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Additive prime number theory

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Note on lecturer

Profesor Alberto Perelli jest uznanym na świecie specjalistą z analitycznej teorii liczb, autorem ponad 90 prac naukowych publikowanych w czołowych czasopismach matematycznych. Jest absolwentem Uniwersytetu w Genui. Studiował również w Pizie pod kierunkiem Enrico Bombieriego. Obecnie jest profesorem zwyczajnym na uniwersytecie w Genui, a poprzednio pracował w Scuola Normale Superiore w Pizie, na Uniwersytecie w Neapolu i w Institute for Advanced Study w Princeton (USA). Jest członkiem komitetów redakcyjnych Acta Arithmetica, Functiones et Approximatio oraz Journal of Combinatorics and Number Theory i organizatorem wielu znaczących konferencji naukowych, między innymi Symposium on Analytic Number Theory (Amalfi 1989) i Analytic Number Theory (Cetraro 2002). Jego zainteresowania naukowe dotyczą w szczególności ogólnej teorii funkcji typu L, addytywnych problemów teorii liczb pierwszych, sum wykładniczych, metod sitowych w zastosowaniu do wielomianów wielu zmiennych oraz rozmieszczenia liczb pierwszych. Prowadzi ożywioną współpracę z wieloma matematykami. Na liście współautorów jego prac znajdują się między innymi: E.Bombieri (Princeton), J.Brüdern (Getynga), A.Granville (Montreal), J.Kaczorowski (Poznań), M.R.Murty (Kingston), J.Pintz (Budapeszt), R.C.Vaughan (Penn State Univ.), T.D.Wooley (Bristol), A.Zaharescu (Urbana), U.Zannier (Piza).

Lectures 1-2. Introduction and Shnirelman's theorem.

The main theme of Additive Number Theory is the representation of integers as sums of elements of a given set of integers. A very classical example is the representation as sums of squares, in particular sums of two squares (Fermat-Euler's theorem and Landau's theorem), sums of three squares (Euler's theorem) and four squares (Lagrange's theorem). Another classical problem is Waring's problem dealing with sums of k-th powers. Here we have a first contribution by Hilbert, and then the famous circle method by Hardy, Ramanujan, Littlewood and Vinogradov. Waring's problem is still the subject of deep researches.

Additive Prime Number Theory deals with the representation of integers as sums of primes or of closely related integers (like prime powers, almost primes, ...). The prototypical example is Goldbach's conjecture: every even integer $n \geqslant 4$ is a sum of two primes. The interest, and the difficulty, of this problem lies in the fact that primes are defined by multiplicative properties, while the problem involves additive properties. Many tools are used in Additive Prime Number Theory; for example

- elementary methods (e.g. sieve methods, combinatorics)
- analytic methods (e.g. exponential sums, L-functions)
- probabilistic methods (e.g. dispersion method)
- harmonic and functional analysis (e.g. in the recent work of Green-Tao).

In these lectures we will mainly use analytic methods, but we will start with an elementary result, namely Shnirelman's theorem. Before entering it, we very briefly describe sieve methods and the circle method.

a) Sieve methods. Let

$\mathcal{A}$ - a finite sequence of integers of cardinality $|\mathcal{A}|$
$\mathcal{P}$ - a set of primes
$z \geqslant 2$ and $P(z) = \prod_{p \leqslant z, p \in \mathcal{P}} p$

and consider the sifting function

$$S(\mathcal{A}, \mathcal{P}, z) = |\{a \in \mathcal{A} : (a, P(z)) = 1\}|.$$

Many interesting number theoretical problems can be expressed by means of the sifting function. We state (notation is explained during the lectures) the following important result.

Theorem 1.1 (Selberg, 1940's)**.**

$$S(\mathcal{A}, \mathcal{P}, z) \leqslant \frac{|\mathcal{A}|}{G(z)} + \sum_{d \leqslant z^2, d | P(z)} 3^{\omega(d)} |r(d)|.$$

b) Circle method. The basic idea is to associate to a finite set $\mathcal{A}$ the Fourier polynomial

$$A(\alpha) = \sum_{a \in \mathcal{A}} e(a\alpha) \qquad\qquad e(x) = e^{2\pi i x}$$

and express the number of representations $R(n)$ of n as a sum of k elements of $\mathcal{A}$ by

$$R(n) = \int_0^1 A(\alpha)^k e(-n\alpha) d\alpha.$$

Then one tries to get suitable info about $A(\alpha)$, and hence on $R(n)$, by suitable info about the set $\mathcal{A}$. Sometimes, the pikes of $A(\alpha)^k$ contribute a dominant term to the integral, and consequently one gets an asymptotic formula for $R(n)$.

