Spectral Analysis and Spectral Synthesis

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Abstract

The roots of classical harmonic analysis go back to ancient times. The pioneer works of Fourier focused on the possibility of expanding functions into a series of basic harmonics. Nowadays researchers on abstract harmonic analysis has been studying functions, measures and spaces of such objects on topological groups. The purpose of this study is to build up these objects from elementary components, from special building bricks. These components serve as typical elements of the spaces in question, a kind of basis. Harmonic analysis means to discover the structure of the given space by finding those building bricks, and harmonic synthesis is the reconstruction process: describe and build up the space from the building bricks. A classical example is the following: the Uniqueness Theorem on the Fourier series of 2π-periodic continuous complex valued functions on the real line says that if the function is nonzero, then it has at least one nonzero Fourier coefficient. This can be reformulated by saying that the smallest translation invariant linear space including the function, and closed under uniform convergence, that is, the variety of the function, contains at least one complex exponential function. From Fejér's famous result on the uniform convergence of Fejér means to the function one derives the stronger property: the complex exponentials in the variety of the function actually span a dense subspace. The first reformulation above is a typical *harmonic analysis* result, while the second one is a *spectral synthesis* theorem. In the abstract setting all things become more general: instead of 2π-periodic continuous complex valued functions on the real line one considers complex valued functions defined on topological groups, and the role of complex exponentials is played by some special functions from a wider class. The tune, however, remains the same: are there any special functions available, and are there sufficiently many of them?

In some sense spectral analysis and synthesis is a generalization of classical harmonic analysis. Namely, one considers arbitrary continuous complex valued functions on locally compact Abelian groups without any growth conditions. Of course, the major part of the technical arsenal from classical harmonic analysis

depending on duality and Fourier transformation is lost. Even the class of elementary components is not rich enough: easy examples show that the family of characters, that is, the dual group is not satisfactory to describe the variety of a given continuous function. In order to extend this family the idea is that one includes all "generalized" characters, which are the common eigenfunctions of all translation operators, and on the other hand, one takes into account the multiplicities of these eigenfunctions by introducing exponential monomials as basic building blocks. From now on the spectral analysis question reads as follows: is there any nonzero exponential monomial in the variety of a given function? And the corresponding synthesis problem is about the size of the set of exponential monomials in a given variety. The famous pioneering result in this area is due to Schwartz. His wonderful theorem states that each continuous complex valued function on the real line is the uniform limit on compact sets of exponential monomials taken from its variety. To prove this theorem Schwartz used deep results and strong tools from complex function theory, and obviously, this machinery is not available on topological groups. It turns out, however, that at least on discrete Abelian groups a whole lot of results are at our command from the theory of commutative rings. Classical theorems on Artin and Noether rings can be combined and incorporated in the theory resulting a method depending on the annihilators of varieties. This algebraic approach has been utilized on discrete Abelian groups and culminates in the characterization of those possessing spectral analysis and spectral synthesis.

The purpose of this series of lectures is to give an introduction in the theory of spectral analysis and spectral synthesis. Although we intend to focus on the case of discrete Abelian groups, however, some non-discrete and non-commutative results may be included, too. The content of this series is planned as follows:

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   (a) Introduction
   (b) Basics from ring theory
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2. Part 2.
   (a) Spectral analysis and synthesis on vector modules
   (b) Varieties, annihilators
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   (a) Exponential monomials
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   (c) Exponential polynomials

4. Part 4.
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5. Part 5.
   (a) Examples and history of spectral synthesis
   (b) Synthesizable varieties
   (c) Spectral synthesis on discrete Abelian groups

1 Part 1

1.1 Introduction

In 1947 Laurent Schwartz proved the following theorem (see [6]):

Theorem 1. *Every continuous complex valued function is the uniform limit on compact sets of a sequence of linear combinations of exponential monomials of the form* $x \mapsto x^n e^{\lambda x}$*, which belong to the smallest translation invariant linear space including the function.*

This result is a pearl of modern analysis. Let $\tau(f)$ denote the intersection of all linear translation invariant spaces of functions, which are closed with respect to uniform convergence on compact sets and include the function f. This set is called the *variety of* f. Schwartz's result states that in the variety of every continuous complex valued functions the exponential monomials span a dense subspace. We note that if the variety of f is the set of all continuous complex valued functions, then this statement follows immediately from the classical Stone–Weierstrass theorem (see e.g. [20]). The real strength of this theorem can be understood in the case if the variety of f is not the whole space of continuous complex valued functions, in which case the function is called *mean periodic*. Indeed, in this case there are nonzero compactly supported measures μ such that the convolution equation $f * \mu = 0$, and it follows from Schwartz's result that the function f can be uniformly approximated on compact sets by a sequence of exponential polynomials, which satisfy the same convolution equations. Here by "exponential polynomial" we mean a linear combination of exponential monomials of the type given in the above theorem.

The concept of variety can be generalized: any translation invariant linear space of continuous complex valued functions on the real line is called a variety if it is closed with respect to the uniform convergence on compact sets. In the classical and modern analysis there are important examples for varieties. For instance, the solution space of a linear homogeneous differential equation of any order with constant coefficients is a variety and the same holds if we replace "differential equation" by "difference equation". The list can be continued by diverse types of systems of partial differential equations, convolution-type functional equations, etc. For varieties the corresponding generalization of Schwartz's theorem says that all exponential monomials span a dense subspace in any variety. This property of the reals is expressed by saying that *spectral synthesis* holds on the reals.

