PRODUCTIVE PROPERTIES IN TOPOLOGICAL GROUPS

M. TKACHENKO

Abstract. According to the celebrated theorem of Comfort and Ross (1966), the product of an arbitrary family of pseudocompact topological groups is pseudocompact. We present an overview of several important generalizations of this result, both of "absolute" and "relative" nature. One of them is the preservation of *functional boundedness* for subsets of topological groups. Also we consider close notions of C*-compactness* and r*-pseudocompactness* for subsets of Tychonoff spaces and establish their productivity in the class of topological groups.

Finally, we give a very brief overview of productivity properties in paratopological and semitopological groups.

1. Introduction

Given a class $\mathcal{C}$ of topological spaces, it is always a good idea to find out whether this class is closed under taking (finite) products, continuous (open, closed) mappings, and passing to (closed, open) subspaces. In other words, it is important to know what the *permanence* properties of the class $\mathcal{C}$ are. In this short course we will be primarily interested in finding *productive* properties in the class of topological groups.

Main Problem. *Given a topological (topological group) property* $\mathcal{P}$*, find out whether* $\mathcal{P}$ *is productive in the class of topological groups.*

In other words, we wonder whether a product $\prod_{i\in I} G_i$ has property $\mathcal{P}$ provided that each factor G_i is a topological group having property $\mathcal{P}$. In fact, the same question makes sense in the wider classes of *paratopological* and *semitopological* groups which will be introduced in Section 5.

This problem has been intensively studied in the case when the factors G_i's are topological spaces. We present below relatively brief lists of productive and non-productive properties (many results from the two lists can be found in [8]):

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"ABSOLUTELY" PRODUCTIVE PROPERTIES FOR ALL SPACES:

- compactness and τ-*boundedness*, for each $\tau \geq \omega$;
- connectedness and pathwise connectedness;
- axioms of separation (T_0 to $T_{3.5}$);
- completeness (realcompactness, Dieudonné-completeness);
- total disconnectedness and zero-dimensionality (i.e., $\mathrm{ind} = 0$);
- calibers, precalibers, the *Knaster* and *Shanin* properties;
- countable cellularity (under the additional assumption of Martin's Axiom plus negation of the Continuum Hypothesis), etc.

We recall that a space X is τ-*bounded*, for an infinite cardinal τ, if the closure in X of every subset of cardinality $\leq \tau$ is compact.

We also say that a space X the *Knaster* property if every uncountable family of open sets in X contains an uncountable subfamily such that every two elements of the subfamily have a non-empty intersection. Similarly, a space X has the *Shanin* property if every uncountable family of open sets in X contains an uncountable subfamily with the finite intersection property. It is clear that the Shanin property is stronger than the Knaster one:

$$\text{Shanin property} \Longrightarrow \text{Knaster property} \Longrightarrow \text{countable cellularity}$$

NON-PRODUCTIVE PROPERTIES IN SPACES:

1) countable compactness (also in topological groups, requires MA);
2) pseudocompactness (**productive** in topological groups);
3) normality (also in topological groups);
4) the Lindelöf property (also in topological groups);
5) countable cellularity (under CH, also in topological groups);
6) countable tightness (also in topological groups);
7) sequentiality (also in topological groups, in some models);
8) Fréchet–Urysohn property (also in topological groups, in some models), etc.

COUNTEREXAMPLES:

A. Novák's example of two subspaces X and Y of $\beta\mathbb{N}$ (see [8, Example 3.10.19]); it serves for the above items 1) and 2).

Even if Novák's construction is well known, it deserves a brief reproduction here. The subspaces X and Y of $\beta\mathbb{N}$ are constructed to satisfy the following three simple conditions:

(i) $|X| = \mathfrak{c}$, where $\mathfrak{c} = 2^\omega$;
(ii) $X \cap Y = \mathbb{N}$;
(iii) X and Y are countably compact.

Let $X_0 = \mathbb{N}$. We will define an increasing sequence $\{X_\alpha : \alpha < \omega_1\}$ of subspaces of $\beta\mathbb{N}$ such that for every $\alpha < \omega_1$, $|X_\alpha| \leq \mathfrak{c}$ and each infinite subset of X_α has an accumulation point in $X_{\alpha+1}$. This is possible since the family $Exp(X_\alpha)$ of all countable subsets of X_α has cardinality $|X_\alpha|^\omega \leq \mathfrak{c}^\omega = \mathfrak{c}$.

Let $X = \bigcup_{\alpha<\omega_1} X_\alpha$. It is clear that $|X| = \mathfrak{c}$. To see that X is countably compact, take an infinite subset A of X. It follows from our definition of X that $A \cap X_\alpha$ is infinite, for some $\alpha < \omega_1$. Let B be a countably infinite subset of $A \cap X_\alpha$. Then B (and, hence, A) has an accumulation in $X_{\alpha+1} \subseteq X$. Thus A has an accumulation point in X, so X is countably compact.

Let $Y = \mathbb{N} \cup (\beta\mathbb{N} \setminus X)$. Since the closure of every infinite subset of $\beta\mathbb{N}$ has cardinality $2^\mathfrak{c}$ and $|X| = \mathfrak{c}$, the subspace Y of $\beta\mathbb{N}$ is also countably compact.

Finally, we claim that the product space $X \times Y$ is neither countably compact nor even pseudocompact. Indeed, let $D = \{(n, n) : n \in \mathbb{N}\}$. The infinite subspace D of $X \times Y$ consists of isolated points and has no accumulation points in $X \times Y$ because of the equality $D = (X \times Y) \cap \Delta$, where the diagonal $\Delta = \{(x, x) : x \in \beta\mathbb{N}\}$ is a closed subspace of $\beta\mathbb{N} \times \beta\mathbb{N}$. This shows that $X \times Y$ is not pseudocompact.

[*Observation:* The subspace Y of $\beta\mathbb{N}$ has cardinality $2^\mathfrak{c}$, but a small modification of the above construction gives a countably compact subspace Z of $\beta\mathbb{N}$ such that $|Z| = \mathfrak{c}$, $\mathbb{N} \subseteq Z$, and $X \times Z$ is not pseudocompact.]

B. The Sorgenfrey line $\mathbb{S}$ is a counterexample to 3) and 4). Indeed, the space $\mathbb{S}$ is regular and Lindelöf, hence normal. However, the product $\mathbb{S} \times \mathbb{S}$ contains a closed discrete subset $C = \{(x, y) \in \mathbb{S} \times \mathbb{S} : x + y = 0\}$ of cardinality $\mathfrak{c}$. Since the spaces $\mathbb{S}$ and $\mathbb{S} \times \mathbb{S}$ are separable, the latter space is neither Lindelöf nor normal.

C. Laver and Galvin's examples serve for 5), see [9].

D. Sequential fan with $2^{\aleph_0}$ spines does the job for 6), 7) and 8) (Arhangel'skii [1] for spaces, Okunev [19] for topological groups).

Fréchet–Urysohn property is not finitely productive even in the class of compact spaces (P. Simon [27]), but sequentiality and countable tightness do remain finitely (and countably) productive in compact spaces (Malykhin [17] for tightness). In fact, countable tightness is productive in the wider class of regular *initially* $\aleph_1$*-compact* spaces (Alan Dow).

WHY TOPOLOGICAL GROUPS?

Topological groups constitute a very interesting subclass of *homogeneous* spaces. The degree of homogeneity in topological groups is even higher than in spaces: *Given two points a, b of a topological group G, there exists a homeomorphism φ of G onto itself such that $\varphi(a) = b$ and $\varphi(b) = a$.* It is worth noting that the *long line* is a homogeneous locally compact space which does not have this property!

Let us mention the following special features of topological groups (to list a few):

a) $T_0 \iff T_{3.5}$ (Pontryagin, [21]);
b) first countable $\iff$ metrizable (Birkhoff–Kakutani's theorem);
c) all σ-compact topological groups have countable cellularity (even more, they have the Knaster property). See [28] and [3, Corollary 5.4.8]);
d) every topological group has a *maximal group extension* ϱG, called the Raĭkov completion of G, and all other group extensions of G are subgroups of ϱG containing G (compare the one-point compactification $\alpha\mathbb{N}$ with the Stone–Čech compactification $\beta\mathbb{N}$ of the discrete space $\mathbb{N}$).