c) Shnirelman's theorem. Shnirelman's theorem was the first unconditional result about Goldbach type problems. It is based on the notion of Shnirelman density of a set $\mathcal{A}$ of non-negative integers. Writing

$$A(x) = \sum_{a \in \mathcal{A}, 1 \leqslant a \leqslant x} 1,$$

Shnirelman's density is

$$\rho(\mathcal{A}) = inf_{n \geqslant 1} \frac{A(n)}{n}.$$

We say that $\mathcal{A}$ is a basis of finite order if $h\mathcal{A} = \mathcal{A} + ... + \mathcal{A} = \{1, 2, 3, ...\}$ for some h. The interest of Shnirelman density lies in the following

Theorem 2.1 (Shnirelman, 1930's)**.** *Let* $0 \in \mathcal{A}$ *and* $\rho(\mathcal{A}) > 0$*. Then* $\mathcal{A}$ *is a basis of finite order.*

The prof is simple and elementary, and is given in the course. The next tool is the following sieve estimate deduced from Theorem 1.1. Let $r(n)$ be the number of representations of n as a sum of two primes. Then

Theorem 2.2.

$$r(n) \ll \frac{n}{\log^2 n} \prod_{p | n} (1 + \frac{1}{p}).$$

From Theorems 2.1 and 2.2 we deduce

Theorem 2.3 (Shnirelman, 1930's). *Every integer* $n \geqslant 2$ *is the sum of a bounded number of primes.*

The proof of Theorem 2.3 first uses Theorem 2.2 and the Cauchy-Schwarz inequality to prove that the integers which are a sum of two primes have positive density, then Theorem 2.1 provides the full result. It is now known that 7 primes suffice to produce all integers, and from the Goldbach conjecture the best possible result, namely 3 primes suffice, follows.

Lectures 3-4. L-functions and distribution of primes.

In these two lectures we will briefly outline the main aspects of the theory of the Remann zeta function $\zeta(s)$ and of the Dirichlet L-functions $L(s,\chi)$, as well as their applications to the distribution of primes.

a) Riemann zeta function. $\zeta(s)$ is defined for $\sigma > 1$ $(s = \sigma + it)$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1},$$

so the logaritmic derivative is

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

where $\Lambda(n)$ is the von Mangoldt function. Hence $-\frac{\zeta'}{\zeta}(s)$ is important in the distribution of primes thanks to the elementary equivalence between

$$\pi(x) = |\{p \leqslant x\}| \sim \frac{x}{\log x} \qquad \text{and} \qquad \psi(x) = \sum_{n \leqslant x} \Lambda(n) \sim x$$

(and between quantitative versions as well). The basic properties of $\zeta(s)$ are given by

Theorem 3.1 (Riemann, 1850's)**.** *$\zeta(s)$ is meromorphic on $\mathbb{C}$ with only a simple pole at $s = 1$. Moreover, writing*

$$\Phi(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s),$$

we have that $\Phi(s)$ is meromorphic on $\mathbb{C}$ with only simple poles at $s = 0$ and $s = 1$ and satisfies the functional equation

$$\Phi(s) = \Phi(1-s).$$

Theorem 3.1 and the non-vanishing for $\sigma > 1$ provide the well known notions of trivial zeros, non-trivial zeros, critical line and critical strip. The vertical distribution of the zeros (which are important in applications since the zeros of $\zeta(s)$ become simple poles of the logarithmic derivative) is rather well understood thanks to the Riemann - von Mangoldt formula, while the horizontal distribution is still at an unsatisfactory stage. Indeed, only a weak zero-free region is known for $\zeta(s)$, while the famous Riemann Hypothesis asserts that $\zeta(s) \neq 0$ for $\sigma > 1/2$. The importance of the zeta zeros in the distribution of primes is clear from the following explicit formula: for $2 \leqslant T \leqslant x$

$$\psi(x) = x - \sum_{|\gamma| \leqslant T} \frac{x^\rho}{\rho} + O\Big(\frac{x \log^2 x}{T}\Big).$$