Although for the first look Schwartz's theorem is an approximation theorem, but actually it is a very strong existence theorem, too. Indeed, it implies that every nonzero variety contains exponential monomials. In the above mentioned examples it is far from being obvious that all those systems of equations do have exponential monomial solutions. The property of the reals that every nonzero variety contains exponential monomials is referred by saying that *spectral analysis* holds on the reals.

The basic concepts behind spectral analysis and spectral synthesis are the following: linear space, translation invariance, continuous function, uniform convergence on compact sets, exponential monomial. Apart the last one it is clear that they make sense on any topological group considering complex valued functions on it. It turns out that also a reasonable concept of exponential monomials can be introduced on commutative topological groups. This means that one can consider varieties in this more general situation and one can ask if spectral analysis or spectral synthesis holds. Moreover, one can study spectral analysis and spectral synthesis just for a single particular variety instead of for all varieties. It is not very difficult to see that in this more general situation varieties are nothing but the solution spaces of systems of a special type of functional equations: the so-called convolution-type functional equations. In this interpretation spectral analysis for such a system means that there are nonzero exponential monomial solutions of the system, and spectral synthesis says that there are sufficiently many nonzero exponential monomial solutions of the system. Here "sufficiently many" means that the solution space of the system is uniquely determined by the exponential monomial solutions, and it can be reconstructed from those. In the presence of spectral synthesis we can construct a solution method as follows: first we *analyze* the solution space, by finding all exponential monomial solutions, and then we *synthesize* the solution space by finding the closure of the linear span of these solutions.

Extension of Schwartz's spectral synthesis theorem to the *lattice group* $\mathbb{Z}^n$ ($\mathbb{Z}$ denotes the set of integers) is due to Marcel Lefranc in [9]. In the volumes [21, 29] the author presents several applications of spectral analysis and synthesis. In [27] Miklós Laczkovich and Gábor Székelyhidi characterized those discrete Abelian groups which possess spectral analysis, and in the 2007 paper [30] Miklós Laczkovich and the present author obtained the full characterization of discrete Abelian groups having spectral synthesis. In this course we intend to present the background, the methods and the applications of spectral analysis and synthesis.

1.2 Basics from ring theory

We shall use basic concepts and facts from commutative ring theory. As reference books we depend on the terminology and notation of some standard volumes like [11, 14, 15, 18, 19, 22].

By a *ring* we always mean a commutative ring with identity. In any ring R we shall use the standard notation $A+B$ and $A\cdot B$ for the *ideal sum* and the *ideal product*, resp. of the ideals A, B, which denote the smallest ideal containing all sums $a+b$, resp. $a \cdot b$ with a in A and b in B. In particular, the *ideal power* is defined recursively, in the natural way with the convention $A^0 = R$.

A proper ideal in a ring is called a *maximal ideal*, if it is not properly contained in any proper ideal. The *nilradical*, or simply *radical* of an ideal is the set of all elements having a positive power belonging to the ideal. An ideal in a ring is called *prime*, if it contains the product of two elements if and only if at least one of them belongs to the ideal. Every maximal ideal is prime. An ideal is called *primary*, if it contains a product of two elements the one of which not belonging to the ideal, then the other belongs to the radical of the ideal. Clearly, each prime ideal is primary, and the radical of each primary ideal is prime. Further, each ideal is primary, if its radical is maximal. A ring is called *local ring*, if it has exactly one maximal ideal. A ring is called *semi-local*, if it has finitely many maximal ideals.

If R is a ring, H is a subset in R, and A is an R-module, then we use the notation $H \cdot A$, or HA for the set of all elements in A of the form $h \cdot a$ with h in H and a in A.

The ideals I, J in the ring R are called *co-prime* if their sum is R: $I + J = R$. A family of ideals in R is called *pairwise co-prime*, if any two ideals in the family are co-prime. We have the following simple result.

Theorem 2. *In the ring R let $I_1, I_2, \ldots, I_k$ be pairwise co-prime ideals. Then we have*

$$I_1 \cap I_2 \cap \cdots \cap I_k = I_1 \cdot I_2 \cdot \cdots \cdot I_k \, .$$

An important result is the following.

Theorem 3. *(Chinese Remainder Theorem) Let R be a ring, and let $I_1, I_2, \ldots, I_n$ be pairwise co-prime ideals in R. Then R/I has a direct decomposition in the form*

$$R/I \cong R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_n \, .$$

Conversely, if R/I has a direct decomposition of the form

$$R/I \cong R_1 \oplus R_2 \oplus \cdots \oplus R_n$$

with some rings $R_1, \ldots, R_n$, then there are pairwise co-prime ideals $I_1, \ldots, I_n$ in R such that $I = I_1 \cap I_2 \cap \cdots \cap I_n$ and $R_k \cong R/I_k$ for $k = 1, 2, \ldots, n$.

A ring is called *Noether ring*, if it satisfies the *ascending chain condition* for the ideals: any ascending chain of ideals terminates after finitely many steps. In a Noether ring every ideal is finitely generated. A ring is called an *Artin ring*, if it satisfies the *descending chain condition* for the ideals: any descending chain of ideals terminates after finitely many steps. Every Artin ring is Noether ring, but the converse is not true in general. Analogously, a module is called Noether module, resp. Artin module, if it satisfies the ascending, resp. descending chain condition for the submodules. It is not true in general, that Artin modules are Noether modules.

