is the generality, simplicity and applicability of our approach that is useful.

## 5 Applications

The lectures will focus on applications within number theory. There is a neat concept of convergent sieves, and their theory turns out to be nothing more than an instructive example for the general theory. The details have appeared in [2], so that we shall give a brief sketch only.

In a very different direction, one can apply the theory to multiplicative functions. One might ask whether a multiplicative function in  $\mathcal{L}$  is limit-periodic. Recall that  $\mu^2$  is limit-periodic, but  $\mu$  is not, so a somewhat complicated answer is to be expected. Elliott and Daboussi gave a simple criterion. We also have one, via the singular series. If time permits, we work out the relation between the two.

Finally, we consider the implications on conjecture K of Hardy and Littlewood. One considers the number  $r_k(n)$  of solutions of the diophantine equation

$$n = x_1^k + x_2^k + \dots + x_k^k$$

in natural numbers. A deep result of Hooley will help us in the process of deciding whether  $r_k$  is limit-periodic. The question is easy for  $k = 2$  because  $r_2$  is not even in  $\mathcal{L}$ . But for  $k \geq 3$ , we do not know this, and one is well advised now to expect that  $r_k$  is in  $\mathcal{L}$ . Even without deciding this question, we can establish that  $r_k$  is not limit-periodic for  $k \geq 2$ .

## 6 The distribution in arithmetic progressions

In prime number theory, there are two classical results asserting that the Riemann hypothesis for Dirichlet  $L$ -functions is essentially correct on average. One of these is the Barban-Davenport-Halberstam theorem. This is the estimate

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left| \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p - \frac{x}{\phi(q)} \right|^2 \ll Qx \log x + x^2 (\log x)^{-A},$$

that is valid for any fixed positive number  $A$ . For  $Q$  near  $x$ , this can be refined to an asymptotic formula known as the Montgomery-Hooley theorem. For a distributed  $f$ , we may study the analogous variance

$$V_f(x, Q) = \sum_{q \leq Q} q \sum_{\substack{a=1 \\ (a,q)=1}}^q \left| \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} f(n) - \eta_f(q, a)x \right|^2$$

Note the weight  $q$  here. For  $Q$  near  $x$ , we will find that

$$V_f(x, Q) = \frac{1}{2} (\|f\|^2 - \mathfrak{S}(f)) Q^2 x + o(x^3).$$

Thus, the variance  $V_f$  is smaller than suggested by square-root cancellation if and only if  $f$  is limit-periodic. Results of this type, relating the “noise” in the distribution in arithmetic progressions with specific properties of  $f$ , seem to be entirely new.

The other average result in prime number theory is the Bombieri-Vinogradov theorem (see Davenport’s book for details). One can formulate a version for distributed functions. We intend to discuss this briefly if time permits.

In the last lecture, some open problems and further generalisations will be mentioned to stimulate further work on the topics discussed.

## **7 Disclaimer**

We intend to develop our theory in full, with complete proofs of all the relevant results in sections 2–4. For the other sections, only brief sketches will be given. However, according to the pre-education of the audience, the course will be adjusted with respect to the amount of detail provided in the lectures. Accordingly, we shall feel free to add further material or omit some of the more advanced topics, as time progresses.