(here $\rho = \beta + i\gamma$ denotes the generic zero of $\zeta(s)$). For example, from the standard zero-free region one gets the PNT in the form

$$\psi(x) = x + O(xe^{-c \log x})$$

(with some constant $c > 0$), while the Riemann Hypothesis implies that

$$\psi(x) = x + O(\sqrt{x} \log^2 x).$$

b) Dirichlet L-functions. To study primes in AP, Dirichlet introduced the characters of the groups $\mathbb{Z}_q^*$ of the invertible residues (mod q) which, thanks to the orthogonality relations, for $(a, q) = 1$ give rise to the indicator of the AP $n \equiv a \pmod q$ as follows:

$$\frac{1}{\phi(q)} \sum_\chi \bar{\chi}(a)\chi(n).$$

Hence, in order to extend the results from the Riemann zeta function theory, the Dirichlet L-functions are defined for $\sigma > 1$ by

$$L(s, \chi) = \sum_{n=1}^\infty \frac{\chi(n)}{n^s} = \prod_p \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1}$$

and hence

$$-\frac{L'}{L}(s) = \sum_{n=1}^\infty \frac{\Lambda(n)\chi(n)}{n^s}.$$

The theory of L-functions for a fixed primitive character is quite similar to that of $\zeta(s)$. In particular, such L-functions are entire and satisfy a functional equation similar to the functional equation of $\zeta(s)$. Moreover, the General Riemann Hypothesis is expected to hold, i.e. $L(s, \chi) \neq 0$ for $\sigma > 1/2$.

c) Uniformity problems. More interesting and definitely more important in applications are the results uniform with respect to χ (mod q). Roughly speaking, as a general rule one may replace t by qt in the individual estimates, thus obtaining uniform results. However, there is an important exception regarding the horizontal distribution of zeros. By a "technical" problem, one cannot exclude the existence of a real zero inside the standard zero-free region if χ is a real character. This creates a lot of problems in the "uniform" theory of L-functions, and especially in their applications.

There are, however, results giving some control on such exceptional zeros.

Theorem 3.2 (Landau, Page, Siegel, 1920's-1930's). *There is a constant $c > 0$ such that for every $Q \geqslant 1$ the region*

$$\sigma > 1 - \frac{c}{\log Q(|t| + 2)}$$

contains at most one zero of

$$\prod_{q \leqslant Q} \prod_{\chi - prim.} L(s, \chi).$$

Moreover, if such a zero exists then it is real, simple and < 1, and is associated with a real character. Further, if $\tilde{\beta}$ (mod q) is such a zero and $\varepsilon > 0$ is arbitrary, there exists a constant $c(\varepsilon) > 0$ such that

$$\tilde{\beta} < 1 - \frac{c(\varepsilon)}{q^{\varepsilon}}.$$

Theorem 3.2 is very useful. In the lectures we will present several applications to the distribution of primes in AP's, both individual (Siegel-Walfisz theorem) and on average (Bombieri's theorem, Gallagher's prime number theorem).

Lectures 5-8. Exponential sums over primes.

The classical exponential sum over primes is ($\alpha \in \mathbb{R}$)

$$S(\alpha) = \sum_{n \leqslant x} \Lambda(n) e(n\alpha),$$

and in this chapter we will present several methods to get non-trivial bounds for it. The trivial bound (e.g. coming from Chebyshev bound for $\psi(x)$) is

$$S(\alpha) \ll x,$$

while by the Parseval identity we see that "on average" over $\alpha \in [0,1]$

$$S(\alpha) \ll \sqrt{x \log x}.$$

To get an idea on the size of $S(\alpha)$ we consider $\alpha = a/q$, in which case a heuristic argument based on the distribution of primes in AP suggests that

$$S(a/q) \ll \frac{x}{\phi(q)}.$$

In other words, we expect $S(\alpha)$ to be "large" when α is close to a rational with "small" denominator, and "small" when α is close to a rational with "large" denominator.

a) Bilinear forms. In general, bounds for sums of type

$$S(\alpha) = \sum_{n \leqslant x} \Lambda(n) f(n)$$

when $f(n)$ is "oscillatory" can be obtained by means of a reduction to suitable bilinear forms. I.M.Vinogradov was the first to use this idea, and he reduced to forms of type

$$\sum_{m} \sum_{n, mn \leqslant x} a_m b_n f(mn)$$

with suitable restrictions on m, n. In practice, the general case is of type

$$\sum_{m \leqslant M} \sum_{n, mn \leqslant x} a_m b_n f(mn),$$

and we single out the two basic bilinear forms of such type, i.e.