Given an Abelian group G the *group algebra* of G is the set $\mathbb{C}G$ of all finitely supported complex functions on G equipped with the linear operations and with the *convolution* $\mu * \nu$ defined by

$$\mu * \nu(x) = \sum_{y \in G} \mu(x - y)\nu(y)$$

for eah x in G. Then $\mathbb{C}G$ is a commutative complex algebra with identity δ_0, where 0 denotes the zero element of the group G and δ_0 is the characteristic function of the singleton $\{0\}$. More generally, δ_y denotes the characteristic function of the singleton $\{y\}$ for each y in G. The elements of the group algebra can be considered as finitely supported complex measures on G, where μ acts on the subset A of G by the formula

$$\mu(A) = \sum_{a \in A} \mu(a)\,.$$

Hence in this case $\mathbb{C}G$ is identified with the space $\mathcal{M}_c(G)$ of all finitely supported complex measures on G. In addition, if $\mathcal{C}(G)$ denotes the set of all complex valued functions on G equipped with the linear operations and with the topology of pointwise convergence, then $\mathcal{C}(G)$ is a locally convex topological vector space, and its dual $\mathcal{C}(G)^*$ can be identified with $\mathbb{C}G$ using the pairing

$$\mu(f) = \sum_{y \in G} f(y)\mu(y) \tag{1}$$

for each f in $\mathcal{C}(G)$ and μ in $\mathbb{C}G$.

1.3 Vector modules, group representations and actions

Let R be a ring and let X be a topological vector space. We say that X is a *vector module over* R if X, as an Abelian group, is a module over R, and for each r in R the mapping $x \mapsto r \cdot x$ is a continuous linear operator on X. If R has a unit e, then we require that the corresponding linear operator $x \mapsto e \cdot x$ is the identity operator on X. We remark that if no topology is specified on X, then we always consider it with the discrete topology. By a *vector submodule*, or simply a *submodule* of a vector module we mean a linear subspace, which is also a vector module over R, with the same meaning of $r \cdot x$, of course. A closed vector submodule is called a *variety*. The intersection of any nonempty family of submodules, resp. varieties is a submodule, resp. variety. For any x in X the smallest submodule, resp. variety is the intersection of all submodules, resp. varieties including x, which is called the *submodule, resp. variety generated by the element* x.

If X is a topological vector space, then $\mathcal{L}(X)$ denotes the algebra of all continuous linear mappings, that is, the linear operators on X. On $\mathcal{L}(X)$ one usually considers the strong operator topology, hence all topological concepts on this space refer to that topology. In this topology a generalized sequence $(A_i)_{i \in I}$ of operators converges to the operator A if and only if the generalized sequence $\big(A_i(x)\big)_{i \in I}$ converges to $A(x)$ in X for each x in X.

Let X be a topological vector space and R a ring. By a *representation* of the ring R on X we mean a homomorphism of R into $\mathcal{L}(X)$. If R is a topological ring and this homomorphism is continuous, then we call it a *continuous* *representation*. If R has a unit, then we require that it is mapped onto the identity operator. Similarly, if an algebra $\mathcal{A}$ is given, then a *representation* of this algebra on X we mean a representation of the ring on X, which is also a homomorphism of the linear space structure of $\mathcal{A}$. Continuity of an algebra representation is meant in the obvious way.

Let X be a topological vector space and G a group. By a *representation* of G on X we mean a homomorphism of G into $\mathcal{L}(X)$, where the unit element of G is mapped onto the identity operator. If G is a topological group and the homomorphism is continuous, then we call it a *continuous representation*. If T denotes this homomorphism, then let $\mathcal{A}_T$ denote the subalgebra in $\mathcal{L}(X)$ generated by the image of G under T. Obviously, $\mathcal{A}_T$ is the set of all finite linear combinations of operators of the form $T(g)$ with g in G. Then X is a vector module over the operator algebra $\mathcal{A}_T$. We say that this vector module is *induced by the representation* T. Submodules are those linear subspaces which are invariant under all operators $T(g)$ with g in G. We may call them G-*invariant subspaces*, however these depend not just on G, but rather on T. We remark, that if T is a representation of G on X, then we may write T_g instead of $T(g)$.

Let E be a topological space and suppose that a topological group G is given, which acts continuously on E. This means, that a continuous map $\pi : G \times E \to E$ is given with $\pi\big(g_1, \pi(g_2, a)\big) = \pi(g_1 g_2, a)$ and $\pi(e, a) = a$ for each a in E and g_1, g_2 in G, where e denotes the unit element of G. The map π will be referred to as an *action* of G and the function $x \mapsto \pi(g, x)$ will be denoted by π_g for each g in G.

Let X be an arbitrary topological vector space of complex valued functions on E. Suppose that this function space is π-*invariant*, which means that for any f in X and g in G the function $T_\pi(g)f$, defined by

$$T_\pi(g)f(a) = f\big(\pi(g, a)\big) \tag{2}$$

whenever a is in E, belongs to X. Suppose, moreover, that $f \mapsto T_\pi(g)f$ is a linear operator on X. Then $T_\pi : g \mapsto T_\pi(g)$ is a representation of G on X. In this case we say that this representation is *induced by the action* π.

This means that if a continuous action of G on E is given, which induces the representation T_π of G on the π-invariant function space X, then X becomes a vector module over the algebra $\mathcal{A}_{T_\pi}$. The vector submodules of X are exactly those linear subspaces of X, which are π-invariant. If an action π of G on E is given and X is a π-invariant function space on E, then we may call X a π-*module*.

2 Part 2

2.1 Spectral analysis and synthesis on vector modules

Let X be a vector module over the algebra $\mathcal{A}$. We say that $\mathcal{A}$-*spectral analysis*, or simply *spectral analysis* holds on X, if every nonzero submodule in X has a nonzero finite dimensional submodule. We say that $\mathcal{A}$-*spectral synthesis*, or *spectral synthesis* holds on X, if for each submodule X_0 in X the sum of all finite dimensional vector submodules of X_0 is dense in X_0. If $\mathcal{A}$ is of the form $\mathcal{A}_A$, or $\mathcal{A}_T$, resp. $\mathcal{A}_{T_\pi}$ as above, then we speak about A-spectral analysis and A-spectral synthesis, or T-spectral analysis and T-spectral synthesis, resp. π-spectral analysis and π-spectral synthesis. Clearly, if X is nonzero, then $\mathcal{A}$-spectral synthesis implies $\mathcal{A}$-spectral analysis on X.