As usual, if G is a *dense* subgroup of a topological group H, we say that H is an extension of G. It is known that if H_1 and H_2 are maximal group extensions of a topological group G, then there exists a topological isomorphism $\varphi\colon H_1 \to H_2$ such that $\varphi(x) = x$ for each $x \in G$ [3, Theorem 3.6.14]. Hence the Raĭkov completion of a topological group is essentially unique.

Summing up, the above items a)–d) give a hope that some non-productive topological properties become productive in the class of topological groups. This is indeed the case!

2. Around the Comfort–Ross' theorem.

Problem 1. *Do there exist Novák type topological groups?*

We ask for two countably compact topological groups G and H such that $G \times H$ is not pseudocompact (or at least fails to be countably compact).

Does normality help? We have in mind the simple fact that a pseudocompact normal space is countably compact. Therefore, if both normality and pseudocompactness were finitely productive in topological groups, the product of two *normal* and *countably compact* topological groups would also be countably compact. Unfortunately, *the product of two countably compact, hereditarily normal topological groups can* *simultaneously fail to be countably compact and normal.* This fact can be deduced from the combination of [11] and [7]. All known constructions of this kind of groups, however, depend on additional axiomatic assumptions like CH or MA.

It turns out that *the product of two normal, countably compact topological groups need not be normal, in ZFC alone.* Indeed, let Σ be the Σ-product of ω_1 copies of the circle group $\mathbb{T}$, considered as a subgroup of $\mathbb{T}^{\omega_1}$. Clearly both groups Σ and $\mathbb{T}^{\omega_1}$ are normal and countably compact, and the Stone–Čech compactification of Σ is $\mathbb{T}^{\omega_1}$. Hence Tamano's theorem [8, Theorem 5.1.38] implies that the product space $\Sigma \times \mathbb{T}^{\omega_1}$ is not normal.

Historically, the first example of a Hausdorff topological group which is not normal was given by A.H. Stone in 1948. It was the group $\mathbb{Z}^{\omega_1}$ with the usual Tychonoff product topology, where $\mathbb{Z}$ is the discrete group of integers.

In what follows we will frequently work with *precompact* topological groups.

Definition 2.1. *A topological group G is called* <u>*precompact*</u> *if for every open neighborhood U of the neutral element in G, one can find a finite set $K \subseteq G$ such that $KU = G$ (equivalently, $UK = G$).*

Precompact topological groups admit a nice characterization given by Weil in 1937.

Theorem 2.2. (A. Weil, [36]) *A topological group G is precompact if and only if it is topologically isomorphic to a dense subgroup of a compact topological group.*

Weil's characterization of precompact groups yields the following simple fact:

Corollary 2.3. *A topological group G is precompact iff ϱG is compact.*

A clear relation between precompactness and pseudocompactness in topological group was found by Comfort and Ross in [6]:

Proposition 2.4. *Every pseudocompact topological group is precompact.*

It is a good idea to imitate Novák's approach and construct two countably compact subgroups, say, H_1 and H_2 of a "big" compact topological group G such that their intersection $H_1 \cap H_2$ is countable and infinite. If Δ is the diagonal in $G \times G$, then

$$S = \Delta \cap (H_1 \times H_2)$$

is a closed, countable, infinite subgroup of $H_1 \times H_2$. Hence S cannot be countably compact—otherwise it would be compact and would have uncountable cardinality. Since S is closed in $H_1 \times H_2$, the product group $H_1 \times H_2$ is not countably compact either.

There is, however, a serious obstacle for a construction of Novák type topological groups. We recall that the spaces X and Y in Novák's example were defined as subspaces of the compact space $\beta\mathbb{N}$, and all infinite closed subsets of the latter space have cardinality $2^{\mathfrak{c}}$. In particular, $\beta\mathbb{N}$ does not contain non-trivial convergent sequences. Compact topological groups are very different in this respect, by the Ivanovskii–Kuz'minov theorem [16] proved in 1959:

Theorem 2.5. *Every compact topological group G is dyadic. Therefore, if G is infinite, it contains non-trivial convergent sequences.*

Let us deduce the second claim of the above theorem from the first one. Since G is dyadic, there exists a continuous mapping $f\colon D^\tau \to G$ of the generalized Cantor discontinuum D^τ onto G, where $D = \{0,1\}$ and τ is an infinite cardinal. Denote by Σ the corresponding Σ-product of the factors lying in D^τ. Then Σ is a dense ω-*bounded* and ω-*monolithic* subspace of D^τ, i.e., the closure of every countable subset of Σ is a compact metrizable subspace of Σ. Hence $f(\Sigma)$ is also a dense ω-bounded and ω-monolithic subspace of G. If G is infinite, so is $f(\Sigma)$. Thus the latter space contains non-trivial convergent sequences.

In fact, compact groups have a stronger property which can be established by methods of the Pontryagin–van Kampen duality theory: *Every infinite compact topological group contains a non-discrete compact metrizable subgroup.*

Conclusion: *To construct, 'a la Nováк', two countably compact topological groups whose product fails to be countably compact, we have to find an infinite countably compact topological group without non-trivial convergent sequences.*

There are several subclasses of pseudocompact spaces in which pseudocompactness becomes productive. One of these classes was found by I. Glicksberg:

Theorem 2.6. (Glicksberg, [10]) *A Cartesian product of locally compact, pseudocompact spaces is pseudocompact.*

Let us now look at topological groups.

Fact. *A locally compact pseudocompact topological group is compact.*

Therefore, Glicksberg's theorem does not give anything new for topological groups. Nevertheless, Comfort and Ross proved in 1966 the following remarkable result which is a starting point for numerous generalizations:

Theorem 2.7. (COMFORT and ROSS, [6])] *The Cartesian product of arbitrarily many pseudocompact topological groups is pseudocompact.*

SKETCH OF THE PROOF. Every pseudocompact topological group is precompact, by Proposition 2.4. It turns out that a precompact topological group H is pseudocompact iff H intersects each non-empty G_δ-set in the Raĭkov completion ϱH of H, i.e., H is G_δ-*dense* in ϱH. The latter property of precompact topological groups is productive, as well as precompactness itself. □

Corollary 2.8. *The product of any two countably compact topological groups is pseudocompact.*

We come once again to the problem whether the conclusion about pseudocompactness of the product in Corollary 2.8 can be strengthened to countable compactness. It turns out that this is impossible, at least in ZFC:

Example 2.9. (E. van Douwen, [7]) *Under Martin's Axiom, there exist countably compact topological groups H_1 and H_2 such that the product $H_1 \times H_2$ is not countably compact.*

IDEA OF THE CONSTRUCTION. First, van Douwen uses MA to produce an infinite countably compact topological group G (a dense subgroup of the compact Boolean group $\{0,1\}^{\mathfrak{c}}$, where $\mathfrak{c} = 2^\omega$) such that G does not contain non-trivial convergent sequences. In fact, the group G as well as every infinite closed subset of G have the same cardinality $\mathfrak{c}$ (the latter property of G follows from MA). Once the group G is constructed, the rest is a simple application of Novák's idea.

Take a countable infinite subgroup S of G and construct by recursion of length $\mathfrak{c}$ two countably compact subgroups H_1 and H_2 of G such that $H_1 \cap H_2 = S$. Then $H_1 \times H_2$ fails to be countably compact. □

For a deeper insight into the matter of productivity of pseudocompactness in topological groups, one may wish to find out the 'real' reason of this phenomenon.

Problem 2. *Do topological groups possess a special topological property that makes pseudocompactness productive?*

In 1982, A. Chigogidze answered this question affirmatively. He used the following interesting concept introduced earlier by E.V. Ščepin.