Type I: $B_I = \sum_{m \leqslant M} a_m \sum_{n, mn \leqslant x} f(mn)$

Type II: B_{II} the general case.

A simple treatment of B_I and B_{II} leads to

$$B_I \ll \sum_{m \leqslant M} |a_m| \Big| \sum_{n \leqslant x/m} f(mn) \Big|$$

and

$$B_{II} \ll \Delta(\sum_{m \leqslant M} |a_m|^2)^{1/2} (\sum_{n \leqslant x/M} |b_n|^2)^{1/2}$$

with (roughly)

$$\Delta \ll \Big(max_{n \leqslant x/M} \sum_{n' \leqslant x/M} \Big| \sum_{m \leqslant M} f(mn) \overline{f(mn')} \Big| \Big)^{1/2}.$$

Note that assuming that $|a_m|, |b_n|, |f(n)| \leqslant 1$ (as we may in practice) we have that the trivial bound for $S(\alpha)$ is x and also the trivial bound for B_I and B_{II} is x. So, in order to get non-trivial bounds for $S(\alpha)$ is enough to get non-trivial bounds for such bilinear forms. Moreover, an inspection of B_I and B_{II} shows that non-trivial bounds are to be expected provided

- Type I: $f(n)$ is oscillatory and M is not too large
- Type II: $f(n)$ has quasi-orthogonality property and M is not too large nor to small.

In the case $f(n) = e(n\alpha)$ we need the following basic lemma.

Lemma 4.1. *Let* $(a, q) = 1$ *and* $|\alpha - a/q| \leqslant 1/q^2$. *Then*

$$\sum_{n \leqslant T} min\Big(\frac{x}{n}, \|\alpha n\|^{-1}\Big) \ll \Big(\frac{x}{T} + T + q\Big) \log 2qT.$$

From Lemma 4.1 we easly get

Theorem 4.1. *Let* $(a, q) = 1$ *and* $|\alpha - a/q| \leqslant 1/q^2$. *Then*

$$\sum_{m \leqslant M} a_m \sum_{n \leqslant x/m} e(\alpha mn) \ll (max_{m \leqslant M} |a_m|) \Big(\frac{x}{q} + M + q\Big) \log 2qM$$

$$\sum_{m \leqslant M} a_m \sum_{n \leqslant x/m} b_n e(\alpha mn) \ll$$
$$\ll \Big(\sum_{m \leqslant M} |a_m|^2 \Big)^{1/2} \Big(\sum_{n \leqslant x/M} |b_n|^2 \Big)^{1/2} \Big(M + \frac{x}{q} + \frac{x}{M} + q \Big)^{1/2} \log^{1/2} qx.$$

b) Vaughan's method. Vinogradov reduced the exponential sum over primes to bilinear forms first by an application of the sieve of Erathostenes and then by means of complicated elementary arguments. His result was

Theorem 4.2 (I.M.Vinogradov, 1930's)**.** *Let* $(a, q) = 1$ *and* $|\alpha - a/q| \leqslant 1/q^2$. *Then for suitable* $c_1, c_2 > 0$ *we have*

$$\sum_{p\leqslant x} e(\alpha p) \ll \Big(\frac{x}{\sqrt{q}} + \sqrt{qx} + xe^{-c_1\sqrt{\log x}}\Big)\log^{c_2} x.$$

Vaughan found a much simpler method leading to a sharper result. His method is based on Vaughan's identity

$$-\frac{\zeta'}{\zeta}(s) = F(s) - \zeta(s)F(s)G(s) - \zeta'(s)G(s) + \Big(-\frac{\zeta'}{\zeta}(s) - F(s)\Big)(1 - \zeta(s)G(s)),$$

which holds for any $F(s)$ and $G(s)$. Vaughan's choice was

$$F(s) = \sum_{m\leqslant U} \Lambda(m)m^{-s} \qquad\qquad G(s) = \sum_{d\leqslant V} \mu(d)d^{-s}$$

which are, respectively, approximations to $-\zeta'(s)/\zeta(s)$ and $1/\zeta(s)$. Comparing coefficients of both sides one gets an expression of $\Lambda(n)$ as a sum of four terms. Hence, summing over $n \leqslant x$, after a rearrangement one gets