1. Obviously spectral synthesis holds for each nonzero finite dimensional vector module.

2. Let X be a nonzero topological vector space over a field F. Then, as we have seen, X is a vector module over the algebra of scalar operators and submodules are exactly the linear subspaces of X. It follows that spectral synthesis holds on X, as every subspace is the sum of all finite dimensional subspaces of it.

3. This example shows the connection between spectral analysis and the invariant subspace problem (see e.g. [24], [28]). Let X be a topological vector space and A a linear operator in $\mathcal{L}(X)$. As we have seen, X is a vector module over the algebra $\mathcal{A}_A$ and the submodules are exactly the A-invariant linear subspaces of X. Hence A-spectral analysis for X is equivalent to the existence of a nonzero finite dimensional invariant subspace of A.

4. Suppose that X is a Banach space and A is a compact operator on X. By the spectral theory of compact operators X is the sum of A-invariant subspaces and each eigensubspace corresponding to a nonzero element of the spectrum of A is finite dimensional. It follows that A-spectral analysis holds on X. Moreover, A-spectral synthesis holds on X if and only if the kernel of A is finite dimensional.

The following theorem is of fundamental importance.

Theorem 4. *Let X be a vector module over the algebra $\mathcal{A}$. Spectral analysis holds on X if and only if for each submodule X_0 there is a positive integer n and there are linearly independent vectors $x_1, x_2, \dots, x_n$ in X_0 such that*

$$Ax_i = \sum_{j=1}^{n} \lambda_{i,j}(A)x_j \tag{3}$$

holds for $i = 1, 2, \ldots, n$ *and for each* A *in* $\mathcal{A}$ *with some functions* $\lambda_{i,j} : \mathcal{A} \to \mathbb{C}$. *In particular, they satisfy the system of functional equations*

$$\lambda_{i,j}(AB) = \sum_{k=1}^{n} \lambda_{k,j}(A)\lambda_{i,k}(B) \tag{4}$$

for each A, B *in* $\mathcal{A}$ *and for* $i, j = 1, 2, \ldots, n$.

Vectors satisfying a system of equations of the form (3) are called matrix elements. More exactly, an element x in X is called a *matrix element* if it is contained in a finite dimensional submodule. Clearly, an element of a vector module is a matrix element if and only if it generates a finite dimensional variety.

Theorem 5. *Let* X *be a vector module over the algebra* $\mathcal{A}$. *Spectral analysis holds on* X *if and only if each nonzero submodule of* X *contains a matrix element. Spectral synthesis holds on* X *if and only if in each submodule* X_0 *the matrix elements in* X_0 *span a dense subspace in* X_0.

In the case of commutative $\mathcal{A}$ we have the following theorem.

Theorem 6. *Let* X *be a vector module over the commutative algebra* $\mathcal{A}$. *Spectral analysis holds on* X *if and only if in each nonzero submodule there exists a common eigenvector for* $\mathcal{A}$.

2.2 Varieties, annihilators

Let for each y in G and f in $\mathcal{C}(G)$ the mapping $\pi_l : G \times \mathcal{C}(G) \to \mathcal{C}(G)$ be defined by

$$\pi_l(y, f)(x) = f(yx)\,,$$

whenever x is in G. The mapping $f \mapsto \pi(y, f)$ will be denoted by λ_y and it will be called the *left translation* of G. Similarly, we define the *right translation* ρ_y of G using the action $\pi_r : G \times \mathcal{C}(G) \to \mathcal{C}(G)$ defined by

$$\pi_r(y, f)(x) = f(xy^{-1})\,,$$

whenever x is in G. In the commutative case we call the right (and left) translation simply "translation by y", and we denote it by τ_y. According to these actions from the left and right we shall use expressions like *left invariant* and *right invariant* with the obvious meaning.

Using the above terminology, a *left variety* is a closed left invariant submodule of $\mathcal{C}(G)$ with respect to the algebra of linear operators generated by all left translations λ_y with y in G. The *right variety* has similar meaning. Although in the commutative case the left and right translations corresponding to a fixed element are different, however, invariance with respect to all left translations or to all right translations are obviously equivalent. Hence in that case we can refer closed submodules simply as *varieties*: they are closed linear subspaces of $\mathcal{C}(G)$ invariant under all translations. We use the terms *left synthesisable*, *right* *synthesisable*, *synthesisable*, *left spectral analysis left spectral synthesis*, *right spectral analysis*, *right spectral synthesis*, and *spectral analysis*, *spectral synthesis* in the obvious sense. In case of locally compact groups, when using this terminology we mean it in general with respect to the given actions of G over $\mathcal{C}(G)$, that is, we shall consider $\mathcal{C}(G)$ as a module over the algebra of linear operators generated by the corresponding translation operators.

Whenever H is a subset in $\mathcal{C}(G)$, then $\tau(H)$ denotes the intersection of all varieties containing H, which is obviously a variety, and it is called the *variety generated by* H. In particular, if $H = \{f\}$, a singleton, then we write $\tau(f)$ for H, and we call it the *variety of* f.

Let X be a vector module over the operator algebra $\mathcal{A}$, and let H be a subset in X. The *annihilator* of H is the set $H^\perp$ of all elements r in $\mathcal{A}$ satisfying $r \cdot x = 0$ for each x in H. We have the following statements.