Definition 2.10. (E.V. Ščepin) *A space X is called κ-metrizable if it admits a non-negative "distance" function $\varrho(x, \overline{O})$ from a point $x \in X$ to a regular closed subset $\overline{O}$ of X satisfying the following conditions:*

1) $x \in \overline{O} \Longleftrightarrow \varrho(x, \overline{O}) = 0$;
2) *if U, V are open in X and $U \subseteq V$ then $\varrho(x, \overline{V}) \leq \varrho(x, \overline{U})$ for each $x \in X$;*
3) *$\varrho(x, \overline{O})$ is continuous in the first argument;*
4) $\varrho(x, \overline{\bigcup \gamma}) = \inf\{\varrho(x, \overline{\bigcup \mu}) : \mu \subseteq \gamma,\ |\mu| < \aleph_0\}$ *for any family γ of open sets in X.*

The class of κ-metrizable spaces is very wide and has nice permanence properties:

Proposition 2.11. (Ščepin, [25])

(a) *All metrizable spaces are κ-metrizable.*
(b) *A dense (open, regular closed) subspace of a κ-metrizable space is κ-metrizable.*
(c) *The product of any family of κ-metrizable spaces is κ-metrizable.*
(d) *Every locally compact topological group is κ-metrizable.*

Items (b) and (d) of Proposition 2.11 together with the Weil's characterization of precompact groups given in Theorem 2.2 imply that all precompact (hence pseudocompact) topological groups are κ-metrizable.

Here is a generalization of the Comfort–Ross theorem obtained by Chigogidze:

Theorem 2.12. (Chigogidze, [5]) *A Cartesian product of κ-metrizable pseudocompact spaces is pseudocompact.*

IDEA OF THE PROOF. If X is κ-metrizable and pseudocompact, then βX is also κ-metrizable (Chigogidze). Let $\{X_i : i \in I\}$ be a family of pseudocompact κ-metrizable spaces. Each X_i is G_δ-dense in βX_i, and the latter compact space is κ-metrizable. Hence the product space $\prod_{i \in I} \beta X_i$ is κ-metrizable by (c) of Proposition 2.11, and $X = \prod_{i \in I} X_i$ is G_δ-dense in $\prod_{i \in I} \beta X_i$.

Every regular closed subset of a κ-metrizable space is a zero-set. In other words, every κ-metrizable space is *perfectly κ-normal* or, equivalently, is an *Oz-space.* It is not difficult to verify that a G_δ-dense subspace of an Oz-space is C-embedded, so X is C-embedded in the compact space $\prod_{i \in I} \beta X_i$. This implies that X is pseudocompact. □

Theorem 2.12 implies several results established earlier by Trigos–Arrieta and Uspenskij:

Theorem 2.13. *Suppose that X_i is a pseudocompact subspace of a topological group H_i, for $i \in I$. Then $\prod_{i\in I} X_i$ is pseudocompact in each of the following cases:*

(1) (Trigos–Arrieta, [34]) *X_i is regular closed in H_i for each $i \in I$;*
(2) (Uspenskij, [35]) *X_i is a retract of H_i for each $i \in I$;*
(3) (Uspenskij, [35]) *X_i is a G_δ-set in H_i for each $i \in I$.*

Proof. To deduce (1), consider the inclusions $X_i \hookrightarrow H_i \hookrightarrow \varrho H_i$ and notice that the closure of X_i in ϱH_i, say, $\overline{X_i}$ is a regular closed subset of ϱH_i. Since X_i is pseudocompact and the group ϱH_i is Raĭkov complete, the set $\overline{X_i}$ is compact. Hence the group ϱH_i is locally compact. It now follows from (d) of Proposition 2.11 that the group ϱH_i is κ-metrizable, and so is its regular closed subset $\overline{X_i}$. Then X_i is κ-metrizable as a dense subspace of $\overline{X_i}$ and, therefore, the product space $\prod_{i\in I} X_i$ is pseudocompact by Theorem 2.12.

For 2) and 3), we note that βX_i is a *Dugundji compact space* for each $i \in I$ (Uspenskij) and hence is κ-metrizable (Ščepin). The rest of the argument is identical to the one given above. □

It is worth noting that in Theorem 2.13, item (3) implies (1). Indeed, let C be the subgroup of ϱH_i generated by $\overline{X_i}$. Since $\overline{X_i}$ is compact and has a non-empty interior in ϱH_i, the subgroup C is open in ϱH_i and σ-compact. It remains to note that every regular closed subset of a σ-compact group is a zero-set [29, Theorem 1]. So $\overline{X_i}$ is a zero-set in C and in ϱH_i.

Finally, we recall that a compact space X is *Dugundji* if for every closed subspace Y of a zero-dimensional compact space Z, every continuous mapping $f\colon Y \to X$ admits a continuous extension $\tilde{f}\colon Z \to X$ (see [20]). It turns out that every Dugundji compact space has a *multiplicative* lattice of continuous open mappings onto compact metrizable spaces, and this property characterizes the class of compact Dugundji spaces [26].

3. Relative pseudocompactness

In this section we consider 'relative' properties, i.e., properties of how a subspace is placed in the whole space, and how certain relative properties behave with respect to the product operation.

Let us start with precompact subsets of topological groups. A subset X of a topological group G is called *precompact* if for every open neighborhood U of identity in G there exists a finite set $K \subseteq G$ such that $X \subseteq K \cdot U \cap U \cdot K$.

The following two facts are quite easy to verify (see [3, Section 3.7]).

Proposition 3.1. *A subset X of a topological group G is precompact in G iff $\mathrm{cl}_{\varrho G}X$ is compact.*

Proposition 3.2. *Let X_α be a precompact subset of a topological group G_α, $\alpha \in A$. Then $\prod_{\alpha\in A} X_\alpha$ is precompact in $\prod_{\alpha\in A} G_\alpha$.*

In the following definition we introduce one of the most interesting relative properties. It has a close relation with pseudocompactness.

Definition 3.3. *A subset Y of a space X is functionally bounded in X if for every continuous function $f\colon X \to \mathbb{R}$, the image $f(Y)$ is bounded in $\mathbb{R}$.*

Every pseudocompact subspace $Y \subseteq X$ is functionally bounded in X, but not vice versa. Indeed, denote by ω_1+1 the space of all ordinals $\le \omega_1$ endowed with the order topology. Let us take

$$Y = \omega \times \{\omega_1\} \subseteq (\omega + 1) \times (\omega_1 + 1) \setminus \{(\omega, \omega_1)\} = X.$$

Then Y is an infinite, closed, discrete, functionally bounded subset of X. Clearly, Y is not pseudocompact.

It is also clear that

$$X \text{ is functionally bounded in itself} \iff X \text{ is pseudocompact.}$$

It is worth mentioning that the product of functionally bounded sets $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ need not be functionally bounded in $X_1 \times X_2$ (let $Y_1 = X_1$ and $Y_2 = X_2$ in Novák's example).

It turns out that the Comfort–Ross theorem follows from the next 'relative' version of it:

Theorem 3.4. (Tkachenko, [31]) *Let B_i be a functionally bounded subset of a topological group G_i, where $i \in I$. Then $\prod_{i\in I} B_i$ is functionally bounded in $\prod_{i\in I} G_i$.*

Note that the groups G_i in Theorem 3.4 are NOT assumed to satisfy any restriction. The Comfort–Ross theorem is immediate from Theorem 3.4 if one takes $B_i = G_i$ for each $i \in I$.

Corollary 3.5. *Let X and Y be functionally bounded subsets of a topological group G. Then:*

a) *$X \cdot Y$ is functionally bounded in G;*
b) *if $\langle X\rangle$ is dense in G then the cellularity of G is countable and G has the Knaster property.*

Once again, we can try to find a 'topological' reason for the productivity of functional boundedness in topological groups. The explanation of this phenomenon can be given in the following 'advanced' terms:

Every Hausdorff topological group has an ω-directed lattice of open continuous mappings onto Dieudonné-complete spaces.

This is exactly the property which makes functional boundedness stable under the product operation. Let us explain briefly the terms and basic ideas behind the above sentence.

Suppose that G is a topological group. We will call a subgroup N of G *admissible* if there exists a sequence $\{U_n : n \in \omega\}$ of open neighborhoods of the neutral element in G such that

(a) $U_n = U_n^{-1}$ for each $n \in \omega$;
(b) $U_n^3 \subset U_n$ for each $n \in \omega$;
(c) $N = \bigcap_{n\in\omega} U_n$.

Denote by $\mathcal{A}_G$ the family of all admissible subgroups of G. It is easy to verify that the family $\mathcal{A}_G$ is closed under countable intersections, i.e., $\bigcap \gamma \in \mathcal{A}_G$ for every countable family $\gamma \subset \mathcal{A}_G$. For every $N \in \mathcal{A}_G$, let $\pi_N : G \to G/N$ be the quotient mapping of G onto the left coset space G/N. Then the mapping π_N is open.