$$S(\alpha) = \sum_{j=1}^{4} S_j(\alpha)$$

with

$$S_1(\alpha) = \sum_{m\leqslant U} \Lambda(m)e(m\alpha)$$

$$S_2(\alpha) = \sum_{m\leqslant UV} a(m) \sum_{n\leqslant x/m} e(\alpha mn)$$

$$S_3(\alpha) = \sum_{m\leqslant V} \mu(m) \sum_{n\leqslant x/m} \log n e(\alpha mn)$$

$$S_4(\alpha) = \sum_{U<m<x/V} \sum_{V<n\leqslant x/m} \Lambda(m)b(n)e(\alpha mn)$$

with certain $a(m)$ and $b(n)$. One can see that the first sum is trivial, the second and third (after partial summation to remove the smooth $\log n$) are Type I bilinear forms, and the fourth is a Type II bilinear form. Hence the general theory applies, and after some computations one gets

Theorem 4.3 (Vaughan, 1970's)**.** *Let* $(a,q)=1$ *and* $|\alpha-a/q|\leqslant 1/q^2$. *Then*

$$S(\alpha)\ll (\frac{x}{\sqrt{q}}+\sqrt{qx}+x^{4/5})\log^{7/2}x.$$

Density method. The above bounds are elementary and interesting, but unfortunately little sensitive with respect to the horizontal distribution of the zeros of L-functions. Hardy-Littlewood were the first to devise a technique for bounding exponential sums explicitly depending on the zeros of L-functions. Unfortunately they could prove only conditional results, but the method was refined by Linnik and then by Montgomery. We present here Montgomery's version, based on the density theorems for L-functions. We start with $\alpha=a/q$, a rational number. Thanks to the periodicity of the complex exponential, this gives the possibility of connecting $S(a/q)$ with the primes in arithmetic progressions (mod q), and hence with character sums over primes. After few computations one arrives to

$$\begin{aligned} S(a/q)\ll{}& \frac{x}{\phi(q)}+\frac{\sqrt{q}}{\phi(q)}\log^2 x max_{0\leqslant\sigma\leqslant 1} max_{1\leqslant T\leqslant\sqrt{x}}\frac{x^\sigma}{T}\sum_\chi N(\sigma,T,\chi)\\ &+O(\sqrt{qx}\log^2 qx)\end{aligned}$$

where $N(\sigma,T,\chi)$ counts the number of zeros $\rho=\beta+i\gamma$ of $L(s,\chi)$ in the rectangle $\sigma\leqslant\beta\leqslant 1$ and $|\gamma|\leqslant T$. Thanks to large sieve bounds for this quantity one finally gets

Theorem 4.4 (Montgomery, 1970's)**.** *For* $(a,q)=1$ *and a certain constant* $c>0$ *we have*

$$S(a/q)\ll\left(\frac{x}{\sqrt{q}}+\sqrt{qx}+x^{3/4}q^{1/10}\right)\log^c x$$

Then Montgomery devised an ingenious diophantine argument to pass from a/q to the real numbers close to a/q, thus getting bounds for $S(\alpha)$ with $|\alpha-a/q|\leqslant 1/q^2$ of similar quality to Theorem 4.3.

We have seen several ways of getting bounds for exponential sums, and the two terms $x/\sqrt{q}$ and $\sqrt{qx}$ are always present. This is not a coincidence, since these are the only two terms remaining in the bound for $S(\alpha)$ if the General Riemann Hypothesis is assumed.

Lectures 9-10. Vinogradov's three primes theorem.

In this chapter we will prove the famous three primes theorem by I.M.Vinogradov.

a) Set-up of the circle method. The Farey fractions of order Q are the rational numbers a/q with $(a,q)=1$ and $q \leqslant Q$. One defines the Farey arcs of order Q at the rational a/q, denoted by $\mathfrak{M}_{q,a}$, which form a non-overlapping dissection of the unit interval $(1/(Q+1), 1+1/(Q+1)]$, and whose measure is roughly $1/qQ$. The ternary Goldbach problem concerns the representation of odd integers n as

$$n = p_1 + p_2 + p_3.$$

Due to analytic reasons, as usual it is more convenient to use the weighted number of representations

$$R(n) = \sum_{n=n_1+n_2+n_3} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3).$$