Theorem 7. *Let X be a vector module over the operator algebra $\mathcal{A}$, and let H be a subset in X. Then the annihilator of H is an ideal in $\mathcal{A}$.*

Theorem 8. *Let X be a vector module over the operator algebra $\mathcal{A}$ and let K be a subset in $\mathcal{A}$. Further let*

$$K^\perp = \{x : r \cdot x = 0 \ \textit{for all} \ r \ \textit{in} \ K\}.$$

Then $K^\perp$ is a submodule in X.

The submodule $K^\perp$ is called the *annihilator* of the set K. It is easy to see that we have the inclusions $K^{\perp\perp} \supseteq K$ and $H^{\perp\perp} \supseteq H$ for each subset K in $\mathcal{A}$ and H in X. In general, we have the following theorems.

Theorem 9. *Let G be a locally compact group, and let V be a variety in $\mathcal{C}(G)$. Then $V^{\perp\perp} = V$.*

Theorem 10. *Let G be a discrete Abelian group. Then $I^{\perp\perp} = I$ holds for every ideal I in $\mathcal{M}_c(G)$.*

2.3 Exponentials and modified differences

Let G be an Abelian group. A function $f : G \to \mathbb{C}$ is called *normed*, if $f(0) = 1$. A normed continuous function $m : G \to \mathbb{C}$ on the locally compact G is called an *exponential* if its variety is one dimensional.

Theorem 11. *Let G be a locally compact Abelian group, and let $f : G \to \mathbb{C}$ be a function. Then the following statements are equivalent.*

1. *f is an exponential.*
2. *f is a continuous homomorphism of G into the multiplicative group of nonzero complex numbers.*
3. *f is a common normed continuous eigenfunction of all translation operators.*

We shall use *modified differences*: given a function f on the Abelian group G and an element y in G we define

$$\Delta_{f;y} = \delta_{-y} - f(y)\,\delta_0\,.$$

For the products of modified differences we use the notation

$$\Delta_{f;y_1,y_2,\ldots,y_{n+1}} = \Pi_{i=1}^{n+1}\Delta_{f;y_i}\,,$$

for any natural number n and for each $y_1, y_2, \ldots, y_{n+1}$ in G. On the right hand side Π is meant as a convolution product.

For each function $f : G \to \mathbb{C}$ the ideal in $\mathbb{C}G$ generated by all modified differences of the form $\Delta_{f:y}$ with y in G is denoted by M_f. We use the following terminology: in any ring R a maximal ideal M is called *exponential maximal ideal*, if R/M is the complex field.

Theorem 12. *Let G be an Abelian group, and let $f : G \to \mathbb{C}$ be a function. The ideal M_f is proper if and only if f is an exponential. In this case $M_f = \tau(f)^\perp$ is an exponential maximal ideal.*

Theorem 13. *Let G be a locally compact Abelian group. Then the function $f : G \to \mathbb{C}$ is an exponential if and only if it is normed, and its annihilator is an exponential maximal ideal.*

If $G = \mathbb{R}$ is the additive group of the reals equipped with the euclidean topology, then every exponential has the form $x \mapsto e^{\lambda x}$ with some complex number λ. Another familiar example is $G = \mathbb{Z}$, the additive group of the integers equipped with the discrete topology, where the general form of the exponentials is $n \mapsto \lambda^n$ with some nonzero complex number λ. On every commutative topological group bounded exponentials are called *characters*. Sometimes exponentials are called *generalized characters*.

3 Part 3

3.1 Exponential monomials

If G is an Abelian group, then the function $f : G \to \mathbb{C}$ is called a *generalized exponential monomial*, if its annihilator includes a positive power of an exponential maximal ideal.

Theorem 14. *Let G be an Abelian group. Then the nonzero function $f : G \to \mathbb{C}$ is a generalized exponential monomial if and only if $\mathbb{C}G/\tau(f)^{\perp}$ is a local ring with nilpotent exponential maximal ideal.*

It follows that for each nonzero generalized exponential monomial there exists a unique exponential m and a natural number n such that

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} * f = 0 \tag{5}$$

holds for each $y_1, y_2, \dots, y_{n+1}$ in G. The smallest n with this property is called the *degree* of the generalized exponential monomial, and we say that it is *associated with the exponential* m. If $m \equiv 1$, then we call it a *generalized polynomial*. In this case the annihilator of m is the *augmentation ideal* of the group algebra, in other words, the annihilator of the variety consisting of all constant functions.

Theorem 15. *Let G be an Abelian group. The annihilator of each generalized exponential monomial is a primary ideal.*

Generalized polynomials are characterized by Fréchet's Functional Equation:

$$\Delta_{1;y_1,y_2,\dots,y_{n+1}} * f = 0\,, \tag{6}$$

which was studied by several mathematicians under various assumptions ([1, 2, 3, 4, 5]). Generalized polynomials play a fundamental role in the theory of linear functional equations. If in (6) we have $y_1 = y_2 = \cdots = y_{n+1}$, then we write

$$\Delta_{1;y}^{n+1} * f = 0\,. \tag{7}$$

Obviously, (6) implies (7). By the result [12] of D. Ž. Djokovič the converse is also true. The interested reader should consult with [21] and the references given there. The description of generalized polynomials is possible with the help of complex homomorphisms. For the modified difference $\Delta_{1;y}$ we shall use the notation Δ_y, and we call it simply *difference*.