One of the basic facts we are going to use is that the space G/N is *submetrizable* for each $N \in \mathcal{A}_G$, i.e., G/N admits a coarser metrizable topology:

Lemma 3.6. *If N is an admissible subgroup of a Hausdorff topological group G, then the quotient space G/N is submetrizable.*

Proof. There exists a sequence $\{U_n : n \in \omega\}$ of open neighborhoods of the identity in G such that $U_n^{-1} = U_n$ and $U_{n+1}^3 \subset U_n$ for each $n \in \omega$, and $N = \bigcap_{n\in\omega} U_n$. We define

$$\mathcal{V}_n = \{(x, y) \in G \times G : x^{-1}y \in U_n\},\ n \in \omega.$$

It follows from Theorem 8.1.10 of [8] that there exists a continuous pseudometric d on G such that

$$\mathcal{V}_{n+1} \subset \{(x, y) \in G \times G : d(x, y) < 2^{-n-1}\} \subset \mathcal{V}_n,$$

for each $n \in \omega$. Obviously $d(x, y) = 0$ if $x^{-1}y \in N$. Thus there exists a metric d^* on G/N satisfying $d^*(\pi(x), \pi(y)) = d(x, y)$ for all $x, y \in G$, where $\pi : G \to G/N$ is the naturel quotient mapping. The continuity of the metric d^* on the quotient space G/N follows from the fact that π is an open continuous mapping. $\square$

Therefore, it follows from Lemma 3.6 that G/N is *Dieudonné-complete* for each $N \in \mathcal{A}_G$ or, equivalently, G/N is homeomorphic to a closed subspace of a product of metrizable spaces [8, 8.5.13]. In its turn, this implies that *the closure of every functionally bounded subset of G/N is compact.* In addition, we see that

$$\{\pi_N : N \in \mathcal{A}_G\}$$

is an ω-*directed* family of open mappings onto Dieudonné-complete spaces, as we claimed above.

The next lemma is a crucial step towards the proof of Theorem 3.4. This lemma says us that functional boundedness of subsets of a topological group G is 'reflected' in quotient spaces G/N, with $N \in \mathcal{A}_G$.

Lemma 3.7. *A subset B of a topological group G is functionally bounded in G if and only if $\pi_N(B)$ is functionally bounded in the quotient space G/N, for each admissible subgroup N of G.*

Proof. Suppose that B is not bounded in G. Then there exists a locally finite family $\{V_n : n \in \omega\}$ of open subsets of G each element of which intersects B. For each $n \in \omega$, we pick a point $x_n \in V_n \cap B$ and an open neighborhood W_n of the identity in G such that $x_n W^2 \subset V_n$. Let also H_n be an admissible subgroup of G with $H_n \subset W_n$. Then $H = \bigcap_{n\in\omega} H_n$ is an admissible subgroup of G. For the quotient mapping $\pi_n \colon G \to G/H_n$ we have

$$\pi_n^{-1}\pi_n(x_n W_n) = x_n W_n H_n \subset x_n W_n^2 \subset V_n.$$

Consequently $\pi^{-1}\pi(x_n W_n) \subset V_n$ for each $n \in \omega$, where $\pi \colon G \to G/H$ is the quotient mapping. The mapping π is open, hence the previous inclusion implies that the family $\{\pi(x_n W_n) : n \in \omega\}$ of open sets in G/H is locally finite in G/H. Evidently, each element of this family meets $\pi(B)$, hence $\pi(B)$ is not bounded in G/H. This proves the lemma. □

Let us turn back to the proof of Theorem 3.4. Suppose that B_i is a functionally bounded subset of a topological group G_i, where $i \in I$. Let $B = \prod_{i\in I} B_i$. Suppose for a contradiction that B is not functionally bounded in $G = \prod_{i\in I} G_i$. By Lemma 3.7, there exists an admissible subgroup N of G such that $\pi_N(B)$ fails to be functionally bounded in G/N. Since N is admissible, we can find, for every $i \in I$, an admissible subgroup N_i of the group G_i such that $\prod_{i\in I} N_i \subset N$. [Notice that our choice of the subgroups N_i's can be made in a way that $N_i \neq G_i$ for at most countably many $i \in I$; however, we won't use this observation.]

Put $N^* = \prod_{i\in I} N_i$ and let $\varphi \colon G \to G/N^*$ be the canonical quotient mapping. Since $N^* \subset N$, there exists a continuous mapping $f \colon G/N^* \to G/N$ satisfying the equality $\pi_N = f \circ \varphi$. Hence the set $\varphi(B)$ is not functionally bounded in G/N^*.

For every $i \in I$, let $\pi_i \colon G_i \to G_i/N_i$ be the canonical quotient mapping. Let also $\pi = \prod_{i\in I} \pi_i$ be the Cartesian product of the mappings π_i's. Then π is a continuous open mapping of the group $G = \prod_{i\in I} G_i$ onto $\prod_{i\in I} G_i/N_i$. Since $N^* = \prod_{i\in I} N_i$, consider the canonical mapping i of G/N^* onto $\prod_{i\in I} G_i/N_i$ which satisfies the equality $\pi = i \circ \varphi$.

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & G/N^* \\ & \searrow^{\pi} & \downarrow{\scriptstyle i} \\ & & \prod_{i\in I} G_i/N_i \end{array}$$

Since the mappings π and φ are continuous and open, i is a homeomorphism. Hence $\pi(B) = i(\varphi(B))$ cannot be functionally bounded in $\prod_{i\in I} G_i/N_i$.

It is clear that $\pi_i(B_i)$ is a functionally bounded subset of G_i/N_i, for each $i \in I$. Since the space G_i/N_i is Dieudonné-complete, the closure of $\pi_i(B_i)$ in G_i/N_i, say, K_i is compact. We conclude, therefore, that the set $\pi(B)$ is contained in the compact subset $K = \prod_{i\in I} K_i$ of the space $\prod_{i\in I} G_i/N_i$ and, hence, is functionally bounded in this space. This contradiction completes the proof of Theorem 3.4.

M. Hušek [15] gave an alternative proof of Theorem 3.4 via factorization of continuous functions on weakly Lindelöf subgroups of Cartesian products.

Let us consider another concept of relative boundedness:

Definition 3.8. *A subset Y of a Tychonoff space X is C-compact (equiv., hyperbounded) in X if the image $f(Y)$ is compact, for any continuous real-valued function f on X.*

It is immediate from the above definition that

pseudocompact $\Longrightarrow$ C-compact $\Longrightarrow$ functionally bounded

Examples. a) Evidently $\mathbb{N}$ is functionally bounded in $\alpha\mathbb{N}$, the one-point compactification of $\mathbb{N}$, but $\mathbb{N}$ is not C-compact in $\alpha\mathbb{N}$. Further, let D be an uncountable discrete space. Then D is C-compact in αD, but D is not pseudocompact.

b) C-compactness is strictly stronger than functional boundedness in topological groups. Indeed, let $Y = \{1/n : n \in \mathbb{N}^+\}$ be a subspace of $G = \mathbb{R}$. Then Y is functionally bounded in G, but it obviously fails to be C-compact in G.

Proposition 3.9. (S. Hernández and M. Sanchis, [13]) *A functionally bounded subset Y of a topological group H is C-compact in H if and* *only if Y is G_δ-dense in $cl_{\varrho H}Y$, where ϱH is the Raĭkov completion of H.*

Proof. Let Y be a functionally bounded subset of H. Then the closure of Y in ϱH, say, $\overline{Y}$ is compact. Therefore, if Y is not G_δ-dense in $\overline{Y}$, we can find a continuous, non-negative, real-valued function f on $\overline{Y}$ such that $f(x_0) = 0$ for some $x_0 \in \overline{Y} \setminus Y$ and $f(y) > 0$ for each $y \in Y$. Let g be a continuous extension of f over ϱH and h the restriction of g to H. Then $h(y) = f(y) > 0$ for each $y \in Y$, while zero is an accumulation point for $h(Y) = g(Y)$ in $\mathbb{R}$, so $h(Y)$ is not compact and Y is not C-compact in H.