Let n be a large odd integer, let Q be a large integer which will be chosen later on, $1 < P < Q$ be another integer parameter again to be chosen later on and

$$S(\alpha) = \sum_{m \leqslant n} \Lambda(m) e(m\alpha).$$

Due to the heuristic concerning pikes of $S(\alpha)$, by means of the Farey dissection of order Q we split the unit interval into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ as

$$\mathfrak{M} = \cup_{q \leqslant P} \cup_{(a,q)=1} \mathfrak{M}_{q,a} \qquad\qquad \mathfrak{m} = \cup_{P<q \leqslant Q} \cup_{(a,q)=1} \mathfrak{M}_{q,a}.$$

Consequently

$$R(n) = \int_0^1 S(\alpha)^3 e(-n\alpha)d\alpha = \int_{\mathfrak{M}} S(\alpha)^3 e(-n\alpha)d\alpha + \int_{\mathfrak{m}} S(\alpha)^3 e(-n\alpha)d\alpha$$
$$= R_{\mathfrak{M}}(n) + R_{\mathfrak{m}}(n),$$

say.

b) Major arcs. On the major arcs we use an approximation of $S(\alpha)^3$ depending on the distribution of primes in AP's. Indeed, for $a/q+\eta \in \mathfrak{M}_{q,a}$ we first have

$$S(a/q+\eta) = \frac{\mu(q)}{\phi(q)}T(\eta) + R(\eta,q,a),$$

where

$$R(\eta,q,a) = \frac{1}{\phi(q)}\sum_{\chi}\chi(a)\tau(\bar{\chi})\Big(\sum_{m\leqslant n}\Lambda(m)\chi(m)e(m\eta) - \delta_\chi T(\eta)\Big) + O(\log^2 qn)$$

and

$$T(\eta) = \sum_{m\leqslant n} e(m\alpha).$$

Here δ_χ is 1 if χ is the principal character, and is 0 otherwise. Now choose

$$P = \log^A n \qquad\qquad Q = n/\log^A n$$

with an arbitrarily large $A>0$. Hence by the Siegel-Walfisz theorem we get ($c_1>0$ suitable)

$$R(\eta,q,a) \ll ne^{-c_1\sqrt{\log n}}$$

uniformly for $a/q+\eta \in \mathfrak{M}_{q,a}$, $(a,q)=1$ and $q\leqslant P$. Therefore

$$S(a/q+\eta)^3 = \frac{\mu(q)}{\phi(q)^3}T(\eta)^3 + O(n^3e^{-c_2\sqrt{\log n}})$$

and hence

$$R_{\mathfrak{M}}(n) = \sum_{q\leqslant P}\frac{\mu(q)}{\phi(q)^3}\sum_{(a,q)=1}e(-an/q)\int_{\mathfrak{M}_{q,a}-a/q}T(\eta)^3e(-n\eta)d\eta + O(n^2e^{-c_3\sqrt{\log n}}).$$

Thanks to the nice behaviour of $T(\alpha)$ we may extend the integral to the whole unit interval with a relatively small error, and then compute the resulting integral by means of the associated diophantine problem. In this way we get

$$R_{\mathfrak{M}}(n) = \frac{1}{2}n^2\sum_{q\leqslant P}\frac{\mu(q)}{\phi(q)^3}c_q(-n) + O(n^2\log^{-A} n),$$

where

$$c_q(-n) = \sum_{(a,q)=1} e(-an/q)$$

is the Ramanujan sum. Moreover, thanks to the decay of the summands we may extend the sum to ∞, again with a relatively small error. Further, thanks to the multiplicativity in q of the Ramanujan sum we transform the series into an infinite product by Euler's identity, thus getting

$$R_{\mathfrak{M}}(n) = \frac{1}{2}n^2 \prod_p \Big(1 - \frac{c_p(-n)}{(p-1)^2}\Big) + O(n^2 \log^{1-A} n)$$

$$= \frac{1}{2}n^2 \prod_{p|n} \Big(1 - \frac{1}{(p-1)^2}\Big) \prod_{p\nmid n} \Big(1 + \frac{1}{(p-1)^2}\Big) + O(n^2 \log^{1-A} n)$$

$$= \frac{1}{2}n^2 \mathfrak{S}(n) + O(n^2 \log^{1-A} n),$$

say, thanks to Hölder's formula for $c_q(-n)$. One can check that $\mathfrak{S}(n) \neq 0$ and even $1 \ll \mathfrak{S}(n) \ll 1$.