The function $f : G \to \mathbb{C}$ is called an *exponential monomial*, if it is a generalized exponential monomial and $\tau(f)$ is finite dimensional. It turns out that exponential monomials are exactly the matrix elements which play the fundamental role in spectral analysis and spectral synthesis on Abelian groups.

Theorem 16. *Let G be an Abelian group. The function $f : G \to \mathbb{C}$ is an exponential monomial if and only if it is an indecomposable matrix element.*

We can characterize exponential monomials in terms of their annihilators as follows in the next two theorems.

Theorem 17. *Let G be an Abelian group. The function $f : G \to \mathbb{C}$ is an exponential monomial if and only if $\mathbb{C}G/\tau(f)^{\perp}$ is a local Noether ring with nilpotent exponential maximal ideal.*

Theorem 18. *Let G be an Abelian group. The function $f : G \to \mathbb{C}$ is an exponential monomial if and only if $\mathbb{C}G/\tau(f)^{\perp}$ is a local Artin ring with exponential maximal ideal.*

3.2 Polynomials

Exponential monomials corresponding to the exponential 1 are called *polynomials*. Clearly, they are exactly those generalized polynomials, whose variety is finite dimensional. In other words, polynomials are those solutions of Fréchet's Functional Equation (6), resp. (7), which generate a finite dimensional variety. For a more explicit description of polynomials we use the following theorems, which are important and interesting in itself, too (see [17, 16]).

Theorem 19. *Let G be an Abelian group and n a positive integer. Let $a : G \to \mathcal{L}(\mathbb{C}^n)$ be a mapping satisfying*

$$a(x+y) = a(x) + a(x)a(y) + a(y) \tag{8}$$

for each x, y in G, further we assume that $a(x)$ is strictly upper triangular, that is, $a_{i,j}(x) = 0$ for $i \leq j$ $(i, j = 1, 2, \dots, n)$ and x in G. Then the function $A : G \to \mathcal{L}(\mathbb{C}^n)$ defined by

$$A(x) = \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n}\, a(x)^n \tag{9}$$

satisfies

$$A(x+y) = A(x) + A(y), \quad A(x)A(y) = A(y)A(x) \tag{10}$$

for each x, y in G.

Theorem 20. *Let G be an Abelian group and n a positive integer. Suppose that $F : G \to \mathcal{L}(\mathbb{C}^n)$ satisfies*

$$F(x+y) = F(x)F(y)\,, \tag{11}$$

whenever x, y is in G, further $F(x)$ is regular for each x. Then there exists a regular matrix S in $\mathcal{L}(\mathbb{C}^n)$, and there exist positive integers $k_1, k_2, \dots, k_l$ with $k_1 + k_2 + \dots + k_l = n$, exponentials $m_1, m_2, \dots, m_l$ and mappings $M_j, A_j : G \to \mathcal{L}(\mathbb{C}^{k_j})$ such that

1. *A_j is strictly upper triangular and satisfies* (10) *for each x, y in G,*

2. *$M_j(x) = m_j(x) \exp A_j(x)$ is upper triangular for each x in G,*

3. $F(x) = S^{-1} diag\big(M_1(x), M_2(x), \ldots, M_l(x)\big) S$ *for each* x *in* G.

Conversely, every function $F : G \to \mathcal{L}(\mathbb{C}^n)$ *having the listed properties satisfies equation* (11).

Using these auxiliary results, in the following theorem we completely describe polynomials on Abelian groups with the help of additive and multiplicative homomorphisms.

Theorem 21. *Let* G *be an Abelian group, and let* $f : G \to \mathbb{C}$ *be a polynomial. Then there exists a natural numbers* k*, there exists a polynomial* $P : \mathbb{C}^k \to \mathbb{C}$*, and homomorphisms* a_j $(j = 1, 2, \ldots, k)$ *of* G *into the additive group of complex numbers such that*

$$f(x) = P\big(a_1(x), a_2(x), \ldots, a_k(x)\big) \tag{12}$$

holds for each x *in* G. *Conversely, every function of the given form is a polynomial.*

3.3 Exponential polynomials

Finite sums of exponential monomials are called *exponential polynomials*. The following theorem is obvious.

Theorem 22. *Let* G *be an Abelian group. The function* $f : G \to \mathbb{C}$ *is an exponential polynomial if and only if it is a generalized exponential polynomial and* $\tau(f)$ *is finite dimensional.*

It is clear that in the group-situation exponential polynomials are exactly the functions what we called earlier matrix elements. Similarly to exponential monomials, exponential polynomials can be characterized in terms of ring-theoretical properties of their annihilators.

Theorem 23. *Let* G *be an Abelian group. Then* $f : G \to \mathbb{C}$ *is an exponential polynomial if and only if* $\mathbb{C}G/\tau(f)^{\perp}$ *is an Artin ring with exponential maximal ideals. Equivalently,* f *is an exponential polynomial if and only if* $\mathbb{C}G/\tau(f)^{\perp}$ *is a semi-local Noether ring with exponential maximal ideals and nilpotent Jacobson radical.*

As consequences of theorems 19 and 20 we get the following generalization of Theorem 21, which is a complete description of exponential polynomials.