Conversely, suppose that Y is G_δ-dense in $\overline{Y}$ and take an arbitrary continuous, non-negative, real-valued function f on H. Then the restriction of f to Y is uniformly continuous with respect to the two-sided uniformity of the group H (see [30, Corollary 2.29]). It is clear that the two-sided uniformly of H is the restriction of the two-sided uniformity of ϱH to H. Therefore, $f \restriction Y$ extends to a continuous function g on $\overline{Y}$. Since Y is G_δ-dense in $\overline{Y}$, we conclude that $f(Y) = g(Y) = g(\overline{Y})$ is a compact subset of $\mathbb{R}$. Hence Y is C-compact in H. □

Novák's example shows that C-compactness is not productive. However, the situation is different in the class of topological groups. It turns out that C-compactness behaves in topological groups similarly to functional boundedness:

Theorem 3.10. (Hernández and Sanchis, [13]) *Let Y_i be a C-compact subset of a topological group G_i, where $i \in I$. Then the set $Y = \prod_{i\in I} Y_i$ is C-compact in $G = \prod_{i\in I} G_i$.*

Proof. The set Y is functionally bounded in G by Theorem 3.4. According to [3, Corollary 3.6.23], $\varrho(\prod_{i\in I} G_i) \cong \prod_{i\in I} \varrho G_i$. Therefore, the closure of Y in ϱG is homeomorphic to $\prod_{i\in I} cl_{\varrho G_i} Y_i$. It remains to apply Proposition 3.9. □

Notice that Theorem 3.10 also implies the Comfort–Ross theorem (let $Y_i = G_i$ for each $i \in I$).

Exercise 3.11. Let B be a C-compact subset of a topological group G. Then B is C-compact in the subgroup $\langle B\rangle$ of G generated by B. (*Hint:* see [14, Proposition 3.7].)

4. If one of the factors is not a group

In the case of two factors, both Theorem 3.4 and Theorem 3.10 (on the productivity of functional boundedness and C-compactness, resp.)

admit a more general form. In what follows the second factor will be a space, not necessarily a topological group.

We start with functional boundedness and then describe advances obtained in the case of C-compactness.

Proposition 4.1. (Tkachenko, [32]) *Let A be a functionally bounded subset of a topological group G and B a functionally bounded subset of a space Y. Then $A \times B$ is functionally bounded in $G \times Y$.*

The proof of Proposition 4.1 requires the next definition.

Definition 4.2. *A subset B of a space X is strongly bounded in X if every infinite family of open subsets of X meeting B contains an infinite subfamily $\{U_n : n \in \omega\}$ which has the following property:*

$(*)$ *For each filter $\mathcal{F}$ consisting of infinite subsets of ω,*

$$\bigcap_{F\in\mathcal{F}} cl_X\left(\bigcup_{n\in F} U_n\right) \neq \emptyset.$$

It is clear from Definition 4.2 that every strongly bounded subset of X is functionally bounded in X. The lemma below is the crucial step towards the proof of Proposition 4.1.

Lemma 4.3. *Let X_0 and X_1 be spaces, Φ a filter of infinite subsets of ω, and $x_0 \in X_0$. Suppose also that $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ are families of open sets in X_0 and X_1, respectively, such that $x_0 \in cl_{X_0}(\bigcup_{n\in F} U_n)$ for each $F \in \Phi$ and $\{V_n : n \in \omega\}$ satisfies $(*)$ of Definition 4.2. Then there exists a point $x_1 \in X_1$ such that $(x_0, x_1) \in cl_{X_0\times X_1}(\bigcup_{n\in F} U_n \times V_n)$ for each $F \in \Phi$.*

Proof. Let $\mathcal{B}$ be a neighborhood base for x_0 in X_0. For $U \in \mathcal{B}$ and $F \in \Phi$, let $S(U, F) = \{n \in F : U \cap U_n \neq \emptyset\}$. Then the family

$$\mathcal{S} = \{S(U, F) : U \in \mathcal{B},\ F \in \Phi\}$$

consists of infinite subsets of ω and has the finite intersection property. Denote by $\mathcal{F}$ the filter on ω generated by $\mathcal{S}$. Since the family $\{V_n : n \in \omega\}$ satisfies $(*)$, there exists a point $x_1 \in X_1$ which is in the closure of the set $\bigcup_{n\in P} V_n$, for each $P \in \mathcal{F}$. Then for every $F \in \Phi$, the point (x_0, x_1) is in the closure of the set $\bigcup_{n\in F} U_n \times V_n$. Indeed, take a neighborhood $U \times V$ of (x_0, x_1) in $X_0 \times X_1$, where $U \in \mathcal{B}$. Then $V \cap V_n \neq \emptyset$ for some $n \in S(U, F)$ and hence $(U \times V) \cap (U_n \times V_n) \neq \emptyset$. This proves the lemma. □

Corollary 4.4. *If B_0 is a functionally bounded subset of a space X_0 and B_1 is a strongly bounded subset of a space X_1, then $B_0 \times B_1$ is functionally bounded in $X_0 \times X_1$.*

Proof. Consider an infinite family $\{O_n : n \in \omega\}$ of open sets in $X_0 \times X_1$ such that $O_n \cap (B_0 \times B_1) \neq \emptyset$ for each $n \in \omega$. We can assume without loss of generality that each O_n has the rectangular form $U_n \times V_n$. Hence $U_n \cap B_0 \neq \emptyset$ and $V_n \cap B_1 \neq \emptyset$ for each $n \in \omega$. Since B_1 is strongly bounded in X_1, the family $\{V_n : n \in \omega\}$ contains an infinite subfamily satisfying $(*)$ of Definition 4.2. We can assume that this subfamily is $\{V_n : n \in \omega\}$ itself.

As B_0 is functionally bounded in X_0, the family $\{U_n : n \in \omega\}$ has an accumulation point in X_0, say, x_0. For every neighborhood U of x_0, let $F_U = \{n \in \omega : U \cap U_n \neq \emptyset\}$. Denote by Φ the filter on ω generated by the family $\{F_U : U \text{ is a neighborhood of } x_0 \text{ in } X_0\}$. Clearly Φ consists of infinite sets. According to Lemma 4.3, there exists a point $x_1 \in X_1$ such that (x_0, x_1) is in the closure of $\bigcup_{n\in F} U_n \times V_n$ for each $F \in \Phi$. This means that (x_0, x_1) is an accumulation point of the family $\{U_n \times V_n : n \in \omega\}$. Hence $B_0 \times B_1$ is functionally bounded in $X_0 \times X_1$. □

Exercise 4.5. Let B_i be a strongly bounded subset of a space X_i, where $i \in I$. Prove that $\prod_{i\in I} B_i$ is strongly bounded in $\prod_{i\in I} X_i$. (Hint: modify the proof of [18, Theorem 3.4].)

In topological groups, 'functionally bounded' and 'strongly bounded' coincide:

Lemma 4.6. *Every functionally bounded subset of a topological group is strongly bounded.*

Proof. Let B be a bounded subset of a topological group G. Denote by $\mathcal{A}$ the family of admissible subgroups of G. For every $N \in \mathcal{A}$, let π_N be the canonical quotient mapping of G onto the (left) coset space G/N.

Suppose that $\{U_k : k \in \omega\}$ is a family of open subsets of G each of which meets B. For every $k \in \omega$, pick a point $x_k \in U_k \cap B$. As the family $\mathcal{A}$ generates the topology of G (this was shown implicitly in the proof of Lemma 3.7), we can find, for every $k \in \omega$, an element $N_k \in \mathcal{A}$ and an open subset $V_k \subset G/N_k$ such that $x_k \in \pi_{N_k}^{-1}(V_k) \subset U_k$. We choose $N \in \mathcal{A}$ with $N \subseteq N_k$ for every $k \in \omega$. There exist open subsets W_k of the space $Y = G/N$ such that $x_k \in \pi_N^{-1}(W_k) \subset U_k$ for $k \in \omega$. The set $\pi_N(B)$ is bounded in the Dieudonné-complete space Y, hence $K = cl_Y \pi_N(B)$ is compact. Hence for each filter Φ on ω, the set $\bigcap_{P\in\Phi} cl_Y(\bigcup_{k\in P} W_k)$ is not empty. Since the mapping π_N is open, we see that $\pi_N^{-1}(cl_Y W) = cl_G \pi_N^{-1}(W)$ for each open set $W \subseteq Y$. In particular, the intersection

$$\bigcap_{P\in\Phi} cl_G\left(\bigcup_{k\in P} U_k\right) \supseteq \bigcap_{P\in\Phi} cl_G\left(\bigcup_{k\in P} \pi_N^{-1}(W_k)\right) = \pi_N^{-1}\left(\bigcap_{P\in\Phi} cl_Y\left(\bigcup_{k\in P} W_k)\right)\right)$$

is not empty. This completes the proof of the lemma. □

Corollary 4.7. *If G is a pseudocompact topological group and Y is a pseudocompact space, then $G \times Y$ is pseudocompact.*

We conclude, therefore, that pseudocompact topological groups are "strongly" pseudocompact.