c) Conclusion of the proof. From the bounds for $S(\alpha)$ in Ch.4 and the choice of P and Q we get

$$S(\alpha) \ll n \log^{-A/2+c} n$$

($c > 0$ suitable) uniformly for $\alpha \in \mathfrak{m}$. Hence

$$R_{\mathfrak{m}}(n) \ll max_{\alpha\in\mathfrak{m}}|S(\alpha)| \int_0^1 |S(\alpha)|^2 d\alpha \ll n^2 \log^{-A/2+c+1} n.$$

Combining the major arcs treatment with the above minor arcs estimate we finally obtain

Theorem 5.1 (I.M.Vinogradov, 1930's)**.** *For every $A > 0$*

$$R(n) = \frac{1}{2}n^2 \mathfrak{S}(n) + O(n^2 \log^{-A} n).$$

In particular, every sufficiently large odd integer is a sum of three primes.

Note that the constant in the O-symbol is ineffective due to the use of the Siegel-Wafisz theorem. An effective version can be obtaned by using instead a weaker PNT for AP. However, in such a case the constants involved are so large that computers can't be used to check the representability of the remaining odd integers.

Lectures 11-12. Binary problems.

In this chapter we will discuss the exceptional set for the binary Goldbach problem, as well as a binary problem involving square-free numbers which can be successfully treated by the circle method. Note that a treatment of the binary Goldbach problem by the circle method is definitely delicate. Indeed, one can prove (in some cases assuming GRH) that for rather general minor arcs $\mathfrak{m}$

$$\int_{\mathfrak{m}} |S(\alpha)|^2 d\alpha \gg n \log n,$$

therefore in order to get any result one has to be able to get some cancellation from the term $e(-n\alpha)$ appearing in $r_{\mathfrak{m}}(n)$.

a) The standard exceptional set. Choosing $\mathfrak{M}$ and $\mathfrak{m}$ essentially as in Ch.5 it is not difficult to show that for any $A > 0$

$$r_{\mathfrak{M}}(n) = n\mathfrak{S}(n) + O(n \log^{-A} n)$$

uniformly for $n \leqslant X$, where

$$\mathfrak{S}(n) = \prod_{p \nmid n} \Big(1 - \frac{1}{(p-1)^2}\Big) \prod_{p \mid n} \Big(1 + \frac{1}{p-1}\Big)$$

$$S(\alpha) = \sum_{n \leqslant X} \Lambda(n) e(\alpha n)$$

$$r(n) = \sum_{n = n_1 + n_2} \Lambda(n_1)\Lambda(n_2) = \Big(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\Big) S(\alpha)^2 e(-n\alpha) d\alpha = r_{\mathfrak{M}}(n) + r_{\mathfrak{m}}(n).$$

The treatment of the minor arcs depends on the following mean-square bound

$$\sum_{n \leqslant X} |r_{\mathfrak{m}}(n)|^2 \ll \int_{\mathfrak{m}} |S(\alpha)|^4 d\alpha \ll max_{\alpha \in \mathfrak{m}} |S(\alpha)|^2 \int_0^1 |S(\alpha)|^2 d\alpha \ll X^3 \log^{-A} X.$$

Hence the following result follows

Theorem 6.1 (Chudakov, van der Corput, Estermann, Heilbronn, 1930's). *For any $A > 0$ we have*

$$\sum_{n \leqslant X} |r(n) - n\mathfrak{S}(n)|^2 \ll X^3 \log^{-A} X.$$

From this, by a standard use of Chebyshev's inequality one gets that

$$r(n) = n\mathfrak{S}(n) + O(n \log^{-A} n)$$

for all but $O(X \log^{-A} X)$ even integers $n \leqslant X$. In particular

$$E(X) \ll X \log^{-A} X,$$

where $E(X)$ is the exceptional set in the binary Goldbach problem, i.e. the number of even integers up to X which are not the sum of two primes. Note that Goldbach's conjecture implies that $E(X) = 1$.

b) Montgomery-Vaughan's method. An inspection of the techniques presented above shows that the bound for $E(X)$ we obtained is due to the level $P = \log^A X$ of the major arcs, which in turn is forced by the use of the Siegel-Walfisz theorem. If we could successfully treat larger major arcs, then better bounds for $E(X)$ would follow. Montgomery-Vaughan devised a way to do this, which unfortunately is quite complicated. We present here only the very basic strategy of their method.

Step 1. Choose $P = X^\delta$ with a small $\delta > 0$, and consequently $Q = X^{1-\delta}$. By the Landau-Page theorem, we consider (roughly) the P-exceptional character $\tilde{\chi}$ (mod $\tilde{q}$) with exceptional zero $\tilde{\beta}$ (if it exists; if not, even better).