Theorem 24. *Let* G *be an Abelian group, and let* $f : G \to \mathbb{C}$ *be an exponential polynomial. Then there exists natural numbers* n, k*, and for each* $i = 1, 2, \ldots, n; j = 1, 2, \ldots, k$ *there exists a polynomial* $P_i : \mathbb{C}^k \to \mathbb{C}$*, an exponential* m_i*, and a homomorphism* a_j *of* G *into the additive group of complex numbers such that*

$$f(x) = \sum_{i=1}^{n} P_i\big(a_1(x), a_2(x), \ldots, a_k(x)\big) m_i(x) \tag{13}$$

holds for each x *in* G. *Conversely, every function of the given form is an exponential polynomial.*

4 Part 4

4.1 The torsion free rank

Let G be an Abelian group. The *rank* of G is the smallest cardinality of a generating set of G. An element x in G is called of *finite order*, or a *torsion element*, if it generates a finite subgroup. This is exactly the case if there is a positive integer n with $n \cdot x = 0$. The smallest n with this property is called the *order* of x. An element, which is not of finite order is called an element of *infinite order*. Hence x is an element of infinite order if and only if the subgroup generated by x is isomorphic to the additive group of integers. We call G a *torsion group*, if every element of G is of finite order. We call G *torsion free*, if every nonzero element of G is of infinite order.

Theorem 25. *Let G be an Abelian group. All torsion elements of G form a subgroup T, and G/T is torsion free.*

We call the *torsion free rank* of the Abelian group G the *cardinality* of a maximal *linearly independent system* of elements. Independence is meant over the integers, that is, the set $x_1, x_2, \dots, x_n$ of elements is called *linearly independent*, if an equation

$$k_1x_1 + k_2x_2 + \cdots + k_nx_n = 0$$

with some integers $k_1, k_2, \dots, k_n$ implies $k_1 = k_2 = \cdots = k_n = 0$. Obviously, the elements of any linearly independent set are of infinite order.

Theorem 26. *The torsion free rank of an Abelian group G is κ if and only if G has a subgroup H isomorphic to $\mathbb{Z}^{(\kappa)}$ such that G/H is a torsion group.*

It turns out that the torsion free rank of G is infinite if and only if it has a subgroup isomorphic to the group $\mathbb{Z}^{(\omega)}$. For an arbitrary Abelian group G we use the notation $Hom(G, \mathbb{C})$ for the linear space of all complex homomorphisms of G into the additive group of complex numbers. We recall that $Hom(G, \mathbb{C})$ is the set of all additive functions on G. The following result enlightens the connection of the dimension of $Hom(G, \mathbb{C})$ to the torsion free rank of G (see also in [26]).

Theorem 27. *Let G be an Abelian group. The torsion free rank of G is finite if and only if $Hom(G, \mathbb{C})$ is a finite dimensional vector space, and in this case its dimension is equal to the torsion free rank of G.*

4.2 The polynomial ring

Theorem 28. *All generalized polynomials on an Abelian group form a ring.*

The ring of all generalized polynomials on the Abelian group G is called the *polynomial ring* of G and is denoted by $\mathcal{P}(G)$.

Theorem 29. *Let G be an Abelian group and let T denote its torsion subgroup. Then the polynomial rings of G and G/T are isomorphic.*

Theorem 30. *Let G be an Abelian group, and let H be a free subgroup such that G/H is a torsion group. Then the polynomial rings of G and H are isomorphic.*

The following theorem characterizes Abelian groups of finite torsion free rank in terms of the polynomial ring (see [31]).

Theorem 31. *Let G be an Abelian group. Then the polynomial ring of G is Noetherian if and only if the torsion free rank of G is finite.*

The following two theorems give additional information about the relation of the torsion free rank and the existence of generalized exponential polynomials, which are not exponential polynomials.

Theorem 32. *The torsion free rank of an Abelian group is finite if and only if every generalized polynomial on the group is a polynomial.*

Theorem 33. *The torsion free rank of an Abelian group is finite if and only if every bi-additive function is a bi-linear function of additive functions.*

We have the following two corollaries.

Corollary 1. *The torsion free rank of an Abelian group is finite if and only if every generalized exponential monomial on the group is an exponential monomial.*

Corollary 2. *The torsion free rank of an Abelian group is finite if and only if every generalized exponential polynomial on the group is an exponential polynomial.*

4.3 Spectral analysis on discrete Abelian groups

Let G be an Abelian group, and let V be a variety on G. According to the terminology used in Part 2 we say that spectral analysis holds for V, if every nonzero subvariety of V has a nonzero finite dimensional subvariety. In particular, spectral analysis holds for every finite dimensional variety. Also, we say that spectral analysis holds on G, if spectral analysis holds for each variety on G, or equivalently, if spectral analysis holds for $\mathcal{C}(G)$. The following theorem holds.

Theorem 34. *Let G be an Abelian group, and let V be a variety on G. Then the following statements are equivalent.*

1. *There is a nonzero finite dimensional subvariety in V.*
2. *There is an exponential in V.*
3. *There is a nonzero exponential monomial in V.*
4. *There is a nonzero exponential polynomial in V.*

We have the obvious corollary.

Corollary 3. *Let G be an Abelian group, and let V be a variety on G. Then the following statements are equivalent.*

1. *Spectral analysis holds for V.*
2. *Each nonzero subvariety of V contains an exponential.*
3. *Each nonzero subvariety of V contains a nonzero exponential monomial.*
4. *Each nonzero subvariety of V contains a nonzero exponential polynomial.*

The following consequences are easy to derive.

Corollary 4. *If spectral analysis holds on an Abelian group, then it holds on every subgroup of it, too.*

Corollary 5. *If spectral analysis holds on an Abelian group, then it holds on every homomorphic image of it, too.*

The following theorem characterizes varieties possessing spectral analysis in terms of their annihilator.

Theorem 35. *Let G be an Abelian group, and let V be a variety on G. Spectral analysis holds for V if and only if every maximal ideal including its annihilator is exponential.*

It follows immediately the corollary.

Theorem 36. *Spectral analysis holds on an Abelian group if and only if every maximal ideal of its group algebra is exponential.*

The following fundamental result of M. Laczkovich and G. Székelyhidi in [27] characterizes those discrete Abelian groups having spectral analysis in terms of their torsion free rank.