Notice that combining Lemma 4.6 and Exercise 4.5, one obtains an alternative proof of Theorem 3.4.

Our next step is to consider and comment on the following problem:

Problem 3. *Does Proposition 4.1 hold for C-compactness in place of functional boundedness?*

In other words, we wonder whether the product of two C-compact subsets is C-compact in the product of two spaces provided that one of the spaces is a topological group.

In view of Proposition 4.1, the following definition comes naturally.

Definition 4.8. *A subset A of a space X is strongly functionally bounded in X if $A \times B$ is functionally bounded in $X \times Y$ for any pair (Y, B), where B is a functionally bounded subset of the space Y.*

Similarly, one defines strongly C-compact subsets of a space X, etc. Reformulating Proposition 4.1, we can say that functionally bounded subsets of a topological group are strongly functionally bounded. Similarly, Corollary 4.4 says that every strongly bounded subset of a space is strongly functionally bounded.

Therefore, Problem 3 actually asks whether C-compact subsets of topological groups are strongly C-compact.

The following notion helps to solve Problem 3.

Definition 4.9. (Arhangel'skii and Genedi, [2]) *A subset A of a space X is relatively pseudocompact (r-pseudocompact, for short) in X if every infinite family of open sets in X meeting A has a cluster point in A.*

The implications below are almost immediate from Definition 4.9:

$$\text{pseudocompact} \Longrightarrow r\text{-pseudocompact} \Longrightarrow C\text{-compact}$$

The following examples help us to distinguish between several concepts of (relative) boundedness.

Examples.

a) Let $\mathcal{A}$ be a maximal almost disjoint family of infinite subsets of ω and let $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$ be the Mrówka–Isbell space. Then $\Psi(\mathcal{A})$ is pseudocompact. Since $\mathcal{A}$ is a maximal almost disjoint family (and each point of ω is isolated in $\Psi(\mathcal{A})$), the set $\mathcal{A}$ is r-pseudocompact in $\Psi(\mathcal{A})$. However, $\mathcal{A}$ is discrete and uncountable $\Longrightarrow \mathcal{A}$ is not pseudocompact.
b) If D is an uncountable discrete space, then D is C-compact in the one-point compactification αD of D, but D is not r-pseudocompact in αD.
c) r-pseudocompactness is not finitely productive (Novák's example).

Theorem 4.10. (Hernández, Sanchis, Tkachenko, [14]) *Every C-compact subset of a topological group G is r-pseudocompact in G. Therefore, the notions of C-compactness and r-pseudocompactness coincide for subsets of topological groups.*

Proof. We know that every r-pseudocompact subset of a space is C-compact in the space. So, let B be a C-compact subset of the group G and H be the subgroup of G generated by B. By Exercise 3.11, B is C-compact in H. Therefore, it suffices to show that B is r-pseudocompact in H.

Let ϱH be the Raĭkov completion of the group H. Then $K = cl_{\varrho H} B$ is a compact subset of ϱH, so the subgroup $\widetilde{H}$ of ϱH generated by K is σ-compact and contains H as a dense subgroup. Hence $\widetilde{H}$ is an *Oz-space*, i.e., every regular closed subset of $\widetilde{H}$ is a zero-set (see [29, Theorem 1]).

Let $\{U_n : n \in \omega\}$ be a sequence of open sets in H such that each U_n meets B. For every $n \in \omega$, take an open set V_n in $\widetilde{H}$ such that $V_n \cap H = U_n$ and consider the open sets $O_n = \bigcup_{k \geq n} V_n$, where $k \in \omega$. It is clear that $O_n \supseteq O_{n+1}$ and $O_n \cap K \supseteq U_n \cap B \neq \emptyset$ for each $n \in \omega$. We know that each $F_n = cl_{\widetilde{H}} O_n$ is a zero-set in $\widetilde{H}$, and so is the set $F = \bigcap_{n \in \omega} F_n$. Since K is compact, the intersection $F \cap K$ is non-empty. It also follows from the C-compactness of B in H that B is G_δ-dense in $K = cl_{\varrho H} B$ (see Proposition 3.9). Therefore, $F \cap B \neq \emptyset$ and we can take a point $x \in F \cap B$. Since each U_n is dense in V_n, it follows from the definition of the sets F_n's and our choice of the point x that the sequence $\{U_n : n \in \omega\}$ accumulates at x. Hence B is r-pseudocompact in H and in G. □

Therefore, Theorems 4.10 and 3.10 together imply that the Cartesian product of r-pseudocompact subsets of topological groups is r-pseudocompact in the product of groups.

The following result *almost* solves Problem 3:

Theorem 4.11. (Hernández–Sanchis–Tkachenko, [HST]) *An r-pseudocompact subset of a topological group G is strongly r-pseudocompact in G. In other words, if A is r-pseudocompact in a topological group G and B is r-pseudocompact in a space Y, then $A \times B$ is r-pseudocompact in $G \times Y$.*

The proof of Theorem 4.11 is based on the following result characterizing strongly r-pseudocompact sets in a way similar to the one used in Definition 4.2 for strongly bounded subsets. Its proof which goes like in [4, Proposition 1] and hence is omitted.

Theorem 4.12. *Let A be a subset of a space X. The following conditions are equivalent:*

1. *A is strongly r-pseudocompact in X;*
2. *every infinite family of pairwise disjoint open subsets of X meeting A contains an infinite subfamily $\{U_n\}_{n\in\omega}$ such that*

$$A \cap \bigcap_{F\in\mathcal{F}} cl_X \left(\bigcup_{n\in F} U_n \right) \neq \emptyset,$$

   for each filter $\mathcal{F}$ of infinite subsets of ω;
3. *for every pseudocompact space Y, $A \times Y$ is r-pseudocompact in $X \times Y$.*

One can change condition (2) of Theorem 4.12 by applying it to a family of *pairwise distinct* open subsets of X, thus adding another equivalent condition to the above list of (1)–(3).

We also need a simple lemma:

Lemma 4.13. *Let A be a C-compact subset of a space X and $p\colon X \to Y$ be a continuous mapping of X onto a submetrizable space Y. Then $p(A)$ is a compact subset of Y.*

Proof. Since A is functionally bounded in X, the set $B = p(A)$ is functionally bounded in Y. Every submetrizable space is Dieudonné-complete, so the closure of B in Y, say K is a compact subset of Y. It remains to verify that $B = K$, i.e., B is closed in Y.

Suppose for a contradiction that B is not closed in Y and pick a point $y_0 \in K\setminus B$. Since Y is submetrizable, it has countable pseudocharacter. Hence there exists a continuous real-valued function f on Y such that

$f \geq 0$ and $f(y) = 0$ if and only if $y = y_0$. Let $g = f \circ p$. Then g is a continuous real-valued function on X, $g(x) > 0$ for each $y \in A$, but $0 \in \overline{g(A)}$. Hence $g(A)$ is not compact, thus contradicting our assumption about A. Thus $p(A) = K$ is compact. □

In view of Theorem 4.12, Theorem 4.11 will follow if we prove the next lemma (which is close to Lemma 4.6):

Lemma 4.14. *Let A be an r-pseudocompact subset of a topological group G and $\{U_k : k \in \omega\}$ a sequence of pairwise disjoint open subsets of G each of which meets A. Then the set*

$$A \cap \bigcap_{F \in \mathcal{F}} cl_G \left(\bigcup_{k \in F} U_k \right)$$

is not empty, for each filter $\mathcal{F}$ of infinite subsets of ω.