Step 2. In the treatment of the major arcs consider the term coming from such exceptional zero as a secondary main term, thus computing explicitly its contribution, without introducing bounds for it. Thanks to Gallagher's prime number theorem, which contains in itself the Deuring-Heilbronn phenomenon (roughly: the existence of an exceptional zero implies that the other zeros of L-functions are better distributed), the contribution to major arcs of the remaining zeros is under control.

Step 3. Thanks to the higher level of the major arcs, the mean-square average of $r_{\mathfrak{m}}(n)$ over $n \leqslant X$ (see the previous section) shows that $r_{\mathfrak{m}}(n)$ can be large only for fewer n's, i.e. for a smaller exceptional set.

Step 4. By arithmetic arguments we show that, notwithstanding the presence of the secondary main term, $r_{\mathfrak{M}}(n)$ is still sufficiently large to dominate over $r_{\mathfrak{m}}(n)$, apart from the n's in another exceptional set, again smaller than the standard one.

The final result is

Theorem 6.2 (Montgomery-Vaughan, 1970's). *There exists a (small) constant $\delta > 0$ such that*

$$E(X) \ll X^{1-\delta}.$$

Note that the bound for the exceptional set is smaller, but the asymptotic formula for non-exceptional n's is lost.

c) Another binary problem. We end these lectures with the problem of representing integers as sums of two square-free numbers. This problem was solved, and an asymptotic formula for the number of representations was obtained, by means of elementary methods in the 1940's. Here we sketch a successful approach by the circle method. Let

$$S(\alpha) = \sum_{m \leqslant n} \mu(m)^2 e(\alpha m)$$

be the associated exponential sum. Based on the distribution of square-free numbers in AP's we consider the approximation

$$S(\alpha) = S^*(\alpha) + \Delta(\alpha),$$

where

$$S^*(a/q + \eta) = \frac{G(q)}{\zeta(2)} T(\eta)$$

and $G(q)$ is a kind of Gauss sum associated with the problem. Moreover, we choose the level of major and minor arcs as

$$P = n^{2/5} \qquad\qquad Q = n^{3/5}$$

and write as usual

$$r(n) = \sum_{n=n_1+n_2} \mu(n_1)^2 \mu(n_2)^2 = \Big(\int_{\mathfrak{M}} + \int_{\mathfrak{m}} \Big) S(\alpha)^2 e(-n\alpha) d\alpha = r_{\mathfrak{M}}(n) + r_{\mathfrak{m}}(n).$$

We deal first with the minor arcs estimate, which is performed in a completely different way with respect to the case of the primes. Moreover, a new phenomenon appears in this case. From suitable average results on the distribution of square-free numbers in AP's we get the following major arcs bounds.

Lemma 6.1. *For every $\varepsilon > 0$ we have*

$$\int_{\mathfrak{M}} |\Delta(\alpha)|^2 d\alpha \ll n^{4/5+\varepsilon} \qquad\qquad \int_{\mathfrak{M}} |S^*(\alpha)\Delta(\alpha)| d\alpha \ll n^{4/5+\varepsilon}.$$

Lemma 6.2. *For every $\varepsilon > 0$ we have*

$$\int_{\mathfrak{M}} |S^*(\alpha)|^2 d\alpha = \frac{n}{\zeta(2)} + O(n^{4/5+\varepsilon}).$$

Now note that

$$\int_0^1 |S(\alpha)|^2 d\alpha = \sum_{m \leqslant n} \mu(m)^2 = \frac{n}{\zeta(2)} + O(n^{1/2+\varepsilon}),$$

hence using Lemmas 6.1 and 6.2, by subtraction we immediately get

$$\int_{\mathfrak{m}} |S(\alpha)|^2 d\alpha \ll n^{4/5+\varepsilon}.$$

Therefore, the L^2-norm of $S(\alpha)$ is concentrated on the major arcs !

Such a good minor arcs bound allows to carry over the circle method paradigm, thus getting

Theorem 6.3 (Brüdern-Granville-Perelli-Vaughan-Wooley, 1990's)**.** *For every $\varepsilon > 0$ we have*

$$r(n) = n\mathfrak{S}(n) + O(n^{4/5+\varepsilon}).$$

Here $\mathfrak{S}(n)$ is the appropriate singular series.