Theorem 37. *Spectral analysis holds on an Abelian group if and only if its torsion free rank is less than the continuum.*

5 Part 5

5.1 Examples and history of spectral synthesis

Let G be an Abelian group, and let V be a variety on G. Recall that V is called synthesizable, if the finite dimensional subvarieties of V span a dense subvariety in V. In particular, every finite dimensional variety is synthesizable. We say that spectral synthesis holds for V, if every subvariety of it is synthesizable. Clearly, spectral synthesis holds for every finite dimensional variety. We say that spectral synthesis holds on G, if spectral synthesis holds for each variety on G, or equivalently, if spectral synthesis holds for $\mathcal{C}(G)$. Obviously, spectral synthesis for a variety implies spectral analysis for it.

The first general result on non-discrete spectral synthesis is due to L. Schwartz and was published in [6], where the following theorem was proved.

Theorem 38. *(L. Schwartz) Spectral synthesis holds on the reals.*

In 1954 B. Malgrange proved the following result (see [7]).

Theorem 39. *(B. Malgrange) For any nonzero linear partial differential operator $P(D)$ in $\mathbb{R}^n$ spectral synthesis holds for the solution space of the partial differential equation $P(D)f = 0$.*

This result has been generalized by L. Ehrenpreis in 1955 by proving the following theorem (see [8]). We recall that an ideal is called *principal*, if it is generated by a single element. In the theorem $\mathcal{E}(\mathbb{C}^n)$ denotes the Schwartz space of complex valued functions on $\mathbb{C}^n$.

Theorem 40. *(L. Ehrenpreis) Spectral synthesis holds for each variety in $\mathcal{E}(\mathbb{C}^n)$, whose annihilator is a principal ideal.*

The first general result on discrete Abelian groups, published in 1958 by M. Lefranc, was the following theorem (see [9]).

Theorem 41. *(M. Lefranc) Spectral synthesis holds on $\mathbb{Z}^n$.*

In 1965 R. J. Elliott published a result (see [10]) claiming that spectral synthesis holds on every Abelian group. However, in 1987 Z. Gajda pointed out that Elliott's proof had a gap. Finally, 17 years later it turned out (see [23]) that not just Elliott's proof was defective, but, in fact, his theorem was false.

In 1975 D. I. Gurevič presented the following result (see [13]), which was the first negative result in the non-discrete case.

Theorem 42. *(D. I. Gurevič) Spectral synthesis fails to hold on $\mathbb{R}^n$, if $n \geq 2$.*

5.2 Synthesizable varieties

The following theorems characterize synthesizable varieties in terms of their annihilator.

Theorem 43. *Let G be an Abelian group, and let V be a variety on G. Then V is synthesizable if and only if $V^\perp = \bigcap_{V^\perp \subseteq I} I$, where the intersection is extended to all ideals I containing $V^\perp$ such that $\mathbb{C}G/I$ is a local Artin ring with exponential maximal ideal.*

Applying this theorem we get the following result.

Theorem 44. *Let G be an Abelian group, and let V be a variety on G. Then V is synthesizable if and only if $\mathbb{C}G/V^\perp$ is embedded into a direct product of local Artin rings with exponential maximal ideal.*

The following theorem gives a necessary condition for the synthesizability of indecomposable varieties.

Theorem 45. *Let G be an Abelian group and V a variety on G. If V is indecomposable, and spectral synthesis holds for V, then $\mathbb{C}G/V^\perp$ is a local Artin ring.*

An easy consequence follows.

Theorem 46. *Let G be an Abelian group, and let $f : G \to \mathbb{C}G$ be a generalized exponential monomial. Then $\tau(f)$ is synthesizable if and only if f is an exponential monomial.*

The following corollaries are important from the point of view of non-synthesizable varieties.

Corollary 6. *Let G be an Abelian group, and let V be a variety on G. If V contains a generalized exponential monomial, which is not an exponential monomial, then spectral synthesis fails to hold for V.*

Corollary 7. *If spectral synthesis holds on an Abelian group, then every generalized exponential polynomial on this group is an exponential polynomial.*

Using our previous results we have the failure of spectral synthesis on Abelian groups with infinite torsion free rank – disproving Elliott's above mentioned result. This was proved originally in [23].

Corollary 8. *Spectral synthesis fails to hold on any Abelian group of infinite torsion free rank.*

5.3 Spectral synthesis on discrete Abelian groups

The above results suggest a strong connection between spectral synthesis and the non-existence of "pathological" polynomials on an Abelian group. A reasonable question arises (see [23]): is the finiteness of the torsion free rank is sufficient for spectral synthesis on an Abelian group?

Using the result 41 the following result can be obtained.

Theorem 47. *Spectral synthesis holds on every finitely generated Abelian group.*

A natural question at this point is the following: is there a non-finitely generated Abelian group possessing spectral synthesis? The following result gives an answer (see [25]).

Theorem 48. *Spectral synthesis holds on every Abelian torsion group.*

The following theorem completely characterizes discrete Abelian groups with spectral synthesis.

Theorem 49. *(M. Laczkovich–L. Székelyhidi) Spectral synthesis holds on an Abelian group if and only if its torsion free rank is finite.*

For the proof the reader is referred to [30]. This theorem obviously implies the following result.

Theorem 50. *If spectral synthesis holds on two Abelian groups, then it holds on their direct sum, too.*

Conversely, this theorem, together with the theorem of Lefranc 41 and Theorem 48, implies Theorem 49, as it is easy to see. Unfortunately, a simple direct proof of Theorem 50 has not been found so far.

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