Proof. It is clear that A is C-compact in G. As in the proof of Lemma 4.6, denote by $\mathcal{A}$ the family of admissible subgroups of G. For every $N \in \mathcal{A}$, let π_N be the canonical quotient mapping of G onto the (left) coset space G/N.

For every $k \in \omega$, pick a point $x_k \in U_k \cap A$. As the family $\mathcal{A}$ generates the topology of G, we can find, for every $k \in \omega$, an element $N_k \in \mathcal{A}$ and an open subset $V_k \subset G/N_k$ such that $x_k \in \pi_{N_k}^{-1}(V_k) \subset U_k$. We choose $N \in \mathcal{A}$ with $N \subseteq N_k$ for every $k \in \omega$. There exist open subsets W_k of the space G/N such that $x_k \in \pi_N^{-1}(W_k) \subset U_k$ for $k \in \omega$. It is clear that $\pi_N(x_k) \in W_k \cap \pi_N(A) \neq \emptyset$, for each $k \in \omega$.

The space $Y = G/N$ is submetrizable by Lemma 3.6, so the set $B = \pi_N(A)$ is compact in view of Lemma 4.13. Hence for each filter $\mathcal{F}$ of infinite subsets of ω, the set

$$B \cap \bigcap_{P \in \mathcal{F}} cl_Y (\bigcup_{k \in P} W_k)$$

is not empty. Since the mapping π_N is open, we see that $\pi_N^{-1}(cl_Y W) = cl_G \pi_N^{-1}(W)$ for each open set $W \subseteq Y$. In particular, we have:

$$\bigcap_{P \in \mathcal{F}} cl_G \left(\bigcup_{k \in P} U_k \right) \supseteq \bigcap_{P \in \mathcal{F}} cl_G \left(\bigcup_{k \in P} \pi_N^{-1}(W_k) \right) = \pi_N^{-1} \left(\bigcap_{P \in \mathcal{F}} cl_Y \left(\bigcup_{k \in P} W_k) \right) \right).$$

Therefore,

$$A \cap \bigcap_{P \in \mathcal{F}} cl_G \left(\bigcup_{k \in P} U_k \right) \neq \emptyset.$$

This proves the lemma. □

The above lemma shows that an r-pseudocompact subset of a topological group satisfies condition (2) of Theorem 4.12, so r-pseudocompact subsets of topological groups are strongly r-pseudocompact, i.e., we have proved Theorem 4.11.

Theorem 4.11 does not solve the problem on the productivity of C-compactness since C-compactness of subsets of a space is strictly weaker than r-pseudocompactness. The following result does solve Problem 3 and, hence, complements Theorem 4.11.

Theorem 4.15. (Hernández–Sanchis–Tkachenko, [14]) *C-compact subsets of a topological group G are strongly C-compact in G.*

Again, we can reformulate Theorem 4.15 as follows:

$$A \text{ is } C\text{-compact in a topological group } G \text{ and } B \text{ is } C\text{-compact in } Y \\ \Longrightarrow A \times B \text{ is } C\text{-compact in } G \times Y.$$

BASIC IDEAS AND FACTS FOR THE PROOF OF THEOREM 4.15.

All results that follow were proved in [14]. In what follows υX stands for the Hewitt realcompactification of a given space X (see [8, Section 3.11]).

Proposition 4.16. *Let A be a functionally bounded subset of a topological group G and B a functionally bounded subset of a space Y. Then every bounded continuous real-valued function f on $G \times Y$ admits a continuous extension over $cl_{\upsilon G} A \times cl_{\upsilon Y} B$.*

The crucial step towards the proof of Theorem 4.15 is the following non-trivial fact which is based on Proposition 4.16:

Proposition 4.17. *Let A be a functionally bounded subset of a topological group H and B a functionally bounded subset of a space Y. Then the following relative distribution law is valid:*

$$cl_{\upsilon(H\times Y)}(A \times B) \cong cl_{\upsilon H} A \times cl_{\upsilon Y} B.$$

SKETCH OF THE PROOF OF THEOREM 4.15. Suppose that A is a C-compact subset of a topological group H and B is a C-compact subset of a space Y. Let $f\colon H \times Y \to \mathbb{R}$ be a continuous function. Denote by f^υ a continuous extension of f over $\upsilon(H \times Y)$. By Proposition 4.17, $(A\times B)^* = cl_{\upsilon(H\times Y)}(A\times B)$ is naturally homeomorphic to the product $A^* \times B^* = cl_{\upsilon H} A \times cl_{\upsilon Y} B$. Since A is C-compact in H, it follows that A is G_δ-dense in A^*. Similarly, B is G_δ-dense in B^*. Therefore, $A \times B$ is G_δ-dense in $(A\times B)^* = A^* \times B^*$. Since the latter set is compact and the real line $\mathbb{R}$ is first countable, we conclude that

$$f(A \times B) = f^\upsilon(A \times B) = f^\upsilon((A \times B)^*)$$

is a compact subset of $\mathbb{R}$. Thus, $A \times B$ is C-compact in $H \times Y$. □

It is worth mentioning that Proposition 4.17 is valid for the Dieudonné completion μX instead of the Hewitt completion υX:

Proposition 4.18. *Let A be a functionally bounded subset of a topological group G and B a functionally bounded subset of a Tychonoff space Y. Then*

$$cl_{\mu(G\times Y)}(A \times B) \cong cl_{\mu G}A \times cl_{\mu Y}B.$$

Indeed, the above proposition follows from Proposition 4.17 and the fact given below.

Fact. *Let $\varphi\colon \mu X \to \upsilon X$ be a continuous extension of the identity embedding $X \hookrightarrow \upsilon X$. Then φ is injective. In particular, for every functionally bounded subset K of X, the compact sets $K^\mu = cl_{\mu X}K$ and $K^\upsilon = cl_{\upsilon X}K$ are naturally homeomorphic.*

The relative distributive law also holds for infinite products of functionally bounded subsets of topological groups.

Theorem 4.19. (Hernández–Sanchis–Tkachenko, [HST]) *Suppose that $G = \prod_{i\in I} G_i$ is a Cartesian product of topological groups. If B_α is a functionally bounded subset of G_i and $B_i^* = cl_{\upsilon G_i}B_i$ for each $i \in A$, then*

$$cl_{\upsilon G}(\prod_{i\in I} B_i) \cong \prod_{i\in I} B_i^*.$$

5. Paratopological groups and semitopological groups

A *semitopological* group is a group G with topology such that the left and right translations in G are continuous. In other words, a semitopological group is a group with *separately continuous* multiplication.

A *paratopological* group is a group G with topology such that multiplication in G is *jointly* continuous, i.e., continuous as a mapping of $G \times G$ to G.

Clearly, we have the following implications:

$$\text{'topological'} \Longrightarrow \text{'paratopological'} \Longrightarrow \text{'semitopological'}$$

The standard example of a paratopological group is the Sorgenfrey line $\mathbb{S}$ with its usual topology and the sum operation. Notice that the space $\mathbb{S}$ is first countable, regular, and hereditarily Lindelöf, but it is not metrizable. Hence the Birkhoff–Kakutani metrization theorem is not valid for paratopological groups.

It is an interesting open problem to find out which of the results established for topological groups remain valid for paratopological or even semitopological groups. This is a (part of) very vast area of Topological Algebra which attracts attention of many researches. It would be really great to answer this general question in the framework of our course, i.e., to extend (some of) the aforementioned results to paratopological/semitopological groups or to show that such an extension is impossible.

The work in this direction was started several years ago and nowadays we have some information on the subject. No doubt, one of the most attracting problems is to extend the Comfort–Ross theorem (Theorem 2.7) to paratopological groups. However, such an extension does not present any difficulty after the following fact established by Reznichenko [23]:

Theorem 5.1. *Every pseudocompact paratopological group is a topological group.*

We recall that pseudocompactness is defined for Tychonoff spaces only. However, one can slightly modify the definition of pseudocompactness in order to obtain a wider class of spaces, without any separation restriction. Let us recall that a space X is *feebly compact* if every locally finite family of open sets in X is finite. It is clear that in the class of Tychonoff spaces, feeble compactness and pseudocompactness coincide. Therefore, feeble compactness is a correct extension of the notion of pseudocompactness to the class of all topological spaces.

One of most interesting results on feeble compactness in paratopological groups was recently obtained by Ravsky in [22]. He proved that the Comfort–Ross theorem remains valid in the class of paratopological groups:

Theorem 5.2. *The Cartesian product of an arbitrary family of feebly compact paratopological groups is feebly compact.*

It should be mentioned that feebly compact Hausdorff paratopological groups need not be topological groups [22], so Theorem 5.2 is a 'proper' extension of the Comfort–Ross theorem.

The next natural step would be an extension of Ravsky's theorem to pseudocompact semitopological groups. However, this is impossible in view of the following result proved by Hernández and Tkachenko in [12]:

Theorem 5.3. *There exist two pseudocompact Boolean semitopological groups G and H such that the product $G \times H$ is not pseudocompact.*

As usual, a group is called *Boolean* if every element of the group distinct from the identity has order 2. Notice that in a Boolean group, inversion is the identity mapping. Hence inversion in a Boolean semitopological group is a homeomorphism. Semitopological groups with the latter property are called *quasitopological groups.* Therefore, Theorem 5.3 states that there exist two pseudocompact quasitopological groups whose product fails to be pseudocompact.

Surprisingly, we know almost nothing about functionally bounded subsets of paratopological groups. Here is the main problem:

Problem 4. Suppose that B_i is a functionally bounded subset of a (Tychonoff, regular, or Hausdorff) paratopological group, where $i = 1, 2$. Is the product $B_1 \times B_2$ functionally bounded in $G_1 \times G_2$?

Also, the above problem is open in the case of infinite products of functionally bounded subsets of paratopological groups. We do not know either what happens if one replaces functional boundedness by C-compactness or r-pseudocompactness in Problem 4.

The recent survey articles [24] by M. Sanchis and [33] by the author contain a wealth of information on this and close subjects, as well as many open problems, new and old.

References

[1] A. V. Arhangel'skii, The frequency spectrum of a topological space and the product operation, *Trans. Moscow Math. Soc.* **40** (1981), 163–200. Russian original in: *Trudy Moskov. Mat. Obshch.* **40** (1979), 171–206.

[2] A. V. Arhangel'skii and H. M. M. Genedi, Properties of placement type: relative strong pseudocompactness, *Proc. Steklov Inst. Math.* no. 3 (1993), 25–27.

[3] A. V. Arhangel'skii and M. G. Tkachenko, *Topological Groups and Related Structures*, Atlantis Series in Mathematics, Vol. I, Atlantis Press and World Scientific, Paris–Amsterdam 2008.

[4] J. L. Blasco and M. Sanchis, On the product of two b_f-spaces, *Acta Math. Hung.* **62** no. 1-2 (1993), 111–118.

[5] A. C. Chigogidze, On κ-metrizable spaces, *Russian Math. Surveys* **37** (1982), 209–210. Russian original in: *Uspekhy Mat. Nauk* **37** (1982), 241–242.

[6] W.W. Comfort and K. A. Ross, Pseudocompactness and uniform continuity in topological groups, *Pacific J. Math.* **16** (1966), 483–496.

[7] E. K. Douwen, van, The product of two countably compact topological groups, *Trans. Amer. Math. Soc.* **262** (1980), 417–427.

[8] R. Engelking, *General Topology*, Heldermann Verlag, Berlin 1989.

[9] F. Galvin, Chain condition and products, *Fund. Math.* **108** (1980), 33–48.

[10] I. Glicksberg, Stone–Čech compactifications of products, *Trans. Amer. Math. Soc.* **90** (1959), 369–382.

[11] A. Hajnal and I. Juhász, A separable normal topological group need not be Lindelöf, *General Topol. Appl.* **6** (1976), 199–205.

[12] C. Hernández and M. G. Tkachenko, Three examples of pseudocompact quasitopological groups, *Topol. Appl.* **153** no. 18 (2006), 3615–3620.

[13] S. Hernández and M. Sanchis, G_δ-open functionally bounded subsets in topological groups, *Topol. Appl.* **53** (1993), 289–299.

[14] S. Hernández, M. Sanchis, and M. Tkachenko, Bounded sets in spaces and topological groups, *Topol. Appl.* **101** (2000), 21–43.

[15] M. Hušek, Productivity of properties of topological groups, *Topol. Appl.* **44** (1992), 189–196.

[16] V. Kuz'minov, On a hypothesis of P. S. Alexandrov in the theory of topological groups, *Doklady Akad. Nauk SSSR* **125** (1959), 727–729 (*in Russian*).

[17] V. I. Malykhin, On the tightness and Souslin number in $exp\, X$ and in the product spaces, *Soviet Math. Dokl.* **203** (1972), 1001–1003.

[18] Noble, N., Countably compact and pseudocompact products, *Czech. Math. J.* **19** (1969) no. 2, 390–397.

[19] O. G. Okunev, A remark on the tightness of products, *Comment. Math. Univ. Carolin.* **37** (1996), 397–399.

[20] A. Pełczyński, Linear extensions, linear averagings and their applications, Dissertationes Math. 58 (Warszawa, PWN, 1968).

[21] Pontryagin, L. S., *Continuous groups*, third edition, "Nauka", Moscow 1973.

[22] O. V. Ravsky, Pseudocompact paratopological groups that are topological, arXiv:1003.5343v3 [math.GR], April 7, 2012. http://arxiv.org/abs/1003.5343

[23] E. A. Reznichenko, Extensions of functions defined on products of pseudocompact spaces and continuity of the inverse in pseudocompact groups, *Topol. Appl.* **59** (1994), 233–244.

[24] M. Sanchis, Problems on bounded subsets, Questions and Answers in General Topology **28** (2010), 65–79.

[25] E. V. Ščepin, On topological products, groups and a new class of spaces containing metrizable ones, *Dokl. AN SSSR* **226** (1976), 527–529 (*in Russian*).

[26] E. V. Ščepin, Topology of limit spaces of uncountable inverse spectra, *Russian Math. Surveys* **31** (1976), 175–207. Russian original in: *Uspekhy Mat. Nauk* **31** (1976), 191–226.

[27] P. Simon, A compact Fréchet space whose square is not Fréchet, *Comment. Math. Univ. Carol.* **21** (1980) no. 4, 749–753.

[28] M. G. Tkachenko, The Souslin property in free topological groups over compact spaces, *Mat. Zametki* **34** (1983), no. 4, 601–607 (in Russian). English translat.: *Math. Notes* **34** (1983), 790–793.

[29] M. G. Tkachenko, On boundedness and pseudocompactness in topological groups, *Math. Notes* **41** (1987), 229–231. Russian original in: *Mat. Zametki* **41** (1987), 400–405.

[30] M. G. Tkachenko, Compactness type properties in topological groups, *Czech. Math. J.* **38** (1988), 324–341.

[31] M. G. Tkachenko, Generalization of a theorem of Comfort and Ross, *Ukrainian Math. J.* **41** (1989), 334–338. Russian original in: *Ukrain. Mat. Zh.* **41** (1989), 377–382.

[32] M. G. Tkachenko, Generalization of a theorem of Comfort and Ross. II, *Ukrainian Math. J.* **41** (1990), 802–806. Russian original in: *Ukrain. Mat. Zh.* **41** (1989), 939–944.

[33] M. Tkachenko, Paratopological and semitopological groups vs topological groups, Recent Progress in General Topology III, Elsevier, North Holland, to appear.

[34] F. J. Trigos-Arrieta, Products of locally pseudocompact topological groups, Papers on general topology and applications (Brookville, NY, 1990), 182–187, *Ann. New York Acad. Sci.* **659**, New York Acad. Sci., New York 1992.
[35] V. V. Uspenskij, Topological groups and Dugundji compacta, *Math. USSR Sbornik* **67** (1990), 555–580. Russian original in: *Matem. Sbornik* **180** (1989), 1092–1118.
[36] A. Weil, *Sur les Espaces à Structure Uniforme et sur la Topologie Générale,* Hermann & Cie, Paris. Publ. Math. Univ. Strasbourg 1937.

(M. Tkachenko)
Departamento de Matemáticas
Universidad Autónoma Metropolitana
Av. San Rafael Atlixco 186
Col. Vicentina, C.P. 09340
Del. Iztapalapa, Mexico, D.F.
E-mail address: mich@xanum.uam.mx